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Research article

Physical vs mathematical origin of the extended KdV and mKdV equations

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Abstract: The higher-order Korteweg-de Vries (KdV) and modified KdV (mKdV) equations are derived from a physical model describing a three-component plasma composed of cold fluid ions and two species of Boltzmann electrons at different temperatures. While the higher-order KdV equation is well established, the corresponding mKdV equation is typically derived using the system's integrability properties. In this work, we present the extended mKdV equation, derived directly from the physical system, offering a fundamentally different form from its integrable counterpart. We explore the connections between the two equations via Miura transformations and analyze their solutions within the framework of asymptotic integrability.

Keywords: extended Korteweg-de Vries equation; extended modified KdV equation; Miura transformations of higher-order equations; asymptotic integrability; plasma dynamics
Mathematics Subject Classification: 35B20, 35B40, 35C08, 35Q51, 35Q53, 35Q55, 76X05

1. Introduction

The study of nonlinear wave phenomena has been a cornerstone of mathematical physics. An important model governing weakly nonlinear and weakly dispersive long waves, described by the field $u(x,t)$, is the Korteweg-de Vries (KdV) equation,

$$u_t + q_1 u u_x + u_{xxx} = 0, \tag{1.1}$$

where q_1 is a real constant (this constant can be trivially scaled out of the equation but is kept here for reasons that will become apparent later), and subscripts denote partial derivatives. This model, originally derived in the context of shallow water waves, has been a benchmark for understanding nonlinear wave phenomena, finding extensive applications in various physical contexts, such as shallow water wave dynamics [1, 2], plasma physics [3], nonlinear lattices [4], and so on.

An important variant of the KdV model is the modified KdV (mKdV) equation, where the quadratic nonlinear term uu_x is replaced by the cubic nonlinear term u^2u_x, namely:

$$u_t + r_1 u^2 u_x + u_{xxx} = 0, \tag{1.2}$$

where r_1 is, again, a real constant. Much like the KdV equation, the mKdV equation finds many applications in various fields, ranging from fluid mechanics [5, 6], plasmas [7, 8], and nonlinear optics with few-optical-cycle pulses [9, 10] to traffic flow [11].

Both the KdV and mKdV equations are completely integrable via the Inverse Scattering Transform (IST) [12]. Moreover, the equations are linked through the Miura transformation [5, 13], which allows solutions of one equation to be mapped onto the other. This transformation not only highlights the deep connection between these systems but may also provide a pathway for deriving extended versions of the equations—incorporating higher-order dispersive and nonlinear terms—while preserving their integrable structure [14, 15]. The relevant extended versions of the KdV and mKdV models are first, the extended KdV (eKdV) equation [16], which reads:

$$u_t + q_1 uu_x + u_{xxx} + \varepsilon\left(q_2 u^2 u_x + q_3 u_x u_{xx} + q_4 uu_{xxx} + q_5 u_{xxxxx}\right) = 0, \tag{1.3}$$

where q_j $(j = 2, 3, 4, 5)$ are constants and $\varepsilon \ll 1$. Second, the extended mKdV (emKdV) equation is of the form [17, 18]:

$$u_t + r_1 u^2 u_x + u_{xxx} + \varepsilon\left(r_2 u_x^3 + r_3 u^4 u_x + r_4 uu_x u_{xx} + r_5 u^2 u_{xxx} + r_6 u_{xxxxx}\right) = 0, \tag{1.4}$$

where r_j $(j = 2, 3, \ldots, 6)$ are constants.

The need for higher-order equations arises from the limitations of simpler systems in accurately capturing the complexities of nonlinear wave phenomena. In physical problems, by systematically retaining higher-order terms in the perturbation expansion, extended equations provide corrections to wave speeds, amplitudes, and stability properties that align more closely with experimental and observational data. In this way, the eKdV equation has been derived from a variety of physical systems, including shallow water waves [16, 19], solid mechanics [20], nonlinear optics of nematic liquid crystals [21], and plasma dynamics [22, 23]. Importantly, the eKdV retains many properties of its integrable counterpart, such as soliton solutions, modulation theory solutions for dispersive shock waves (also known as "undular bores" in fluid mechanics [24]), and conservation laws. These properties make the eKdV equation a powerful tool for understanding the limitations of simpler models. It is thus not surprising that it has been used to describe complex solitary wave interactions [25, 26], distinct regimes of shallow water dispersive shock propagation in the presence of surface tension effects [27, 28], the transition from nonlocal to local effects in the evolution of defocusing nematic resonant dispersive shocks [21], and resonant soliton radiation in shallow water and optical media [29]. Recent studies have also explored its asymptotic integrability [30–32] and connections to higher-order hierarchies, further emphasizing its significance in both theoretical and applied contexts.

Our scope in this work is to delve into the physical and mathematical origin of the eKdV and emKdV equations. To be more specific, we aim to investigate whether the emKdV equation (1.4), which is directly connected with the eKdV equation (1.3) via a Miura map as mentioned above, results—as a higher-order correction of the regular mKdV equation (1.2)—in a physical problem. To address this question, we consider a model of a plasma of cold positive ions in the presence of a two-temperature electron population. Employing the reductive perturbation method [33, 34], we show that while the eKdV equation (1.3) can indeed result as a higher-order correction of the KdV equation (1.1), this is not the case with the higher-order mKdV equation: indeed, after deriving the regular mKdV equation (1.2), at the next order of approximation, we obtain a higher-order mKdV equation that is not of the form of Eq (1.4). Hence, unlike previous studies where the mKdV equation and its higher-order extensions were primarily derived as mathematical constructs, this study grounds a novel, physically relevant form of an emKdV equation. Thus, our study bridges the gap between physical and mathematical derivations of higher-order evolution equations and reveals nonlinear interactions that are absent in standard formulations.

The extended KdV and mKdV equations derived in this work not only enhance our understanding of plasma dynamics but also provide a platform for exploring the asymptotic integrability of higher-order systems. Connections between the extended equations via Miura transformations further highlight the intricate relationship between these nonlinear systems and their integrable counterparts. We thus provide a comprehensive framework for analyzing these equations, with implications for both theoretical and applied research.

The paper is organized as follows: In Section 2, we present the model and derive the KdV and eKdV equations. Section 3 is devoted to the derivation of the mKdV and emKdV equations. In Section 4, we present Miura map connections between the regular and extended KdV and mKdV models, while in Section 5, we establish an asymptotic integrability argument for the emKdV model derived in this work. Finally, in Section 6 we summarize our conclusions and discuss perspectives for future work.

2. Model, KdV and eKdV equations

Our analysis is based on a set of equations describing a three-component plasma comprising cold fluid ions and two Boltzmann electron species at different temperatures [7]. The system consists of the continuity and momentum equations for the cold ions and Poisson's equation coupling the electrostatic potential to the plasma densities. In normalized/dimensionless form, the model under consideration expressed in (1 + 1)-dimensions reads:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0, \tag{2.1a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial x} = 0, \tag{2.1b}$$

$$\frac{\partial^2 \varphi}{\partial x^2} + n - f\exp(\alpha_c\varphi) - (1-f)\exp(\alpha_h\varphi) = 0. \tag{2.1c}$$

Here, n and u refer to the ion density and fluid velocity, respectively; ϕ is the electrostatic potential; and f is the fractional charge density of the cool electrons. The temperatures T_c and T_h of the Boltzmann electrons are expressed through $\alpha_c = T_{\text{eff}}/T_c$ and $\alpha_h = T_{\text{eff}}/T_h$ for the cool and hot species, respectively, whereas the effective temperature is given by $T_{\text{eff}} = T_cT_h/[fT_h+(1-f)T_c]$, such that $f\alpha_c+(1-f)\alpha_h = 1$.

Our goal is to reduce Eq (2.1) to a single nonlinear evolution equation whose solutions are known and thus can asymptotically represent the solutions of the original system. To do so, we identify the equilibrium solution $n = n_0$ (where n_0 is the equilibrium density), $u = 0$, and $\phi = 0$, and consider the following asymptotic expansions of the unknown fields:

$$n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \varepsilon^3 n_3 + \dots, \tag{2.2a}$$
$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots, \tag{2.2b}$$
$$\varphi = \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \varepsilon^3 \varphi_3 + \dots, \tag{2.2c}$$

where $0 < \varepsilon \ll 1$ is a formal small parameter. Of particular interest is the linearized system occurring at $O(\varepsilon)$:

$$\frac{\partial \eta_1}{\partial t} + \eta_0 \frac{\partial u_1}{\partial x} = 0, \tag{2.3a}$$
$$\frac{\partial u_1}{\partial t} + \frac{\partial \phi_1}{\partial x} = 0, \tag{2.3b}$$
$$\frac{\partial^2 \phi_1}{\partial x^2} - [f(\alpha_c - \alpha_h) + \alpha_h]\,\phi_1 + \eta_1 = 0. \tag{2.3c}$$

The dispersion relation of the above system can be obtained upon considering plane wave solutions of the form $\exp[i(kx - \omega t)]$, where k and ω denote the wavenumber and frequency, respectively. Substituting, we obtain

$$\omega^2 = \frac{k^2}{k^2 + [f(\alpha_c - \alpha_h) + \alpha_h]}, \tag{2.4}$$

where $\eta_0 = 1$, so that $O(1)$ is also satisfied. Focusing our analysis on long waves (i.e., waves with small wave number k), we substitute [3, 35] $k = \varepsilon^p k$ (with $p > 0$), where the exponent p is unknown (to be determined). Hence, the phase $\theta = kx - \omega t$ of the plane waves is written as

$$\theta = kx - \omega t = kx - (a_1 k + a_2 k^3)t = \varepsilon^p k(x - a_1 t) - \varepsilon^{3p} k^3 a_2 t, \tag{2.5}$$

where $a_1 = [f(\alpha_c - \alpha_h) + \alpha_h]^{-1/2}$ and $a_2 = 0.5[f(\alpha_c - \alpha_h) + \alpha_h]^{-3/2}$. As such, a natural rescaling based on Eq (2.5) is

$$\xi = \varepsilon^p (x - ct), \quad \tau = \varepsilon^{3p} \beta t, \tag{2.6}$$

where c is the velocity and β is an auxiliary parameter. These new variables ξ and τ are "slow", in the sense that it needs a large change in x and t in order to change ξ and τ appreciably. The value of p can be determined upon requiring the leading-order dispersion and nonlinearity terms of the system (2.1a)–(2.1c)—for the considered form of the asymptotic expansions in Eqs (2.2a)–(2.2c)—to be of the same order; this way, the perturbation scheme leads to a reduced model (which turns out to be the KdV equation in this case) that can support soliton solutions. This "maximal balance" condition [1] leads to $p = 1$, and hence, the slow variables become

$$\xi = \varepsilon^{1/2}(x - ct), \quad \tau = \varepsilon^{3/2} \beta t. \tag{2.7}$$

Note that the above slow variables are consistent with the similarity of the asymptotic behavior of the KdV equation, which holds for a coordinate system satisfying $\zeta \propto (x - ct)/t^{1/3} = \text{const.}$ [1]; hence, the asymptotic behavior along the direction defined by (2.7) is expected to be the same as that of the KdV equation.

To proceed, we assume that the perturbations around the equilibrium solution, n_j, u_j and ϕ_j (with $j = 1, 2, \ldots$) in (2.2a)–(2.2c) depend on the slow variables (2.7), with the velocity c to be determined in a self-consistent manner (also, the value of β will be chosen below). Substitute back to Eq (2.1) and collect the different orders of the parameter ε. At the lowest order, $O(1)$, we obtain the value of the equilibrium density, $n_0 = 1$, while at $O(\varepsilon)$ we obtain

$$n_{1\xi} - \left[f\left(\alpha_c - \alpha_h\right) + \alpha_h\right]\varphi_{1\xi} = 0, \tag{2.8}$$

and at $O(\varepsilon^{3/2})$ we obtain

$$n_0 u_{1\xi} - c n_{1\xi} = 0, \tag{2.9a}$$
$$-c u_{1\xi} + \varphi_{1\xi} = 0. \tag{2.9b}$$

The compatibility of Eqs (2.8) and (2.9) yields the velocity c:

$$c^2 = \frac{n_0}{f\left(\alpha_c - \alpha_h\right) + \alpha_h}. \tag{2.10}$$

Nonlinear equations arise at the next orders, $O(\varepsilon^2)$ and $O(\varepsilon^{5/2})$. Differentiating with respect to ξ the equations at order $O(\varepsilon^2)$, we find

$$n_{2\xi} - \left[\alpha_h^2 + f\left(\alpha_c^2 - \alpha_h^2\right)\right]\varphi_1\varphi_{1\xi} - \left[f\left(\alpha_c - \alpha_h\right) + \alpha_h\right]\varphi_{2\xi} = 0, \tag{2.11}$$

so that the fields n_2, u_2, and φ_2, are eliminated from the system at $O(\varepsilon^{5/2})$:

$$\frac{\beta n_0}{c^2}\varphi_{1\tau} + n_0 u_{2\xi} - c n_{2\xi} + \frac{2n_0}{c^3}\varphi_1\varphi_{1\xi} = 0, \tag{2.12a}$$
$$\frac{\beta}{c}\varphi_{1\tau} - c u_{2\xi} + \frac{\varphi_1\varphi_{1\xi}}{c^2} + \varphi_{2\xi} = 0. \tag{2.12b}$$

Then, using (2.11) and eliminating the fields n_2, u_2, and φ_2, we obtain the regular KdV equation:

$$\varphi_{1\tau} + c_1\varphi_1\varphi_{1\xi} + \varphi_{1\xi\xi\xi} = 0, \tag{2.13}$$

where we have chosen $\beta = c^3/(2n_0)$, and the nonlinearity coefficient c_1 is given by

$$c_1 = \frac{1}{n_0}\left[3(\alpha_c - \alpha_h)^2 f^2 - (\alpha_c - \alpha_h)\left(\alpha_h\left(n_0 - 6\right) + \alpha_c n_0\right) f - \left(n_0 - 3\right)\alpha_h^2\right]. \tag{2.14}$$

To extend the analysis to higher-order, i.e., derive an extended KdV equation, we take into account the higher-order of approximation, at $O(\varepsilon^{7/2})$, and obtain

$$\beta n_{2\tau} + \frac{\partial}{\partial\xi}\left(u_1 n_2 + u_2 n_1 + n_0 u_3 - c n_3\right) = 0, \tag{2.15a}$$
$$\beta u_{2\tau} + \frac{\partial}{\partial\xi}\left(u_2 u_1 - c u_3 + \varphi_3\right) = 0. \tag{2.15b}$$

Proceeding as above, we differentiate the equations at $O(\varepsilon^3)$ with respect to ξ and obtain

$$\frac{c}{2}u_{2\tau} + \frac{c^2}{2n_0}n_{2\tau} + \frac{2}{c^2}\varphi_1\varphi_{1\tau} + \frac{2n_0}{c^3}u_2\varphi_{1\xi} + \frac{1}{c^2}n_2\varphi_{1\xi}$$

$$-\frac{\left(\alpha_h^3+\left(\alpha_c^3-\alpha_h^3\right)f\right)c^6-10n_0}{2c^6}\varphi_1^2\varphi_{1\xi}-\left[\alpha_h^2+\left(\alpha_c^2-\alpha_h^2\right)f\right]\varphi_2\varphi_{1\xi}$$
$$-\frac{\left[\alpha_h^2+\left(\alpha_c^2-\alpha_h^2\right)f\right]c^4-3n_0}{c^4}\varphi_1\varphi_{2\xi}-\frac{\left[\alpha_h+\left(\alpha_c-\alpha_h\right)f\right]c^2-n_0}{c^2}\varphi_{3\xi}+\varphi_{2\xi\xi\xi}=0. \tag{2.16}$$

Next, define $\Phi=\varphi_1+\varepsilon\varphi_2$ and eliminate the fields n_3, u_3, and φ_3 from the higher-order equations to conclude with the eKdV equation:

$$\Phi_t+c_1\Phi\Phi_\xi+\Phi_{\xi\xi\xi}+\varepsilon\left(c_2\Phi^2\Phi_\xi+c_3\Phi_\xi\Phi_{\xi\xi}+c_4\Phi\Phi_{\xi\xi\xi}+c_5\Phi_{\xi\xi\xi\xi\xi}\right)=0, \tag{2.17}$$

where

$$c_2=\frac{3\left[\alpha_h^2+\left(\alpha_c^2-\alpha_h^2\right)f\right]^2c^8-2n_0\left[\alpha_h^3+\left(\alpha_c^3-\alpha_h^3\right)f\right]c^6+2n_0\left[\alpha_h^2+\left(\alpha_c^2-\alpha_h^2\right)f\right]c^4-3n_0^2}{4n_0c^6}, \tag{2.18a}$$
$$c_3=\frac{-9\left[\alpha_h^2+\left(\alpha_c^2-\alpha_h^2\right)f\right]c^4+19n_0}{4n_0c^2}, \tag{2.18b}$$
$$c_4=\frac{-3\left[\alpha_h^2+\left(\alpha_c^2-\alpha_h^2\right)f\right]c^4+n_0}{2n_0c^2}, \tag{2.18c}$$
$$c_5=\frac{3c^2}{4n_0}. \tag{2.18d}$$

The solutions of the regular KdV equation, Eq (2.13), are well known and will be used to construct the solutions of the extended KdV equation, Eq (2.17), above. This will be done in a section below, as we now turn our attention to the derivation of the mKdV and extended mKdV equations.

3. mKdV and emKdV equations

As above, the starting system is Eq (2.1). We use the asymptotic expansions:

$$n=n_0+\varepsilon n_1+\varepsilon^2n_2+\varepsilon^3n_3+\dots, \tag{3.1a}$$
$$u=\varepsilon u_1+\varepsilon^2u_2+\varepsilon^3u_3+\dots, \tag{3.1b}$$
$$\varphi=\varepsilon\varphi_1+\varepsilon^2\varphi_2+\varepsilon^3\varphi_3+\dots, \tag{3.1c}$$

where the perturbations of the equilibrium solution now depend on the new slow variables:

$$\xi=\varepsilon(x-ct),\quad \tau=\varepsilon^3\beta t. \tag{3.2}$$

Substitute back to the system, and, similarly to the previous section, we obtain at different orders the following results. At order $O(1)$, we find $n_0=1$. At $O(\varepsilon)$, we obtain

$$n_{1\xi}-\left[f\left(\alpha_c-\alpha_h\right)+\alpha_h\right]\varphi_{1\xi}=0, \tag{3.3}$$

and at $O(\varepsilon^2)$ we obtain

$$u_{1\xi}-cn_{1\xi}=0, \tag{3.4a}$$

$$-cu_{1\xi} + \varphi_{1\xi} = 0. \tag{3.4b}$$

The compatibility of the above equations leads again to the velocity c:

$$c^2 = \frac{1}{(\alpha_c - \alpha_h)f + \alpha_h}, \tag{3.5}$$

as before. Continuing along the same lines, we get at $O(\varepsilon^2)$, after differentiating with respect to ξ,

$$n_{2\xi} - \left[f(\alpha_c^2 - \alpha_h^2) + \alpha_h^2\right]\varphi_1\varphi_{1\xi} - \left[f(\alpha_c - \alpha_h) + \alpha_h\right]\varphi_{2\xi} = 0, \tag{3.6}$$

and at $O(\varepsilon^3)$:

$$u_{2\xi} - cn_{2\xi} + \frac{\partial}{\partial\xi}(n_1u_1) = 0, \tag{3.7a}$$

$$\varphi_{2\xi} - cu_{2\xi} + u_1u_{1\xi} = 0. \tag{3.7b}$$

These lead to the following compatibility condition:

$$3(\alpha_c - \alpha_h)^2 f^2 - (\alpha_c - \alpha_h)(a_c - 5\alpha_h) f + 2\alpha_h^2 = 0. \tag{3.8}$$

Contrary to the derivation of the KdV equation, here, and for the sake of consistency, the constants of the original system have to satisfy the above relation, Eq (3.8). As such, obtaining an mKdV equation is more challenging and restrictive than the KdV equation. Note that throughout our analysis, this is found to be true for any of the mKdV properties (solutions and Miura transformations).

Continuing the analysis, at $O(\varepsilon^3)$ we obtain—after differentiating with respect to ξ—the following equation:

$$n_{3\xi} - \frac{1}{2}\left[f(\alpha_c^3 - \alpha_h^3) + \alpha_h^3\right]\varphi_1^2\varphi_{1\xi} - \left[f(\alpha_c^2 - \alpha_h^2) + \alpha_h^2\right](\varphi_{1\xi}\varphi_2 + \varphi_1\varphi_{2\xi}) - \left[f(\alpha_c - \alpha_h) + \alpha_h\right]\varphi_{3\xi} + \varphi_{3\xi\xi\xi} = 0, \tag{3.9}$$

and at $O(\varepsilon^4)$ we obtain the system:

$$\beta n_{1\tau} + \frac{\partial}{\partial\xi}(n_2u_1 + n_1u_2 + u_3 - cn_3) = 0, \tag{3.10a}$$

$$\beta u_{1\tau} + \frac{\partial}{\partial\xi}(u_1u_2 - cu_3 + \varphi_3) = 0. \tag{3.10b}$$

Eliminating the fields n_3, u_3, φ_3, and using the condition (3.8), we obtain the mKdV equation:

$$\varphi_{1\tau} - c_1\varphi_1^2\varphi_{1\xi} + \varphi_{1\xi\xi\xi} = 0, \tag{3.11}$$

where $\beta = c^3/2$ and the coefficient c_1 is given by

$$c_1 = \frac{\left[\left(\alpha_c^3 - \alpha_h^3\right)f + \alpha_h^3\right]c^6 - 15}{2c^6}. \tag{3.12}$$

At the next order of approximation, $O(\varepsilon^5)$, we obtain the following equations:

$$\beta n_{2\tau} + \frac{\partial}{\partial\xi}(n_3u_1 + n_2u_2 + n_1u_3 + u_4 - cn_4) = 0, \tag{3.13a}$$

$$\beta u_{2\tau} + \frac{\partial}{\partial \xi}\left(u_1 u_3 + \frac{1}{2}u_2 - c u_4 + \varphi_4\right) = 0, \tag{3.13b}$$

as well as

$$n_4 - \frac{\left(\alpha_c^4 - \alpha_h^4\right) f + \alpha_h^4}{24}\varphi_1^4 - \frac{\left(\alpha_c^2 - \alpha_h^2\right) f + \alpha_h^2}{2}\varphi_2^2 - \left[\left(\alpha_c^2 - \alpha_h^2\right) f + \alpha_h^2\right]\varphi_3\varphi_1 - \frac{\left(\alpha_c^3 - \alpha_h^3\right) f + \alpha_h^3}{2}\varphi_2\varphi_1^2 - \left[\left(\alpha_c - \alpha_h\right) f + \alpha_h\right]\varphi_4 + \varphi_{2\xi\xi\xi} = 0. \tag{3.14}$$

Proceeding as in the case of the eKdV equation, we differentiate the above equation with respect to ξ, define $\Phi = \phi_1 + \varepsilon\phi_2$, and eliminate n_3, u_3, and ϕ_3. In this way, we end up with the following extended mKdV equation:

$$\Phi_{1\tau} + c_1\Phi^2\Phi_\xi + \Phi_{\xi\xi\xi} + \varepsilon\left(c_2\Phi^3\Phi_\xi + c_3\Phi_\xi\Phi_{\xi\xi} + c_4\Phi\Phi_{\xi\xi\xi}\right) = 0, \tag{3.15}$$

where the coefficients c_2, c_3, and c_4 are given by

$$c_2 = -\frac{1}{6c^8}\left\{\left[\left(c^2\alpha_c - 14\right)\alpha_c^3 - \left(c^2\alpha_h - 14\right)\alpha_h^3\right]c^6 f + \left(c^2\alpha_h - 14\right)c^6\alpha_h^3 + 105\right\}, \tag{3.16a}$$

$$c_3 = -\frac{2}{c^2}, \tag{3.16b}$$

$$c_4 = -\frac{4}{c^2}. \tag{3.16c}$$

This is a rather unexpected result. Indeed, as can be readily seen, the form of the emKdV equation (3.15) differs significantly from Eq (1.4), which can be derived from the eKdV via a Miura map, as will be shown below. In particular, there are no higher-order terms that yield resonant nonlinear dispersive wave solutions, such as resonant solitary and dispersive shock waves. This is due to the pure convexity or concavity of the associated linear dispersion relation [36, 37], in contrast to the one governed by Eq (1.4).

4. The Miura map

Consider the regular form of the KdV equation:

$$\Phi_\tau + c_1\Phi\Phi_\xi + \Phi_{\xi\xi\xi} = 0, \tag{4.1}$$

which can be shown to be converted to the relative mKdV equation:

$$\tilde{\Phi}_\tau + \tilde{c}_1\tilde{\Phi}^2\tilde{\Phi}_\xi + \tilde{\Phi}_{\xi\xi\xi} = 0, \tag{4.2}$$

under the Miura map:

$$\Phi_\tau + c_1\Phi\Phi_\xi + \Phi_{\xi\xi\xi} = \left(2A\tilde{\Phi} + B\frac{\partial}{\partial\xi}\right)\left(\tilde{\Phi}_\tau + \tilde{c}_1\tilde{\Phi}^2\tilde{\Phi}_\xi + \tilde{\Phi}_{\xi\xi\xi}\right) = 0, \tag{4.3}$$

where $\Phi = A\tilde{\Phi}^2 + B\tilde{\Phi}_\xi$, $A = \tilde{c}_1/c_1$ and $B^2 = -6\tilde{c}_1/c_1^2$.

However, when the extended systems are considered, while such a map may still exist, certain restrictions have to apply. In particular, the extended KdV (denoted below as eKdV[Φ]), Eq (2.17), can be mapped to an extended mKdV (denoted below as emKdV[$\tilde{\Phi}$]) [15],

$$\text{emKdV}[\tilde{\Phi}] = \tilde{\Phi}_\tau + \tilde{c}_1\tilde{\Phi}^2\tilde{\Phi}_\xi + \tilde{\Phi}_{\xi\xi\xi} + \varepsilon\left(\tilde{c}_2\tilde{\Phi}_\xi^3 + \tilde{c}_3\tilde{\Phi}^4\tilde{\Phi}_\xi + \tilde{c}_4\tilde{\Phi}\tilde{\Phi}_\xi\tilde{\Phi}_{\xi\xi} + \tilde{c}_5\tilde{\Phi}^2\tilde{\Phi}_{\xi\xi\xi} + \tilde{c}_6\tilde{\Phi}_{\xi\xi\xi\xi\xi}\right) = 0, \quad (4.4)$$

which is very different from Eq (3.15), as

$$\text{eKdV}[\Phi] = \text{eKdV}[A\tilde{\Phi}^2 + B\tilde{\Phi}_x] = (2A\tilde{\Phi} + B\partial_x)(\text{emKdV})[\tilde{\Phi}], \quad (4.5)$$

where $A = \tilde{c}_1/c_1$, $B^2 = -6\tilde{c}_1/c_1^2$, as before. However, certain conditions now apply not only for the constants of the emKdV equation but also for the original eKdV equation; these are

$$\tilde{c}_2 = \frac{2\tilde{c}_1 c_2}{c_1^2}, \quad \tilde{c}_3 = \frac{\tilde{c}_1^2 c_2}{c_1^2}, \quad \tilde{c}_4 = \frac{8\tilde{c}_1 c_2}{c_1^2}, \quad \tilde{c}_5 = \frac{2\tilde{c}_1 c_2}{c_1^2}, \quad \tilde{c}_6 = \frac{6c_2}{5c_1^2}, \quad (4.6)$$

and for the eKdV equation:

$$c_3 = \frac{4c_2}{c_1}, \quad c_4 = \frac{2c_2}{c_1}, \quad c_5 = \frac{6c_2}{5c_1^2}. \quad (4.7)$$

Note that these are consistent with the findings of Ref. [15].

5. Asymptotic integrability and solitons

In this section, we will discuss the possibility of connecting the above derived eKdV (2.17) and emKdV (3.15) equations with their regular counterparts, the KdV and mKdV equations, respectively, employing the concept of asymptotic integrability. The latter refers to the transformation of a complicated, higher-order evolution equation (which may not be exactly solvable or integrable in the strict mathematical sense) to a simpler, integrable system [30–32]. Importantly, the connection of higher-order nonlinear evolution equations with their lower-order integrable counterparts with the asymptotic integrability argument allows for the derivation of solutions of such higher-order equations—see also Refs. [38] and [18] for asymptotic soliton solutions of the eKdV and emKdV equations, respectively.

For the eKdV equation, which we write here with general coefficients,

$$\Phi_t + c_1\Phi\Phi_\xi + \Phi_{\xi\xi\xi} + \varepsilon\left(c_2\Phi^2\Phi_\xi + c_3\Phi_\xi\Phi_{\xi\xi} + c_4\Phi\Phi_{\xi\xi\xi} + c_5\Phi_{\xi\xi\xi\xi\xi}\right) = 0, \quad (5.1)$$

we introduce the transformation:

$$\Phi = \Psi + \varepsilon\left(\lambda_1\Psi^2 + \lambda_2\Psi_{\xi\xi} + \lambda_3\Psi_\xi\int\Psi d\xi + \lambda_4\xi\Psi_{\xi\xi\xi} + \lambda_5\xi\Psi\Psi_\xi\right), \quad (5.2)$$

where

$$\lambda_1 = \frac{-18c_2 + 3c_1c_4 + 2c_1^2c_5}{18c_1}, \ \lambda_2 = \frac{-6c_2 + c_1c_3 - c_1^2c_5}{2c_1^2}, \lambda_3 = \frac{-3c_4 + 4c_1c_5}{9}, \ \lambda_4 = -\frac{c_5}{3}, \ \lambda_5 = -\frac{c_1c_5}{3}, \quad (5.3)$$

so that

$$\Psi_t + c_1\Psi\Psi_\xi + \Psi_{\xi\xi\xi} = 0. \quad (5.4)$$

The above KdV system is IST integrable, and its solutions may now be used to approximate the solutions of the eKdV equation. For example, consider the single soliton solution of Eq (5.4):

$$\Psi(\xi,\tau) = \frac{12\eta^2}{c_1}\,\mathrm{sech}^2[\eta(\xi - 4\eta^2\tau) + \xi_0], \tag{5.5}$$

where η and ξ_0 are $O(1)$ real parameters. Then the $O(\varepsilon)$ correction based on Eq (5.2) reads:

$$\begin{aligned}\Phi(\xi,\tau) = {} & \frac{12\eta^2\left[c_1 + 4\eta^2\left(\lambda_2 c_1 - 6\lambda 3\right)\varepsilon\right]}{c_1^2}\,\mathrm{sech}^2[\eta(\xi - 4\eta^2\tau) + \xi_0] \\ & - \frac{96\eta^5\lambda_4\varepsilon}{c1}\xi\,\mathrm{sech}^2[\eta(\xi - 4\eta^2\tau) + \xi_0]\tanh[\eta(\xi - 4\eta^2\tau) + \xi_0] \\ & + \frac{72\eta^4\left(2\lambda_1 - c_1\lambda_2 + 4\lambda_3\right)\varepsilon}{c_1^2}\,\mathrm{sech}^4[\eta(\xi - 4\eta^2\tau) + \xi_0] \\ & + \frac{288\eta^5\left(c_1\lambda_4 - \lambda_5\right)\varepsilon}{c_1^2}\xi\,\mathrm{sech}^4[\eta(\xi - 4\eta^2\tau) + \xi_0]\tanh[\eta(\xi - 4\eta^2\tau) + \xi_0]. \end{aligned} \tag{5.6}$$

Here it should be noted that, in principle, a similar procedure could be used to identify other decaying approximate solutions of the eKdV equation from relevant solutions of the KdV equation. Such solutions include the rational solutions of the KdV (see, e.g., Refs. [39–42]) and are particularly relevant because they are connected with rogue waves [43]—especially in the context of the complex KdV equation [44, 45].

In a similar manner, consider the extended mKdV equation

$$\Phi_{1\tau} + c_1\Phi_1^2\Phi_{1\xi} + \Phi_{1\xi\xi\xi} + \varepsilon\left(c_2\Phi^3\Phi_\xi + c_3\Phi_\xi\Phi_{\xi\xi} + c_4\Phi\Phi_{\xi\xi\xi}\right) = 0, \tag{5.7}$$

and introduce the transformation

$$\Phi = \Psi + \varepsilon\left(\lambda_1\Psi^2 + \lambda_2\Psi_\xi\int\Psi d\xi\right), \tag{5.8}$$

where now

$$\lambda_1 = \frac{-c_3 + c_4}{6}, \quad \lambda_2 = -\frac{c_4}{3}, \quad 3c_2 - c_1c_3 + c_4/3 = 0. \tag{5.9}$$

This results in the integrable mKdV equation:

$$\Psi_t + c_1\Psi^2\Psi_\xi + \Psi_{\xi\xi\xi} = 0. \tag{5.10}$$

Notably, an additional restriction between the equation's coefficients needs to be held in order for the reduction to the simpler equation, also consistent with the method of Ref. [18] to asymptotically approximate the soliton of Eq (1.4).

As in the case of the KdV, we proceed with the single soliton solution of the mKdV Eq (5.10), which is of the form

$$\Psi(\xi,\tau) = \sqrt{\frac{6}{c_1}}\eta\,\mathrm{sech}[\eta(\xi - \eta^2\tau) + \xi_0]. \tag{5.11}$$

Then substituting this solution into Eq (5.8) leads to the $O(\varepsilon)$ correction:

$$\Phi(\xi,\tau) = \sqrt{\frac{6}{c_1}}\eta\,\mathrm{sech}[\eta(\xi-\eta^2\tau)+\xi_0] + \frac{6\eta^2\lambda_1\varepsilon}{c_1}\mathrm{sech}^2[\eta(\xi-\eta^2\tau)+\xi_0]$$
$$+ \frac{6\eta^2\lambda_2\varepsilon}{c_1}\cot^{-1}(\sinh[\eta(\xi-\eta^2\tau)+\xi_0])\,\mathrm{sech}[\eta(\xi-\eta^2\tau)+\xi_0]\tanh[\eta(\xi-\eta^2\tau)+\xi_0]. \quad (5.12)$$

Similarly to the above discussion, one should expect that rational solutions of the emKdV equation could also be found by pertinent ones existing in the mKdV equation (see, e.g., Refs. [39, 46–49]). Furthermore, relevant considerations could also be extended to the case of rogue waves thanks to the connections of the mKdV model with the nonlinear Schrödinger equation [50, 51].

6. Conclusions

In this work, we have derived and analyzed extended Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) equations from a physical model describing a three-component plasma composed of cold fluid ions and two species of Boltzmann electrons at different temperatures. While we manage to recover the "usual" higher-order KdV system, our analysis provides a fundamentally different derivation compared to the conventional mKdV equation obtained through integrability considerations. Through the analysis of this new formulation, we have explored its structural properties and solutions, offering insights into its connection with the extended KdV equation via Miura transformations.

One of the significant aspects of our study is the broader implication of these equations in the context of nonlinear wave theory. The KdV equation is widely regarded as a universal equation for weakly nonlinear, weakly dispersive wave systems. This universality arises due to its emergence in a diverse range of physical settings, from shallow water waves to plasma dynamics, optical fibers, and liquid crystals. A natural extension of this perspective is to investigate whether the extended mKdV equation we have derived could also share similar universal properties. Given that the standard mKdV equation appears in various contexts, including nonlinear optics and fluid dynamics, its extended version might exhibit a similarly broad applicability. Future work should focus on identifying physical systems where this equation naturally arises and exploring its integrability and solution structures in greater depth. Additionally, further investigations into asymptotic integrability and constructing the higher KdV system (if that exists) that is connected, through a Miura transformation, to the physically relevant mKdV system may provide useful information towards the "universal" character of the equation. Finally, the construction of rational and, when relevant, rogue wave solutions for the extended models considered in this work constitutes a very interesting future direction.

Author contributions

S. Baqer, T. P. Horikis and D. J. Frantzeskakis: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review & editing. All authors contributed equally to this article.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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