AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(4): 9202–9236.
DOI: 10.3934/math.2025423
Received: 26 January 2025
Revised: 30 March 2025
Accepted: 11 April 2025
Published: 22 April 2025

Research article

A weighted networked eco-epidemiological model with nonlinear p-Laplacian

Ling Zhou*, **Guoqing Ding and Zuhan Liu**

School of Mathematical Science, Yangzhou University, Yangzhou 225002, China

* **Correspondence:** Email: zhoul@yzu.edu.cn.

Abstract: This paper investigates a eco-epidemiological model with a graph p-Laplacian ($p \geq 2$). We first overcome the difficulties caused by the nonlinearity of the p-Laplacian and show the existence and uniqueness of the global solution to the system. By the approach of Lyapunov functions and the comparison principle, we show that the trivial equilibrium, the disease-free equilibrium without predators, the coexisting disease-free equilibrium, the prey's endemic equilibrium, and the coexisting endemic equilibrium are asymptotically stable under the given conditions. With numerical simulations, we apply our generalized weighed graph to the Watts-Strogatz network, which illustrates the effect of population mobility.

Keywords: network; p-Laplacian; eco-epidemiological model; global stability
Mathematics Subject Classification: 35B40, 35K51, 35R35, 92B05

1. Introduction

In this paper, we study a weighted networked eco-epidemiological model with p-Laplacian ($p \geq 2$)

$$\begin{cases} \dfrac{\partial S}{\partial t} - D_1 \Delta_\omega^p S = rS\left(1 - \dfrac{S+I}{k}\right) - \dfrac{\beta S I}{1+\alpha I} - \dfrac{p_1 S Y}{m+S} - h_1 S, & (x,t) \in V \times (0,+\infty), \\ \dfrac{\partial I}{\partial t} - D_2 \Delta_\omega^p I = \dfrac{\beta S I}{1+\alpha I} - d_1 I - \dfrac{p_2 I Y}{m+I} - h_2 I, & (x,t) \in V \times (0,+\infty), \\ \dfrac{\partial Y}{\partial t} - D_3 \Delta_\omega^p Y = -d_2 Y + q\dfrac{p_1 S Y}{m+S} + q\dfrac{p_2 I Y}{m+I}, & (x,t) \in V \times (0,+\infty), \\ S(x,0) = S_0(x),\ I(x,0) = I_0(x),\ Y(x,0) = Y_0(x), & x \in V. \end{cases} \tag{1.1}$$

Here, $S(x,t)$ represents the density of susceptible prey in vertex $x \in V$ at time t, $I(x,t)$ represents the density of infected prey in vertex $x \in V$ at time t, and $Y(x,t)$ represents the density of predators in vertex $x \in V$ at time t. Δ_ω^p is the graph p-Laplacian operator defined in (1.2), $D_i (i = 1,2,3)$ represent the diffusion rate of individuals among vertices, r is the intrinsic birth rate of susceptible prey, k is the carrying capacity of the prey environment, α is the inhibition effect, and β is the force of infection. The model consumes susceptible and infected prey following a Holling Type-II functional response with the predation coefficients p_1 and p_2 and the half-saturation constant m. Here, q $(0 < q \leq 1)$ is the assimilation efficiency of predators, d_1 is the death rate of infected prey, and d_2 is the natural death rate of predators, wherease h_1 and h_2 are the harvesting capabilities for susceptible and infected prey, respectively. In addition, the system satisfies the initial conditions $S_0(x) \geq (\not\equiv) 0$, $I_0(x) \geq 0$, $Y_0(x) \geq 0$.

A network is mathematically described as a undirected graph $G = (V, E)$, which contains a set V of vertices and a set E of edges (x, y) connecting vertex x and vertex y. If vertices x and y are connected by an edge (also called adjacent), we write $x \sim y$. G is called a finite-dimensional graph if it has a finite number of edges and vertices. G is called connected if for every pair of vertices x and y, a sequence (called a path) of vertices $x_0 = x, x_1, \ldots, x_n = y$ exists such that x_{j-1} and x_j are adjacent for $j = 1, \ldots, n$. A graph is weighted if each adjacent x and y is assigned a weight function $\omega(x, y)$. Here, $\omega : V \times V \to [0, +\infty)$ satisfies $\omega(x, y) = \omega(y, x)$ and $\omega(x, y) > 0$ if and only if $x \sim y$. Throughout this paper, $G = (V, E)$ is assumed to be a connected weighted finite-dimensional graph with $V = \{1, 2, \ldots, n\}$. Next, we introduce the discrete p-Laplacian operators defined on a network.

Definition 1.1. *For a function $v : V \to \mathbb{R}$ and $p \in (2, +\infty)$, the discrete p-Laplacian Δ_ω^p on V is defined by*

$$\Delta_\omega^p v(x) := \sum_{y \sim x, y \in V} |v(y) - v(x)|^{p-2}(v(y) - v(x))\omega(x, y). \tag{1.2}$$

When $p = 2$, it is called the discrete Laplacian $\Delta_\omega := \Delta_\omega^2$ on V, which is defined by

$$\Delta_\omega v(x) := \sum_{y \sim x, y \in V} (v(y) - v(x))\omega(x, y). \tag{1.3}$$

Here, we should emphasize that the discrete p-Laplacian operator Δ_ω^p $(p > 2)$ is actually nonlinear, which is different from the classical Laplacian Δ or the discrete Laplacian Δ_ω.

Definition 1.2. *For a function $D_\omega : V \to [0, +\infty)$, the degree $D_\omega(x)$ is defined by*

$$D_\omega(x) := \sum_{y \sim x, y \in V} \omega(x, y). \tag{1.4}$$

In the following, we briefly introduce the developments of the eco-epidemiological system and the inspirations of our problem.

Ecology and epidemiology have traditionally been regarded as two separate fields of study, but theoretical and empirical research has found that various populations within most ecosystems are potentially susceptible to various infectious diseases to varying degrees, necessitating the study of the impact of epidemiological parameters in the ecological domain. Currently, eco-epidemiology has emerged as an important branch of biomathematics, integrating issues from both ecology and epidemiology.

Over the past decades, scholars have integrated infectious diseases into predator-prey models. Anderson and May [1] pioneered this interdisciplinary research, examining parasite infections in prey and predators. They emphasized the role of disease mechanisms in altering basic reproductive numbers and systems' persistence. Temple [2] experimentally found that predators disproportionately target unhealthy prey, with this bias increasing as prey become harder to capture. Subsequent studies, including Xiao and Chen [3], explored predator-prey models with disease in the prey, focusing on equilibrium existence and stability. Packer [4] discussed predator removal's impact on diseased prey, while Holt [5] provided an example where predation enhanced disease spread. These works highlighted the profound influence of disease and predators on predator-prey systems. Further research, such as Upadhyay [6] on complex functional responses and Liu [7] on diseased prey with induced defenses, mainly focused on Hopf bifurcations, with limited discussions on disease transmission and prevalence. Wang and Yao [8] proposed an eco-epidemiological prey-predator model incorporating infectious diseases in prey, demonstrating the stability of equilibria in the diffusive model under Neumann boundary conditions and proving the uniqueness and global stability under Dirichlet constraints. Pandey et al. [9] investigated a delayed eco-epidemic model with environmental noise, showing how chaos can be controlled through adjustments in predator growth and stochasticity within Holling-type interactions. Dey et al. [10] modeled predator-prey-pathogen dynamics with non-local disease transmission, revealing that larger transmission ranges reduce spatial complexity and stabilize uniform states by analyzing bifurcations, Turing instability, and integro-differential system dynamics.

Apart from the impact of infectious diseases on biological populations, the influence of human harvesting on predator-prey models is also crucial. These models play a pivotal role in bioeconomics, particularly in managing renewable resources. By determining the maximum sustainable yield, targeted harvesting can maintain ecosystem balance and generate economic benefits. Early works by Brauer et al. [11, 12] explored the effects of harvesting on predators and prey, highlighting the risks of overharvesting predators or prey, which can lead to ecosystem collapse and species extinction. They also considered simultaneous harvesting of both, identifying harvest rates that ensure global stability (coexistence). Dai et al. [13] studied the dynamics of two interacting species harvested independently at constant rates, emphasizing the risk of species extinction due to unbalanced harvesting rates and the inadequacy of equilibrium points in preventing overexploitation. Cristiano et al. [14] considered the global stability of a two-predator system with human harvesting of both predators.

In recent decades, recognizing both the economic benefits of human harvesting and the practice of culling infected populations during epidemics, scholars have incorporated harvesting into epidemic-predator-prey systems. Bairagi et al. [15] studied the impact of prey harvesting rates on an epidemic-predator model, under the assumption that predators exclusively prey on infected prey, focusing on local stability of equilibria. Bhattacharyya and Mukhopadhyay [16] extended this model, considering predators that prey on both susceptible and infected prey, as described in the following system (1.5):

$$\begin{cases} \dfrac{dS}{dt} = rS\left(1 - \dfrac{S+I}{k}\right) - \dfrac{\beta SI}{1+\alpha I} - \dfrac{p_1 SY}{m+S} - h_1 S, \\ \dfrac{dI}{dt} = \dfrac{\beta SI}{1+\alpha I} - d_1 I - \dfrac{p_2 IY}{m+I} - h_2 I, \\ \dfrac{dY}{dt} = -d_2 Y + q\dfrac{p_1 SY}{m+S} + q\dfrac{p_2 IY}{m+I}. \end{cases} \tag{1.5}$$

Population mobility stands out as a noteworthy scenario in epidemic transmission dynamics, and some differential equation models have emerged as pivotal tools in modeling the outbreak and dissemination of epidemic diseases, (refer to [17–20] and references therein). The dispersal equation for v without any growth term is

$$\frac{\partial v}{\partial t} = \nabla \cdot (D(v, \nabla v)\nabla v),$$

where v is the population density. $D(v, \nabla v)$ is the population's random motility set to be a constant, which leads to linear isotropic diffusion. During the past few years, many authors have paid much attention to the Laplacian diffusion when $D = 1$, see for example such as Allen et al. [21,22], Yang et al. [23], Lin and Zhu [24], Li et al. [25], Lei et al. [26], and the references therein. However, in ecosystems, the distribution and migration of species are influenced by various factors, including environmental resources, competitive relationships, predator-prey relationships, etc. These factors often exhibit complex nonlinear relationships, making it difficult for traditional linear models to accurately describe them. We consider the other typical nonlinear relation $D(v, \nabla v) = |\nabla v|^{p-2}$. If $p > 2$, the nonlinear diffusion $\Delta^p v := \nabla \cdot (|\nabla v|^{p-2} v)$ is called the slow p-Laplacian diffusion, whereas $1 < p < 2$, it is called the fast p-Laplacian diffusion. The p-Laplacian operator is distinguished by its isotropic nature, making it particularly suited for analyzing rotationally symmetric problems. Its fundamental mathematical behavior depends critically on the exponent parameter p: the operator reduces to the classical Laplacian when $p = 2$, while deviations from this value introduce intrinsic nonlinearity with distinct analytical features. A key characteristic emerges at nodal points of solutions, where the operator exhibits singular behavior, presenting unique challenges in the study of partial differential equations. This nonlinear operator has proven indispensable for modeling multiscale phenomena across disciplines. In physics, it captures non-Fickian diffusion processes, turbulent flow dynamics, and phase transitions that linear approximations fail to describe. Engineering applications leverage its structure to simulate complex fluid–structure interactions and characterize materials with strain-rate-dependent properties. Within biological contexts, the operator demonstrates remarkable versatility: ecological models employ it to quantify population dispersal under environmental heterogeneity, epidemiological studies incorporate its formalism to refine predictions of infection spread through dynamic contact networks, and biophysical applications utilize its framework to model anomalous tissue growth mechanisms. In summary, the p-Laplacian is a powerful and indispensable tool in physics, biology, and engineering (see for example Oruganti et al. [27], Liu et al. [28], Yang et al. [29] and Li et al. [30]).

Traditional transmission models assume that the population is uniformly mixed, implying an equal probability of contact between any two individuals within the population. This clearly cannot fully reflect the realistic characteristics of disease transmission. For instance, the geographical dissemination of COVID-19 from city to city and from country to country primarily occurred through air travel, wherein passengers traverse predefined flight routes. In fact, the spread of infectious diseases among humans can be viewed as a network-based transmission behavior that follows certain patterns.

Therefore, investigating the impact of network structure on transmission behavior naturally becomes a crucial topic. Complex network theory serves as a powerful tool to address this significant issue. Over the past two decades, the propagation of epidemics in heterogeneous networks has garnered significant attention. We refer to a comprehensive review by Pastor-Satorras et al. [31] and the references therein. Recently, the classical Laplacian Δ has been substituted by the discrete Laplacian Δ_ω in graph Laplacian problems, and various methods and techniques have been developed to study the existence and qualitative properties of the solutions [32–41].

For the eco-epidemiological model without a graph p-Laplacian, Bhattacharyya et al. [16] have studied the uniform upper bound and asymptotic behavior of the solution, and Rao et al. investigated the stability of the equilibria further. In this paper, our main aim was to study the effect of the graph on infectious disease transmission and the spreading of predators. We expect that the results are also applicable to the eco-epidemiological model with the discrete p-Laplacian. First, we overcome the difficulties caused by the nonlinear p-Laplacian $\Delta_\omega^p (p > 2)$ operator and obtain the unique local solution of the system (1.1). We then extend the local solution to the maximal time by the means of a priori estimation. After that, under the given initial conditions, we prove that the equilibria are asymptotically stable by the method of the comparison principle, the Lyapunov function, and the Lemmas obtained in Section 3. Here are our main conclusions.

Theorem 1.1. *System (1.1) possesses a unique solution for all* $t \in [0, +\infty)$.

For convenience, throughout this paper, we write

$$S_A = k(1 - \frac{h_1}{r}),\ \ S_B = \frac{md_2}{qp_1 - d_2},\ \ Y_B = \frac{q}{d_2} S_B \left[r(1 - \frac{S_B}{k}) - h_1 \right],\ \ \varphi(x) = \frac{qx}{m+x},$$

$$R_0 = \frac{r}{h_1},\ \ R_1 = \frac{S_A}{S_B},\ \ R_1^* = \frac{qp_1}{d_2},\ \ R_2 = \frac{d_2}{p_1\varphi(S_C) + p_2\varphi(I_C)},\ \ R_3 = \frac{\beta S_A}{d_1 + h_2},$$

where S_C, I_C are positive constants and satisfy

$$\begin{cases} g_1(S_C, I_C, 0) = 0, \\ g_2(S_C, I_C, 0) = 0. \end{cases}$$

Theorem 1.2. *If* $R_0 \leq 1$, *then* $E_0 = (0, 0, 0)$ *is globally asymptotically stable.*

Theorem 1.3. *Suppose* $R_0 > 1$ *and* $R_3 < 1$. *If* $R_1 < 1$, *then* $E_A = (S_A, 0, 0)$ *is globally asymptotically stable.*

Theorem 1.4. *Suppose* $R_1^* > 1$, $R_3 < 1$, *and* $p_1 < \frac{r(m+S_B)^2}{kY_B}$. *If* $R_1 > 1$, $\min_{x\in V} S_0(x) \geq S_B$, *and* $\min_{x\in V} Y_0(x) \geq Y_B$, *then* $E_B = (S_B, 0, Y_B)$ *is globally asymptotically stable.*

Theorem 1.5. *Suppose* $R_3 > 1$. *If* $R_2 > 1$, $\min_{x\in V} S_0(x) \geq S_C$, *and* $\min_{x\in V} I_0(x) \geq I_C$, *then* $E_C = (S_C, I_C, 0)$ *is globally asymptotically stable.*

Theorem 1.6. *Suppose that the positive equilibrium* $E^* = (S^*, I^*, Y^*)$ *exists, with* $p_1 < \frac{r(m+S^*)^2}{kY^*}$ *and* $p_2 < \frac{\alpha\beta S^*(m+I^*)^2}{Y^*(1+\alpha I^*)^2}$. *If* $\min_{x\in V} S_0(x) \geq S^*$, $\min_{x\in V} I_0(x) \geq I^*$, *and* $\min_{x\in V} Y_0(x) \geq Y^*$, *then* E^* *is globally asymptotically stable.*

The rest of the article is organized as follows. In Section 2, we overcome the difficulties caused by the nonlinear operator p-Laplacian $\Delta_\omega^p (p \geq 2)$ and establish the global existence and uniqueness of the solution to the system (1.1) via the coupled upper–lower solutions approach and a prior estimate. In Section 3, by the method of Lyapunov functions and the comparison principle, we investigate the asymptotical behavior of the system (1.1) under the given conditions. In Section 4, we carry out numerical simulations to confirm our analytical findings and illustrate the small-time dynamical behavior. Discussions and conclusions are given in Section 5.

2. Existence and uniqueness

In this section, we first use the method of coupled upper and lower solutions to establish the local existence and uniqueness of the solution to the system (1.1). We then give a prior estimate of the solutions to (1.1), which is useful to extend the local solution to the maximal time.

For the sake of simplicity, throughout this paper, we write

$$g(v) := \big(g_1(v_1, v_2, v_3),\ g_2(v_1, v_2, v_3),\ g_3(v_1, v_2, v_3)\big),$$

where

$$\begin{cases} g_1(v_1, v_2, v_3) = rv_1\left(1 - \dfrac{v_1 + v_2}{k}\right) - \dfrac{\beta v_1 v_2}{1 + \alpha v_2} - \dfrac{p_1 v_1 v_3}{m + v_1} - h_1 v_1, \\ g_2(v_1, v_2, v_3) = \dfrac{\beta v_1 v_2}{1 + \alpha v_2} - d_1 v_2 - \dfrac{p_2 v_2 v_3}{m + v_2} - h_2 v_2, \\ g_3(v_1, v_2, v_3) = -d_2 v_3 + q\dfrac{p_1 v_1 v_3}{m + v_1} + q\dfrac{p_2 v_2 v_3}{m + v_2}. \end{cases}$$

Definition 2.1. *Suppose that for each* $x \in V$, $\tilde{v}_i(x, \cdot), \hat{v}_i(x, \cdot) \in C([0, T])\,(i = 1, 2, 3)$ *are differentiable in* $(0, T]$ *for each* $x \in \Omega$. *A pair of functions* $\tilde{v} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$, $\hat{v} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$ *are called the coupled upper and lower solutions of the system (1.1) if* $\tilde{v} \geq \hat{v} \geq 0$ *and if*

$$\begin{cases} \dfrac{\partial \tilde{v}_1}{\partial t} - D_1 \Delta_\omega^p \tilde{v}_1 \geq g_1(\tilde{v}_1, \hat{v}_2, \hat{v}_3), & (x, t) \in V \times (0, T], \\ \dfrac{\partial \tilde{v}_2}{\partial t} - D_2 \Delta_\omega^p \tilde{v}_2 \geq g_2(\tilde{v}_1, \tilde{v}_2, \hat{v}_3), & (x, t) \in V \times (0, T], \\ \dfrac{\partial \tilde{v}_3}{\partial t} - D_3 \Delta_\omega^p \tilde{v}_3 \geq g_3(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3), & (x, t) \in V \times (0, T], \\ \dfrac{\partial \hat{v}_1}{\partial t} - D_1 \Delta_\omega^p \hat{v}_1 \leq g_1(\hat{v}_1, \tilde{v}_2, \tilde{v}_3), & (x, t) \in V \times (0, T], \\ \dfrac{\partial \hat{v}_2}{\partial t} - D_2 \Delta_\omega^p \hat{v}_2 \leq g_2(\hat{v}_1, \hat{v}_2, \tilde{v}_3), & (x, t) \in V \times (0, T], \\ \dfrac{\partial \hat{v}_3}{\partial t} - D_3 \Delta_\omega^p \hat{v}_3 \leq g_3(\hat{v}_1, \hat{v}_2, \hat{v}_3), & (x, t) \in V \times (0, T], \\ \hat{v}_i(x, 0) \leq v_{i0}(x) \leq \tilde{v}_i(x, 0) \quad \textit{for } i = 1, 2, 3, & x \in V. \end{cases} \tag{2.1}$$

For a given pair of coupled upper and lower solutions, denoted as $\tilde{v}$ and $\hat{v}$, we set

$$\Pi_i := \{\, v_i(x, \cdot) \in C([0, T]) \,\big|\, \hat{v}_i \leq v_i \leq \tilde{v}_i \}, \quad \Pi := \{\, v \,\big|\, \hat{v} \leq v \leq \tilde{v} \} = \Pi_1 \times \Pi_2 \times \Pi_3.$$

There constants K_i for $i = 1, 2, 3$ exist such that

$$K_i + \frac{\partial g_i}{\partial v_i}(v) \geq 0, \quad \text{for } v \in \Pi.$$

For each $i = 1, 2, 3$, we define

$$G_i(v_1, v_2, v_3) = K_i v_i + g_i(v_1, v_2, v_3). \tag{2.2}$$

We consider the system

$$\begin{cases} \dfrac{\partial v_i}{\partial t} - D_i \Delta_\omega^p v_i + K_i v_i = G_i(v_1, v_2, v_3) \quad \text{for } i = 1,2,3, & (x,t) \in V \times (0,T], \\ v_i(x,0) = v_{i0}(x) \quad \text{for } i = 1,2,3, & x \in V. \end{cases} \tag{2.3}$$

Then System (2.3) is equivalent to System (1.1) in a finite time interval.

By using $\underline{v}^{(0)} = \hat{v}$ and $\overline{v}^{(0)} = \tilde{v}$ as the initial iterations, we can construct the sequences $\{\overline{v}^{(m)}\}_{m=1}^{\infty}$ and $\{\underline{v}^{(m)}\}_{m=1}^{\infty}$ from the scalar equations' iteration process

$$\begin{cases} \dfrac{\partial \overline{v}_1^{(m)}}{\partial t} - D_1 \Delta_\omega^p \overline{v}_1^{(m)} + K_1 \overline{v}_1^{(m)} = G_1(\overline{v}_1^{(m-1)}, \underline{v}_2^{(m-1)}, \underline{v}_3^{(m-1)}), & (x,t) \in V \times (0,T], \\ \dfrac{\partial \overline{v}_2^{(m)}}{\partial t} - D_2 \Delta_\omega^p \overline{v}_2^{(m)} + K_2 \overline{v}_2^{(m)} = G_2(\overline{v}_1^{(m-1)}, \overline{v}_2^{(m-1)}, \underline{v}_3^{(m-1)}), & (x,t) \in V \times (0,T], \\ \dfrac{\partial \overline{v}_3^{(m)}}{\partial t} - D_3 \Delta_\omega^p \overline{v}_3^{(m)} + K_3 \overline{v}_3^{(m)} = G_3(\overline{v}_1^{(m-1)}, \overline{v}_2^{(m-1)}, \overline{v}_3^{(m-1)}), & (x,t) \in V \times (0,T], \\ \dfrac{\partial \underline{v}_1^{(m)}}{\partial t} - D_1 \Delta_\omega^p \underline{v}_1^{(m)} + K_1 \underline{v}_1^{(m)} = G_1(\underline{v}_1^{(m-1)}, \overline{v}_2^{(m-1)}, \overline{u}_3^{(m-1)}), & (x,t) \in V \times (0,T], \\ \dfrac{\partial \underline{v}_2^{(m)}}{\partial t} - D_2 \Delta_\omega^p \underline{v}_2^{(m)} + K_2 \underline{v}_2^{(m)} = G_2(\underline{v}_1^{(m-1)}, \underline{v}_2^{(m-1)}, \overline{v}_3^{(m-1)}), & (x,t) \in V \times (0,T], \\ \dfrac{\partial \underline{v}_3^{(m)}}{\partial t} - D_3 \Delta_\omega^p \underline{v}_3^{(m)} + K_3 \underline{v}_3^{(m)} = G_3(\underline{v}_1^{(m-1)}, \underline{v}_2^{(m-1)}, \underline{v}_3^{(m-1)}), & (x,t) \in V \times (0,T], \\ \overline{v}_i^{(m)}(x,0) = \underline{v}_i^{(m)}(x,0) = v_{i0}(x) \quad \text{for } i = 1,2,3, & x \in V. \end{cases} \tag{2.4}$$

We now briefly discuss the local existence and uniqueness of a solution for the equation

$$\begin{cases} v_t(x,t) = d\Delta_\omega^p v(x,t) + g(x,t,v(x,t)), & (x,t) \in V \times (0,+\infty), \\ v(x,0) = v_0(x), & x \in V, \end{cases} \tag{2.5}$$

where d is a positive constant, g is locally Lipschitz continuous with respect to v on $\mathbb{R}$.

Let $t_0 > 0$ be fixed and consider a Banach space

$$X_{t_0} = \{v : V \times [0,t_0] \to \mathbb{R} \,|\, v(x,\cdot) \in C[0,t_0] \text{ for each } x \in V\}$$

with the norm $\|v\|_{X_{t_0}} := \max_{x\in V} \max_{0\le t\le t_0} |v(x,t)|$.

It is then easy to see that the operator $D : X_{t_0} \to X_{t_0}$ given by

$$D[v](x,t) := v_0(x) + d\int_0^t \Delta_\omega^p v(x,s)ds + \int_0^t g(x,s,v(x,s))ds, \quad (x,t) \in V \times [0,t_0],$$

is well-defined. In Lemma 2.1, we show that this operator D is contractive on a small closed ball. Hence, we obtain the existence and uniqueness of a solution to the Eq (2.5) in a small time interval $[0,t_0]$, as a consequence of Banach's fixed point theorem.

Lemma 2.1. *Let $g(x,t,v)$ be locally Lipschitz continuous with respect to v on $\mathbb{R}$. Then the operator D is a contraction on the closed ball*

$$B(v_0, 2\|v_0\|_{X_{t_0}}) := \left\{ v \in X_{t_0} \middle| \, \|v - v_0\|_{X_{t_0}} \le 2\|v_0\|_{X_{t_0}} \right\},$$

if t_0 is small enough.

Proof. Consider v_1 and $v_2 \in B(v_0, 2\|v_0\|_{X_{t_0}})$. Since g is locally Lipschitz continuous with respect to v on $\mathbb{R}$, $L > 0$ exists such that

$$|g(x,t,a) - g(x,t,b)| \le L|a-b|, \ a,b \in [-m,m],$$

where $m := 3\|v_0\|_{X_{t_0}}$. Then for any $x \in V \times [0, t_0]$,

$$\begin{aligned}
&\Delta^p_\omega v_1(x,s) - \Delta^p_\omega v_2(x,s)\\
=&\sum_{y\sim x, y\in V}\Big[\,|v_1(y,s)-v_1(x,s)|^{p-2}(v_1(y,s)-v_1(x,s)) - |v_2(y,s)-v_2(x,s)|^{p-2}(v_2(y,s)-v_2(x,s))\Big]\omega(x,y).
\end{aligned}$$

Assume that

$$a_y = v_2(y,s) - v_2(x,s),\ b_y = v_1(y,s) - v_1(x,s),\ J_p(t) = |t|^{p-2}t.$$

Then $|a_y| = |v_2(y,s) - v_2(x,s)| \le 2\|v_2\|_{X_{t_0}} \le 6\|v_0\|_{X_{t_0}}$. Similarly, $|b_y| \le 6\|v_0\|_{X_{t_0}}$.

Since

$$J_p(t) = |t|^{p-2}t = \begin{cases} t^{p-1}, & t \ge 0,\\ -(-t)^{p-1}, & t < 0,\end{cases}$$

by calculation, we have

$$J'_p(t) = \begin{cases} (p-1)t^{p-2}, & t \ge 0,\\ (p-1)(-t)^{p-2}, & t < 0,\end{cases} \ = \ (p-1)|t|^{p-2}.$$

In view of the Lagrange mean value theorem, we have

$$\begin{aligned}
\Big|J_p(b_y) - J_p(a_y)\Big| &= J'_p(c_y)|b_y - a_y|\\
&= (p-1)|c_y|^{p-2}|(v_1(y,s)-v_1(x,s)) - (v_2(y,s)-v_2(x,s))|\\
&\le (p-1)(|a_y|^{p-2} + |b_y|^{p-2})|(v_1(y,s)-v_2(y,s)) - (v_1(x,s)-v_2(x,s))|\\
&\le 4(p-1)6^{p-2}\|v_0\|^{p-2}_{X_{t_0}}\|v_1 - v_2\|_{X_{t_0}}.
\end{aligned}$$

where c_y is some intermediate value between a_y and b_y. Therefore

$$\begin{aligned}
&\Big|\Delta^p_\omega v_1(x,s) - \Delta^p_\omega v_2(x,s)\Big|\\
\le &\sum_{y\sim x, y\in V} |J_p(b_y) - J_p(a_y)|\omega(x,y)\\
\le &4n(p-1)6^{p-2}\|v_0\|^{p-2}_{X_{t_0}}\max_{x,y\in V}\omega(x,y)\|v_1 - v_2\|_{X_{t_0}},
\end{aligned}$$

where n is the number of vertices of the graph. Then

$$\begin{aligned}
&\Big|D[v_1](x,t) - D[v_2](x,t)\Big|\\
=&\left|d\int_0^t (\Delta^p_\omega v_1(x,s) - \Delta^p_\omega v_2(x,s))\,ds + \int_0^t (g(x,s,v_1) - g(x,s,v_2))\,ds\right|\\
\le &4dnt(p-1)6^{p-2}\|v_0\|^{p-2}_{X_{t_0}}\max_{x,y\in V}\omega(x,y)\|v_1 - v_2\|_{X_{t_0}} + Lt\|v_1 - v_2\|_{X_{t_0}}\\
\le &Ct_0\|v_1 - v_2\|_{X_{t_0}},
\end{aligned}$$

where $C := 4dn6^{p-2}(p-1)\|v_0\|^{p-2}_{X_{t_0}}\max_{x,y\in V}\omega(x,y) + L$. Choosing t_0 sufficiently small, we obtain a contraction on the closed ball $B(v_0, 2\|v_0\|_{X_{t_0}})$ into itself. □

From the above, the sequences $\{\overline{v}^{(m)}\}_{m=1}^{\infty}$ and $\{\underline{v}^{(m)}\}_{m=1}^{\infty}$ exist and are unique for a small T. In the following, we aim to show the existence and uniqueness of solutions to System (1.1). In order to do so, we need to give the following lemmas of the graph p-Laplacian equations.

Lemma 2.2. *(Green Formula) For any functions $v_1, v_2 : V \to \mathbb{R}$, we have*

$$2\sum_{x\in V} v_2(x)(-\Delta_\omega^p)v_1(x) = \sum_{x,y\in V} |v_1(y) - v_1(x)|^{p-2}(v_1(y) - v_1(x))(v_2(y) - v_2(x))\omega(x, y).$$

In particular, in the case of $v_1 = v_2$, we have

$$2\sum_{x\in V} v_1(x)(-\Delta_\omega^p)v_1(x) = \sum_{x,y\in V} |v_1(y) - v_1(x)|^p\omega(x, y). \tag{2.6}$$

Moreover, in the case of $v_2 = 1$, we have

$$\sum_{x\in V}(-\Delta_\omega^p)v_1(x) = 0. \tag{2.7}$$

Proof. Using (1.2), we have

$$\begin{aligned}\sum_{x\in V} v_2(x)(-\Delta_\omega^p)v_1(x) &= -\sum_{x\in V} v_2(x)\sum_{y\sim x,y\in V} |v_1(y) - v_1(x)|^{p-2}(v_1(y) - v_1(x))\omega(x, y)\\ &= -\sum_{x,y\in V} v_2(x)|v_1(y) - v_1(x)|^{p-2}(v_1(y) - v_1(x))\omega(x, y)\\ &= -\sum_{x,y\in V} v_2(y)|v_1(y) - v_1(x)|^{p-2}(v_1(x) - v_1(y))\omega(x, y).\end{aligned}$$

From the equality above, we deduce

$$2\sum_{x\in V} v_2(x)(-\Delta_\omega^p)v_1(x) = \sum_{x,y\in V} |v_1(y) - v_1(x)|^{p-2}(v_1(y) - v_1(x))(v_2(y) - v_2(x))\omega(x, y),$$

which completes the proof. □

Lemma 2.3. *(Maximum Principle) Suppose that d is a positive constant and $K(x, t)$ is a bounded function in $V \times [0, T]$ for any $T > 0$. Assume that $v(x, t)$ is continuously differentiable with respect to t in $V \times [0, T]$ and also satisfies*

$$\begin{cases}\dfrac{\partial v}{\partial t} - d\Delta_\omega^p v + Kv \geq 0, & (x, t) \in V \times (0, T],\\ v(x, 0) \geq 0, & x \in V,\end{cases} \tag{2.8}$$

and thus $v(x, t) \geq 0$ in $V \times [0, T]$.

Proof. By setting $\overline{v} = e^{-K_0 t}v(x, t)$, where K_0 is a positive constant satisfying $K_0 + K > 0$, we deduce $\Delta_\omega^p v = \Delta_\omega^p(e^{K_0 t}\overline{v}) = e^{(p-1)K_0 t}\Delta_\omega^p\overline{v}$. Thus we have

$$\frac{\partial\overline{v}}{\partial t} - de^{(p-2)K_0 t}\Delta_\omega^p\overline{v} + (K_0 + K)\overline{v} \geq 0 \quad \text{for } (x, t) \in V \times (0, T]. \tag{2.9}$$

Notice that $\overline{v}(x,t)$ is continuous on $[0,T]$ for each $x \in V$ and V is finite, and thus we can find $(x_0,t_0) \in V \times [0,T]$ such that $\overline{v}(x_0,t_0) := \min_{x\in V} \min_{t\in[0,T]} \overline{v}(x,t)$.

For the case where $t_0 = 0$, we have $\overline{v}(x_0,t_0) = v_0(x_0) \geq 0$. Thus we have $\overline{v}(x,t) \geq 0$ in $V \times [0,T]$, which implies $v(x,t) \geq 0$ in $V \times [0,T]$.

For the case where $t_0 > 0$, the Eq (2.9) implies $\overline{v}(x_0,t_0) \leq \overline{v}(y,t_0)$ for any $y \in V$. In view of the definition of Δ_ω^p, we have

$$\Delta_\omega^p \overline{v}(x_0,t_0) = \sum_{y\sim x_0, y\in V} |\overline{v}(y,t_0) - \overline{v}(x_0,t_0)|^{p-2}(\overline{v}(y,t_0) - \overline{v}(x_0,t_0))\omega(x_0,y) \geq 0. \tag{2.10}$$

Meanwhile, it follows from the differentiability of $\overline{v}(x,t)$ in $(0,T]$ that

$$\frac{\partial \overline{v}}{\partial t}(x_0,t_0) \leq 0. \tag{2.11}$$

By substituting (2.10) and (2.11) into (2.9), we have $(K_0 + K)\overline{v}(x_0,t_0) \geq 0$. Noting that $(K_0 + K) > 0$, we deduce $\overline{v}(x_0,t_0) \geq 0$, which means $\min_{x\in V} \min_{t\in[0,T]} \overline{v}(x,t) \geq 0$. Therefore, we have $\overline{v}(x,t) \geq 0$ in $V \times [0,T]$. That is $v(x,t) \geq 0$ in $V \times [0,T]$. □

Lemma 2.4. *(Strong maximum principle) Suppose that d is a positive constant and $K(x,t)$ is a bounded function in $V \times [0,T]$ for any $T > 0$. Assume that $v(x,t)$ is continuously differentiable with respect to t in $V \times [0,T]$ and also satisfies (2.8). If $v(x^*,0) > 0$ for some $x^* \in V$, then $v(x,t) > 0$ in $V \times (0,T]$.*

Proof. Using the maximum principle above, we have $v(x,t) \geq 0$ in $V\times[0,T]$. By setting $\overline{v} = e^{-K_0 t}v(x,t)$, where K_0 is defined as in the proof of Lemma 2.3, which satisfies $K_0 + K > 0$, we have

$$\Big(\frac{\partial \overline{v}}{\partial t} - de^{(p-2)K_0 t}\Delta_\omega^p \overline{v} + (K_0 + K)\overline{v}\Big)|_{(x^*,t)} \geq 0. \tag{2.12}$$

Notice that $\overline{v}(x,t)$ is continuous on $[0,T]$ for each $x \in V$ and V is finite, and thus we can deduce that $M := \max_{x\in V} \max_{t\in[0,T]} \overline{v}(x,t) < +\infty$. Plugging (1.2) into (2.12), we have

$$\begin{aligned}
\frac{\partial \overline{v}(x^*,t)}{\partial t} &\geq -de^{(p-2)K_0 t} \sum_{y\sim x^*, y\in V} |\overline{v}(y,t) - \overline{v}(x^*,t)|^{p-2}\overline{v}(x^*,t)\omega(x^*,y) - (K_0 + K)\overline{v}(x^*,t) \\
&\geq -d(2M)^{p-2}e^{(p-2)K_0 t} \sum_{y\sim x^*, y\in V} \omega(x^*,y)\overline{v}(x^*,t) - (K_0 + K)\overline{v}(x^*,t) \\
&\geq -d(2M)^{p-2}e^{(p-2)K_0 T} D_\omega(x^*)\overline{v}(x^*,t) - (K_0 + K)\overline{v}(x^*,t) \\
&\geq -[d(2M)^{p-2}e^{(p-2)K_0 T} D_\omega(x^*) + (K_0 + K)]\overline{v}(x^*,t) \ \text{ for } t \in (0,T].
\end{aligned} \tag{2.13}$$

Since $\overline{v}(x^*,0) = v(x^*,0) > 0$, (2.13) implies that

$$\overline{v}(x^*,t) \geq \overline{v}(x^*,0)e^{-[d(2M)^{p-2}e^{(p-2)K_0 T}D_\omega(x^*)+(K_0+K)]t} > 0 \ \text{ for } t \in (0,T]. \tag{2.14}$$

We prove the result by contradiction. If $v(x,t) > 0$ in $V\times(0,T]$ cannot hold, a point $(x_0,t_0) \in V\times(0,T]$ exists such that $v(x_0,t_0) = 0$, which implies $\overline{v}(x_0,t_0) = \min_{x\in V} \min_{t\in[0,T]} \overline{v}(x,t) = 0$. By (2.9), we have

$$\Big(\frac{\partial \overline{v}}{\partial t} - de^{(p-2)K_0 t_0}\Delta_\omega^p \overline{v} + (K_0 + K)\overline{v}\Big)\Big|_{(x_0,t_0)} \geq 0. \tag{2.15}$$

Since $\overline{v}$ is differential with respect to t in $V \times (0, T]$, it follows that $\frac{\partial \overline{v}}{\partial t}|_{(x_0,t_0)} \leq 0$. Thus (2.15) implies that

$$de^{(p-2)K_0 t_0} \Delta_\omega^p \overline{v}(x_0, t_0) \leq \Big(\frac{\partial \overline{v}}{\partial t}\Big|_{(x_0,t_0)} + (K_0 + K)\overline{v}(x_0, t_0) \Big) \leq 0. \tag{2.16}$$

By (1.2), we also have $\Delta_\omega^p \overline{v}(x_0, t_0) \geq 0$. In view of $\overline{v}(x_0, t_0) = 0$ and $\overline{v}(x, t) \geq 0$, we have $\Delta_\omega^p \overline{v}(x_0, t_0) = \sum_{y \sim x_0, y \in V} \overline{v}^{p-1}(y, t_0)\omega(x_0, y) = 0$. This inequality implies that

$$\overline{v}(y, t_0) = 0 \text{ for any } y \in V \text{ and } y \sim x_0. \tag{2.17}$$

On the other hand, since V is connected, for $x^* \in V$, there is a path $x_0 \sim x_1 \sim \cdots \sim x_n = x^*$. By (2.17), we obtain that $\overline{v}(x_1, t_0) = 0$. Employing this argument repeatedly, we induce $\overline{v}(x^*, t_0) = 0$ in order, which contradicts (2.14). The proof is completed. □

Lemma 2.5. *(Lemma B.4 in [42]) For $p > 2$, $J_p(t) := |t|^{p-2}t$, we have*

$$2^{2-p}|b - a|^p \leq (J_p(b) - J_p(a))(b - a), \qquad a,\ b \in \mathbb{R}.$$

Moreover, if $b \geq a$, we have

$$J_p(b - a) \leq 2^{p-2}[J_p(b) - J_p(a)]. \tag{2.18}$$

Lemma 2.6. *(Comparison principle) Assume that (v_1, v_2, v_3) is a solution of the system (1.1). If $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ is continuous with respect to t on $V \times [0, T]$, is differentiable with respect to t in $V \times (0, T]$, and satisfies*

$$\begin{cases} \dfrac{\partial \tilde{v}_1}{\partial t} - D_1 \Delta_\omega^p \tilde{v}_1 \geq g_1(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3), & (x, t) \in V \times (0, T], \\ \dfrac{\partial \tilde{v}_2}{\partial t} - D_2 \Delta_\omega^p \tilde{v}_2 \geq g_2(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3), & (x, t) \in V \times (0, T], \\ \dfrac{\partial \tilde{v}_3}{\partial t} - D_3 \Delta_\omega^p \tilde{v}_3 \geq g_3(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3), & (x, t) \in V \times (0, T], \\ \tilde{v}_1(x, 0) \geq v_{10}(x),\ \tilde{v}_2(x, 0) \geq v_{20}(x),\ \tilde{v}_3(x, 0) \geq v_{30}(x), & x \in V, \end{cases} \tag{2.19}$$

then $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) \geq (v_1, v_2, v_3)$ on $V \times [0, T]$. Moreover, if $(\hat{v}_1, \hat{v}_2, \hat{v}_3)$ satisfies (2.19) by reversing all the inequalities, then $(\hat{v}_1, \hat{v}_2, \hat{v}_3) \leq (v_1, v_2, v_3)$ on $V \times [0, T]$.

Proof. Denote $z_i(x, t) := (\tilde{v}_i(x, t) - v_i(x, t))e^{-Kt}$, where K is a positive constant to be determined later. Notice that $z_1(x, t)$ is continuous on $[0, T]$ for each $x \in V$ and V is finite, we can find $(x_0, t_0) \in V \times [0, T]$ such that

$$z_1(x_0, t_0) \triangleq \min_{x \in V} \min_{t \in [0,T]} z_1(x, t), \tag{2.20}$$

If $t_0 = 0$, we have $z_1(x, t) \geq z_1(x_0, 0) \geq 0$, which implies that $\tilde{v}_1 \geq v_1$. Next, we consider the case where $t_0 > 0$. Equation (2.20) immediately implies that

$$z_1(x_0, t_0) \leq z_1(y, t_0), \text{ for any } y \in V.$$

This is equivalent to

$$\tilde{v}_1(x_0, t_0) - v_1(x_0, t_0) \leq \tilde{v}_1(y, t_0) - v_1(y, t_0), \text{ for any } y \in V. \tag{2.21}$$

and

$$\tilde{v}_1(y,t_0)-\tilde{v}_1(x_0,t_0)\geq v_1(y,t_0)-v_1(x_0,t_0),\ \text{ for any } y\in V. \tag{2.22}$$

Recalling the definition of Δ_ω^p, we have

$$\Delta_\omega^p z_1(x_0,t_0)\geq 0. \tag{2.23}$$

At the same time, from the differentiability of $z_1(x,t)$ in $(0,T]$, we obtain

$$\frac{\partial z_1}{\partial t}(x_0,t_0)\leq 0. \tag{2.24}$$

Note that

$$\begin{aligned}
\Delta_\omega^p z_1(x,t) &= e^{-Kt(p-1)}\Delta_\omega^p(\tilde{v}_1-v_1)(x,t)\\
=&e^{-Kt(p-1)}\sum_{y\sim x,y\in V}\Big|(\tilde{v}_1(y,t)-v_1(y,t))-(\tilde{v}_1(x,t)-v_1(x,t))\Big|^{p-2}\\
&\cdot[(\tilde{v}_1(y,t)-v_1(y,t)))-(\tilde{v}_1(x,t)-v_1(x,t))]\omega(x,y)\\
=&e^{-Kt(p-1)}\sum_{y\sim x,y\in V}\Big|(\tilde{v}_1(y,t)-\tilde{v}_1(x,t))-(v_1(y,t)-v_1(x,t))\Big|^{p-2}\\
&\cdot[(\tilde{v}_1(y,t)-\tilde{v}_1(x,t))-(v_1(y,t)-v_1(x,t))]\omega(x,y),
\end{aligned} \tag{2.25}$$

we have

$$\begin{aligned}
\Delta_\omega^p(\tilde{v}_1-v_1)(x_0,t_0) = &\sum_{y\sim x_0,y\in V}\Big|(\tilde{v}_1(y,t_0)-\tilde{v}_1(x_0,t_0))-(v_1(y,t_0)-v_1(x_0,t_0))\Big|^{p-2}\\
&\cdot[(\tilde{v}_1(y,t_0)-\tilde{v}_1(x_0,t_0))-(v_1(y,t_0)-v_1(x_0,t_0))]\omega(x_0,y).
\end{aligned} \tag{2.26}$$

Assume that

$$b_y := \tilde{v}_1(y,t_0)-\tilde{v}_1(x_0,t_0),\ \ a_y := v_1(y,t_0)-v_1(x_0,t_0) \quad\text{and}\quad J_p(t):=|t|^{p-2}t.$$

In view of (2.22), we have $b_y \geq a_y$ for any $y\sim x_0$ and $y\in V$. Combining this with inequality (2.18) in Lemma 2.5, we deduce that

$$|b_y-a_y|^{p-2}(b_y-a_y)=J_p(b_y-a_y)\leq 2^{p-2}[J_p(b_y)-J_p(a_y)]=2^{p-2}[|b_y|^{p-2}b_y-|a_y|^{p-2}a_y],$$

which implies

$$\begin{aligned}
\Delta_\omega^p(\tilde{v}_1-v_1)(x_0,t_0) &= \sum_{y\sim x_0,y\in V}|b_y-a_y|^{p-2}(b_y-a_y)\omega(x_0,y)\\
&\leq 2^{p-2}\sum_{y\sim x_0,y\in V}[|b_y|^{p-2}b_y-|a_y|^{p-2}a_y]\omega(x_0,y)\\
&= 2^{p-2}\Big[\sum_{y\sim x_0,y\in V}|b_y|^{p-2}b_y\,\omega(x_0,y)-\sum_{y\sim x_0,y\in V}|a_y|^{p-2}a_y\,\omega(x_0,y)\Big]\\
&= 2^{p-2}\Big[\Delta_\omega^p\tilde{v}_1(x_0,t_0)-\Delta_\omega^p v_1(x_0,t_0)\Big].
\end{aligned} \tag{2.27}$$

Combining (2.27) with (2.25), we have

$$\Delta_\omega^p z_1(x_0,t_0) \le 2^{p-2} e^{-Kt_0(p-1)}\Big[\Delta_\omega^p \tilde{v}_1(x_0,t_0) - \Delta_\omega^p v_1(x_0,t_0)\Big]. \tag{2.28}$$

Note that $\tilde{v}_1$ and v_1 satisfy (2.19) and (1.1), respectively, so by calculation, we deduce that

$$\begin{aligned} &2^{p-2} e^{-Kt_0(p-2)} \frac{\partial z_1}{\partial t}(x_0,t_0) - D_1 \Delta_\omega^p z_1(x_0,t_0) \\ \ge\, &2^{p-2} e^{-Kt_0(p-1)}\Big[g_1(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3) - g_1(v_1,v_2,v_3) - K(\tilde{v}_1 - v_1)\Big](x_0,t_0). \end{aligned} \tag{2.29}$$

Substituting (2.23) and (2.24) into (2.29), we deduce that

$$\Big[g_1(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3) - g_1(v_1,v_2,v_3) - K(\tilde{v}_1 - v_1)\Big](x_0,t_0) \le 0. \tag{2.30}$$

Next, we prove $z_1(x_0,t_0) \ge 0$ by contradiction. Suppose, on the contrary, that $(\tilde{v}_1 - v_1)(x_0,t_0) = -\delta < 0$. Choosing

$$K := \frac{|(g_1(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3) - g_1(v_1,v_2,v_3))(x_0,t_0)|}{\delta} + 1,$$

we find that $[g_1(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3) - g_1(v_1,v_2,v_3) - K(\tilde{v}_1 - v_1)](x_0,t_0) > 0$, which contradicts (2.30). Thus, we have

$$z_1(x_0,t_0) = e^{-Kt_0}(\tilde{v}_1 - v_1)(x_0,t_0) \ge 0.$$

In view of (2.20), it follows that $z_1(x,t) \ge 0$ for $(x,t) \in V \times [0,T]$. By a similar argument to $z_i (i = 2,3)$, we can also obtain $z_i(x,t) \ge 0$ for $(x,t) \in V \times [0,T]$. These conclusions imply that $\tilde{v}_i(x,t) \ge v_i(x,t) (i = 1,2,3)$ for $(x,t) \in V \times [0,T]$. Applying a similar process, we have $(\hat{v}_1,\hat{v}_2,\hat{v}_3) \le (v_1,v_2,v_3)$ on $V \times [0,T]$. The proof is completed. □

Lemma 2.7. *The sequences* $\{\overline{v}^{(m)}\}_{m=1}^\infty$ *and* $\{\underline{v}^{(m)}\}_{m=1}^\infty$ *governed by (2.4) admit the limit as* $m \to \infty$. *We write*

$$\lim_{m\to\infty} \overline{v}^{(m)} := \overline{v} \quad \textit{and} \quad \lim_{m\to\infty} \underline{v}^{(m)} := \underline{v}. \tag{2.31}$$

Meanwhile, the following monotonicity property holds:

$$\hat{v} \le \underline{v}^{(m)} \le \underline{v}^{(m+1)} \le \underline{v} \le \overline{v} \le \overline{v}^{(m+1)} \le \overline{v}^{(m)} \le \tilde{v} \quad \textit{for } m = 1,2,\cdots \tag{2.32}$$

for $(x,t) \in V \times [0,T]$. *Moreover, for each* $m = 1,2,\cdots$, $\underline{v}^{(m)}$ *and* $\overline{v}^{(m)}$ *are coupled upper and lower solutions of (1.1).*

Proof. We first prove the following two claims and then use an induction method to prove the theorem.
Claim 1. $\underline{v}_i^{(1)} \ge \underline{v}_i^{(0)}$ for $(x,t) \in V \times [0,T]$ for $i = 1,2,3$.
Proof of Claim 1. Take $z_i(x,t) := (\underline{v}_i^{(1)}(x,t) - \underline{v}_i^{(0)}(x,t))e^{-M_i t}$ for $i = 1,2,3$, where M_i represents positive constants. Notice that $z_i(x,t)$ is continuous on $[0,T]$ for each $x \in V$ and V is finite, and thus we can find $(x_0^i, t_0^i) \in V \times [0,T]$ such that

$$z_i(x_0^i, t_0^i) \triangleq \min_{x\in V} \min_{t\in[0,T]} z_i(x,t). \tag{2.33}$$

If $t_0^i = 0$, we have $z_i(x,t) \ge z_i(x_0^i, 0) = 0$, which implies that $\underline{v}_i^{(1)} \ge \underline{v}_i^{(0)}$. Next, we consider the case where $t_0^i > 0$. Equation (2.33) immediately implies that

$$z_i(x_0^i, t_0^i) \le z_i(y, t_0^i), \text{ for any } y \in V.$$

This is equivalent to

$$\underline{v}_i^{(1)}(x_0^i,t_0^i)-\underline{v}_i^{(0)}(x_0^i,t_0^i)\le \underline{v}_i^{(1)}(y,t_0^i)-\underline{v}_i^{(0)}(y,t_0^i),\ \text{for any } y\in V \tag{2.34}$$

and

$$\underline{v}_i^{(1)}(y,t_0^i)-\underline{v}_i^{(1)}(x_0^i,t_0^i)\ge \underline{v}_i^{(0)}(y,t_0^i)-\underline{v}_i^{(0)}(x_0^i,t_0^i),\ \text{for any } y\in V. \tag{2.35}$$

Recalling the definition of Δ_ω^p, we have

$$\Delta_\omega^p z_i(x_0^i,t_0^i)\ge 0. \tag{2.36}$$

At the same time, from the differentiability of $z_1(x,t)$ in $(0,T]$, we obtain

$$\frac{\partial z_i}{\partial t}(x_0^i,t_0^i)\le 0. \tag{2.37}$$

Note that

$$\begin{aligned}\Delta_\omega^p z_i(x,t) &= e^{-M_i t(p-1)}\Delta_\omega^p(\underline{v}_i^{(1)}-\underline{v}_i^{(0)})(x,t)\\
&=e^{-M_i t(p-1)}\sum_{y\sim x,y\in V}\Big|(\underline{v}_i^{(1)}(y,t)-\underline{v}_i^{(0)}(y,t))-(\underline{v}_i^{(1)}(x,t)-\underline{v}_i^{(0)}(x,t))\Big|^{p-2}\\
&\qquad\cdot[(\underline{v}_i^{(1)}(y,t)-\underline{v}_i^{(0)}(y,t)))-(\underline{v}_i^{(1)}(x,t)-\underline{v}_i^{(0)}(x,t))]\omega(x,y)\\
&=e^{-M_i t(p-1)}\sum_{y\sim x,y\in V}\Big|(\underline{v}_i^{(1)}(y,t)-\underline{v}_i^{(1)}(x,t))-(\underline{v}_i^{(0)}(y,t)-\underline{v}_i^{(0)}(x,t))\Big|^{p-2}\\
&\qquad\cdot[(\underline{v}_i^{(1)}(y,t)-\underline{v}_i^{(1)}(x,t))-(\underline{v}_i^{(0)}(y,t)-\underline{v}_i^{(0)}(x,t))]\omega(x,y),\end{aligned} \tag{2.38}$$

and thus we have

$$\begin{aligned}\Delta_\omega^p(\underline{v}_1^{(1)}-\underline{v}_1^{(0)})(x_0^i,t_0^i)=&\sum_{y\sim x_0^i,y\in V}\Big|(\underline{v}_i^{(1)}(y,t_0^i)-\underline{v}_i^{(1)}(x_0^i,t_0^i))-(\underline{v}_i^{(0)}(y,^i t_0)-\underline{v}_i^{(0)}(x_0^i,t_0^i))\Big|^{p-2}\\
&\cdot[(\underline{v}_i^{(1)}(y,t_0^i)-\underline{v}_i^{(1)}(x_0^i,t_0^i))-(\underline{v}_i^{(0)}(y,t_0^i)-\underline{v}_i^{(0)}(x_0^i,t_0^i))]\omega(x_0^i,y),\end{aligned} \tag{2.39}$$

Assume that

$$b_y=b_y^i:=\underline{v}_i^{(1)}(y,t_0^i)-\underline{v}_i^{(1)}(x_0^i,t_0^i),\ \ a_y=a_y^i:=\underline{v}_i^{(0)}(y,t_0^i)-\underline{v}_i^{(0)}(x_0^i,t_0^i)\quad\text{and}\quad J_p(t):=|t|^{p-2}t.$$

In view of (2.35), we have $b_y\ge a_y$ for any $y\sim x_0^i$ and $y\in V$. Combining this with inequality (2.18) in Lemma 2.5, we deduce that

$$|b_y-a_y|^{p-2}(b_y-a_y)=J_p(b_y-a_y)\le 2^{p-2}[J_p(b_y)-J_p(a_y)]=2^{p-2}[|b_y|^{p-2}b_y-|a_y|^{p-2}a_y],$$

which implies

$$\begin{aligned}\Delta_\omega^p(\underline{v}_i^{(1)}-\underline{v}_i^{(0)})(x_0^i,t_0^i)=&\sum_{y\sim x_0^i,y\in V}|b_y-a_y|^{p-2}(b_y-a_y)\omega(x_0^i,y)\\
&\le 2^{p-2}\sum_{y\sim x_0^i,y\in V}[|b_y|^{p-2}b_y-|a_y|^{p-2}a_y]\omega(x_0^i,y)\\
&=2^{p-2}\Big[\sum_{y\sim x_0^i,y\in V}|b_y|^{p-2}b_y\,\omega(x_0,y)-\sum_{y\sim x_0^i,y\in V}|a_y|^{p-2}a_y\,\omega(x_0^i,y)\Big]\\
&=2^{p-2}\Big[\Delta_\omega^p\underline{v}_i^{(1)}(x_0^i,t_0^i)-\Delta_\omega^p\underline{v}_i^{(0)}(x_0^i,t_0^i)\Big].\end{aligned} \tag{2.40}$$

Combining (2.40) with (2.38), we have

$$\Delta_\omega^p z_i(x_0^i,t_0^i) \le 2^{p-2}e^{-M_it_0(p-1)}\Big[\Delta_\omega^p \underline{v}_i^{(1)}(x_0^i,t_0^i) - \Delta_\omega^p \underline{v}_i^{(0)}(x_0^i,t_0^i)\Big]. \tag{2.41}$$

Note that $\underline{v}_1^{(0)} = \hat{v}_1$ and $\underline{v}_1^{(1)}$ satisfy Systems (2.1) and (2.4), respectively. By calculation, we deduce that

$$2^{p-2}e^{-M_it_0^i(p-2)}\frac{\partial z_i}{\partial t}(x_0^i,t_0^i) - D_i\Delta_\omega^p z_i(x_0^i,t_0^i) \ge -2^{p-2}e^{-M_it_0^i(p-2)}(M_i+K_i)z_i(x_0^i,t_0^i). \tag{2.42}$$

Substituting (2.36) and (2.37) into (2.42), we deduce that

$$z_i(x_0^i,t_0^i) \ge 0. \tag{2.43}$$

In view of (2.33), it follows that $z_i(x,t) \ge 0$ for $(x,t) \in V\times[0,T]$. That is, $\underline{v}_i^{(1)}(x,t) \ge \underline{v}_i^{(0)}(x,t)$ when $(x,t)\in V\times[0,T]$ for $i=1,2,3$. Thus, we complete the proof of Claim 1.

With the help of Claim 1, we have

$$\underline{\mathbf{v}}^{(\mathbf{0})}(x,t) \le \underline{\mathbf{v}}^{(\mathbf{1})}(x,t), \qquad \text{for } (x,t)\in V\times[0,T]. \tag{2.44}$$

Applying a similar process, we also have

$$\overline{\mathbf{v}}^{(\mathbf{0})}(x,t) \ge \overline{\mathbf{v}}^{(\mathbf{1})}(x,t), \qquad \text{for } (x,t)\in V\times[0,T]. \tag{2.45}$$

Claim 2. $\overline{v}_i^{(1)} \ge \underline{v}_i^{(1)}$ for $(x,t)\in V\times[0,T]$ for $i=1,2,3$.
Proof of Claim 2. Denote $w_i(x,t) := (\overline{v}_i^{(1)}(x,t) - \underline{v}_i^{(1)}(x,t))e^{-H_it}$ for $i=1,2,3$, where H_i are positive constants to be determined later. Notice that $w_i(x,t)$ is continuous on $[0,T]$ for each $x\in V$ and V is finite, and thus we can find $(x_1^i,t_1^i)\in V\times[0,T]$ such that

$$w_i(x_1^i,t_1^i) \triangleq \min_{x\in\Omega}\min_{t\in[0,T]} w_i(x,t), \tag{2.46}$$

If $t_0^i = 0$, we have $w_i(x,t) \ge w_i(x_0^i,0) = 0$, which implies that $\overline{v}_i^{(1)} \ge \underline{v}_i^{(1)}$. Next, we consider the case where $t_0^i > 0$. Equation (2.46) immediately implies that

$$w_i(x_1^i,t_1^i) \le w_i(y,t_1^i), \text{ for any } y\in V.$$

This is equivalent to

$$\overline{v}_i^{(1)}(x_1^i,t_1^i) - \underline{v}_i^{(1)}(x_1^i,t_1^i) \le \overline{v}_i^{(1)}(y,t_1^i) - \underline{v}_i^{(1)}(y,t_1^i), \text{ for any } y\in V \tag{2.47}$$

and

$$\overline{v}_i^{(1)}(y,t_1^i) - \overline{v}_i^{(1)}(x_1^i,t_1^i) \ge \underline{v}_i^{(1)}(y,t_1^i) - \underline{v}_i^{(1)}(x_1^i,t_1^i), \text{ for any } y\in V. \tag{2.48}$$

Recalling the definition of Δ_ω^p, we have

$$\Delta_\omega^p w_i(x_1^i,t_1^i) \ge 0. \tag{2.49}$$

At the same time, from the differentiability of $w_i(x,t)$ in $(0,T]$, we obtain

$$\frac{\partial w_i}{\partial t}(x_1^i,t_1^i) \le 0. \tag{2.50}$$

Note that

$$\begin{aligned}\Delta_\omega^p w_i(x,t) &= e^{-H_i t(p-1)}\Delta_\omega^p(\overline{v}_i^{(1)}-\underline{v}_i^{(1)})(x,t)\\ =&e^{-H_i t(p-1)}\sum_{y\sim x, y\in V}\Big|(\overline{v}_i^{(1)}(y,t)-\underline{v}_i^{(1)}(y,t))-(\overline{v}_i^{(1)}(x,t)-\underline{v}_i^{(1)}(x,t))\Big|^{p-2}\\ &\cdot[(\overline{v}_i^{(1)}(y,t)-\underline{v}_i^{(1)}(y,t)))-(\overline{v}_i^{(1)}(x,t)-\underline{v}_i^{(1)}(x,t))]\omega(x,y)\\ =&e^{-H_i t(p-1)}\sum_{y\sim x, y\in V}\Big|(\overline{v}_i^{(1)}(y,t)-\overline{v}_i^{(1)}(x,t))-(\underline{v}_i^{(1)}(y,t)-\underline{v}_i^{(1)}(x,t))\Big|^{p-2}\\ &\cdot[(\overline{v}_i^{(1)}(y,t)-\overline{v}_i^{(1)}(x,t))-(\underline{v}_i^{(1)}(y,t)-\underline{v}_i^{(1)}(x,t))]\omega(x,y),\end{aligned}\tag{2.51}$$

we have

$$\begin{aligned}\Delta_\omega^p(\overline{v}_i^{(1)}-\underline{v}_i^{(1)})(x_1^i,t_1^i) = &\sum_{y\sim x_1^i, y\in V}\Big|(\overline{v}_i^{(1)}(y,t_1^i)-\overline{v}_i^{(1)}(x_1^i,t_1^i))-(\underline{v}_i^{(1)}(y,^i t_1)-\underline{v}_i^{(1)}(x_1^i,t_1^i))\Big|^{p-2}\\ &\cdot[(\overline{v}_i^{(1)}(y,t_1^i)-\overline{v}_i^{(1)}(x_1^i,t_1^i))-(\underline{v}_i^{(1)}(y,t_1^i)-\underline{v}_i^{(1)}(x_1^i,t_1^i))]\omega(x_1^i,y),\end{aligned}\tag{2.52}$$

Take

$$B_y = B_y^i := \overline{v}_i^{(1)}(y,t_1^i)-\overline{v}_i^{(1)}(x_1^i,t_1^i),\ \ A_y = A_y^i := \underline{v}_i^{(1)}(y,t_1^i)-\underline{v}_i^{(1)}(x_1^i,t_1^i) \quad \text{and} \quad J_p(t) := |t|^{p-2}t.$$

In view of (2.48), we have $B_y \geq A_y$ for any $y \sim x_1^i$ and $y \in V$. Combining this with inequality (2.18) in Lemma 2.5, we deduce that

$$|B_y-A_y|^{p-2}(B_y-A_y) = J_p(B_y-A_y) \leq 2^{p-2}[J_p(B_y)-J_p(S_y)] = 2^{p-2}[|B_y|^{p-2}B_y-|A_y|^{p-2}A_y],$$

which implies

$$\begin{aligned}\Delta_\omega^p(\overline{v}_i^{(1)}-\underline{v}_i^{(1)})(x_1^i,t_1^i) &= \sum_{y\sim x_1^i, y\in V}|B_y-A_y|^{p-2}(B_y-A_y)\omega(x_1^i,y)\\ &\leq 2^{p-2}\sum_{y\sim x_1^i, y\in V}[|B_y|^{p-2}B_y-|A_y|^{p-2}A_y]\omega(x_1^i,y)\\ &= 2^{p-2}\Big[\sum_{y\sim x_1^i, y\in V}|B_y|^{p-2}B_y\,\omega(x_1^i,y)-\sum_{y\sim x_1^i, y\in V}|A_y|^{p-2}A_y\,\omega(x_1^i,y)\Big]\\ &= 2^{p-2}\Big[\Delta_\omega^p\overline{v}_i^{(1)}(x_1^i,t_1^i)-\Delta_\omega^p\underline{v}_i^{(1)}(x_1^i,t_1^i)\Big].\end{aligned}\tag{2.53}$$

Combining (2.53) with (2.51), we have

$$\Delta_\omega^p w_i(x_1^i,t_1^i) \leq 2^{p-2}e^{-H_i t_1^i(p-1)}\Big[\Delta_\omega^p\overline{v}_i^{(1)}(x_1^i,t_1^i)-\Delta_\omega^p\underline{v}_i^{(1)}(x_1^i,t_1^i)\Big].\tag{2.54}$$

Note that $\overline{v}_1^{(1)}$ and $\underline{v}_1^{(1)}$ satisfy System (2.4), and thus by calculation, we deduce that

$$\begin{aligned}&2^{p-2}e^{-H_i t_1^i(p-2)}\frac{\partial w_i}{\partial t}(x_1^i,t_1^i)-D_i\Delta_\omega^p w_i(x_1^i,t_1^i)\\ \geq& 2^{p-2}e^{-H_i t_1^i(p-1)}\Big[(-H_i+K_i)(\overline{v}_i^{(1)}-\underline{v}_i^{(1)})+F_i\Big](x_1^i,t_1^i),\end{aligned}\tag{2.55}$$

where

$$F_i = \begin{cases} G_1(\overline{v}_1^{(0)}, \underline{v}_2^{(0)}, \underline{v}_3^{(0)}) - G_1(\underline{v}_1^{(0)}, \overline{v}_2^{(0)}, \overline{v}_3^{(0)}), & i = 1, \\ G_2(\overline{v}_1^{(0)}, \overline{v}_2^{(0)}, \underline{v}_3^{(0)}) - G_2(\underline{v}_1^{(0)}, \underline{v}_2^{(0)}, \overline{v}_3^{(0)}), & i = 2, \\ G_3(\overline{v}_1^{(0)}, \overline{v}_2^{(0)}, \overline{v}_3^{(0)}) - G_3(\underline{v}_1^{(0)}, \underline{v}_2^{(0)}, \underline{v}_3^{(0)}), & i = 3. \end{cases} \tag{2.56}$$

Substituting (2.49) and (2.50) into (2.55), we deduce that

$$\left[(-H_i + K_i)(\overline{v}_i^{(1)} - \underline{v}_i^{(1)}) + F_i\right](x_1^i, t_1^i) \leq 0. \tag{2.57}$$

Next, we prove $w_i(x_1^i, t_1^i) \geq 0$ by contradiction. Suppose that $(\overline{v}_i^{(1)} - \underline{v}_i^{(1)})(x_0^i, t_0^i) = -\tau_i < 0$ on the contrary. Choosing

$$H_i := \frac{1}{\tau_i} \left|F_i(x_1^i, t_1^i)\right| + K_i + 1,$$

we find that $\left[(-H_i + K_i)(\overline{v}_i^{(1)} - \underline{v}_i^{(1)}) + F_i\right](x_1^i, t_1^i) > 0$, which contradicts (2.57). Thus, we have $w_i(x_1^i, t_1^i) = e^{-H_i t_1^i}(\overline{v}_i^{(1)} - \underline{v}_i^{(1)}) \geq 0$. In view of (2.46), it follows that $w_i(x,t) \geq 0$ for $(x,t) \in V \times [0,T]$. That is, $\overline{v}_i^{(1)}(x,t) \geq \underline{v}_i^{(1)}(x,t)$ when $(x,t) \in V \times [0,T]$ for $i = 1,2,3$. Thus, we complete the proof of Claim 2.

With the help of Claim 2, we have

$$\overline{v}^{(1)}(x,t) \leq \underline{v}^{(1)}(x,t) \qquad \text{for } (x,t) \in V \times [0,T]. \tag{2.58}$$

The conclusions from (2.44), (2.45), and (2.58) show that

$$\underline{v}^{(0)} \leq \underline{v}^{(1)} \leq \overline{v}^{(1)} \leq \overline{v}^{(0)}. \tag{2.59}$$

In the following, we demonstrate that $\overline{v}^{(1)}$ and $\underline{v}^{(1)}$ are the coupled upper and lower solutions of (1.1). To do this, it suffices to show that $\overline{v}^{(1)}$ and $\underline{v}^{(1)}$ satisfy (2.1). Using (2.4) and (2.59) for $(x,t) \in V \times (0,T]$, we have

$$\begin{aligned} \frac{\partial \overline{v}_1^{(1)}}{\partial t} - D_1 \Delta_\omega^p \overline{v}_1^{(1)} + K_1 \overline{v}_1^{(1)} &= G_1(\overline{v}_1^{(0)}, \underline{v}_2^{(0)}, \underline{v}_3^{(0)}) \geq G_1(\overline{v}_1^{(0)}, \underline{v}_2^{(1)}, \underline{v}_3^{(1)}) \\ &= G_1(\overline{v}_1^{(1)}, \underline{v}_2^{(1)}, \underline{v}_3^{(1)}) + (G_1(\overline{v}_1^{(0)}, \underline{v}_2^{(1)}, \underline{v}_3^{(1)}) - G_1(\overline{v}_1^{(1)}, \underline{v}_2^{(1)}, \underline{v}_3^{(1)})) \\ &= G_1(\overline{v}_1^{(1)}, \underline{v}_2^{(1)}, \underline{v}_3^{(1)}) + K_1(\overline{v}_1^{(0)} - \overline{v}_1^{(1)}) + \frac{\partial g_1}{\partial v_1}(\xi^{(1)}, \underline{v}_2^{(1)}, \underline{v}_3^{(1)})(\overline{v}_1^{(0)} - \overline{v}_1^{(1)}) \\ &\geq G_1(\overline{v}_1^{(1)}, \underline{v}_2^{(1)}, \underline{v}_3^{(1)}), \end{aligned}$$

where $\xi^{(1)}(x,t)$ is some intermediate value between $\overline{v}_1^{(1)}$ and $\overline{v}_1^{(0)}$. Applying the argument above to $\underline{v}_1^{(1)}$ and the other components, it follows that $\overline{v}^{(1)}$ and $\underline{v}^{(1)}$ satisfy (2.1).

Next, we employ an induction method. By selecting $\overline{v}^{(1)}$ and $\underline{v}^{(1)}$ as the coupled upper and lower solutions $\tilde{v}$ and $\hat{v}$, and following a similar argument as above, we obtain

$$\underline{v}^{(1)} \leq \underline{v}^{(2)} \leq \overline{v}^{(2)} \leq \overline{v}^{(1)}.$$

Therefore, $\overline{v}^{(2)}$ and $\underline{v}^{(2)}$ are the coupled upper and lower solutions of (1.1). The relation (2.32) is derived from the induction principle. Thus, we induce (2.31). The proof is completed. □

Theorem 2.1. *Let $\tilde{v}$ and $\hat{v}$ be a pair of coupled upper and lower solutions of System (2.3) that are bounded on $V \times (0,T]$. Let $(\overline{v}_1, \overline{v}_2, \overline{v}_3)$ and $(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ be given by (2.31). Then $\underline{v} = \overline{v}(\equiv v^*)$ holds, and v^* is the unique solution of System (2.3) for $t \in (0,T]$.*

Proof. Let v be a solution of System (2.3). By Lemma 2.6 and Lemma 2.7, we have

$$\underline{v}^{(m)} \le v \le \overline{v}^{(m)} \quad \text{for } m = 1, 2, \cdots \tag{2.60}$$

for $(x,t) \in V \times [0,T]$. Thus, $\underline{v} \le v \le \overline{v}$ for $(x,t) \in V \times [0,T]$.

In the following, we prove $\underline{v} \equiv \overline{v}$ for $(x,t) \in V \times [0,T]$. In view of Lemma 2.7, since $\hat{v}_1 \le \underline{v}_1 \le \overline{v}_1 \le \tilde{v}_1$ for $(x,t) \in V \times [0,T]$, $\underline{v}_1(x,t)$ and $\overline{v}_1(x,t)$ satisfy the relation

$$\frac{\partial \overline{v}_1}{\partial t} - D_1 \Delta_\omega^p \overline{v}_1 = g_1(\overline{v}_1, \underline{v}_2, \underline{v}_3) \tag{2.61}$$

and

$$\frac{\partial \underline{v}_1}{\partial t} - D_1 \Delta_\omega^p \underline{v}_1 = g_1(\underline{v}_1, \overline{v}_2, \overline{v}_3). \tag{2.62}$$

Claim A. $\overline{v}_1 \equiv \underline{v}_1$ for $(x,t) \in V \times [0,T]$.

Proof of Claim A. In order to prove Claim A, it is sufficient to prove $\overline{v}_1 \le \underline{v}_1$ for $(x,t) \in V \times [0,T]$.

We write $W(x,t) := (\overline{v}_1(x,t) - \underline{v}_1(x,t))e^{-Qt}$, where Q is a positive constant to be determined later. Notice that $W(x,t)$ is continuous on $[0,T]$ for each $x \in V$ and V is finite, and thus we can find $(x_1, t_1) \in V \times [0,T]$ such that

$$W(x_1, t_1) \triangleq \max_{x \in V} \max_{t \in [0,T]} W(x,t), \tag{2.63}$$

If $t_1 = 0$, we have $W(x,t) \le W(x_1, 0) = 0$, which implies that $\overline{v}_1 \le \underline{v}_1$. Next, we consider the case where $t_1 > 0$. Equation (2.63) immediately implies that

$$W(x_1, t_1) \ge W(y, t_1), \text{ for any } y \in V.$$

This is equivalent to

$$\overline{v}_1(x_1, t_1) - \underline{v}_1(x_1, t_1) \ge \overline{v}_1(y, t_1) - \underline{v}_1(y, t_1), \text{ for any } y \in V \tag{2.64}$$

and

$$\overline{v}_1(y, t_1) - \overline{v}_1(x_1, t_1) \le \underline{v}_1(y, t_1) - \underline{v}_1(x_1, t_1), \text{ for any } y \in V. \tag{2.65}$$

Recalling the definition of Δ_ω^p, we have

$$\Delta_\omega^p W(x_1, t_1) \le 0. \tag{2.66}$$

At the same time, from the differentiability of $W(x,t)$ in $(0,T]$, we obtain

$$\frac{\partial W}{\partial t}(x_1, t_1) \ge 0. \tag{2.67}$$

Note that

$$
\begin{aligned}
\Delta_\omega^p W(x,t) &= e^{-Qt(p-1)}\Delta_\omega^p(\overline{v}_1-\underline{v}_1)(x,t)\\
=&e^{-Qt(p-1)}\sum_{y\sim x, y\in V}\Big|(\overline{v}_1(y,t)-\underline{v}_1(y,t))-(\overline{v}_1(x,t)-\underline{v}_1(x,t))\Big|^{p-2}\\
&\cdot[(\overline{v}_1(y,t)-\underline{v}_1(y,t)))-(\overline{v}_1(x,t)-\underline{v}_1(x,t))]\omega(x,y)\\
=&e^{-Qt(p-1)}\sum_{y\sim x, y\in V}\Big|(\overline{v}_1(y,t)-\overline{v}_1(x,t))-(\underline{v}_1(y,t)-\underline{v}_1(x,t))\Big|^{p-2}\\
&\cdot[(\overline{v}_1(y,t)-\overline{v}_1(x,t))-(\underline{v}_1(y,t)-\underline{v}_1(x,t))]\omega(x,y),
\end{aligned}
\tag{2.68}
$$

we have

$$
\begin{aligned}
\Delta_\omega^p(\overline{v}_1-\underline{v}_1)(x_1,t_1) = \sum_{y\sim x_1, y\in V}&\Big|(\overline{v}_1(y,t_1)-\overline{v}_1(x_1,t_1))-(\underline{v}_1(y,t_1)-\underline{v}_1(x_1,t_1))\Big|^{p-2}\\
&\cdot[(\overline{v}_1(y,t_1)-\overline{v}_1(x_1,t_1))-(\underline{v}_1(y,t_1)-\underline{v}_1(x_1,t_1))]\omega(x_1,y),
\end{aligned}
\tag{2.69}
$$

We write

$$
b_{1y} := \overline{v}_1(y,t_1)-\overline{v}_1(x_1,t_1),\quad a_{1y} := \underline{v}_1(y,t_1)-\underline{v}_1(x_1,t_1)\quad \text{and}\quad J_p(t) := |t|^{p-2}t.
$$

In view of (2.65), we have $b_{1y} \le a_{1y}$ for any $y \sim x_1$ and $y \in V$. Combining this with the inequality (2.18) in Lemma 2.5, we deduce that

$$
|b_{1y}-a_{1y}|^{p-2}(b_{1y}-a_{1y}) = J_p(b_{1y}-a_{1y}) \ge 2^{p-2}[J_p(b_{1y})-J_p(a_{1y})] = 2^{p-2}[|b_{1y}|^{p-2}b_{1y}-|a_{1y}|^{p-2}a_{1y}],
$$

which implies

$$
\begin{aligned}
\Delta_\omega^p(\overline{v}_1-\underline{v}_1)(x_1,t_1) &= \sum_{y\sim x_1, y\in V}|b_{1y}-a_{1y}|^{p-2}(b_{1y}-a_{1y})\omega(x_1,y)\\
&\ge 2^{p-2}\sum_{y\sim x_1, y\in V}[|b_{1y}|^{p-2}b_{1y}-|a_{1y}|^{p-2}a_{1y}]\omega(x_1,y)\\
&= 2^{p-2}\Big[\sum_{y\sim x_1, y\in V}|b_{1y}|^{p-2}b_{1y}\,\omega(x_1,y)-\sum_{y\sim x_1, y\in V}|a_{1y}|^{p-2}a_{1y}\,\omega(x_1^i,y)\Big]\\
&= 2^{p-2}\Big[\Delta_\omega^p\overline{v}_1(x_1,t_1)-\Delta_\omega^p\underline{v}_1(x_1,t_1)\Big].
\end{aligned}
\tag{2.70}
$$

Combining (2.70) with (2.68), we have

$$
\Delta_\omega^p W(x_1,t_1) \ge 2^{p-2}e^{-Qt_1(p-1)}\Big[\Delta_\omega^p\overline{v}_1(x_1,t_1)-\Delta_\omega^p\underline{v}_1(x_1,t_1)].
\tag{2.71}
$$

Note that $\overline{v}_1$ and $\underline{v}_1$ satisfy (2.64) and (2.65), respectively, so by calculation, we deduce that

$$
\begin{aligned}
&2^{p-2}e^{-Qt_1(p-2)}\frac{\partial W}{\partial t}(x_1,t_1)-D_1\Delta_\omega^p W(x_1,t_1)\\
\le\, &2^{p-2}e^{-Qt_1(p-1)}\Big[-Q(\overline{v}_1-\underline{v}_1)+g_1(\overline{v}_1,\underline{v}_2,\underline{v}_3)-g_1(\underline{v}_1,\overline{v}_2,\overline{v}_3)\Big](x_1,t_1).
\end{aligned}
\tag{2.72}
$$

Substituting (2.66) and (2.67) into (2.72), we deduce that

$$\Big[-Q(\overline{v}_1-\underline{v}_1)+g_1(\overline{v}_1,\underline{v}_2,\underline{v}_3)-g_1(\underline{v}_1,\overline{v}_2,\overline{v}_3)\Big](x_1,t_1)\geq 0. \tag{2.73}$$

Next, we prove $W(x_1,t_1)\leq 0$ by contradiction. On the contrary, suppose that $(\overline{v}_1-\underline{v}_1)(x_1,t_1)=\delta>0$. Choosing

$$Q:=\frac{|(g_1(\overline{v}_1,\underline{v}_2,\underline{v}_3)-g_1(\underline{v}_1,\overline{v}_2,\overline{v}_3))(x_1,t_1)|}{\delta}+1,$$

we find that $\Big[-Q(\overline{v}_1-\underline{v}_1)+g_1(\overline{v}_1,\underline{v}_2,\underline{v}_3)-g_1(\underline{v}_1,\overline{v}_2,\overline{v}_3)\Big](x_1,t_1)<0$, which contradicts (2.73). Thus, we have $W(x_1,t_1)=e^{-Qt_1}(\overline{v}_1-\underline{v}_1)\leq 0$. In view of (2.63), it follows that $W(x,t)\leq 0$ for $(x,t)\in V\times[0,T]$. That is, $\overline{v}_1(x,t)\leq\underline{v}_1(x,t)$, which implies $\overline{v}_1(x,t)\equiv\underline{v}_1(x,t)$ for $(x,t)\in V\times[0,T]$. Thus we complete the proof of Claim A.

Likewise, by following a similar argument, we can see that $\overline{v}_i(x,t)\equiv\underline{v}_i(x,t)$ for $(x,t)\in V\times[0,T]$ when $i=2,3$. That is, $\underline{v}=\overline{v}(\equiv v^*)$, and v^* is the unique solution of System (2.3) for $t\in(0,T]$. □

Lemma 2.8. *Let (v_1,v_2,v_3) be a solution to System (2.3) defined for $t\in(0,T]$ for some $T\in(0,+\infty)$. Then for $i=1,2,3$, a constant M independent of T exist such that*

$$0\leq v_i(x,t)\leq M \quad \text{for } (x,t)\in V\times[0,T], \tag{2.74}$$

where $M:=\max\{\sum_{i=1}^3\sum_{x\in V}v_{i0}(x),\ \frac{nM'r}{d}\}$, $d:=\min\{h_1,\ d_1+h_2,\ d_2\}$, *and* $M':=\max\{\|v_{10}(x)\|_\infty,\ k\}$.

Proof. By directly using Lemma 2.3, we obtain the positivity of the solution. In the following, we first estimate the upper boundedness of $v_1(x,t)$. From (2.3), we have

$$\frac{\partial v_1}{\partial t}(x,t)-D_1\Delta_\omega v_1\leq rv_1\Big(1-\frac{v_1}{k}\Big),\quad (x,t)\in V\times(0,T].$$

We write $w(x,t)=\Big[\Big(\|v_{10}(x)\|_\infty^{-1}-\frac{1}{k}\Big)e^{-rt}+\frac{1}{k}\Big]^{-1}$ for $(x,t)\in V\times[0,T]$. It is easy to see that $w(x,t)\leq M':=\max\{\|v_{10}(x)\|_\infty,\ k\}$, and $w(x,t)$ satisfies

$$\begin{cases}\dfrac{\partial w}{\partial t}(x,t)-D_1\Delta_\omega w=rw\Big(1-\dfrac{w}{k}\Big), & (x,t)\in V\times(0,T],\\ w(x,0)=\|v_{10}(x)\|_\infty, & x\in V.\end{cases}$$

By the comparison principle, we have $v_1(x,t)\leq w(x,t)\leq M'$.

Let $v=\Sigma_{i=1}^3 v_i$. Then for $(x,t)\in V\times(0,T]$, we have

$$\frac{\partial v}{\partial t}-D_1\Delta_\omega^p v_1-D_2\Delta_\omega^p v_2-D_3\Delta_\omega^p v_3\leq rv_1-h_1v_1-(d_1+h_2)v_2-d_2v_3\leq M'r-dv,$$

Summing the equation above with respect to $x\in V$, in light of Lemma 2.2, we obtain

$$\frac{\partial\sum_{x\in V}v}{\partial t}\leq nM'r-d\sum_{x\in V}v. \tag{2.75}$$

By the ordinary differential equation theory, it is easy to see that

$$\sum_{x\in V}v\leq\max\Big\{\sum_{i=1}^3\sum_{x\in V}v_{i0}(x),\ \frac{nM'r}{d}\Big\}.$$

Hence we complete the proof. □

In fact, Lemma 2.8 establishes the boundedness of solutions for the model (1.1), ensuring the existence of bounded coupled upper and lower solutions to (1.1) on $V \times (0, T]$. Consequently, the unique local solution obtained in Theorem 2.1 can be extended to the maximal time. Therefore, the proof of Theorem 1.1 is completed.

3. Stability of equilibria

In this section, we show the global asymptotic stability of the trivial equilibrium $E_0 = (0, 0, 0)$, the disease-free equilibrium without predators $E_A = (S_A, 0, 0)$, the coexisting disease-free equilibrium $E_B = (S_B, 0, Y_B)$, the prey's endemic equilibrium $E_C = (S_C, I_C, 0)$, and the coexisting endemic equilibrium $E^* = (S^*, I^*, Y^*)$.

Lemma 3.1. *Suppose that for each $x \in V$, $v(x, \cdot) \in C([0, +\infty))$ is differentiable in $(0, +\infty)$. Assume that $d > 0$, $\alpha > 0$, and $\beta > 0$ are constants. If v satisfies*

$$\begin{cases} \dfrac{\partial v}{\partial t} - d\Delta_\omega^p v \geq (\leq) \alpha - \beta v, & (x, t) \in V \times (0, +\infty), \\ v(x, 0) = v_0(x) \geq (\not\equiv) 0, & x \in V, \end{cases} \tag{3.1}$$

or

$$\begin{cases} \dfrac{\partial v}{\partial t} - d\Delta_\omega^p v \geq (\leq) v(\alpha - \beta v), & (x, t) \in V \times (0, +\infty), \\ v(x, 0) = v_0(x) \geq (\not\equiv) 0, & x \in V, \end{cases} \tag{3.2}$$

then

$$\liminf_{t \to +\infty} v(x, t) \geq \frac{\alpha}{\beta} \ (\limsup_{t \to +\infty} v(x, t) \leq \frac{\alpha}{\beta}) \text{ uniformly for } x \in V. \tag{3.3}$$

Moreover, for any given $\varepsilon > 0$, $t_\varepsilon > 0$ exists such that

$$v(x, t) > \frac{\alpha}{\beta} - \varepsilon \ (v(x, t) < \frac{\alpha}{\beta} + \varepsilon), \text{ for } (x, t) \in V \times [t_\varepsilon, +\infty). \tag{3.4}$$

Proof. We only show the case of (3.1) and the case of (3.2) can be proved similarly. Assume that $z(x, t)$ is a solution of the following equation:

$$\begin{cases} \dfrac{\partial z}{\partial t} - d\Delta_\omega^p z = \alpha - \beta z, & (x, t) \in V \times (0, +\infty), \\ z(x, 0) = v_0(x) \not\equiv 0, & x \in V. \end{cases} \tag{3.5}$$

We first show that $z(x, t)$ converges to $\frac{\alpha}{\beta}$ uniformly for $x \in V$.

Since $v_0(x) \not\equiv 0$ in V, the strong maximum principle implies that $z(x, t) > 0$ for $(x, t) \in V \times (0, +\infty)$. For any small $t_1 > 0$, we set $\delta := \min_{x \in V} z(x, t_1)$, and thus $\delta > 0$. Consider $\underline{z}(x, t)$, which satisfies the following equation:

$$\begin{cases} \dfrac{d\underline{z}}{dt} = \alpha - \beta \underline{z}, & x \in V, \ t \in (t_1, +\infty), \\ \underline{z}(x, t_1) = \delta, & x \in V, \end{cases} \tag{3.6}$$

we have

$$\lim_{t \to +\infty} \underline{z}(x, t) = \frac{\alpha}{\beta} \text{ on } x \in V. \tag{3.7}$$

Note that $\Delta_\omega^p \underline{z}(x,t) \equiv 0$, and hence $\underline{z}(x,t)$ is a lower solution of System (3.5) with $t \in [t_1, +\infty)$. The maximum principle implies that $z(x,t) \geq \underline{z}(x,t)$ for $(x,t) \in V \times [t_1, +\infty)$. Combining with (3.7), we obtain

$$\liminf_{t\to+\infty} z(x,t) \geq \frac{\alpha}{\beta} \text{ uniformly for } x \in V. \tag{3.8}$$

On the other hand, following a similar argument, we have

$$\limsup_{t\to+\infty} z(x,t) \leq \frac{\alpha}{\beta} \text{ uniformly for } x \in V. \tag{3.9}$$

Combining (3.8) and (3.9), we deduce that

$$\lim_{t\to+\infty} z(x,t) = \frac{\alpha}{\beta} \text{ uniformly for } x \in V. \tag{3.10}$$

Next, since v satisfies (3.1), the comparison principle implies (3.3), which immediately implies that (3.4) holds. This completes the proof. □

In the proof above, when $\alpha \leq 0$ the limit in (3.10) is 0. Thus, we obtain the following lemma:

Lemma 3.2. *Suppose that for each $x \in V$, $v(x,\cdot) \in C([0,+\infty))$ is differentiable in $(0,+\infty)$. Assume that $d > 0$, $\alpha \leq 0$, and $\beta > 0$ are constants. If v satisfies*

$$\begin{cases} \dfrac{\partial v}{\partial t} - d\Delta_\omega^p v \leq \alpha - \beta v, & (x,t) \in V \times (0,+\infty), \\ v(x,0) = v_0(x) \geq 0, & x \in V, \end{cases} \tag{3.11}$$

then

$$\limsup_{t\to+\infty} v(x,t) \leq 0 \textit{ uniformly for } x \in V. \tag{3.12}$$

Lemma 3.3. *(i) $E_A = (S_A, 0, 0)$ exists if and only if $R_0 > 1$;*
(ii) $E_B = (S_B, 0, Y_B)$ exists if and only if $R_1 > 1$ and $R_1^* > 1$;
(iii) $E_C = (S_C, I_C, 0)$ exists if and only if $R_3 > 1$.

Proof. (i) and (ii) are obvious, so we omit their proofs. Next, we prove (iii). Consider

$$\begin{cases} \dfrac{g_1(S,I,0)}{S} = \dfrac{r}{k}(S_A - S) - \dfrac{r}{k}I - \dfrac{\beta I}{1+\alpha I} = 0, \\ \dfrac{g_2(S,I,0)}{I} = \dfrac{\beta S}{1+\alpha I} - (d_1 + h_2) = 0. \end{cases}$$

From the last equation, we have $S = \frac{d_1+h_2}{\beta}(1+\alpha I)$. Plugging it into the first equation, we obtain

$$S_A - \frac{d_1+h_2}{\beta} = \frac{d_1+h_2}{\beta}\alpha I + I + \frac{k\beta I}{r(1+\alpha I)}.$$

If $R_3 > 1$, then $S_A - \frac{d_1+h_2}{\beta} > 0$. Therefore, the unique positive S_C and I_C exists such that $g_1(S_C, I_C, 0) = 0$ and $g_2(S_C, I_C, 0) = 0$. The proof is completed. □

By the approach of comparison principle, the Lyapunov function, and the lemmas above, we can prove Theorems 1.2–1.6.

Proof of Theorem 1.2. Define a Lyapunov function

$$L(t) = \sum_{x\in V}\left(S + I + \frac{1}{q}Y\right),$$

We can easily find that $L(t) \geq 0$ for all $t \geq 0$, and $L(t) = 0$ if and only if $(S, I, Y) = (0, 0, 0)$. By direct computation, we have

$$\frac{dL}{dt} = \sum_{x\in V}(D_1\Delta_\omega^p S + g_1(S, I, Y)) + \sum_{x\in V}(D_2\Delta_\omega^p I + g_2(S, I, Y)) + \frac{1}{q}\sum_{x\in V}(D_3\Delta_\omega^p Y + g_3(S, I, Y)).$$

By Lemma 2.2, we have

$$\sum_{x\in V}\Delta_\omega^p S = 0, \quad \sum_{x\in V}\Delta_\omega^p I = 0, \quad \sum_{x\in V}\Delta_\omega^p Y = 0.$$

We then deduce that

$$\frac{dL}{dt} = \sum_{x\in V}\left(g_1(S, I, Y) + g_2(S, I, Y) + \frac{1}{q}g_3(S, I, Y)\right) = \sum_{x\in V}\left[-\frac{r}{k}S(S + I) + (r - h_1)S - (d_1 + h_2)I - \frac{d_2}{q}Y\right].$$

Since $R_0 \leq 1$, $r - h_1 \leq 0$. Hence, $\frac{dL}{dt} \leq 0$. By applying the Lyapunov-LaSalle invariance principle, we have $\lim_{t\to+\infty}(S, I, Y) = (0, 0, 0)$. This completes the proof. □

Proof of Theorem 1.3. By Lemma 2.8, we have $(S, I, Y) \geq (0, 0, 0)$. We then find that S satisfies

$$\begin{cases}\dfrac{\partial S}{\partial t} - D_1\Delta_\omega^p S \leq (r - h_1 - \dfrac{r}{k}S)S, & (x, t) \in V \times (0, +\infty),\\ S(x, 0) = S_0(x) \geq 0, & x \in V.\end{cases}$$

Applying Lemma 3.1, for any small $\varepsilon_1 > 0$, $t_1 > 0$ exists such that

$$S(x, t) < S_A + \varepsilon_1, \text{ for } (x, t) \in V \times [t_1, +\infty). \tag{3.13}$$

Moreover, we have

$$\limsup_{t\to+\infty} S(x, t) \leq S_A, \text{ uniformly for } x \in V. \tag{3.14}$$

By $R_3 < 1$, we can choose $\varepsilon_1 \leq \frac{1}{2}\left(\frac{d_1+h_2}{\beta} - S_A\right)$. Plugging (3.14) into System (1.1), we have

$$\begin{cases}\dfrac{\partial I}{\partial t} - D_2\Delta_\omega^p I \leq (\dfrac{\beta S}{1 + \alpha I} - d_1 - h_2)I \leq -\dfrac{1}{2}(d_1 + h_2 - \beta S_A)I, & (x, t) \in V \times (t_1, +\infty),\\ I(x, t)|_{t=t_1} = I(x, t_1) \geq 0, & x \in V.\end{cases}$$

Applying Lemma 3.1, for any small $\varepsilon_2 > 0$, $t_2 > t_1$ exists such that

$$I(x, t) < \varepsilon_2, \text{ for } (x, t) \in V \times [t_2, +\infty). \tag{3.15}$$

Moreover, we have

$$\limsup_{t\to+\infty} I(x, t) \leq 0, \text{ uniformly for } x \in V.$$

Combining this with $I(x,t) \geq 0$, we have

$$\lim_{t\to+\infty} I(x,t) = 0, \text{ uniformly for } x \in V.$$

Since $R_1 < 1$ and $-d_2 + \frac{qp_1 S_B}{m+S_B} = 0$, we have

$$-d_2 + \frac{qp_1(S_A + \varepsilon_1)}{m + (S_A + \varepsilon_1)} < 0.$$

We choose $\varepsilon_2 \leq \frac{m}{2qp_2}[d_2 - \frac{qp_1(S_A+\varepsilon_1)}{m+(S_A+\varepsilon_1)}]$. Plugging (3.15) into System (1.1), we have

$$\begin{cases} \dfrac{\partial Y}{\partial t} - D_3 \Delta_\omega^p I \leq (-d_2 + \dfrac{qp_1(S_A + \varepsilon_1)}{m + (S_A + \varepsilon_1)} + \dfrac{qp_2\varepsilon_2}{m})Y \leq -\dfrac{1}{2}[d_2 - \dfrac{qp_1(S_A + \varepsilon_1)}{m + (S_A + \varepsilon_1)}]Y, & (x,t) \in V \times (t_2, +\infty), \\ Y(x,t)|_{t=t_2} = Y(x,t_2) \geq 0, & x \in V. \end{cases}$$

Applying Lemma 3.1, for any small $\varepsilon_3 > 0$, $t_3 > t_2$ exists such that

$$Y(x,t) < \varepsilon_3, \text{ for } (x,t) \in V \times [t_3, +\infty). \tag{3.16}$$

Moreover, we have

$$\limsup_{t\to+\infty} Y(x,t) \leq 0, \text{ uniformly for } x \in V.$$

Combining this with $Y(x,t) \geq 0$, we have

$$\lim_{t\to+\infty} Y(x,t) = 0, \text{ uniformly for } x \in V.$$

Plugging (3.15) and (3.16) into System (1.1), we have

$$\begin{cases} \dfrac{\partial S}{\partial t} - D_1 \Delta_\omega^p S \geq (r - h_1 - \beta\varepsilon_2 - \dfrac{p_1}{m}\varepsilon_3 - \dfrac{r}{k}\varepsilon_2 - \dfrac{r}{k}S)S, & (x,t) \in V \times (t_3, +\infty), \\ S(x,t)|_{t=t_3} = S(x,t_3) \geq 0, & x \in V. \end{cases}$$

Applying Lemma 3.1, for any small $\varepsilon_4 > 0$, $t_4 > 0$ exists such that

$$S(x,t) > S_A - (1 + \frac{k\beta}{r})\varepsilon_2 - \frac{kp_1}{mr}\varepsilon_3 - \varepsilon_4, \text{ for } (x,t) \in V \times [t_4, +\infty). \tag{3.17}$$

By the arbitrariness of $\varepsilon_2, \varepsilon_3$, and ε_4, it immediately follows that

$$\liminf_{t\to+\infty} S(x,t) \leq S_A, \text{ uniformly for } x \in V. \tag{3.18}$$

Combining this with (3.14), we have

$$\lim_{t\to+\infty} S(x,t) = 0, \text{ uniformly for } x \in V.$$

This completes the proof. □

Proof of Theorem 1.4. By Lemma 2.6 (the comparison principle), we have $(S, I, Y) \geq (S_B, 0, Y_B)$. Define a Lyapunov function

$$L(t) = \sum_{x \in V} \left[\left(S - S_B - S_B \, ln \frac{S}{S_B} \right) + I + \left(Y - Y_B - Y_B \, ln \frac{Y}{Y_B} \right) \right]. \tag{3.19}$$

We can easily find that $L(t) \geq 0$ for all $t \geq 0$, and $L(t) = 0$ if and only if $(S, I, Y) = (S_B, 0, Y_B)$. By direct computation, we have

$$\begin{aligned} \frac{dL}{dt} =& \sum_{x \in V} \left(1 - \frac{S_B}{S}\right) (D_1 \Delta_\omega^p S + g_1(S, I, Y)) + \sum_{x \in V} (D_2 \Delta_\omega^p I + g_2(S, I, Y)) \\ &+ \sum_{x \in V} \left(1 - \frac{Y_B}{Y}\right) (D_3 \Delta_\omega^p Y + g_3(S, I, Y)) . \end{aligned} \tag{3.20}$$

From the definition of Δ_ω^p and Lemma 2.2 (Green formula), it is easy to calculate that

$$\begin{aligned} \sum_{x \in V} \left(1 - \frac{S_B}{S}\right) \Delta_\omega^p S &= - \sum_{x \in V} \frac{S_B}{S} \Delta_\omega^p S = -S_B \sum_{x \in V} \sum_{y \sim x, y \in V} \frac{|S(y) - S(x)|^{p-2} (S(y) - S(x))}{S(x)} \omega(x, y) \\ &= -S_B \sum_{x \in V} \sum_{y \sim x, y \in V} \frac{|S(x) - S(y)|^{p-2} (S(x) - S(y))}{S(y)} \omega(x, y). \end{aligned}$$

Adding the last two equations together, we obtain

$$\begin{aligned} \sum_{x \in V} \left(1 - \frac{S_B}{S}\right) \Delta_\omega^p S &= -\frac{S_B}{2} \sum_{x \in V} \sum_{y \sim x, y \in V} \left(\frac{(S(y) - S(x))}{S(x)} + \frac{(S(x) - S(y))}{S(y)} \right) |S(y) - S(x)|^{p-2} \omega(x, y) \\ &= -\frac{S_B}{2} \sum_{x \in V} \sum_{y \sim x, y \in V} \frac{|S(y) - S(x)|^p}{S(x) S(y)} \omega(x, y) \leq 0. \end{aligned} \tag{3.21}$$

Similar to (3.21), we have

$$\sum_{x \in V} \left(1 - \frac{Y_B}{Y}\right) \Delta_\omega^p Y \leq 0.$$

We then deduce that

$$\frac{dL}{dt} \leq \sum_{x \in V} \left[\left(1 - \frac{S_B}{S}\right) g_1(S, I, Y) + g_2(S, I, Y) + \left(1 - \frac{Y_B}{Y}\right) g_3(S, I, Y) \right]. \tag{3.22}$$

Note that $g_1(S_B, 0, Y_B)/S_B = 0$ and $g_3(S_B, 0, Y_B)/Y_B = 0$, so we have

$$
\begin{aligned}
\frac{dL}{dt} \leq & \sum_{x\in V}\left[(S - S_B)\left(\frac{g_1(S,I,Y)}{S} - \frac{g_1(S_B,0,Y_B)}{S_B}\right) + g_2(S,I,Y) + (Y - Y_B)\left(\frac{g_3(S,I,Y)}{Y} - \frac{g_3(S_B,0,Y_B)}{Y_B}\right)\right] \\
= & \sum_{x\in V}(S - S_B)\left(-\frac{r}{k}(S - S_B + I) - \frac{\beta I}{1+\alpha I} - \frac{p_1 Y_B(S - S_B) - p_1(m + S_B)(Y - Y_B)}{(m+S)(m+S_B)}\right) \\
& + \sum_{x\in V} I\left(\frac{\beta(S - S_B) + \beta S_B}{1+\alpha I} - \frac{p_2(Y - Y_B) + p_2 Y_B}{m+I} - (d_1 + h_2)\right) \\
& + \sum_{x\in V}(Y - Y_B)\left(\frac{q p_1 m(S - S_B)}{(m+S)(m+S_B)} + \frac{q p_2 I}{m+I}\right) \\
= & \sum_{x\in V}\left[\left(\frac{p_1 Y_B}{(m+S)(m+S_B)} - \frac{r}{k}\right)(S - S_B)^2 - \frac{r}{k}(S - S_B) I + \left(\frac{\beta S_B}{1+\alpha I} - (d_1 + h_2)\right) I\right] \\
& + \sum_{x\in V}\left[\frac{p_1}{(m+S)(m+S_B)}(mq - (m + S_B))(S - S_B)(Y - Y_B) + \frac{p_2}{m+I}(q-1) I (Y - Y_B) - \frac{p_2 Y_B I}{m+I}\right].
\end{aligned}
\tag{3.23}
$$

Since $R_3 < 1$ and $R_1 > 1$, then $\frac{\beta S_B}{1+\alpha I} - (d_1 + h_2) < 0$. By $p_1 < \frac{r(m+S_B)^2}{k Y_B}$, we have $\frac{p_1 Y_B}{(m+S)(m+S_B)} - \frac{r}{k} < 0$. By applying the Lyapunov-LaSalle invariance principle, we have $\lim_{t\to+\infty}(S, I, Y) = (S_B, 0, Y_B)$. Thus, the proof is completed. □

Proof of Theorem 1.5. By Lemma 2.6 (the comparison principle), we have $(S, I, Y) \geq (S_C, I_C, 0)$. Define a Lyapunov function

$$
L(t) = \sum_{x\in V}\left[H_1\left(S - S_C - S_C \ln\frac{S}{S_C}\right) + H_2\left(I - I_C - I_C \ln\frac{I}{I_C}\right) + Y\right], \tag{3.24}
$$

where

$$
H_1 = \max\left\{\frac{mq}{m+S_C}, \frac{mq(1+\alpha I_C)}{m+I_C}\right\}, \quad H_2 = \frac{H_1}{1+\alpha I_C}.
$$

We can easily find that $L(t) \geq 0$ for all $t \geq 0$, and $L(t) = 0$ if and only if $(S, I, Y) = (S_C, I_C, 0)$. By direct computation, we have

$$
\begin{aligned}
\frac{dL}{dt} = & H_1\sum_{x\in V}\left(1 - \frac{S_C}{S}\right)(D_1\Delta_\omega^p S + g_1(S,I,Y)) + H_2\sum_{x\in V}\left(1 - \frac{I_C}{I}\right)(D_2\Delta_\omega^p I + g_2(S,I,Y)) \\
& + \sum_{x\in V}(D_3\Delta_\omega^p Y + g_3(S,I,Y)).
\end{aligned}
\tag{3.25}
$$

Similar to (3.21), we have

$$
\sum_{x\in V}\left(1 - \frac{S_C}{S}\right)\Delta_\omega^p S \leq 0 \text{ and } \sum_{x\in V}\left(1 - \frac{I_C}{I}\right)\Delta_\omega^p I \leq 0. \tag{3.26}
$$

We then deduce that

$$
\frac{dL}{dt} \leq \sum_{x\in V}\left[H_1\left(1 - \frac{S_C}{S}\right)g_1(S,I,Y) + H_2\left(1 - \frac{I_C}{I}\right)g_2(S,I,Y) + g_3(S,I,Y)\right]. \tag{3.27}
$$

Note that $g_1(S_C, 0, Y_C)/S_C = 0$ and $g_2(S_C, I_C, 0)/I_C = 0$, and so we have

$$
\begin{aligned}
\frac{dL}{dt} \le & \sum_{x\in V}\left[H_1(S-S_C)\left(\frac{g_1(S,I,Y)}{S}-\frac{g_1(S_C,I_C,0)}{S_C}\right)\right] \\
& +\sum_{x\in V}\left[H_2(I-I_C)\left(\frac{g_2(S,I,Y)}{I}-\frac{g_2(S_C,I_C,0)}{I_C}\right)+g_3(S,I,Y)\right] \\
= & \sum_{x\in V}H_1(S-S_C)\left(-\frac{r}{k}(S-S_C+I-I_C)-\frac{\beta(I-I_C)}{(1+\alpha I)(1+\alpha I_C)}-\frac{p_1Y}{m+S}\right) \\
& +\sum_{x\in V}H_2(I-I_C)\left(\frac{\beta(S-S_C)(1+\alpha I_C)+\beta\alpha S_C(I-I_C)}{(1+\alpha I)(1+\alpha I_C)}-\frac{p_2Y}{m+I}\right) \\
& +\sum_{x\in V}\left(-d_2+\frac{qp_1m(S-S_C)}{(m+S)(m+S_C)}+\frac{qp_1S_C}{m+S_C}+\frac{qp_2m(I-I_C)}{(m+I)(m+I_C)}+\frac{qp_2I_C}{m+I_C}\right) \\
= & \sum_{x\in V}\left[-\frac{r}{k}H_1(S-S_C)^2-\frac{r}{k}H_1(S-S_C)(I-I_C)-\frac{\alpha\beta S_C}{(1+\alpha I)(1+\alpha I_C)}H_2(I-I_C)^2\right] \\
& +\sum_{x\in V}\left[\frac{p_1(mq-(m+S_C)H_1)}{(m+S)(m+S_C)}(S-S_C)Y+\frac{p_2(mq-(m+I_C)H_2)}{(m+I)(m+I_C)}(I-I_C)Y\right] \\
& +\sum_{x\in V}\left[\left(\frac{qp_1S_C}{m+S_C}+\frac{qp_2I_C}{m+I_C}-d_2\right)Y\right].
\end{aligned}
\tag{3.28}
$$

Since $R_2 > 1$, $\frac{qp_1S_C}{m+S_C}+\frac{qp_2I_C}{m+I_C}-d_2 < 0$. By calculation, we have $mq-(m+S_C)H_1 \le 0$ and $mq-(m+I_C)H_2 \le 0$. Applying the Lyapunov-LaSalle invariance principle, we have $\lim_{t\to+\infty}(S, I, Y) = (S_C, I_C, 0)$. This completes the proof. □

Proof of Theorem 1.6. By Lemma 2.6 (the comparison principle), we have $(S, I, Y) \ge (S^*, I^*, Y^*)$. Define a Lyapunov function

$$
L(t)=\sum_{x\in V}\left[H_1\left(S-S^*-S^*\ln\frac{S}{S^*}\right)+H_2\left(I-I^*-I^*\ln\frac{I}{I^*}\right)+\left(Y-Y^*-Y^*\ln\frac{Y}{Y^*}\right)\right], \tag{3.29}
$$

where

$$
H_1=\max\left\{\frac{mq}{m+S^*},\ \frac{mq(1+\alpha I^*)}{m+I^*}\right\},\quad H_2=\frac{H_1}{1+\alpha I^*}.
$$

We can easily find that $L(t) \ge 0$ for all $t \ge 0$, and $L(t) = 0$ if and only if $(S, I, Y) = (S^*, I^*, Y^*)$. By direct computation, we have

$$
\begin{aligned}
\frac{dL}{dt} = & H_1\sum_{x\in V}\left(1-\frac{S^*}{S}\right)(D_1\Delta_\omega^pS+g_1(S,I,Y))+H_2\sum_{x\in V}\left(1-\frac{I^*}{I}\right)(D_2\Delta_\omega^pI+g_2(S,I,Y)) \\
& +\sum_{x\in V}\left(1-\frac{Y^*}{Y}\right)(D_3\Delta_\omega^pY+g_3(S,I,Y)).
\end{aligned}
\tag{3.30}
$$

Similar to (3.21), we have

$$
\sum_{x\in V}\left(1-\frac{S^*}{S}\right)\Delta_\omega^pS\le 0,\quad \sum_{x\in V}\left(1-\frac{I^*}{I}\right)\Delta_\omega^pI\le 0 \ \text{ and } \ \sum_{x\in V}\left(1-\frac{Y^*}{Y}\right)\Delta_\omega^pY\le 0. \tag{3.31}
$$

We then deduce that

$$\frac{dL}{dt} \leq \sum_{x\in V}\left[H_1\left(1-\frac{S^*}{S}\right)g_1(S,I,Y)+H_2\left(1-\frac{I^*}{I}\right)g_2(S,I,Y)+\left(1-\frac{Y^*}{Y}\right)g_3(S,I,Y)\right]. \tag{3.32}$$

Note that $g_1(S^*,I^*,Y^*)/S^*=0$, $g_2(S^*,I^*,Y^*)/I^*=0$ and $g_3(S^*,I^*,Y^*)/Y^*=0$, and so we have

$$\begin{aligned}
\frac{dL}{dt} \leq& \sum_{x\in V}\left[H_1(S-S^*)\left(\frac{g_1(S,I,Y)}{S}-\frac{g_1(S^*,I^*,Y^*)}{S^*}\right)+H_2(I-I^*)\left(\frac{g_2(S,I,Y)}{I}-\frac{g_2(S^*,I^*,Y^*)}{I^*}\right)\right]\\
&+\sum_{x\in V}(Y-Y^*)\left(\frac{g_3(S,I,Y)}{Y}-\frac{g_3(S^*,I^*,Y^*)}{Y^*}\right)\\
=&\sum_{x\in V}H_1(S-S^*)\left(-\frac{r}{k}(S-S^*+I-I^*)+\frac{\beta I^*}{1+\alpha I^*}-\frac{\beta I}{1+\alpha I}+\frac{p_1Y^*}{m+S^*}-\frac{p_1Y}{m+S}\right)\\
&+\sum_{x\in V}H_2(I-I^*)\left(\frac{\beta S}{1+\alpha I}-\frac{\beta S^*}{1+\alpha I^*}+\frac{p_2Y^*}{m+I^*}-\frac{p_2Y}{m+I}\right)\\
&+\sum_{x\in V}q(Y-Y^*)\left(\frac{p_1S}{m+S}-\frac{p_1S^*}{m+S^*}+\frac{p_2I}{m+I}-\frac{p_2I^*}{m+I^*}\right)\\
=&\sum_{x\in V}\left[H_1\left(\frac{p_1Y^*}{(m+S)(m+S^*)}-\frac{r}{k}\right)(S-S^*)^2+H_2\left(\frac{p_2Y^*}{(m+I)(m+I^*)}-\frac{\alpha\beta S^*}{(1+\alpha I)(1+\alpha I^*)}\right)(I-I^*)^2\right]\\
&+\sum_{x\in V}\left[-\frac{r}{k}H_1(S-S^*)(I-I^*)+\frac{p_1(mq-(m+S^*)H_1)}{(m+S)(m+S^*)}(S-S^*)(Y-Y^*)\right]\\
&+\sum_{x\in V}\frac{p_2(mq-(m+I^*)H_2)}{(m+I)(m+I^*)}(I-I^*)(Y-Y^*).
\end{aligned} \tag{3.33}$$

Since $p_1<\frac{r(m+S^*)^2}{kY^*}$ and $p_2<\frac{\alpha\beta S^*(m+I^*)^2}{Y^*(1+\alpha I^*)^2}$, then $\frac{p_1Y^*}{(m+S)(m+S^*)}-\frac{r}{k}<0$ and $\frac{p_2Y^*}{(m+I)(m+I^*)}-\frac{\alpha\beta S^*}{(1+\alpha I)(1+\alpha I^*)}<0$. By calculation, we have $mq-(m+S^*)H_1\leq 0$ and $mq-(m+I^*)H_2\leq 0$. Applying the Lyapunov-LaSalle invariance principle, we have $\lim_{t\to+\infty}(S,I,Y)=(S^*,I^*,Y^*)$. Thus, the proof is completed. □

4. Numerical simulations

In order to deal with the numerical solutions of System (1.1), we denote the adjacent matrix of the graph $\mathcal{G}$ by G and take the mean number of passengers per unit of time moving from x_i to x_j as the weighted matrix. Here, we assume that $(\omega_{ij})_{nn}$ is a symmetric matrix. Set $S(x,t)=U_1(i,t), I(x,t)=U_2(i,t)$, and $Y(x,t)=U_3(i,t)$; we then rewrite System (1.1) as the following ordinary differential equations:

$$\frac{dU_k(i,t)}{dt}=f_k(U_1,U_2,U_3)(i,t)+D_k\sum_{j=1}^{n}L_{ij}U_k(j,t),\quad \text{for } k=1,2,3 \text{ and } i=1,2,\cdots,n, \tag{4.1}$$

where $L \doteq \{L_{ij}\}$ is called the p-Laplacian matrix, which is defined by

$$L_{ij} = \begin{cases} |U_k(j,t) - U_k(i,t)|^{p-2}\,\omega_{ij}, & j \neq i, \\ -\sum_{j=1}^{n} |U_k(j,t) - U_k(i,t)|^{p-2}\,\omega_{ij}, & j = i. \end{cases}$$

During our numerical simulations, we assume that the initial total susceptible prey population is 8000 and the the initial total predator population is 500, which are apportioned equally to 50 nodes. The initial infected prey population is 5, located at the first node.

Theorem 1.4 indicates that in the case of $R_1 > 1$, $R_1^* > 1$, $R_3 < 1$, and $p_1 < \frac{r(m+S_B)^2}{kY_B}$, System (1.1) will converge to disease-free equilibrium at an arbitrary node. In addition to the long-time behavior, it is natural to investigate the small-time behavior for System (1.1) because the total infected population is a key character during the control of the epidemic. Thus, we take the following parameter values (time = $[t]$ = one month):

$$D_i = 1.5(i = 1,2,3),\ \ p = 2,\ \ r = 3,\ \ k = 200,\ \ \alpha = 2.5,\ \ \beta = 0.015,\ \ m = 0.5,$$
$$p_1 = 0.62,\ \ p_2 = 0.86,\ \ h_1 = 1,\ \ h_2 = 2,\ \ d_1 = 0.54,\ \ d_2 = 0.42,\ \ q = 0.68.$$

We assume that the graph $\mathcal{G}$ is a Watts-Strogatz network, where $n = 50$ in the vertices. From a mathematical point of view, the Watts-Strogatz network is an undirected graph where some of the connections between the nodes are determined and others are random. The edge set E is connected with an average degree of 4 and the probability of rewiring a link is 3/20. The average degree of 4 indicates that each node has two other nodes connected to both its left and right sides. The 3/20 probability of rewiring a link indicates that each node has a probability of 3/20 for it to be connected to any other node in the graph. The rewired edge aims to introduce some stochastic routines between two vertices in the model (1.1) due to the fact that the individual population sometimes unavoidably walks randomly.

We describe, in the first panel in Figures 1 and 2, the initial susceptible prey population takes 160 on all nodes and the initial infected prey population takes 5 on the first node and 0 on all the others. Employing a Runge-Kutta scheme for the time integration, we make a portrait of the spatial densities of the susceptible prey population and the infected prey population on the network for time t = 1, 2, 5 in Figures 1 and 2, showing that, for different nodes, the network exhibits a spatially inhomogeneous behavior. However, the effect of network on the infected population cannot be indicated here; therefore, we describe the temporal solution of System (1.1), without considering the network's structure but only arranging the nodes in a vertical column. We simulate, in the left panel of Figure 3, the dynamic behavior of System (1.1) without Laplacian diffusion, i.e., ordinary differential equations. At first, the solution of the infected population rises to a peak, but it then decreases toward 0. Comparatively, we simulate the dynamical behavior of system (1.1) with Laplacian diffusion with a spatial networked structure subject to the Watts-Strogatz small-world network. From the right panel of Figure 3, we see that the solution of the infected population for each node exhibits a dynamic behavior similar to the solution of an ordinary differential equation. However, even though each node has the same parameters as System (1.1), their peaks are different.

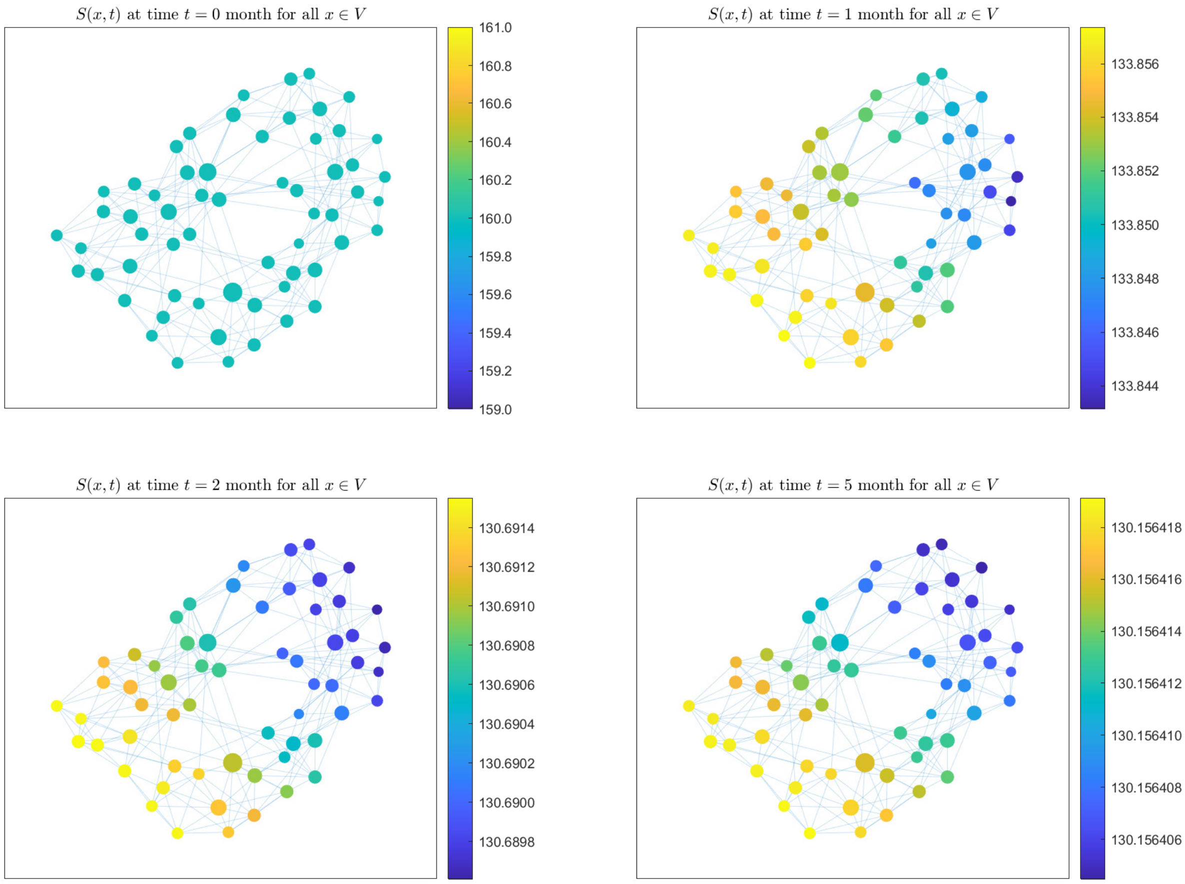

Figure 1. Solutions of S at time points $t = 0$, 1, 2, and 5 in the Watts-Strogatz network. The node size corresponds to the node degree; the node colorbar corresponds to the population.

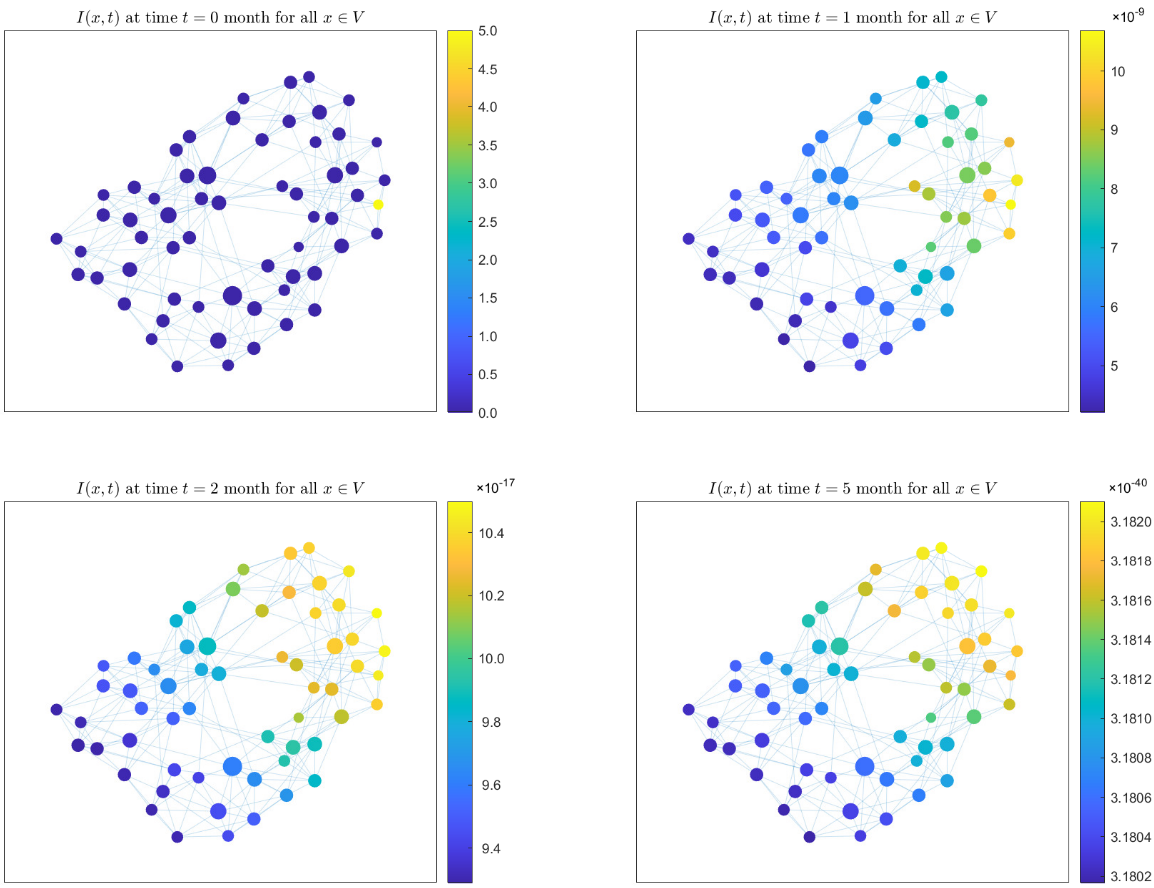

Figure 2. Solutions of I at time points t = 0, 1, 2, and 5 in the Watts-Strogatz network. The node size corresponds to the node degree; the node colorbar corresponds to the population.

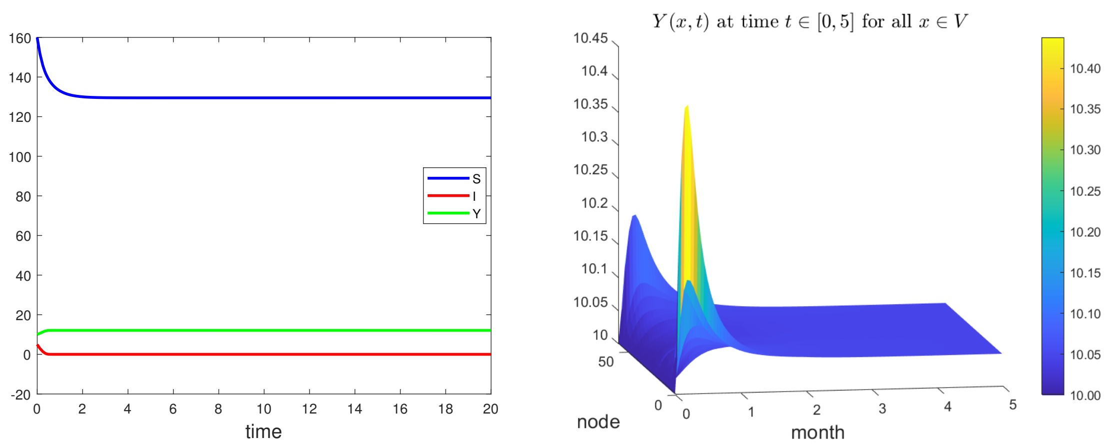

Figure 3. A comparison of dynamical behavior between ordinary differential equations and the weighted networked diffusion system. On the right panel, every grid of the vertical coordinate represents a node.

5. Discussion

A great deal of literature so far on eco-epidemiological transmission dynamics has focused especially on ordinary differential equations. Consequently, uncertainties persist regarding how population mobility influences epidemic outbreaks, necessitating the inclusion of spatial factors within the eco-epidemiological dynamic framework. Laplacian diffusion systems with spatially homogeneous parameters (Murray [17], Yang et al. [23], Lin and Zhu [24]) and spatially heterogeneous parameters (Allen et al. [21, 22]) have been applied to spatio-temporal dynamic systems to study the epidemic spreading along space, as well as the factor of population mobility. Laplacian operators in these studies have been introduced to describe the population diffusion, in which every single individual is assumed to obey Brown's random walk, where the moving direction of population is spontaneous and not affected by the environment. Our eco-epidemiological model in a network is different, however, since the movement of population in each vertex depends on the topological structure of the network. Both the short-term and long-term dynamic behaviors of the eco-epidemiological epidemic model have been studied. To the best of our knowledge, no eco-epidemiological model has used graph p-Laplacian diffusion to describe the spatio-temporal spreading.

With the basic reproduction number $R_1 > 1$, $R_1^* > 1$, $R_3 < 1$, and $p_1 < \frac{r(m+S_B)^2}{kY_B}$, Theorem 1.4 shows that System (1.1) admits a stable coexisting disease-free equilibrium if the initial data of the prey and predator population is large enough (the initial condition is natural). Our results indicate that the existence of a graph p-Laplacian defined on a weighted connected graph does not alter the asymptotic behavior of the eco-epidemiological model. However, our numerical simulations have illustrated that, in the case of stable coexisting disease-free equilibrium when the graph p-Laplacian corresponding to adjacent matrix is subject to the Watts-Strogatz small-world network, the peak infected population admits a spatially inhomogeneous distribution even it is free from disease.

Author contributions

Ling Zhou and Zuhan Liu: Conceptualization, Methodology, Writing-original draft; Guoqing Ding: Validation, Writing-review and editing. The final version of the manuscript was read and approved by all authors.

Use of Generative-AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to express their sincere thanks to the anonymous reviewers for their helpful comments. The work is partially supported by the National Natural Science Foundation of China (11771380) and the Natural Science Foundation of Jiangsu Province (BK20191436).

Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. R. M. Anderson, R. M. May, The invasion, persistence and spread of infectious diseases within animal and plant communities, *Philos. T. R. Soc. B*, **314** (1986), 533–570. https://doi.org/10.1098/rstb.1986.0072
2. S. A. Temple, Do predators always capture substandard individuals disproportionately from prey population? *Ecology*, **68** (1987), 669–674. https://doi.org/10.2307/1938472
3. Y. Xiao, L. Chen, Modeling and analysis of a predator-prey model with disease in the prey, *Math. Biosci.*, **171** (2001), 59–82. https://doi.org/10.1016/S0025-5564(01)00049-9
4. C. Packer, R. D. Holt, P. J. Hudson, K. D. Lafferty, A. P. Dobson, Keeping the herds healthy and alert:implications of predator control for infectious disease, *Ecol. Lett.*, **6** (2010), 797–802. https://doi.org/10.1046/j.1461-0248.2003.00500.x
5. R. D. Holt, M. Roy, Predation can increase the prevalence of infectious disease, *Am. Nat.*, **169** (2007), 690–699. https://doi.org/10.1086/513188
6. R. K. Upadhyay, P. Roy, Spread of a disease and its effect on population dynamics in an eco-epidemiological system, *Commun. Nonlinear Sci.*, **19)** (2014), 4170–4184. https://doi.org/10.1016/j.cnsns.2014.04.016
7. X. Liu, S. Liu, Dynamics of a predator-prey system with inducible defense and disease in the prey, *Nonlinear Anal. Real*, **71** (2023), 103802. https://doi.org/10.1016/j.nonrwa.2022.103802
8. M. Wang, S. Yao, The dynamics of an eco-epidemiological prey-predator model with infectious diseases in prey, *Commun. Nonlinear Sci.*, **132** (2024), 107902. https://doi.org/10.1016/j.cnsns.2024.107902
9. S. Pandey, D. Das, U. Ghosh, S. Chakraborty, Bifurcation and onset of chaos in an eco-epidemiological system with the influence of time delay, *Chaos*, **34** (2024), 063122. https://doi.org/10.1063/5.0177410
10. S. Dey, D. K. Das, S. Ghorai, M. Banerjee, Spatially heterogeneous eco-epidemic model: Stabilizing role of non-local disease transmission, *Commun. Nonlinear Sci.*, **138** (2024), 108238. https://doi.org/10.1016/j.cnsns.2024.108238
11. F. Brauer, A. C. Soudack, Stability regions and transition phenomena for harvested predator-prey systems, *J. Math. Biol.*, **7** (1979), 319–337. https://doi.org/10.1007/BF00275152
12. F. Brauer, A. C. Soudack, Stability regions in predator-prey systems with constant rate prey harvesting, *J. Math. Biol.*, **8** (1979) 55–71. https://doi.org/10.1007/BF00280586
13. C. Dai, M. Tang, Coexistence region and global dynamics of a harvested predator-prey system, *SIAM J. Appl. Math.*, **58** (1998), 193–210. https://doi.org/10.1137/S0036139994275799
14. R. Cristiano, M. M. Henao, D. J. Pagano, Global stability of a Lotka-Volterra piecewise-smooth system with harvesting actions and two predators competing for one prey, *J. Math. Anal. Appl.*, **522** (2023), 126998. https://doi.org/10.1016/j.jmaa.2023.126998

15. N. Bairagi, S. Chaudhuri, J. Chattopadhyay, Harvesting as a disease control measure in an eco-epidemiological system—A theoretical study, *Math. Biosci.*, **217** (2009), 134–144. https://doi.org/10.1016/j.mbs.2008.11.002

16. R. Bhattacharyya, B. Mukhopadhyay, On an eco-epidemiological model with prey harvesting and predator switching: Local and global perspectives, *Nonlinear Anal. Real*, **11** (2010), 3824–3833. https://doi.org/10.1016/j.nonrwa.2010.02.012

17. J. D. Murray, *Mathematical biology II: Spatial models and biomedical applications*, New York: Springer, 2003. https://doi.org/10.1007/b98869

18. R. Cantrell, C. Cosner, *Spatial ecology via reaction-diffusion equations*, John Wiley & Sons, 2004. https://doi.org/10.1002/0470871296

19. K. Kuto, H. Matsuzawa, R. Peng, Concentration profile of the endemic equilibria of a reaction-diffusion-advection SIS epidemic model, *Calc. Var.*, **56** (2017), 112. https://doi.org/10.1007/s00526-017-1207-8

20. J. Cui, X. Tao, H. Zhu, An SIS infection model incorporating media coverage, *Rocky Mountain J. Math.*, **38** (2008), 1323–1334. https://doi.org/10.1216/RMJ-2008-38-5-1323

21. L. J. S. Allen, B. M. Bolker, Y. Lou, A. L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic disease patch model, *SIAM J. Appl. Math.*, **67** (2007), 1283–1309. https://doi.org/10.1137/060672522

22. L. J. S. Allen, B. M. Bolker, Y. Lou, A. L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic reaction diffusion model, *Discrete Cont. Dyn. Syst.*, **21** (2008), 1–20. https://doi.org/10.3934/dcds.2008.21.1

23. F. Yang, W. Li, S. Ruan, Dynamics of a nonlocal dispersal SIS epidemic model with Neumann boundary conditions, *J. Differ. Equ.*, **267** (2019), 2011–2051. https://doi.org/10.1016/j.jde.2019.03.001

24. Z. Lin, H. Zhu, Spatial spreading model and dynamics of West Nile virus in birds and mosquitoes with free boundary, *J. Math. Biol.*, **75** (2017), 1381–1409. https://doi.org/10.1007/s00285-017-1124-7

25. H. Li, R. Peng, Z. Wang, On a diffusive susceptible-infected-susceptible epidemic model with mass action mechanism and birth-death effect: Analysis, simulations, and comparison with other mechanisms, *SIAM J. Appl. Math.*, **78** (2018), 2129–2153. https://doi.org/10.1137/18M1167863

26. C. Lei, F. Li, J. Liu, Theoretical analysis on a diffusive SIR epidemic model with nonlinear incidence in a heterogeneous environment, *Discrete Cont. Dyn. B*, **23** (2018), 4499–4517. https://doi.org/10.3934/dcdsb.2018173

27. S. Oruganti, J. Shi, R. Shivaji, Logistic equation wtih the p-Laplacian and constant yield harvesting, *Abstr. Appl. Anal.*, **9** (2004), 723–727. http://dx.doi.org/10.1155/S1085337504311097

28. C. Liu, P. Li, Global existence for a chemotaxis-haptotaxis model with p-Laplacian, *Commun. Pure Appl. Anal.*, **19** (2020), 1399–1419. https://doi.org/10.3934/cpaa.2020070

29. H. Yang, S. Qiu, L. Xu, Global weak solution in a fully parabolic two-species chemotaxis system with slow p-Laplacian diffusion, *Z. Angew. Math. Phys.*, **73** (2022), 256. https://doi.org/10.1007/s00033-022-01892-9

30. S. Li, C. Liu, Y. Zhao, Global existence for a three-species predator-prey model with slow p-Laplacian diffusion, *J. Math. Anal. Appl.*, **548** (2025), 129426. https://doi.org/10.1016/j.jmaa.2025.129426
31. R. Pastor-Satorras, C. Castellano, P. V. Mieghem, A. Vespignani, Epidemic processes in complex networks, *Rev. Mod. Phys.*, **87** (2015), 925–979. https://doi.org/10.1103/RevModPhys.87.925
32. Y. Du, B. Lou, R. Peng, M. Zhou, The Fisher-KPP equation over simple graphs: Varied persistence states in river networks, *J. Math. Biol.*, **80** (2020), 1559–1616. https://doi.org/10.1007/s00285-020-01474-1
33. F. Bauer, P. Horn, Y. Lin, G. Lippner, D. Mangoubi, S. T. Yau, Li-Yau inequality on graphs, *J. Differential Geom.*, **99** (2015), 359–405. https://doi.org/10.4310/jdg/1424880980
34. S. Y. Chung, M. J. Choi, A new condition for blow-up solutions to discrete semilinear heat equations on networks, *Comput. Math. Appl.*, **74** (2017), 2929–2939. https://doi.org/10.1016/j.camwa.2017.07.030
35. A. Grigoryan, Y. Lin, Y. Yang, Yamabe type equations on graphs, *J. Differ. Equ.*, **261** (2016), 4924–4943. https://doi.org/10.1016/j.jde.2016.07.011
36. C. Tian, S. Ruan, Pattern formation and synchronism in an allelopathic plankton model with delay in a network, *SIAM J. Appl. Dyn. Syst.*, **18** (2019), 531–557. https://doi.org/10.1137/18M1204966
37. Z. Liu, C. Tian, S. Ruan, On a network model of two competitors with applications to the invasion and competition of Aedes albopictus and Aedes aegypti mosquitoes in the United States, *SIAM J. Appl. Math.*, **80** (2020), 929–950. https://doi.org/10.1137/19M1257950
38. C. Tian, Z. Liu, S. Ruan, Asymptotic and transient dynamics of SEIR epidemic models on weighted networks, *Eur. J. Appl. Math.*, **34** (2023), 238–261. https://doi.org/10.1017/S0956792522000109
39. L. Zhou, Z. Liu, Blow up in a p-Laplacian mutualistic model based on graphs, *AIMS Math.*, **8** (2023), 28210–28218. https://doi.org/10.3934/math.20231444
40. Z. Liu, C. Tian, A weighted networked SIRS epidemic model, *J. Differ. Equ.*, **269** (2020), 10995–11019. https://doi.org/10.1016/j.jde.2020.07.038
41. N. C. Minh, D. T. Quyet, A. T. Duong, Liouville-type theorems for systems of elliptic inequalities involving p-Laplace operator on weighted graphs, *Commun. Pure Appl. Anal.*, **24** (2025), 641–660. https://doi.org/10.3934/cpaa.2025008
42. L. Brasco, E. Parini, The second eigenvalue of the fractional p-Laplacian, *Adv. Calc. Var.*, **9(4)** (2016), 323–355. https://doi.org/10.1515/acv-2015-0007

AIMS Press