



Research article

Bayesian and non-Bayesian estimations for a flexible reduced logarithmic-inverse Lomax distribution under progressive hybrid type-I censored data with a head and neck cancer application

Ehab M. Almetwally¹, Ahlam H. Tolba^{2,*}, and Dina A. Ramadan²

¹ Department of Mathematics and Statistics, Faculty of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia; emalmetwally@imamu.edu.sa

² Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 33516, Egypt; dr_ahamdy156@mans.edu.eg, Dinaramadan21@mans.edu.eg

* **Correspondence:** Email: dr_ahamdy156@mans.edu.eg.

Abstract: This article used classical and Bayesian procedures to discuss the statistical inferences to the unknown parameters of the flexible reduced logarithmic-inverse Lomax distribution based on a progressive hybrid type-I censoring (PHT-ICS) approach. The maximum likelihood and Bayesian estimation techniques estimate unknown parameters, reliability, and hazard rate functions. The investigation of Bayes estimators makes use of the gamma priors and symmetric and asymmetric loss functions. Additionally, the classical asymptotic confidence intervals and the credible intervals were explored. A simulation study and a bladder cancer application are presented to illustrate the proposed estimation methods. The flexible reduced logarithmic-inverse Lomax (FRL-IL) distribution's relevance and the given estimators' effectiveness were evaluated using failure data from a sample of 44 patients diagnosed with head and neck cancer. The results show that likelihood based on the Markov chain Monte Carlo (MCMC) method has the smallest mean squared error (MSE) values. In contrast, Bayes estimators with the assumed informative prior outperformed maximum likelihood estimators and Bayes estimators under the squared error loss function in terms of MSEs.

Keywords: flexible reduced logarithmic-inverse Lomax distribution; maximum likelihood estimators; Bayesian method; Markov chain Monte Carlo; progressive hybrid type-I censoring data; head and neck cancer application

Mathematics Subject Classification: 62F15, 62N02

1. Introduction

Data processing and computational methods are two of the most significant sciences, with many applications in key fields including engineering, medicine, and agriculture. The idea of censored samples originated from the need to discover a method for statistical inference for vast data while limiting time and expense by employing tiny samples taken in a particular way. Observed data is censored in nature in several contexts, such as life-testing studies and reliability assessments. The hybrid censoring method, which combines parts of type-I and type-II censoring, was first presented by Epstein [20]. The fact that units cannot be removed from an experiment at any time other than when it is finished is one of the primary issues with type-I, type-II, and hybrid censoring (HC) techniques. Progressive type-II (Pro-II) censoring is a more extensive technique to address this problem. These censoring strategies are thoroughly discussed, along with some recent developments, by Balakrishnan and Cramer [6], Kundu and Howlader [26], El-Saeed et al. [17], and Alsadat et al. [4]. The progressive type-II censoring scheme's drawback is that the experiment might continue for a long time if the units are highly reliable. Therefore, a progressive hybrid censoring scheme (PHT-ICS) was proposed by Kundu and Joarder [27]. A progressive hybrid censoring strategy will limit the experiment's duration to T . Conversely, one limitation of the progressive hybrid censoring method is that it cannot be applied in situations where there are a small number of failures before time T .

The PHT-ICS censoring method is explained briefly below. Consider a lifetime test with n independent and identically distributed units placed on a certain life testing experiment at time zero. Because the integer m ($m < n$) is fixed at the starting point of the test, each random variable, $Y_{1:m:n}, Y_{2:m:n}, \dots, Y_{m:m:n}$, is identically distributed. The lifetime distributions of the n components are denoted by $Y_{1:m:n}, Y_{2:m:n}, \dots, Y_{m:m:n}, Y_{m+1:m:n}, \dots, Y_{J:m:n}$, as the integer $m < n$ is fixed at the start of the test and has the withdrawal scheme $R_1, R_2, \dots, R_m$, so that $\sum_{j=1}^m R_j = n - m$ as shown in Figure 1. The time point T is also predetermined. The R_1 survival units are removed when the test is completed and the first failure occurs. The pro-II methods are followed, but the test will be terminated at the shortest time between T and the time of the m -th failure, whichever occurs first. If $Y_{m:n} < T$, the basic Pro-II censoring scheme is used. Otherwise, if $Y_{m:n} > T$, only J failures occur before the time point T , where $0 < J < m$; then, at the time point T , all remaining R_j^* units are deleted and the experiment terminates.

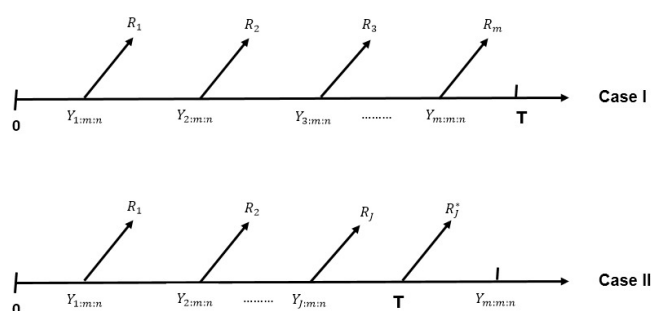


Figure 1. Schematic representation of type-I progressive hybrid censoring.

The two cases as Case I and Case II are given, respectively:

Case I : $\{Y_{1:m:n} < Y_{2:m:n} < \dots < Y_{m:m:n}\}$, if $Y_{m:m:n} < T$.

Case II: $\{Y_{1:m:n} < Y_{2:m:n} < \dots < Y_{m:m:n} < Y_{m+1:m:n} < \dots < Y_{J:m:n}\}$, if $Y_{J:m:n} < T < Y_{J+1:m:n}$, where $Y_{J:m:n} < T < Y_{J+1:m:n} < \dots < Y_{m:m:n}$ and $Y_{J+1:m:n} < \dots < Y_{m:m:n}$ are not observed. The advantage of combining Pro-II and hybrid Type-I is that Pro-II has no time constraints and the researcher may be obligated to a set period for experimenting.

Balakrishnan and Basu [38], Balakrishnan and Ng [7], Childs et al. [9–11], Draper and Guttman [14], Gupta and Kundu [24], Jeong et al. [25], Mansour and Ramadan [32], Dey and Dey [12], and Tolba et al. [41] have all done extensive work on hybrid censoring and its various variations. Fairbanks et al. [21] proposed that PHT-ICS under the lifetime distribution is exponential. As a result, Childs et al. [9] proposed an alternate PHT-ICS that could finish the experiment at a random time $T^* = \max\{Y_{m:m:n}, T\}$. Based on PHT-ICS data, Fatemeh et al. [42], Ramses and Abdellatif [16], and Ramadan et al. [31] proposed classical and Bayesian inference for the Burr type-III distribution and a modified extended exponential distribution. However, many authors have investigated statistical inference for unknown parameters using large data sets [1, 13, 30, 34, 35, 41, 44].

To the best of our knowledge, no researcher has examined the estimations of the aforementioned unknown parameters of the flexible reduced logarithmic-inverse Lomax distribution based on progressive hybrid type-I censoring (PHT-ICS) data, which is what motivated us to start this work. To study entropy estimations, various scholars in the literature employed a variety of censoring schemes, such as enhanced adaptive type-II progressive censoring, adaptive type-II progressive hybrid censoring, generalized type-I progressive hybrid censoring, and progressive First-Failure. Refer to [3, 18, 19, 22, 33, 37, 44] for further information. Some of these researchers just examined classical estimations, while others took into account both Bayesian and classical guesses for the unknown parameters for various lifetime distributions. Therefore, no attempt is made to use progressive regression to estimate the unknown parameters of the flexible reduced logarithmic-inverse Lomax distribution utilizing both Bayesian and conventional estimating techniques. For the unknown parameters of the flexible reduced logarithmic-inverse Lomax distribution, we therefore want to compare the maximum likelihood and Bayesian estimate approaches. There are several reasons why this study uses the FRL-IL distribution. It is frequently used to simulate life statistics, particularly in situations where failure rates are declining or show an inverted bathtub shape. In situations where standard Weibull or exponential distributions might not work, this distribution appropriately depicts the situation. It has several real-world uses in a variety of fields, including risk assessment, survival analysis, and reliability engineering. By guaranteeing a predetermined amount of observed failures, the PHT-ICS approach helps balance cost and efficiency in experimental design, which is another reason why we are interested in employing it. The study's main objective is to recommend classical and Bayesian methods for estimating unknown parameters, reliability, and hazard rate functions based on PHT-ICS samples. The Bayes estimators are derived using the MCMC technique and compared to statistical inference estimators under the SELF and LINEX loss functions. The approximate confidence intervals (ACI) and credible confidence intervals (CCI) [43] using the classical and Bayes methods are also explained.

Despite its high incidence, mortality rates, and associated management costs, head and neck cancer is severely ignored and scientifically underestimated as a public health problem. Although survival rates for patients with other types of head and neck cancer have improved significantly in the last 30 years, the improvement in head and neck cancer has eased, and clinical and research interest in life processes has died down. According to the preceding premise, our study focuses on patients with head

and neck cancer. For our research, we employ a PHT-ICS that satisfies the need to obtain an adequate estimate to limit the patient's experimental time and attempt to prevent experimental patients from failing. Failure data from a lifetime experiment for specific patients must be fitted to a probability model or a statistical lifetime distribution to compute equivalent probabilities. The purpose of this study is to assess the survival function of a set of genuine data sets that include lifetime bladder cancer rates (see Lee and Wang [29]). Buzaridah et al. [8] observed that the failure data sets were very well adapted by the flexible-reduced logarithmic inverse Lomax distribution (FRL-ILD). The cumulative and probability distribution functions (CDF and PDF) of the three parameters FRL-ILD are given as

$$F(y; \theta, \eta, \phi) = 1 - \frac{\log\{1 + \phi - \phi(1 + \frac{\eta}{y})^{-\theta}\}}{\log(1 + \phi)}, \quad (1.1)$$

and

$$f(y; \theta, \eta, \phi) = \frac{\theta\eta\phi y^{-2}(1 + \frac{\eta}{y})^{-(\theta+1)}}{\log(1 + \phi)\{1 + \phi - \phi(1 + \frac{\eta}{y})^{-\theta}\}}, \quad (1.2)$$

where $y > 0$, $\theta, \eta > 0$, and $\phi > -1$.

The functions $S(y)$ and $h(y)$ of the FRL-ILD are specified as

$$S(y; \theta, \eta, \phi) = \frac{\log\{1 + \phi - \phi(1 + \frac{\eta}{y})^{-\theta}\}}{\log(1 + \phi)},$$

and

$$h(y; \theta, \eta, \phi) = \frac{\theta\eta\phi y^{-2}(1 + \frac{\eta}{y})^{-(\theta+1)}}{\log\{1 + \phi - \phi(1 + \frac{\eta}{y})^{-\theta}\}\{1 + \phi - \phi(1 + \frac{\eta}{y})^{-\theta}\}}.$$

The following are the primary reasons for using the FRL-ILD in practice:

- (1) The same ability to change the original distributions by having extra parameters.
- (2) Increasing the efficiency and adaptability of the original distributions.
- (3) It is a nonmonotonic hazard rate function.
- (4) It includes the best fit for unimodal medical care data sets.

Figure 2 shows the PDF and hazard function of the FRL-ILD distribution for various combinations of θ , η , and ϕ . The plots highlight how parameter variations influence the shape and behavior of the curves, demonstrating the model's flexibility in capturing diverse data patterns and risk behaviors.

Table 1 summarizes the survival values (S) and hazard rates (h) at specific quantiles (Q_1 , Q_2 , Q_3) and the mean value ($\bar{y}$) for the FRL-ILD distribution under various parameter combinations of θ , η , and ϕ . The survival values $S(Q_1)$, $S(Q_2)$, and $S(Q_3)$ remain relatively stable across parameter sets, reflecting consistency in survival behavior at these quantiles. However, $S(\bar{y})$ exhibits greater variability, showing a more pronounced sensitivity to parameter changes. The hazard rates, $h(Q_1)$, $h(Q_2)$, $h(Q_3)$, and $h(\bar{y})$, generally decrease as η or ϕ increases, indicating reduced risk with higher parameter values. Notably, $h(Q_1)$ is consistently the highest, representing the greatest risk at the first quantile. The parameter θ also subtly influences survival and hazard rates, shaping the distribution's overall behavior. This table highlights the flexibility of the FRL-ILD model in capturing varying survival and hazard dynamics across different parameter configurations.

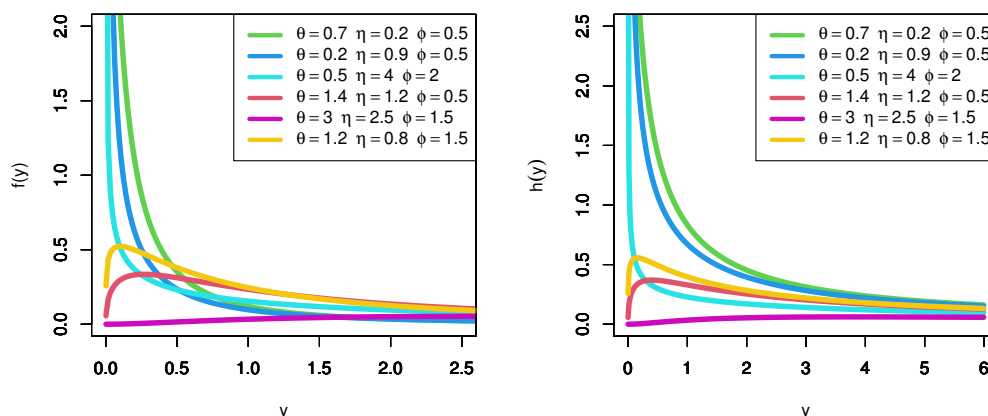


Figure 2. Plots of the PDF and the hazard of the FRL-ILD with different values of θ , η , and ϕ parameters.

Table 1. Survival values and hazard rates of the FRL-ILD distribution for various parameters.

θ	η	Φ	$S(Q_1)$	$S(Q_2)$	$S(\bar{x})$	$S(Q_3)$	$h(Q_1)$	$h(Q_2)$	$h(\bar{x})$	$h(Q_3)$
0.5	0.4	0.3	0.7483	0.4999	0.14071	0.2516	4.8324	2.4137	0.6181	1.1192
		0.9	0.7483	0.4999	0.1411	0.2516	3.7335	1.9353	0.5051	0.9098
		1.5	0.7483	0.4999	0.1412	0.2516	3.1231	1.6491	0.4329	0.7789
		2.1	0.7483	0.4999	0.1412	0.2516	2.7271	1.4543	0.3819	0.6874
		2.7	0.7486	0.4999	0.1411	0.2516	2.4461	1.3110	0.3434	0.6189
	2	0.3	0.7483	0.4999	0.1407	0.2516	0.9665	0.4827	0.1236	0.2238
		0.9	0.7483	0.4999	0.1411	0.2516	0.7467	0.3871	0.1010	0.1820
		1.5	0.7483	0.4999	0.1412	0.2516	0.6246	0.3298	0.0866	0.1558
		2.1	0.7483	0.4999	0.1412	0.2516	0.5454	0.2909	0.0764	0.1375
		2.7	0.7483	0.4999	0.1411	0.2516	0.4892	0.2622	0.0687	0.1238
2	3.2	0.3	0.7483	0.4999	0.1407	0.2516	0.6040	0.3017	0.0773	0.1399
		0.9	0.7483	0.4999	0.1411	0.2516	0.4667	0.2419	0.0631	0.1137
		1.5	0.7483	0.4999	0.1412	0.2516	0.3904	0.2061	0.0541	0.0974
		2.1	0.7483	0.4999	0.1412	0.2516	0.3409	0.1818	0.0477	0.0859
		2.7	0.7483	0.4999	0.1411	0.2516	0.3058	0.1639	0.0429	0.0774
	0.8	0.3	0.7483	0.4999	0.1441	0.2516	0.3708	0.2670	0.0787	0.1370
		0.9	0.7483	0.4999	0.1437	0.2516	0.3131	0.2206	0.0641	0.1121
		1.5	0.7483	0.4999	0.1434	0.2516	0.2768	0.1914	0.0548	0.0963
		2.1	0.7483	0.4999	0.1430	0.2516	0.2512	0.1707	0.0482	0.0852
		2.7	0.7483	0.4999	0.1427	0.2516	0.2318	0.1552	0.0433	0.0768
2.6	2	0.3	0.7483	0.4999	0.14421	0.2516	0.1141	0.0822	0.0242	0.0422
		0.9	0.7483	0.4999	0.1438	0.2516	0.0964	0.0679	0.0197	0.0345
		1.5	0.7483	0.4999	0.1434	0.2516	0.0852	0.0589	0.0169	0.0296
	2.6	2.1	0.7483	0.4999	0.1434	0.2516	0.0773	0.0525	0.0148	0.0262
		2.7	0.7483	0.4999	0.1427	0.2516	0.0713	0.0478	0.0133	0.0236

The document continues as outlined below: Section 2 delineates the likelihood estimates of the model's parameters, reliability, hazard functions, and the formulation of asymptotic confidence intervals. Section 3 presents the Bayesian estimators of unknown model parameters under the SEL and LINEX loss functions, together with Bayes credible intervals. Section 4 presents the failure data for a cohort of 44 head and neck cancer patients as an illustration. In Section 5, we utilize Monte Carlo simulations to quantitatively evaluate the proposed methodologies. Ultimately, Section 6 presents the conclusions.

2. Maximum likelihood estimates (MLEs)

In this section, we derive estimators of the parameters from the data using log-likelihood functions. For additional information on the likelihood method, see Azzalini [5] and Royall [36, 40]. The MLEs have several advantages, such as being asymptotically normally distributed, having the lowest variance asymptotically, being asymptotically unbiased, and satisfying the invariant characteristic. The likelihood function for Case I based on the observed data is provided by

$$L(\theta, \eta, \phi) = k_1 \prod_{i=1}^m f(y_{i:m:n}) [1 - F(y_{i:m:n})]^{R_i}, \quad (2.1)$$

and for Case II, it is

$$L(\theta, \eta, \phi) = k_2 \prod_{i=1}^J f(y_{i:m:n}) [1 - F(y_{i:m:n})]^{R_i} [1 - F(T)]^{R_J^*}, \quad (2.2)$$

where

$$k_1 = \prod_{i=1}^m \left[n - \sum_{k=1}^{i-1} (1 + R_k) \right]$$

and

$$k_2 = \prod_{i=1}^J \left[n - \sum_{k=1}^{i-1} (1 + R_k) \right]$$

both are constant.

Combining two Eqs (2.1) and (2.2), the likelihood function may be rewritten as

$$L(\theta, \eta, \phi) = C \prod_{i=1}^r f(y_{i:m:n}) [1 - F(y_{i:m:n})]^{R_i} [1 - F(T)]^{R_{(J)}^*}, \quad (2.3)$$

where $C = k_1, r = m, R_{(J)}^* = 0$ and $C = k_2, r = J, R_{(J)}^* = R_J^*$ for Case I and Case II, respectively. Let $Y_{1:m:n}, Y_{2:m:n}, \dots, Y_{m:m:n}$, $1 \leq m \leq n$ denote a progressively hybrid type-I censored sample observed from a life study examining n units drawn from a population with the PDF and CDF provided in Eqs (1.1) and (1.2).

From Eq (2.3), the likelihood function is given by

$$L(\theta, \eta, \phi | \underline{y}) \propto \theta^r \eta^r \phi^r \prod_{i=1}^r y_{i:m:n}^{-2} (1 + \eta y_{i:m:n}^{-1})^{-\theta-1} \times \prod_{i=1}^r [1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta}]^{-1}$$

$$\begin{aligned} & \times [\log(1 + \phi)]^{-r} \times \prod_{i=1}^r \left[\log \left(1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta} \right) \right]^{R_i} \\ & \times \prod_{i=1}^r [\log(1 + \phi)]^{-R_i} \left[\log \left(1 + \phi - \phi(1 + \eta T^{-1})^{-\theta} \right) \right]^{R_J^*} [\log(1 + \phi)]^{-R_J^*}, \end{aligned} \quad (2.4)$$

where $y = y_{1:m:n}, y_{2:m:n}, \dots, y_{m:m:n}$.

For the parameters θ , η , and ϕ , the log-likelihood function is calculated as

$$\begin{aligned} \ell = \log L(\theta, \eta, \phi | y) &= r \log \theta + r \log \eta + r \log \phi - r \log [\log(1 + \phi)] - 2 \sum_{i=1}^r \log(y_{i:m:n}) \\ &- (\theta + 1) \sum_{i=1}^r \log \left(\log(1 + \eta y_{i:m:n}^{-1}) \right) - \sum_{i=1}^r \log \left(1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta} \right) \\ &+ \sum_{i=1}^r R_i \log \left[\log \left(1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta} \right) \right] - \sum_{i=1}^r R_i \log [\log(1 + \phi)] \\ &+ R_{(J)}^* \log \left[\log \left(1 + \phi - \phi(1 + \eta T^{-1})^{-\theta} \right) \right] - R_J^* \log [\log(1 + \phi)]. \end{aligned} \quad (2.5)$$

When it comes to θ , η , and ϕ , we can differentiate Eq (2.5) and then bring it to zero. Here are the likelihood equations:

$$\begin{aligned} \frac{\partial \ell}{\partial \theta} &= \frac{r}{\theta} - \sum_{i=1}^r \log \left(\log(1 + \eta y_{i:m:n}^{-1}) \right) - \sum_{i=1}^r \frac{\phi(1 + \eta y_{i:m:n}^{-1})^{-\theta} \log \left(\log(1 + \eta y_{i:m:n}^{-1}) \right)}{\left(1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta} \right)} \\ &+ \sum_{i=1}^r R_i \frac{\phi(1 + \eta y_{i:m:n}^{-1})^{-\theta} \log \left(\log(1 + \eta y_{i:m:n}^{-1}) \right)}{\left(1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta} \right) \left[\log \left(1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta} \right) \right]} \\ &+ \sum_{i=1}^r R_J^* \frac{\phi(1 + \eta T^{-1})^{-\theta} \log \left(\log(1 + \eta T^{-1}) \right)}{\left(1 + \phi - \phi(1 + \eta T^{-1})^{-\theta} \right) \left[\log \left(1 + \phi - \phi(1 + \eta T^{-1})^{-\theta} \right) \right]}, \end{aligned} \quad (2.6)$$

$$\begin{aligned} \frac{\partial \ell}{\partial \eta} &= \frac{r}{\eta} - (\theta + 1) \sum_{i=1}^r \frac{y_{i:m:n}^{-1}}{\left(1 + \eta y_{i:m:n}^{-1} \right)} - \sum_{i=1}^r \frac{\phi \theta y_{i:m:n}^{-1} (1 + \eta y_{i:m:n}^{-1})^{-\theta-1}}{\left(1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta} \right)} \\ &+ \sum_{i=1}^r R_i \frac{\phi \theta y_{i:m:n}^{-1} (1 + \eta y_{i:m:n}^{-1})^{-\theta-1}}{\left(1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta} \right) \left[\log \left(1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta} \right) \right]} \\ &+ \sum_{i=1}^r R_J^* \frac{\phi \theta T^{-1} (1 + \eta T^{-1})^{-\theta} \log \left(\log(1 + \eta T^{-1}) \right)}{\left(1 + \phi - \phi(1 + \eta T^{-1})^{-\theta} \right) \left[\log \left(1 + \phi - \phi(1 + \eta T^{-1})^{-\theta} \right) \right]} \end{aligned} \quad (2.7)$$

and

$$\frac{\partial \ell}{\partial \phi} = \frac{r}{\phi} - \frac{r}{(1 + \phi) \log(1 + \phi)} - \sum_{i=1}^r \frac{1 - \left(1 + \eta y_{i:m:n}^{-1} \right)}{\left(1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta} \right)}$$

$$\begin{aligned}
& + \sum_{i=1}^r R_i \frac{\phi \theta y_{i:m:n}^{-1} (1 + \eta y_{i:m:n}^{-1})^{-\theta-1}}{(1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta}) [\log(1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta})]} - \sum_{i=1}^r R_i \frac{1}{(1 + \phi) \log(1 + \phi)} \\
& + R_J^* \frac{1 - (1 + \eta T^{-1})^{-\theta}}{(1 + \phi - \phi(1 + \eta T^{-1})^{-\theta}) [\log(1 + \phi - \phi(1 + \eta T^{-1})^{-\theta})]} - R_J^* \frac{1}{(1 + \phi) \log(1 + \phi)}. \quad (2.8)
\end{aligned}$$

To obtain estimates for Eqs (2.6)–(2.8), the Newton-Raphson iteration method is used. The normal equations are:

$$\frac{\partial \ell}{\partial \theta} = 0, \quad \frac{\partial \ell}{\partial \eta} = 0, \quad \text{and} \quad \frac{\partial \ell}{\partial \phi} = 0.$$

These equations do not have closed-form solutions. Essam [2] provided a detailed description of the algorithm. Additionally, using the MLEs' invariant condition, the MLEs of $S(y; \theta, \eta, \phi)$ and $h(y; \theta, \eta, \phi)$ may be derived by substituting θ, η , and ϕ by $\hat{\theta}, \hat{\eta}$, and $\hat{\phi}$ as

$$\hat{S}(y; \hat{\theta}, \hat{\eta}, \hat{\phi}) = \frac{\log\{1 + \hat{\phi} - \hat{\phi}(1 + \frac{\hat{\eta}}{y})^{-\hat{\theta}}\}}{\log(1 + \hat{\phi})}, y > 0 \quad (2.9)$$

and

$$h(y; \hat{\theta}, \hat{\eta}, \hat{\phi}) = \frac{\hat{\theta} \hat{\eta} \hat{\phi} y^{-2} (1 + \frac{\hat{\eta}}{y})^{-(\hat{\theta}+1)}}{\log\{1 + \hat{\phi} - \hat{\phi}(1 + \frac{\hat{\eta}}{y})^{-\hat{\theta}}\} \{1 + \hat{\phi} - \hat{\phi}(1 + \frac{\hat{\eta}}{y})^{-\hat{\theta}}\}}, y > 0. \quad (2.10)$$

Analytical representations of the MLEs for the parameters are not available. It is therefore impossible to derive their true distributions. However, we may find confidence intervals for the parameters (θ, η, ϕ) using the asymptotic distribution of the MLE. The MLEs of the asymptotic variance-covariance matrix of (θ, η, ϕ) are given as

$$\hat{I}(\hat{\theta}, \hat{\eta}, \hat{\phi}) = \begin{pmatrix} -\frac{\partial^2 \ell}{\partial \theta^2} & -\frac{\partial^2 \ell}{\partial \theta \partial \eta} & -\frac{\partial^2 \ell}{\partial \theta \partial \phi} \\ -\frac{\partial^2 \ell}{\partial \eta \partial \theta} & -\frac{\partial^2 \ell}{\partial \eta^2} & -\frac{\partial^2 \ell}{\partial \eta \partial \phi} \\ -\frac{\partial^2 \ell}{\partial \phi \partial \theta} & -\frac{\partial^2 \ell}{\partial \phi \partial \eta} & -\frac{\partial^2 \ell}{\partial \phi^2} \end{pmatrix}_{(\theta, \eta, \phi) = (\hat{\theta}, \hat{\eta}, \hat{\phi})}.$$

The estimated asymptotic variance-covariance matrix $[\hat{V}]$ for the maximum likelihood estimators is obtained by inverting the observed information matrix $\hat{I}(\theta, \eta, \phi)$. Alternatively

$$V(\theta, \eta, \phi) = \hat{I}^{-1}(\hat{\theta}, \hat{\eta}, \hat{\phi}) = \begin{pmatrix} \text{Var}(\hat{\theta}) & \text{cov}(\hat{\theta}, \hat{\eta}) & \text{cov}(\hat{\theta}, \hat{\phi}) \\ \text{cov}(\hat{\eta}, \hat{\theta}) & \text{Var}(\hat{\eta}) & \text{cov}(\hat{\eta}, \hat{\phi}) \\ \text{cov}(\hat{\phi}, \hat{\theta}) & \text{cov}(\hat{\phi}, \hat{\eta}) & \text{Var}(\hat{\phi}) \end{pmatrix}.$$

It is widely recognized that $(\hat{\theta}, \hat{\eta}, \hat{\phi})$ is approximately multivariate normal with mean (θ, η, ϕ) and covariance matrix $I^{-1}(\theta, \eta, \phi)$ given certain regularity conditions, as noted by Lawless [28]. The $100(1 - \vartheta)\%$ two-sided confidence intervals for θ, η , and ϕ can be expressed as

$$\widehat{\theta}_i \pm Z_{\frac{\vartheta}{2}} \sqrt{\text{Var}(\widehat{\theta}_i)}, \quad \widehat{\eta}_i \pm Z_{\frac{\vartheta}{2}} \sqrt{\text{Var}(\widehat{\eta}_i)}, \quad \text{and} \quad \widehat{\phi}_i \pm Z_{\frac{\vartheta}{2}} \sqrt{\text{Var}(\widehat{\phi}_i)}.$$

The percentile of the standard normal distribution that corresponds to a right-tail probability of $\frac{\vartheta}{2}$ is $Z_{\frac{\vartheta}{2}}$.

To calculate the ACIs of $S(y; \theta, \eta, \phi)$ and $h(y; \theta, \eta, \phi)$, which depend on the parameters θ , η , and ϕ , respectively, the delta method is applied to approximate the variances of $\hat{S}(y; \theta, \eta, \phi)$ and $\hat{h}(y; \theta, \eta, \phi)$; for more information on the delta method, see Greene [23]. The variance of $\hat{S}(y; \theta, \eta, \phi)$ and $\hat{h}(y; \theta, \eta, \phi)$ can be estimated using this method, as well.

$$\hat{\sigma}_{\hat{S}(y)}^2 = [\nabla \hat{S}(y)]^T [\hat{V}] [\nabla \hat{S}(y)] \text{ and } \hat{\sigma}_{\hat{h}(y)}^2 = [\nabla \hat{h}(y)]^T [\hat{V}] [\nabla \hat{h}(y)].$$

The two-sided confidence intervals for $S(y; \theta, \eta, \phi)$ and $h(y; \theta, \eta, \phi)$, which are equal to $100(1 - \vartheta)\%$, can be formulated as

$$\hat{S}(y) \pm Z_{\frac{\vartheta}{2}} \sqrt{\hat{\sigma}_{\hat{S}(y)}^2} \text{ and } \hat{h}(y) \pm Z_{\frac{\vartheta}{2}} \sqrt{\hat{\sigma}_{\hat{h}(y)}^2}.$$

3. Bayesian technique

This section discusses Bayesian estimation, which uses SE and LINEX loss functions to estimate parameters θ , η , and ϕ as well as reliability functions $S(y; \theta, \eta, \phi)$, and $h(y; \theta, \eta, \phi)$. Independent informative prior distributions express the parameters' prior understanding. It is presumed that θ and η are independent parameters that adhere to gamma prior distributions, while the parameter ϕ has a non-informative prior where $\phi > -1$.

$$\begin{aligned} \pi_1(\theta) &\propto \theta^{a_1-1} e^{-b_1\theta} \quad \theta > 0, a_1 > 0, b_1 > 0, \\ \pi_2(\eta) &\propto \eta^{a_2-1} e^{-b_2\eta} \quad \eta > 0, a_2 > 0, b_2 > 0, \\ \pi_3(\phi) &\propto \frac{1}{\phi}, \quad \phi > 0. \end{aligned} \quad (3.1)$$

By integrating the previous Eq (3.1) with the likelihood function, Eq (2.6) yields the posterior distribution of θ, η , and ϕ defined by $\pi^*(\theta, \eta, \phi | \underline{y})$, which is represented as follows:

$$\pi^*(\theta, \eta, \phi | \underline{y}) = \frac{\pi_1(\theta) \pi_2(\eta) \pi_3(\phi) L(\theta, \eta, \phi | \underline{y})}{\int_0^\infty \int_0^\infty \int_0^\infty \pi_1(\theta) \pi_2(\eta) \pi_3(\phi) L(\theta, \eta, \phi | \underline{y}) d\theta d\eta d\phi}. \quad (3.2)$$

$$L(\rho, \hat{\rho}) = (\hat{\rho} - \rho)^2.$$

The Bayes estimates (BEs) of any function of θ, η , and ϕ , such as $g(\theta, \eta, \phi)$ under the SEL, can be calculated as

$$\hat{g}_{BS}(\theta, \eta, \phi | \underline{y}) = E_{\theta, \eta, \phi | \underline{y}}(g(\theta, \eta, \phi)),$$

where

$$E_{\theta, \eta, \phi | \underline{y}}(g(\theta, \eta, \phi)) = \frac{\int_0^\infty \int_0^\infty \int_0^\infty g(\theta, \eta, \phi) \pi_1(\theta) \pi_2(\eta) \pi_3(\phi) L(\theta, \eta, \phi | \underline{y}) d\theta d\eta d\phi}{\int_0^\infty \int_0^\infty \int_0^\infty \pi_1(\theta) \pi_2(\eta) \pi_3(\phi) L(\theta, \eta, \phi | \underline{y}) d\theta d\eta d\phi}. \quad (3.3)$$

The following steps are used to determine the LINEX loss function $L(\Delta)$ for a parameter named ρ :

$$L(\Delta) = (e^{c\Delta} - c\Delta - 1), \quad c \neq 0, \quad \Delta = \hat{\rho} - \rho.$$

Thus, when applying the LINEX loss function, the BEs of a function $g(\theta, \eta, \phi)$ are

$$\hat{g}_{BL}(\theta, \eta, \phi | \underline{y}) = -\frac{1}{c} \log \left[E \left(e^{-cg(\theta, \eta, \phi)} | \underline{y} \right) \right], \quad c \neq 0,$$

where

$$E \left(e^{-cg(\theta, \eta, \phi)} | \underline{y} \right) = \frac{\int_0^\infty \int_0^\infty \int_0^\infty e^{-cg(\theta, \eta, \phi)} \pi_1(\theta) \pi_2(\eta) \pi_3(\phi) L(\theta, \eta, \phi | \underline{y}) d\theta d\eta d\phi}{\int_0^\infty \int_0^\infty \int_0^\infty \pi_1(\theta) \pi_2(\eta) \pi_3(\phi) L(\theta, \eta, \phi | \underline{y}) d\theta d\eta d\phi}. \quad (3.4)$$

It should be focused on that there is no mathematical solution to calculating the numerous integrals in Eqs (3.3) and (3.4). The MCMC method creates samples from the joint posterior density function in Eq (3.2) by using the M-H samplers approach from Eq (3.2), where the joint posterior distribution can be defined as

$$\begin{aligned} \pi^*(\theta, \eta, \phi | \underline{y}) &\propto \theta^{r+a_1-1} \eta^{r+a_2-1} \phi^{-r} e^{-b_1\theta-b_2\eta} \\ &\prod_{i=1}^r y_{i:m:n}^{-2} (1 + \eta y_{i:m:n}^{-1})^{-\theta-1} \prod_{i=1}^r [1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta}]^{-1} \\ &\times [\log(1 + \phi)]^{-r} \times \prod_{i=1}^r [\log(1 + \phi - \phi(1 + \eta y_{i:m:n}^{-1})^{-\theta})]^{R_i} \\ &\times \prod_{i=1}^r [\log(1 + \phi)]^{-R_i} [\log(1 + \phi - \phi(1 + \beta T^{-1})^{-\theta})]^{R_j^*} [\log(1 + \phi)]^{R_j^*}. \end{aligned} \quad (3.5)$$

The algorithm for M-H within Gibbs sampling is as follows:

- (1) Establish initial values $(\theta^{(0)}, \eta^{(0)}, \phi^{(0)})$.
- (2) Determine $j = 1$.
- (3) Employing the subsequent M-H method, use the normal distributions for the proposal to produce $\theta^{(j)}, \eta^{(j)}$, and $\phi^{(j)}$.

$$N(\theta^{(j-1)}, \text{var}(\theta)), N(\eta^{(j-1)}, \text{var}(\eta)), \text{ and } N(\phi^{(j-1)}, \text{var}(\phi)),$$

and from the main diagonal in the inverse Fisher information matrix (2), we obtained $\text{var}(\theta)$, $\text{var}(\eta)$, and $\text{var}(\phi)$.

- (4) Formulate a proposal:

$$\theta^* \sim N(\theta^{(j-1)}, \text{var}(\theta)), \quad \eta^* \sim N(\eta^{(j-1)}, \text{var}(\eta)),$$

and ϕ^* from $N(\phi^{(j-1)}, \text{var}(\phi))$.

- (i) The acceptance probabilities are

$$\left. \begin{aligned} \omega_{\theta} &= \min \left[1, \frac{\pi_1^*(\theta^*|\eta^{(j-1)}, \phi^{(j-1)}, y)}{\pi_1^*(\theta^{(j-1)}|\eta^{(j-1)}, \phi^{(j-1)}, y)} \right], \\ \omega_{\eta} &= \min \left[1, \frac{\pi_2^*(\eta^*|\theta^{(j)}, \phi^{(j-1)}, y)}{\pi_2^*(\eta^{(j-1)}|\theta^{(j)}, \phi^{(j-1)}, y)} \right], \\ \omega_{\phi} &= \min \left[1, \frac{\pi_3^*(\phi^*|\theta^{(j)}, \eta^{(j)}, y)}{\pi_3^*(\phi^{(j-1)}|\theta^{(j)}, \eta^{(j)}, y)} \right]. \end{aligned} \right\}.$$

- (ii) Generate u_1, u_2 , and u_3 from a uniform (0, 1) distribution.
 (iii) If $u_1 < \omega_{\theta}$, accept the proposal and set $\theta^{(j)} = \theta^*$, otherwise set $\theta^{(j)} = \theta^{(j-1)}$.
 (iv) If $u_2 < \omega_{\eta}$, accept the proposal and set $\eta^{(j)} = \eta^*$, otherwise set $\eta^{(j)} = \eta^{(j-1)}$.
 (v) If $u_3 < \omega_{\phi}$, accept the proposal and set $\phi^{(j)} = \phi^*$, otherwise set $\phi^{(j)} = \phi^{(j-1)}$.
 (5) The reliability and hazard rate functions are computed as

$$\left\{ \begin{aligned} S^{(j)}(y; \theta, \eta, \phi) &= \frac{\log\{1 + \phi^{(j)} - \phi^{(j)}(1 + \frac{\eta^{(j)}}{y})^{-\theta^{(j)}}\}}{\log(1 + \phi^{(j)})}; & y \geq 0, \\ h^{(j)}(y; \theta, \eta, \phi) &= \frac{\theta^{(j)}\eta^{(j)}\phi^{(j)}y^{-2}(1 + \frac{\eta^{(j)}}{y})^{-(\theta^{(j)}+1)}}{\log\{1 + \phi^{(j)} - \phi^{(j)}(1 + \frac{\eta^{(j)}}{y})^{-\theta^{(j)}}\}\{1 + \phi^{(j)} - \phi^{(j)}(1 + \frac{\eta^{(j)}}{y})^{-\theta^{(j)}}\}}; & y \geq 0. \end{aligned} \right.$$

- (6) Set $j = j + 1$.
 (7) Steps (3)–(6) are repeat N times and obtain $\theta^{(j)}, \eta^{(j)}, \phi^{(j)}, S^{(j)}(y)$, and $h^{(j)}(y)$, $j = 1, 2, \dots, N$.
 (8) Compute the CCIs of $(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) = (\theta, \eta, \phi, S(y), h(y))$.

As $\psi_k^{(1)} < \psi_k^{(2)} \dots < \psi_k^{(N)}$, then the $100(1 - \vartheta)\%$ CCIs of ψ_k is

$$(\psi_{k(N-\vartheta/2)}, \psi_{k(N-(1-\vartheta/2))}).$$

The first M simulated alternatives are eliminated to guarantee convergence and reduce the impact of the initial value selection. The chosen samples are $\psi_k^{(j)}$, $j = M + 1, \dots, N$, sufficiently high N . The SEL function is utilized to generate the estimated boundary estimations of ψ_k .

$$\hat{\psi}_k = \frac{1}{N - M} \sum_{j=M+1}^N \psi_k^{(j)}.$$

The estimated Bayes estimates for ψ_k , according to the LINEX loss function, derived from Eq (3.4), are

$$\hat{\psi}_k = \frac{-1}{c} \log \left[\frac{1}{N - M} \sum_{j=M+1}^N e^{-c \psi_k^{(j)}} \right], k = 1, 2, 3, 4, 5.$$

4. Real data analysis

There are many billions of living cells in the human body. In the first few years of life, normal cells divide more quickly so that the person can grow. Cells in a particular body region start to proliferate uncontrollably when cancer develops. Malignant (cancer) cells proliferate in the tissues of the bladder in a condition known as bladder cancer.

4.1. Application for head and neck cancer data

Efron [15] reported a dataset that includes survival times for 44 patients diagnosed with head and neck cancer. The dataset is presented as follows:

12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776.

A combination of radiation and chemotherapy was administered to these patients. Finding the best-fitting probability distribution function that describes the patients' survival durations accurately is the main goal of using this dataset for parametric analysis. It is possible to evaluate the dataset and find the right probability distribution by using statistical modeling techniques like goodness-of-fit testing and maximum likelihood estimation. In survival analysis, this is an essential stage since it allows us to analyze the pattern of survival times, estimate important parameters, and make probabilistic predictions regarding patient outcomes. By shedding light on therapy efficacy and patient prognosis, the results of this analysis can aid in medical research. Clinical decision-making and treatment planning rely heavily on survival probability models, which in turn rely on such modeling tools.

The findings of this section indicate that the FRL-ILD is a suitable distribution for this dataset, as illustrated in Figure 3. The Kolmogorov-Smirnov (K-S) distance between the empirical distribution of the failure data and the FRL-ILD is 0.0742, with a P-value of 0.9538 and parameters ($\theta = 4.6221$, $\eta = 30.0988$, $\phi = -0.7035$).

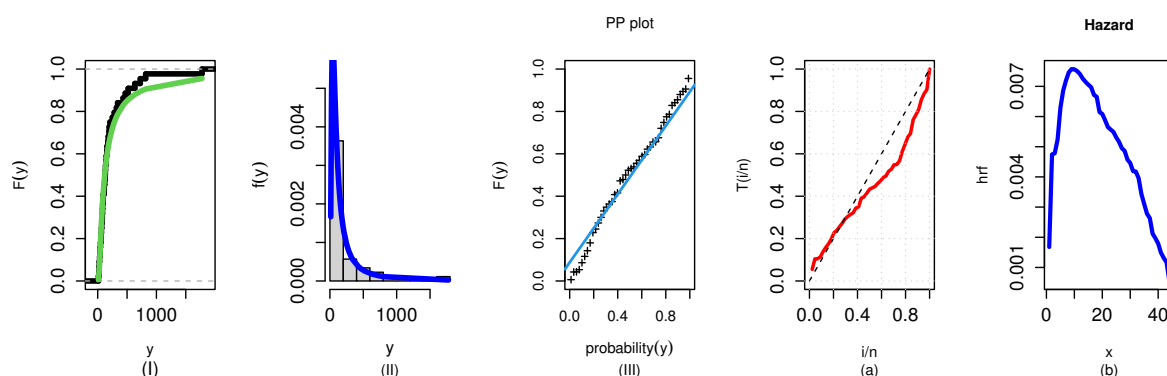


Figure 3. The FRL-ILD fitted for the real data set.

To illustrate the discussed method, a PHT-ICS is generated from this dataset under different schemes. The number of removals is determined based on the removal process, as shown below:

- For $T = 250$, and $m = 35$:

Removals (R): 9, 0.

Survival Times (Y): 12.20, 23.56, 23.74, 31.98, 37.00, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 74.47, 78.26, 92.00, 94.00, 110.00, 112.00, 119.00, 127.00, 130.00, 133.00, 140.00, 173.00, 179.00, 194.00, 209.00, 249.00, 281.00, 319.00, 339.00, 469.00, 519.00, 633.00, 817.00, 1776.00.

- For $T = 250$ and $m = 35$:

Removals (R): 0, 9.

Survival Times (Y): 12.20, 23.56, 23.74, 25.87, 31.98, 37.00, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 74.47, 78.26, 81.43, 84.00, 92.00, 94.00, 110.00, 112.00, 119.00, 127.00, 130.00, 133.00, 140.00, 146.00, 155.00, 159.00, 173.00, 179.00, 194.00, 195.00, 209.00, 249.00, 281.00

- For $T = 250$ and $m = 35$:

Removals (R): 5, 0, 4.

Survival Times (Y): 12.20, 23.56, 23.74, 31.98, 37.00, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 74.47, 78.26, 92.00, 94.00, 110.00, 112.00, 119.00, 127.00, 130.00, 133.00, 140.00, 146.00, 155.00, 159.00, 173.00, 179.00, 194.00, 209.00, 249.00, 281.00, 319.00, 339.00, 469.00, 519.00.

Table 2 presents the MLEs for the parameters θ , η , and ϕ , as well as the functions $S(x)$ and $h(x)$, derived from progressive hybrid type-I censored data, alongside the Bayesian estimates corresponding to SEL loss functions for the same parameters. This section presents the MLEs for θ , η , and ϕ , together with the functions $S(T)$ and $h(T)$, derived from PHT-ICS data using the specified estimation methods. Table 2 illustrates the MLEs and BEs related to SEL loss functions for the model parameters and the functions S and h .

According to Table 2, the estimators did a good job since the estimate values are near each other with different schemes. The Figure 4 profile indicates that the MLEs are unique and represent the actual maximum for the parameters θ , η , ϕ , $S(y)$, and $h(y)$. Figure 5 shows the posterior density function plots and the trace plots of the unknown parameters θ , η , and ϕ functions using the MCMC method, additionally plotting the profile log-likelihood plots. All censoring schemes are plotted in the same way, and it has been discovered that the trace plots of all censoring methods converge strongly.

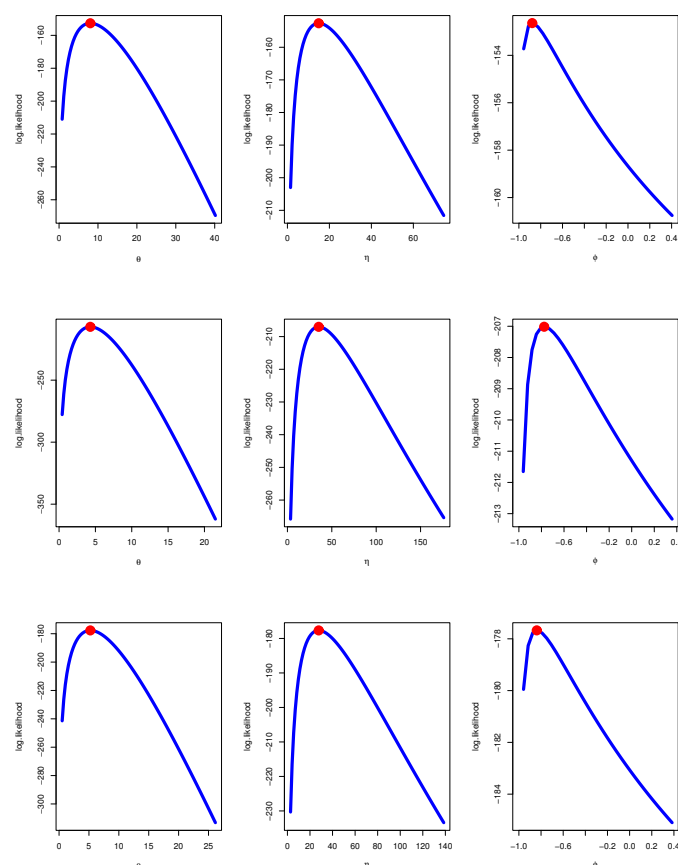


Figure 4. For the parameters θ , η , and ϕ the profile log-likelihood graphs are presented for MLEs with different cases: Data 1.

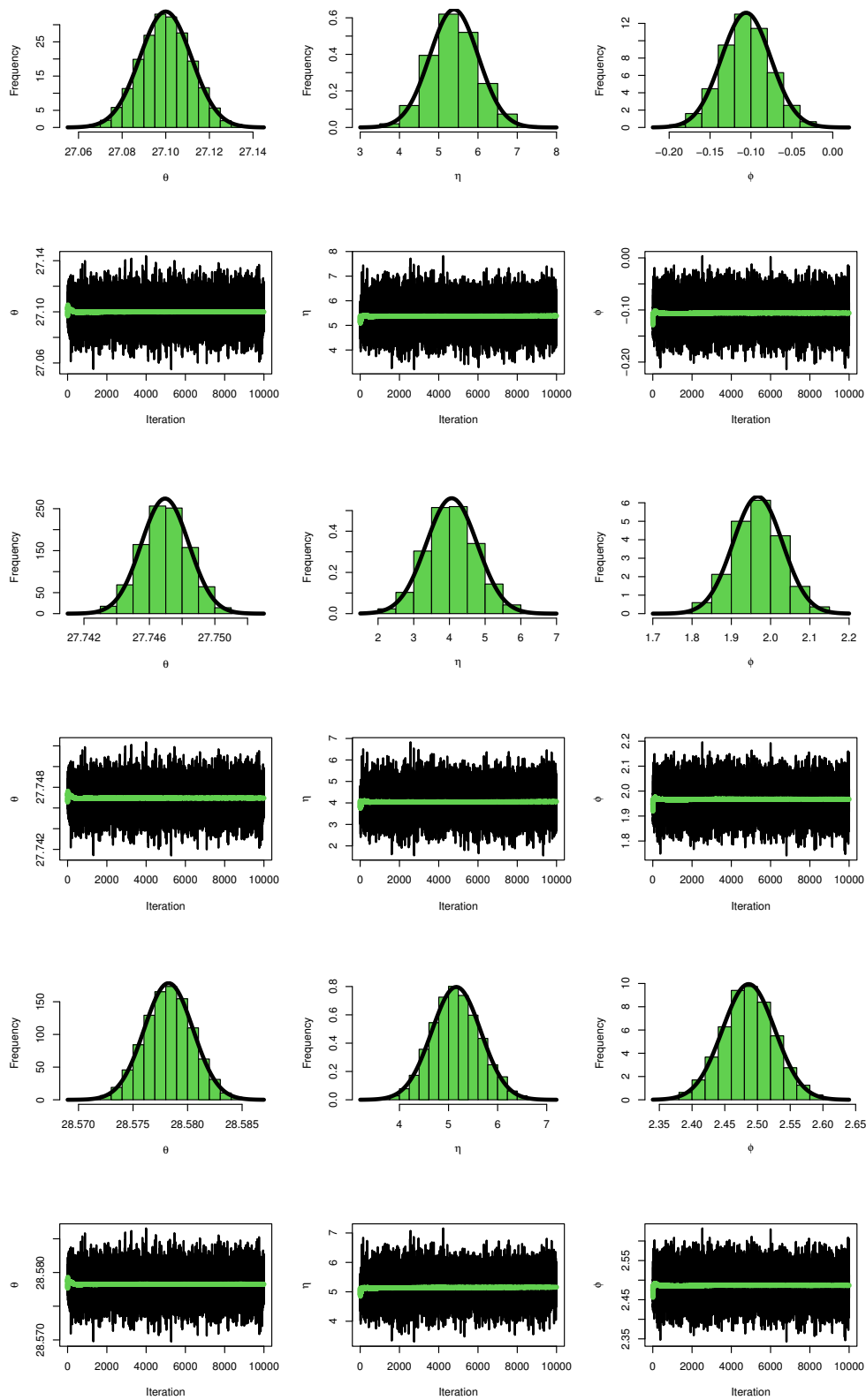


Figure 5. The tracing graphs and posterior density plots for the parameters θ , η , and ϕ with different cases: Data 1.

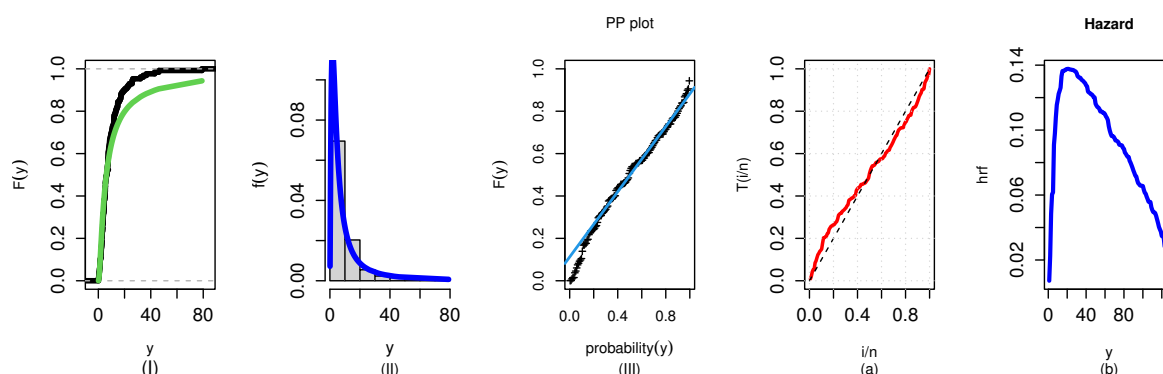
Table 2. MLE and Bayesian estimations for parameters and the reliability functions: Data 1.

$m = 35$		MLE				Bayesian				
T	Scheme	Estimates	StEr	Lower	Upper	Estimates	StEr	Lower	Upper	
250	1	θ	8.0349	1.0827	5.9129	5.9129	27.0999	0.0118	26.0767	27.4222
		η	14.8982	3.1062	8.8100	8.8100	5.3882	0.6175	4.7390	6.0769
		Φ	-0.8778	0.0392	-0.9368	-0.9368	-0.1059	0.0302	-0.1663	-0.0464
		S	0.1880		0.0690	0.7256	0.4252		0.3771	0.4728
		h	0.0037		0.0039	0.0019	0.0030		0.0031	0.0028
250	2	θ	4.3017	0.3805	3.5560	3.5560	27.7470	0.0015	27.1441	27.9707
		η	35.2240	6.5434	22.3990	22.3990	4.0629	0.7123	2.7670	5.5534
		Φ	-0.7761	0.0680	-0.8992	-0.8992	1.9673	0.0629	1.8414	2.0916
		S	0.2735		0.1176	0.9020	0.4930		0.3784	0.5938
		h	0.0034		0.0038	0.0008	0.0024		0.0028	0.0021
250	3	θ	5.2304	0.4715	4.3062	4.3062	28.5783	0.0022	27.5738	28.8248
		η	27.7156	4.6806	18.5417	18.5417	5.1578	0.5005	4.2231	6.1661
		Φ	-0.8385	0.0463	-0.9170	-0.9170	2.4865	0.0401	2.4063	2.5657
		S	0.2402		0.1119	0.8818	0.5938		0.5302	0.6507
		h	0.0036		0.0038	0.0009	0.0021		0.0023	0.0019

4.2. Application for bladder cancer

Malignant (cancer) cells proliferate in the tissues of the bladder in a condition known as bladder cancer. An uncensored data set representing the number of months that 128 bladder cancer patients experienced remission is considered in this subsection. Lee and Wang [29] previously researched these data.

The findings of this subsection indicate that the FRL-ILD is a suitable distribution for dataset 2, as illustrated in Figure 6. The K-S distance between the empirical distribution of the failure data 2 and the FRL-ILD is 0.11166, with a P-value of 0.08219 and parameters ($\theta = 2.3528, \eta = 3.7933, \phi = -0.7746$). Figure 6 discussed empirical CDF, histogram with PDF, PP plot, TTT [39], and estimated hazard.

**Figure 6.** The FRL-ILD fitted for the real data set 2.

To illustrate the discussed method, a PHT-ICS is generated from this dataset 2 under different schemes. The number of removals is determined based on the removal process, as shown below:

As shown in Table 3, the estimators performed well, with the estimated values being close to one another across different schemes. The profile in Figure 7 demonstrates that the MLEs are unique and accurately correspond to the actual maxima for the parameters θ , η , ϕ , $S(y)$, and $h(y)$. Figure 8 presents the posterior density function and trace plots for the unknown parameters θ , η , and ϕ using the MCMC method, along with the profile log-likelihood plots. All censoring schemes are displayed similarly, revealing that the trace plots for all censoring methods exhibit strong convergence.

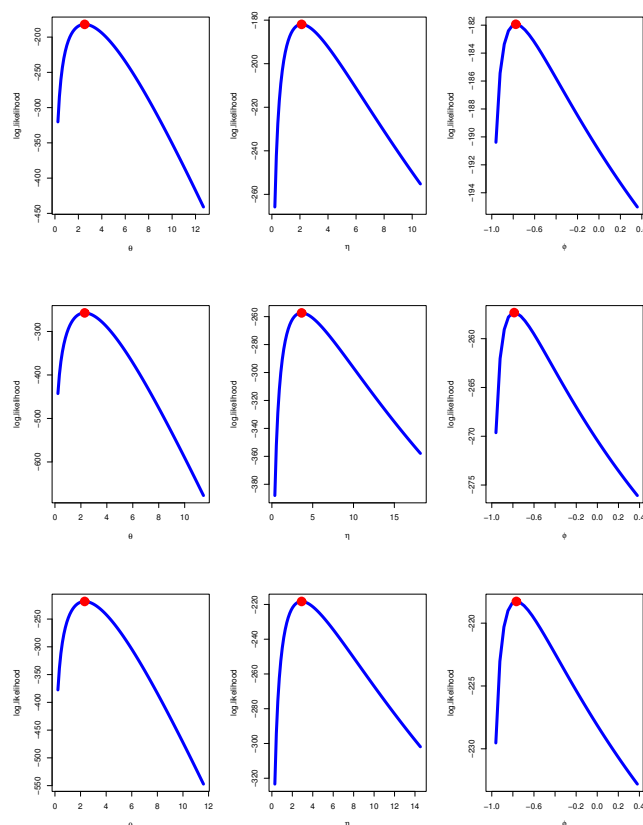


Figure 7. For the parameters θ , η , and ϕ the profile log-likelihood graphs are presented for MLEs with different cases: Data 2.

Table 3. MLE and Bayesian estimations for the parameters and the reliability functions: Data 2.

$m = 100$		MLE				Bayesian				
T	Scheme	Estimates	StEr	Lower	Upper	Estimates	StEr	Lower	Upper	
9	1	θ	2.5354	0.6139	2.5233	2.5233	41.3159	0.0617	41.1941	41.4325
		η	2.1173	1.1357	2.0950	2.0950	0.0472	0.0113	0.0261	0.0702
		Φ	-0.7761	0.2458	-0.7809	-0.7809	0.3446	0.0246	0.2954	0.3933
		S	0.2595		0.2544	0.2647	0.2190		0.1261	0.3099
		h	0.0925		0.0929	0.0922	0.0961		0.1028	0.0892
9	2	θ	2.2896	0.4616	2.2806	2.2806	38.7067	0.0464	38.6150	38.7943
		η	3.6387	1.9652	3.6002	3.6002	0.0705	0.0196	0.0339	0.1103
		Φ	-0.7879	0.2507	-0.7928	-0.7928	0.5903	0.0251	0.5401	0.6400
		S	0.3578		0.3513	0.3642	0.3086		0.1632	0.4366
		h	0.0833		0.0838	0.0827	0.0883		0.0995	0.0777
9	3	θ	2.3203	0.5121	2.3103	2.3103	37.1221	0.0514	37.0204	37.2192
		η	2.9063	1.6038	2.8748	2.8748	0.0545	0.0080	0.0396	0.0708
		Φ	-0.7669	0.2722	-0.7723	-0.7723	1.7397	0.0273	1.6852	1.7937
		S	0.3132		0.3072	0.3192	0.2973		0.2281	0.3640
		h	0.0872		0.0877	0.0868	0.0854		0.0914	0.0795

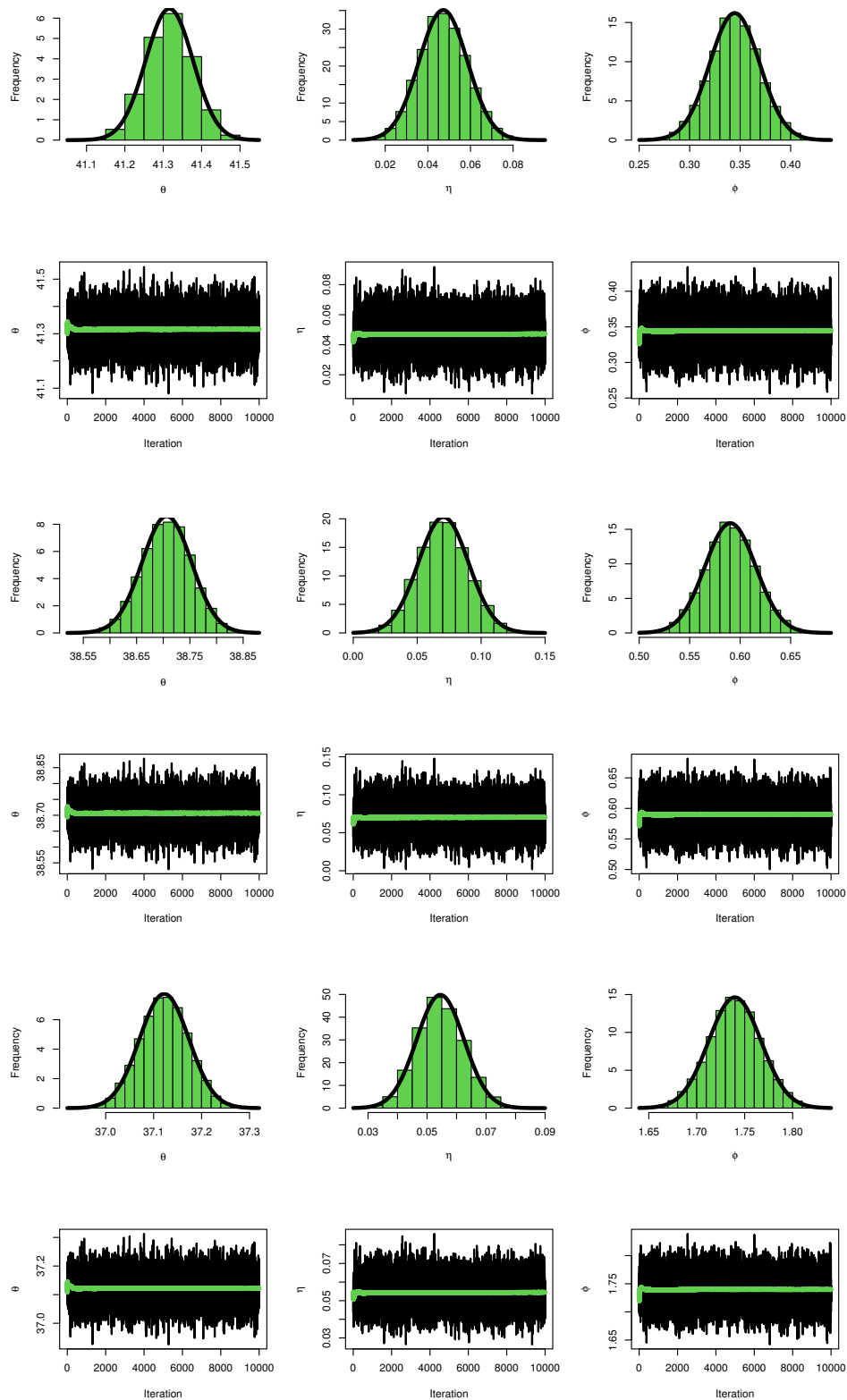


Figure 8. The tracing graphs and posterior density plots for the parameters θ , η , and ϕ with different cases Data 2.

5. Simulation study

To analyze the effects of various estimates, MCMC simulations are performed. The MCMC is used to compare the performance of the proposed estimators of the model parameters, reliability, and hazard rate functions of the FRL-ILD. We repeated progressively hybrid type-I censored data from FRL-ILD 10000 times, with initial values $\theta_o = 0.6$, $\eta_o = 0.04$ and $\phi_o = 0.7$. The comparison of the different methods of the likely-to-result estimators of θ , η , ϕ , $S(y)$ and $h(y)$, at $y = 0.4$, has been examined in terms of the MSE, which is measured for $k = 1, 2, 3, 4, 5$ ($\psi_1 = \theta$, $\psi_2 = \eta$, $\psi_3 = \phi$, $\psi_4 = S(0.4)$, $\psi_5 = h(0.4)$), as $MSE(\psi_k) = \frac{1}{M} \sum_{i=1}^M (\hat{\psi}_k^{(i)} - \psi_k)^2$, where $M = 10000$ is the number of simulated samples. The ACLs and CPs of the 95% ACIs and Bayes CCI are compared (CPs). The following progressive schemes are considered in this study:

I: $R_1 = n - m, R_i = 0$ for $i \neq 1$.

II: $R_{\frac{m}{2}} = R_{\frac{m}{2}+1} = \frac{n-m}{2}$, $R_i = 0$ for $i \neq \frac{m}{2}$ and $i \neq \frac{m}{2} + 1$.

III: $R_m = n - m, R_i = 0$ for $i \neq m$.

Tables 4–13 show the average estimates obtained by ML and BE, along with MSEs, whereas Tables 14–17 show the average 95% ACI and Bayes CCI, as well as their lengths and CP.

Specifically, to obtain a PHT-ICS sample of size m from the FRL-ILD, after assigning T , n , and R , do the following procedure:

Step 1: Set the specified actual values of FRL-ILD(θ, η, ϕ).

Step 2: Obtain an ordinary progressive type-II censored sample as:

(a) Simulate u independent items (say $u_1, u_2, \dots, u_m$) from the uniform $U(0, 1)$ distribution.

(b) Set

$$H_i = u_i \left(i + \sum_{l=m-i+1}^m R_l \right)^{-1}, \quad \text{for } i = 1, 2, \dots, m.$$

(c) Set

$$U_i = 1 - H_m H_{m-1} \cdots H_{m-i+1}, \quad \text{for } i = 1, 2, \dots, m.$$

(d) Collect a progressive type-II censored sample (with size m) from the FRL-ILD(θ, η, ϕ) distribution by setting

$$y_i = \frac{1}{\eta} \left(\left[\frac{-1}{\phi} \left(e^{U_i \log(1+\phi)} - 1 - \phi \right) \right]^{\frac{-1}{\theta}} - 1 \right)^{-1}, \quad i = 1, 2, \dots, m.$$

Step 3: Determine d at predetermined T .

Step 4: Discard the remaining sample ($Y_{j+1}, \dots, Y_m$) when $T < Y_m$.

Step 5: Use the truncated $f(y; \theta, \eta, \phi)[1 - F(y_{d+1}; \theta, \eta, \phi)]^{-1}$ distribution to obtain ($Y_{j+1}, \dots, Y_m$) order statistics with size $n - j - \sum_{l=1}^d R_l - 1$.

To perform the necessary calculations for the Bayes point (or interval) estimates, different priors are assigned to the hyperparameters using the elective hyperparameter method. Following the Bayes MCMC procedure outlined in Section 3, the initial 2000 iterations (out of a total of 12000) are discarded to minimize the influence of the starting values. Once 10000 target PHT-ICS samples are collected, the ‘maxLik’ and ‘coda’ packages are utilized in R 4.3.0 software to compute the maximum likelihood and Bayes MCMC estimates, along with their 95% ACI and HPD interval estimates for θ , η , and ϕ .

Table 4. Bayesian and Non-Bayesian estimation for the parameter θ with $\theta_0 = 0.6$.

(n, m)	CS	T	MLE	SEL	LINEX	
					$c = -2$	$c = 2$
(30, 20)	I	10	1.2914 (0.0275)	1.2344 (0.0421)	1.239 (0.0416)	1.2298 (0.0426)
			1.277 (0.0264)	1.2092 (0.0442)	1.2154 (0.0434)	1.2031 (0.0451)
			1.2868 (0.029)	1.1479 (0.0696)	1.1609 (0.0665)	1.1352 (0.0732)
	II	15	1.2891 (0.0271)	1.229 (0.046)	1.2338 (0.0454)	1.2243 (0.0466)
			1.2904 (0.0292)	1.2235 (0.0433)	1.2293 (0.0426)	1.2179 (0.044)
			1.3001 (0.0286)	1.1686 (0.0715)	1.1804 (0.0679)	1.157 (0.0757)
	III	10	1.3262 (0.0305)	1.2679 (0.0371)	1.2719 (0.0369)	1.2639 (0.0373)
			1.3228 (0.0299)	1.2641 (0.0356)	1.2688 (0.0354)	1.2596 (0.0358)
			1.3191 (0.0292)	1.1953 (0.0545)	1.2044 (0.525)	1.1863 (0.0571)
(40, 30)	I	15	1.3256 (0.0292)	1.2791 (0.0331)	1.2828 (0.0331)	1.2754 (0.0332)
			1.324 (0.0304)	1.2535 (0.0394)	1.2584 (0.0389)	1.2488 (0.0398)
			1.3142 (0.0281)	1.1961 (0.0477)	1.2052 (0.046)	1.1872 (0.0495)

Table 5. Bayesian & and-Bayesian estimation for the parameter θ with $\theta_0 = 0.6$.

(n, m)	CS	T	MLE	SEL	LINEX	
					$c = -2$	$c = 2$
(50, 20)	I	10	1.1197 (0.0518)	1.1219 (0.0534)	1.1222 (0.0534)	1.1215 (0.0533)
			1.2091 (0.0393)	1.2137 (0.0422)	1.2141 (0.0424)	1.2133 (0.0421)
			1.2868 (0.029)	1.1479 (0.0696)	1.1609 (0.0665)	1.1352 (0.0732)
	II	15	1.197 (0.0473)	1.1089 (0.0486)	1.1091 (0.0486)	1.1086 (0.0486)
			1.1759 (0.032)	1.1787 (0.0332)	1.179 (0.0333)	1.1785 (0.0331)
			1.2399 (0.0400)	1.2408 (0.0425)	1.2413 (0.0426)	1.2404 (0.0424)
	III	10	1.303 (0.0285)	1.2397 (0.0369)	1.2434 (0.0364)	1.2361 (0.0374)
			1.2781 (0.0257)	1.1897 (0.0458)	1.1955 (0.0440)	1.1839 (0.0477)
			1.3150 (0.0296)	1.1026 (0.0901)	1.1189 (0.0831)	1.0871 (0.0974)
(60, 30)	I	15	1.2931 (0.0274)	1.2401 (0.0421)	1.2442 (0.0414)	1.2361 (0.0429)
			1.2976 (0.0284)	1.2290 (0.0512)	1.2359 (0.0499)	1.2224 (0.0526)
			1.2786 (0.0254)	1.0952 (0.0752)	1.110 (0.0707)	1.0805 (0.0803)

Table 6. Bayesian and Non-Bayesian estimation for the parameter η with $\eta_o = 0.04$.

(n, m)	CS	T	MLE	SEL	LINEX	
					$c = -2$	$c = 2$
(30, 20)	I	10	1.2682 (0.2011)	1.8905 (3.1894)	1.9978 (5.3335)	1.9978 (1.2679)
			1.2715 (0.2109)	2.638 (2.638)	3.0833 (2.113)	3.0833 (1.4343)
			1.1543 (0.2905)	2.2273 (4.6835)	2.4579 (8.6185)	2.4579 (1.8334)
	II	15	1.2768 (0.2005)	2.2751 (1.119)	2.528 (1.73)	2.528 (1.3909)
			1.263 (0.2122)	1.9527 (2.4939)	2.0788 (4.4067)	2.0788 (1.0498)
			1.0124 (0.3509)	2.7104 (1.215)	3.282 (2.38)	3.282 (1.124)
	III	10	1.127 (0.2198)	2.2322 (2.073)	2.5308 (2.119)	2.5308 (2.2955)
			1.1249 (0.2273)	1.6808 (0.8812)	1.758 (1.4478)	1.758 (0.456)
			1.0824 (0.2604)	2.0968 (4.3863)	2.2951 (5.9567)	2.2951 (1.8153)
(40, 30)	I	15	1.1255 (0.2175)	1.5964 (0.4632)	1.6443 (0.626)	1.6443 (0.3015)
			1.1284 (0.2208)	1.7677 (2.0704)	1.8663 (3.3522)	1.8663 (0.8434)
			1.0788 (0.2681)	1.9386 (2.5533)	2.0866 (4.2943)	2.0866 (0.9973)

Table 7. Bayesian and Non-Bayesian estimation for the parameter η with $\eta_o = 0.04$.

(n, m)	CS	T	MLE	SEL	LINEX	
					$c = -2$	$c = 2$
(50, 20)	I	10	1.523 (0.0642)	1.5234 (0.1559)	1.549 (0.158)	1.5439 (0.1549)
			1.2091 (0.1235)	1.4578 (0.1877)	1.484 (0.2092)	1.484 (0.1829)
			1.1534 (0.0963)	1.4967 (0.2048)	1.5283 (0.215)	1.5283 (0.2001)
	II	15	1.5277 (0.0847)	1.5546 (0.1445)	1.5795 (0.1485)	1.5795 (0.1429)
			1.4765 (0.0806)	1.4904 (0.1496)	1.5216 (0.1737)	1.5216 (0.1394)
			1.4018 (0.1144)	1.4197 (0.1812)	1.4516 (0.1937)	1.4516 (1755)
	III	10	1.2646 (0.1805)	1.8730 (2.3974)	1.9866 (4.2976)	1.9866 (0.8197)
			1.2539 (0.240)	2.0760 (2.1149)	2.2170 (3.2600)	2.217 (0.442)
			1.1234 (0.2926)	2.4333 (4.4366)	2.7259 (7.5232)	2.7259 (2.0884)
(60, 30)	I	15	1.2816 (0.1633)	2.0747 (7.750)	2.2558 (14.959)	2.2558 (2.3633)
			1.2977 (0.1882)	2.0898 (2.0155)	2.2268 (3.0711)	1.8663 (0.8434)
			1.1161 (0.3099)	2.2434 (3.2894)	2.4516 (5.8091)	2.4516 (1.6321)

Table 8. Bayesian and Non-Bayesian estimation for the parameter ϕ with $\phi_0 = 0.7$.

(n, m)	CS	T	MLE	SEL	LINEX	
					$c = -2$	$c = 2$
(30, 20)	I	10	0.7825 (0.0971)	0.8556 (0.1528)	0.862 (0.1882)	0.862 (0.1272)
			0.8053 (0.0785)	0.9113 (0.6091)	0.9283 (1.2493)	0.9283 (0.3042)
			0.8573 (0.0498)	1.0969 (0.2372)	1.1207 (0.3265)	1.1207 (0.1669)
	II	15	0.7854 (0.0941)	0.8697 (0.219)	0.8777 (0.2889)	0.8777 (0.1623)
			0.8005 (0.0847)	0.8829 (0.1196)	0.8882 (0.1291)	0.8882 (0.1096)
			0.869 (0.0397)	1.1812 (4.2313)	1.2645 (12.9046)	1.2645 (1.1755)
	III	10	0.8292 (0.0627)	0.9217 (0.3702)	0.9281 (0.4746)	0.9281 (0.1663)
			0.8353 (0.0594)	0.9171 (0.0748)	0.9213 (0.0816)	0.9213 (0.07)
			0.8727 (0.0404)	1.1071 (0.3565)	1.1312 (0.505)	1.1312 (0.2041)
(40, 30)	I	15	0.8283 (0.0656)	0.8905 (0.0689)	0.8928 (0.0701)	0.8928 (0.0677)
			0.8355 (0.0603)	0.9285 (0.1184)	0.9344 (0.1468)	0.9344 (0.0884)
			0.8644 (0.0417)	1.0563 (0.1588)	1.0703 (0.1962)	1.0703 (0.1135)

Table 9. Bayesian and Non-Bayesian estimation for the parameter ϕ with $\phi_0 = 0.7$.

(n, m)	CS	T	MLE	SEL	LINEX	
					$c = -2$	$c = 2$
(50, 20)	I	10	0.407 (0.4652)	0.4788 (0.5767)	0.5391 (0.5824)	0.5391 (0.5952)
			0.4543 (0.4047)	0.583 (0.6015)	0.6765 (0.7279)	0.6765 (0.5749)
			0.4027 (0.04531)	0.4839 (0.5613)	0.5599 (0.5469)	5599 (0.6006)
	II	15	0.3499 (0.5228)	0.4741 (0.6269)	0.5461 (0.6538)	0.5461 (0.6319)
			0.3975 (0.4739)	0.459 (0.6097)	0.5222 (0.6260)	0.5222 (0.6229)
			0.5141 (0.03452)	0.6605 (0.4415)	0.7372 (0.4513)	0.7372 (0.4546)
	III	10	0.8228 (0.0633)	0.9090 (0.1173)	0.9145 (0.1341)	0.9145 (0.1007)
			0.8605 (0.0445)	0.9691 (0.0554)	0.9741 (0.0587)	0.9741 (0.0526)
			0.8635 (0.0442)	1.2758 (0.4824)	1.3237 (0.6305)	1.3237 (0.3444)
(60, 30)	I	15	0.8153 (0.0766)	0.9069 (0.0160)	0.9151 (0.1976)	0.9151 (0.1261)
			0.8253 (0.0725)	0.9395 (0.1034)	0.9475 (0.1121)	0.9475 (0.0959)
			0.8767 (0.0369)	1.2472 (0.7691)	1.2816 (0.9776)	1.2816 (0.3713)

Table 10. Bayesian and Non-Bayesian estimation for the $S(y)$ with $y = 0.4$.

(n, m)	CS	T	MLE	SEL	LINEX	
					$c = -2$	$c = 2$
(50, 20)	I	10	0.8369 (0.0046)	0.8292 (0.0065)	0.8299 (0.0063)	0.8299 (0.0067)
			0.8446 (0.0033)	0.8451 (0.0051)	0.8451 (0.0052)	0.8451 (0.0052)
			0.8442 (0.0035)	0.8295 (0.0078)	0.831 (0.0071)	0.8310 (0.097)
	II	15	0.8314 (0.0054)	0.8289 (0.0068)	0.8294 (0.0067)	0.8294 (0.0069)
			0.8466 (0.0034)	0.8389 (0.0059)	0.8399 (0.0056)	0.8399 (0.0065)
			0.8560 (0.0027)	0.8523 (0.0040)	0.8529 (0.039)	0.8529 (0.0041)
	III	10	0.4825 (0.0044)	0.4499 (0.0045)	0.4506 (0.0045)	0.4506 (0.0045)
			0.4817 (0.00445)	0.4356 (0.0040)	0.4366 (0.0039)	0.4366 (0.0040)
			0.4817 (0.0036)	0.4322 (0.0037)	0.4344 (0.0036)	0.4344 (0.0038)
	I	15	0.4869 (0.0039)	0.4446 (0.0035)	0.4453 (0.0034)	0.4453 (0.0035)
			0.4796 (0.0049)	0.4309 (0.0060)	0.4319 (0.0059)	0.4319 (0.0061)
			0.4917 (0.0044)	0.4391 (0.0053)	0.4410 (0.0052)	0.4410 (0.0053)

Table 11. Bayesian and Non-Bayesian estimation for the $S(y)$ with $y = 0.4$.

(n, m)	CS	T	MLE	SEL	LINEX	
					$c = -2$	$c = 2$
(30, 20)	I	10	0.4981 (0.0069)	0.4627 (0.0064)	0.4635 (0.0064)	0.4635 (0.0064)
			0.4958 (0.0062)	0.4565 (0.0094)	0.4576 (0.0095)	0.4576 (0.0094)
			0.4906 (0.0058)	0.4405 (0.0069)	0.4425 (0.0068)	0.4425 (0.007)
	II	15	0.4968 (0.0066)	0.4634 (0.0125)	0.4644 (0.0064)	0.4644 (0.0124)
			0.4938 (0.0065)	0.4539 (0.0067)	0.4548 (0.0067)	0.4548 (0.0068)
			0.4977 (0.0057)	0.4531 (0.0143)	0.4555 (0.016)	0.4555 (0.0132)
	III	10	0.4869 (0.0045)	0.4497 (0.0078)	0.4503 (0.0078)	0.4503 (0.0078)
			0.4863 (0.0044)	0.4437 (0.0044)	0.4443 (0.0044)	0.4443 (0.0044)
			0.4814 (0.0039)	0.429 (0.0049)	0.4311 (0.0063)	0.4311 (0.0049)
	I	15	0.4874 (0.0046)	0.4475 (0.0045)	0.4481 (0.0045)	0.4481 (0.0046)
			0.4858 (0.0044)	0.4448 (0.0044)	0.4455 (0.0043)	0.4455 (0.0044)
			0.486 (0.0041)	0.4351 (0.0041)	0.4363 (0.004)	0.4363 (0.0042)

Table 12. Bayesian and Non-Bayesian estimation for the parameter $h(y)$ with $y = 0.4$.

(n, m)	CS	T	MLE	SEL	LINEX	
					$c = -2$	$c = 2$
(30, 20)	I	10	0.2594 (0.0035)	0.2991 (0.0053)	0.3 (0.0055)	0.3 (0.0052)
			0.2592 (0.0036)	0.3056 (0.0066)	0.307 (0.0069)	0.307 (0.0064)
			0.257 (0.0044)	0.3104 (0.0097)	0.3131 (0.0102)	0.3131 (0.0092)
	II	15	0.2604 (0.0034)	0.3018 (0.0062)	0.303 (0.0064)	0.303 (0.006)
			0.2619 (0.0038)	0.3084 (0.0075)	0.3098 (0.0078)	0.3098 (0.0073)
			0.2455 (0.0044)	0.2952 (0.0069)	0.2974 (0.0072)	0.2974 (1.1755)
	III	10	0.2608 (0.0026)	0.3045 (0.0045)	0.3053 (0.0046)	0.3053 (0.0044)
			0.2609 (0.0027)	0.3086 (0.0046)	0.3095 (0.0048)	0.3095 (0.0045)
			0.2608 (0.0029)	0.316 (0.0061)	0.3177 (0.0064)	0.3177 (0.0059)
	I	15	0.2605 (0.0026)	0.306 (0.0044)	0.3067 (0.0045)	0.3067 (0.0043)
			0.2614 (0.0027)	0.3075 (0.0046)	0.3084 (0.0047)	0.3084 (0.0045)
			0.2571 (0.003)	0.3096 (0.005)	0.3112 (0.0052)	0.3112 (0.0048)

Table 13. Bayesian and Non-Bayesian estimation for the $h(y)$ with $y = 0.4$.

(n, m)	CS	T	MLE	SEL	LINEX	
					$c = -2$	$c = 2$
(50, 20)	I	10	0.4221 (0.0172)	0.4427 (0.0311)	0.448 (0.0359)	0.448 (0.028)
			0.4126 (0.018)	0.4233 (0.0325)	0.4267 (0.0341)	0.4267 (0.031)
			0.4134 (0.016)	0.45 (0.0425)	0.4815 (0.1568)	0.4815 (0.035)
	II	15	0.4355 (0.0224)	0.4396 (0.0298)	0.4433 (0.0318)	0.4433 (0.0284)
			0.4127 (0.0163)	0.4324 (0.0317)	0.4489 (0.0622)	0.4489 (0.028)
			0.3987 (0.0146)	0.4068 (0.0234)	0.4106 (0.0249)	0.4106 (0.0223)
(60, 30)	I	10	0.2687 (0.0026)	0.3044 (0.004)	0.3053 (0.0041)	0.3053 (0.0039)
			0.2655 (0.003)	0.3161 (0.0052)	0.3173 (0.0054)	0.3173 (0.005)
			0.2623 (0.0030)	0.3063 (0.0045)	0.3089 (0.0048)	0.3089 (0.0043)
	II	15	0.2653 (0.0021)	0.3112 (0.0033)	0.3121 (0.0034)	0.3121 (0.0032)
			0.2732 (0.0033)	0.3281 (0.0078)	0.3295 (0.0081)	0.3295 (0.0075)
			0.2514 (0.0034)	0.3029 (0.0063)	0.3053 (0.0066)	0.3053 (0.006)

Table 14. ACL and CP of 95% ACIs for the parameters θ , η and ϕ .

(n, m)	CS	T	θ		η		ϕ	
			MLE	MCMC	MLE	MCMC	MLE	MCMC
(30, 20)	I	10	5.6529 (0.9617)	0.1998 (0.9568)	2.0383 (0.957)	0.7687 (0.9466)	5.4298 (0.9627)	0.1389 (0.9294)
			4.4694 (0.9631)	0.2286 (0.9565)	3.4167 (0.9515)	1.5139 (0.9729)	6.6455 (0.9564)	0.1801 (0.9734)
			8.5822 (0.937)	0.3508 (0.9382)	3.0283 (0.9578)	1.2839 (0.9456)	11.7588 (0.9486)	0.3498 (0.9655)
	I	15	7.8707 (0.9692)	0.2046 (0.9557)	4.4399 (0.9361)	1.0811 (0.9712)	5.6912 (0.9661)	0.1498 (0.9558)
			10.6509 (0.9438)	0.2256 (0.9359)	6.1915 (0.9407)	0.877 (0.944)	6.0646 (0.9521)	0.1561 (0.9522)
			23.0997 (0.9577)	0.3316 (0.9369)	4.252 (0.9312)	1.8173 (0.932)	14.0675 (0.9554)	0.4375 (0.9512)
(40, 30)	I	10	8.3426 (0.9687)	0.1849 (0.9309)	5.8395 (0.9685)	1.1809 (0.932)	5.0596 (0.9359)	0.1424 (0.9354)
			9.5162 (0.9732)	0.1993 (0.9264)	8.6481 (0.9598)	0.693 (0.9374)	5.1176 (0.9669)	0.1395 (0.9362)
			5.1507 (0.937)	0.2867 (0.934)	6.9679 (0.9694)	1.1422 (0.9259)	9.484 (0.9377)	0.3143 (0.9448)
	I	15	8.532 (0.9558)	0.1806 (0.9493)	6.7448 (0.9427)	0.5848 (0.9747)	4.5751 (0.9704)	0.1177 (0.9675)
			10.013 (0.975)	0.2042 (0.9289)	9.9955 (0.9393)	0.7744 (0.9633)	5.3558 (0.9366)	0.1509 (0.9732)
			3.8788 (0.9316)	0.2864 (0.9496)	4.5241 (0.9687)	1.011 (0.937)	8.6553 (0.9644)	0.2759 (0.9537)

Table 15. ACL and CP of 95% CIs for the parameters θ , η and ϕ .

(n, m)	CS	T	θ		η		ϕ	
			MLE	MCMC	MLE	MCMC	MLE	MCMC
(50, 20)	I	10	2.3832 (0.9523)	0.054 (0.9634)	21.0137 (0.9557)	0.4334 (0.9485)	35.8958 (0.9357)	0.6381 (0.9235)
			2.7704 (0.9785)	0.059 (0.9335)	21.0137 (0.9454)	0.4197 (0.9854)	41.0347 (0.9369)	0.1801 (0.7884)
			2.3984 (0.945)	0.0511 (0.9844)	22.7283 (0.9782)	0.5144 (0.9056)	40.9238 (0.9856)	0.7526 (0.9557)
	I	15	2.2377 (0.9692)	0.0473 (0.9557)	20.4591 (0.9361)	0.4528 (0.9712)	35.9984 (0.9661)	0.6856 (0.9558)
			2.4405 (0.9438)	0.0493 (0.9359)	20.4951 (0.9407)	0.4654 (0.944)	37.3409 (0.9521)	0.6612 (0.9522)
			2.9478 (0.9577)	0.0631 (0.9369)	22.411 (0.9312)	0.5154 (0.932)	46.0535 (0.9554)	0.8151 (0.9512)
(60, 30)	I	10	8.7379 (0.9235)	0.1708 (0.9439)	20.2582 (0.9465)	0.7596 (0.9832)	4.9573 (0.9849)	0.1377 (0.9158)
			10.8805 (0.9235)	0.2225 (0.9874)	24.8189 (0.9685)	0.9966 (0.9721)	6.0161 (0.9335)	0.176 (0.9559)
			20.445 (0.9785)	0.3831 (0.9784)	33.7699 (0.9471)	1.4883 (0.9365)	13.6826 (0.9668)	0.5387 (0.9448)
	I	15	9.5762 (0.9874)	0.1799 (0.9887)	21.6671 (0.9336)	0.9442 (0.9746)	5.0167 (0.9868)	0.1428 (0.9654)
			10.5869 (0.9882)	0.2319 (0.9912)	23.9899 (0.9128)	0.9224 (0.9775)	6.0736 (0.9228)	0.1864 (0.9328)
			18.7393 (0.9168)	0.3696 (0.9456)	31.4035 (0.9872)	1.2581 (0.9537)	13.1471 (0.9842)	0.4717 (0.9758)

Table 16. ACL and CP of 95% ACIs for the parameters $S(y)$ and $h(y)$.

(n, m)	CS	T	$S(y)$		$h(y)$	
			MLE	MCMC	MLE	MCMC
(30, 20)	I	10	0.4145 (0.9445)	0.0865 (0.9639)	0.3631 (0.9592)	0.0928 (0.9291)
			0.4837 (0.936)	0.1009 (0.932)	0.5998 (0.9677)	0.1106 (0.9735)
			0.3298 (0.9724)	0.1441 (0.9295)	0.3391 (0.9276)	0.1563 (0.9297)
(30, 20)	I	15	0.4741 (0.966)	0.0923 (0.9536)	0.4773 (0.9632)	0.099 (0.9588)
			0.3886 (0.9446)	0.098 (0.9675)	0.3827 (0.9455)	0.1089 (0.9473)
			0.3338 (0.9613)	0.1414 (0.9521)	0.4532 (0.9466)	0.1423 (0.9452)
(40, 30)	I	10	0.3983 (0.9722)	0.0776 (0.9492)	0.3661 (0.9646)	0.0843 (0.9678)
			0.3027 (0.9454)	0.0821 (0.9748)	0.2863 (0.9712)	0.0897 (0.9441)
			0.2839 (0.9328)	0.1178 (0.9723)	0.2463 (0.9448)	0.126 (0.9571)
(40, 30)	I	15	0.3173 (0.9642)	0.0761 (0.9314)	0.2669 (0.9263)	0.0829 (0.9337)
			0.3079 (0.9429)	0.0858 (0.9734)	0.3008 (0.9691)	0.0935 (0.9273)
			0.2818 (0.9573)	0.1116 (0.9757)	0.2379 (0.9387)	0.1192 (0.9711)

Table 17. ACL and CP of 95% CIs for the parameters $S(y)$ and $h(y)$.

(n, m)	CS	T	$S(y)$		$h(y)$	
			MLE	MCMC	MLE	MCMC
(50, 20)	I	10	0.2198 (0.9634)	0.0623 (0.9558)	0.0902 (0.9245)	0.1617 (0.9192)
			0.1724 (0.9428)	0.0582 (0.9447)	0.1203 (0.9887)	0.1562 (0.9356)
			0.3298 (0.958)	0.1673 (0.9425)	0.1304 (0.9685)	0.2186 (0.9792)
(50, 20)	I	15	0.2237 (0.9859)	0.0591 (0.9381)	0.1069 (0.9584)	0.1513 (0.9335)
			0.1721 (0.9884)	0.0702 (0.9632)	0.1172 (0.9534)	0.1844 (0.9753)
			0.1627 (0.9341)	0.0653 (0.9119)	0.119 (0.9228)	0.1733 (0.9718)
(60, 30)	I	10	0.3506 (0.9224)	0.0808 (0.9854)	0.3052 (0.9911)	0.0863 (0.9884)
			0.3392 (0.9885)	0.0953 (0.9447)	0.36 (0.9535)	0.1057 (0.9336)
			0.2709 (0.9663)	0.1523 (0.9458)	0.4885 (0.9338)	0.1578 (0.9748)
(60, 30)	I	15	0.3486 (0.9865)	0.0814 (0.9789)	0.3227 (0.9685)	0.089 (0.9228)
			0.3438 (0.9393)	0.0955 (0.9915)	0.3402 (0.9552)	0.1109 (0.9273)
			0.261 (0.9474)	0.1438 (0.9636)	0.4167 (0.9781)	0.1496 (0.9822)

Tables 4–17 result in the following conclusions.

- When we compared the BEs and MLEs based on the MCMC algorithm, they have the smallest MSE values (Tables 4–13).
- We reported that the BEs under the SEL function with the assumption informative prior performed better than the MLEs. Finally, regarding the MSEs, the BEs that employ informative priors outperform the BEs based on the MLEs.
- Tables 14–17 exhibit the ALs and CPs for 95% CIs estimates for the parameters $\theta, \eta, \phi, S(y)$ and $h(y)$. The ALs of ACI for $\theta, \eta, \phi, S(y)$, and $h(y)$ obtained with MLE are somewhat longer than the comparable lengths determined using MCMC. This is valid for the parameters $\theta, \eta, \phi, S(y)$ and $h(y)$. Furthermore, the ALs of all CIs decrease as the effective sample size (m) increases. We tabulate the CP for the interval estimates of $\theta, \eta, \phi, S(y)$, and $h(y)$ in the tables. As can be seen, most of the CPs for various censoring schemes are below the nominal criterion of 95%.
- In general, as T increases, they become more efficient estimators.
- In particular, MSE values of all estimators decrease as n increases.
- In investigating the effect of various censoring schemes, we discovered that for θ, η , and ϕ parameters, the MSE values in the case of all removals are smaller at the first failure than the other two schemes, and for η parameters, it is smaller at the last failure than the other two schemes.

6. Conclusions

This work uses a progressive hybrid Type-I censored sample for the parameters of the FRL-ILD that match the failure data well for a real-life application of head and neck cancer data. Bayesian inference procedures and MLE procedures are discussed. The average lengths and coverage probabilities for 95% CI estimates for the parameters $\theta, \eta, \phi, S(y)$, and $h(y)$ were obtained with MLE. The Bayesian method offers Bayesian estimates using the Markov chain Monte Carlo sampling methodology. This is accomplished using two loss functions: squared error and general entropy loss functions. Additionally, we investigate Bayes' credible intervals. Monte Carlo simulations are performed to assess the estimation efficiency of the proposed inferential approaches. A numerical comparison of the Bayes and ML estimations is made. The computational research showed that all estimators perform well with increased sample sizes. The study demonstrates that it is reasonable to estimate the failure probability of the actual two data sets with the lifetime of bladder cancer that was previously stated using the distribution that has been presented. The posterior density function plots and trace plots using the MCMC method for the unknown parameters θ, η , and ϕ have been shown in Figures 5 and 8 to show the diagnostic tests for the MCMC method. Some possible directions for future research include: (1) analyzing entropy measure estimation in a competing risks model; (2) analyzing entropy measure estimation problems in the context of accelerated life tests; and (3) investigating traditional estimation techniques other than maximum likelihood, such as weighted least squares, least squares, and maximum product of spacings methods, among others.

Author contributions

Ehab M. Almetwally: Methodology, Software, Formal analysis, Writing-original draft, Writing-review and editing; Ahlam H. Tolba: Conceptualization, Methodology, Validation,

Investigation, Resources, Writing-original draft, Writing-review and editing; Dina A. Ramadan: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Data curation, Writing-original draft, Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2501).

Conflict of interest

The authors declare no conflicts of interest.

References

1. S. Aboul-Fotouh Salem, O. E. Abo-Kasem, A. Abdelgaied Khairy, Inference for generalized progressive Hybrid type-II censored Weibull lifetimes under competing risk data, *Comput. J. Math. Stat. Sci.*, **3** (2024), 177–202.
2. E. A. Ahmed, Estimation of some lifetime parameters of generalized Gompertz distribution under progressively type-II censored data, *Appl. Math. Model.*, **39** (18), 5567–5578. <https://doi.org/10.1016/j.apm.2015.01.023>
3. R. Alotaibi, M. Nassar, Z. A. Khan, W. A. Alajlan, A. Elshahhat, Entropy evaluation in inverse Weibull unified hybrid censored data with application to mechanical components and head-neck cancer patients, *AIMS Math.*, **10** (2025), 1085–1115. <https://doi.org/10.3934/math.2025052>
4. N. Alsadat, D. A. Ramadan, E. M. Almetwally, A. H. Tolba, Estimation of some lifetime parameter of the unit half logistic-geometry distribution under progressively type-II censored data, *J. Radiation Res. Appl. Sci.*, **16** (2023), 100674. <https://doi.org/10.1016/j.jrras.2023.100674>
5. A. Azzalini, *Statistical inference based on the likelihood*, New York: CRC Press, 1996. <https://doi.org/10.1201/9780203738627>
6. N. Balakrishnan, E. Cramer, *The art of progressive censoring*, New York: Birkhäuser, 2014. <https://doi.org/10.1007/978-0-8176-4807-7>
7. N. Balakrishnan, H. T. Ng, *Precedence-type tests and applications*, Hoboken: John Wiley & Sons, 2006.
8. M. M. Buzaridah, D. A. Ramadan, B. S. El-Desouky, Flexible reduced logarithmic-inverse lomax distribution with application for bladder cancer, *Open J. Model. Simul.*, **9** (2021), 351–369. <https://doi.org/10.4236/ojmsi.2021.94023>

9. A. Childs, B. Chandrasekar, N. Balakrishnan, Exact likelihood inference for an exponential parameter under progressive hybrid censoring schemes, In: *Statistical Models and Methods for Biomedical and Technical Systems*, 2008. https://doi.org/10.1007/978-0-8176-4619-6_23
10. A. Childs, B. Chandrasekar, N. Balakrishnan, D. Kundu, Exact likelihood inference based on Type-I and type-II hybrid censored samples from the exponential distribution, *Ann. Inst. Stat. Math.*, **55** (2003), 319–330. <https://doi.org/10.1007/BF02530502>
11. A. Childs, N. Balakrishnan, B. Chandrasekar, Exact distribution of the MLEs of the parameters and of the quantiles of two-parameter exponential distribution under hybrid censoring, *Statistics*, **46** (2012), 441–458. <https://doi.org/10.1080/02331888.2010.538476>
12. S. Dey, T. Dey, On progressively censored generalized inverted exponential distribution, *J. Appl. Stat.*, **41** (2014), 2557–2576. <https://doi.org/10.1080/02664763.2014.922165>
13. S. Dey, S. Singh, Y. M. Tripathi, A. Asgharzadeh, Estimation and prediction for a progressively censored generalized inverted exponential distribution, *Stat. Methodology*, **32** (2016), 185–202. <https://doi.org/10.1016/j.stamet.2016.05.007>
14. N. Draper, I. Guttman, Bayesian analysis of hybrid life tests with exponential failure times, *Ann. Inst. Stat. Math.*, **39** (1987), 219–225. <https://doi.org/10.1007/BF02491461>
15. B. Efron, Logistic regression, survival analysis, and the Kaplan-Meier curve, *J. Amer. Stat. Assoc.*, **83** (1988), 414–425.
16. A. R. El-Saeed, A. D. Abdellatif, Point and interval estimation of reliability and entropy for generalized exponential distribution under generalized type-II Hybrid censoring scheme, *Comput. J. Math. Stat. Sci.*, **4** (2025), 96–138.
17. A. R. El-Saeed, A. T. Ramadan, N. Alsadat, H. Alohal, A. H. Tolba, Analysis of progressive type-II censoring schemes for generalized power unit half-logistic geometric distribution, *AIMS Math.*, **8** (2023), 30846–30874. <https://doi.org/10.3934/math.20231577>
18. A. R. El-Saeed, M. K. Ruidas, A. H. Tolba, Estimation and Bayesian prediction for new version of Xgamma distribution under progressive type-II censoring, *Symmetry*, **17** (2025), 457.
19. E. A. Elsherpieny, A. Abdel-Hakim, Statistical analysis of Alpha-Power exponential distribution using unified Hybrid censored data and its applications, *Comput. J. Math. Stat. Sci.*, **4** (2025), 283–315.
20. B. Epstein, Estimation from life test data, *Technometrics*, **2** (1960), 447–454.
21. K. Fairbanks, R. Madsen, R. Dykstra, A confidence interval for an exponential parameter from a hybrid life test, *J. Amer. Stat. Assoc.*, **77** (1982), 137–140.
22. M. M. Hasaballah, O. S. Balogun, M. E. Bakr, Bayesian inference for the parameters of the generalized logistic distribution under a combined framework of generalized type-I and type-II hybrid censoring schemes with application to physical data, *AIP Adv.*, **15** (2025), 015330. <https://doi.org/10.1063/5.0249742>
23. W. H. Greene, The behavior of the fixed effects estimator in nonlinear models, *NYU Libraries, economics working papers*, 2002.
24. R. D. Gupta, D. Kundu, Hybrid censoring schemes with exponential failure distribution, *Commun. Stat.-Theory Meth.*, **27** (1998), 3065–3083. <https://doi.org/10.1080/03610929808832273>

25. H. S. Jeong, J. I. Park, B. J. Yum, Development of (r, T) hybrid sampling plans for exponential lifetime distributions, *J. Appl. Stat.*, **23** (1996), 601–608. <https://doi.org/10.1080/02664769623964>
26. D. Kundu, H. Howlader, Bayesian inference and prediction of the inverse Weibull distribution for type-II censored data, *Comput. Stat. Data Anal.*, **54** (2010), 1547–1558. <https://doi.org/10.1016/j.csda.2010.01.003>
27. D. Kundu, A. Joarder, Analysis of type-II progressively hybrid censored data, *Comput. Stat. Data Anal.*, **50** (2006), 2509–2528. <https://doi.org/10.1016/j.csda.2005.05.002>
28. J. F. Lawless, *Statistical models and methods for lifetime data*, New York: John Wiley & Sons, 1982.
29. E. T. Lee, J. Wang, *Statistical methods for survival data analysis*, New York: John Wiley & Sons, 2003.
30. M. Mahmoud, M. M. Nassar, M. A. Aefa, Parameter estimation for a mixture of inverse chen and inverse compound Rayleigh distributions based on type-II hybrid censoring scheme, *J. Stat. Appl. Prob.*, **10** (2021), 467–485. <http://doi.org/10.18576/jsap/100216>
31. M. A. W. Mahmoud, D. A. Ramadan, M. M. M. Mansour, Estimation of lifetime parameters of the modified extended exponential distribution with application to a mechanical model, *Commun. Stat.-Simul. Comput.*, **51** (2022), 7005–7018. <https://doi.org/10.1080/03610918.2020.1821887>
32. M. Mansour, D. A. Ramadan, Statistical inference to the parameters of the modified extended exponential distribution under the type-II hybrid censoring scheme, *Basic Science Engineering*, 2020.
33. B. Modok, S. Chowdhury, A. Kundu, Control charts for monitoring Weibull quantile under generalized hybrid and progressive censoring schemes, *Stat. Comput.*, **35** (2025), 49. <https://doi.org/10.1007/s11222-025-10581-2>
34. R. A. Mohamed, A. H. Tolba, E. M. Almetwally, D. A. Ramadan, Inference of reliability analysis for type ii half logistic Weibull distribution with application of bladder cancer, *Axioms*, **11** (2022), 386. <https://doi.org/10.3390/axioms11080386>
35. A. T. Ramadan, A. H. Tolba, B. S. El-Desouky, A unit half-logistic geometric distribution and its application in insurance, *Axioms*, **11** (2022), 676. <https://doi.org/10.3390/axioms11120676>
36. Royall, *Statistical evidence: a likelihood paradigm*, New York: CRC press, 1997.
37. A. S. Samia, E. Osama, A. K. Amarat, Statistical analysis of the inverse weibull distribution based on a generalized asymptotic mixed i-type census with competing risks, *Pakistan J. Stat.*, **40** (2024), 53–80.
38. A. K. Singh, The exponential distribution-theory, methods and applications, *Technometrics*, **39** (1997), p341. <https://doi.org/10.1080/00401706.1997.10485138>
39. M. S. Suprawhardana, S. Prayoto, Total time on test plot analysis for mechanical components of the RSG-GAS reactor, *Atom Indones*, **25** (1999), 81–90.
40. A. H. Tolba, Bayesian and non-Bayesian estimation methods for simulating the parameter of the Akshaya distribution, *Comput. J. Math. Stat. Sci.*, **1** (2022), 13–25.

41. A. H. Tolba, E. M. Almetwally, D. A. Ramadan, Bayesian estimation of a one-parameter Akshaya distribution with progressively type II censored data, *J. Stat. Appl. Prob.*, **11** (2022), 565–579. <http://doi.org/10.18576/jsap/110216>
42. F. Valizadeh Gamchi, O. Gürünlü Alma, R. Arabi Belaghi, Classical and Bayesian inference for Burr type-III distribution based on progressive type-II hybrid censored data, *Math. Sci.*, **13** (2019), 79–95. <https://doi.org/10.1007/s40096-019-0281-9>
43. T. Q. Xu, Y. P. Chen, Two-sided M-Bayesian credible limits of reliability parameters in the case of zero-failure data for exponential distribution, *Appl. Math. Model.*, **38** (2014), 2586–2600. <https://doi.org/10.1016/j.apm.2013.10.054>
44. C. P. Yadav, M. S. Panwar, Parametric inference of inverse Maxwell distribution under type-I progressively hybrid censoring scheme, *Life Cycle Reliab. Saf. Eng.*, **14** (2024), 19–35. <https://doi.org/10.1007/s41872-024-00279-1>



AIMS Press

© 2025 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)