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Research article

Modified methods to solve interval-valued Fermatean fuzzy multi-criteria decision-making problems

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Abstract: In a recently published paper, two methods were proposed to solve interval-valued Fermatean fuzzy multi-criteria decision-making problems (those in which the rating value of each alternative over each criterion is represented by an interval-valued Fermatean fuzzy number). In this paper, some numerical examples are considered to show that these existing methods fail to find the correct ranking of the alternatives. Also, the reasons for the failure of these existing methods are pointed out. Furthermore, new methods are proposed to solve the interval-valued Fermatean fuzzy multi-criteria decision-making problems by modifying existing methods. Moreover, the proposed modified methods are illustrated with the help of numerical examples. Finally, the ranking of the alternatives of the two existing real-life interval-valued Fermatean fuzzy multi-criteria decision-making problems is obtained by the proposed methods.

Keywords: interval-valued Fermatean fuzzy sets; score function; distance measure; multi-criteria decision-making problems
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1. Introduction

In the last few years, several extensions of the fuzzy set [1] have been proposed in the literature. In 2019, by generalizing the existing definitions [2] of a Pythagorean fuzzy set and a Pythagorean fuzzy number, Senapati and Yager [3] proposed the definitions of a Fermatean fuzzy set and a Fermatean fuzzy number. On the same direction, by generalizing the existing definitions [4] of an interval-valued Pythagorean fuzzy set and interval-valued Pythagorean fuzzy number, Jeevaraj [5] proposed the definitions of an interval-valued Fermatean fuzzy set (IVFFS) and an interval-valued Fermatean fuzzy number (IVFFN).

Jeevaraj [5] also proposed a method for comparing two interval-valued Fermatean fuzzy numbers (IVFFNs). Before this definition of an IVFFN, there was no ranking method in the literature to compare IVFFNs. Also, interval-valued intuitionistic fuzzy numbers and interval-valued Pythagorean fuzzy numbers are special types of IVFFNs, considered by Jeevaraj [5] to show the advantage of the proposed ranking method. The author then showed that existing methods [6−10] for comparing interval-valued intuitionistic fuzzy numbers failed to distinguish the distinct interval-valued intuitionistic fuzzy numbers, and other existing methods [4,11] for comparing interval-valued Pythagorean fuzzy numbers failed to distinguish the considered distinct interval-valued Pythagorean fuzzy numbers. However, the proposed ranking method did not fail to distinguish the considered distinct interval-valued intuitionistic and interval-valued Pythagorean fuzzy numbers. The author also proved mathematically that their proposed ranking method will never fail to distinguish two distinct IVFFNs.

Using the proposed ranking method, Jeevaraj [5] also proposed two methods to solve interval-valued Fermatean fuzzy multi-criteria decision-making (IVFFMCDM) problems. In the first method, an IVFFMCDM problem is solved by transforming it into its equivalent crisp multi-criteria decision-making (MCDM) problem. In the second method, an IVFFMCDM problem is solved without transforming it into its equivalent crisp MCDM problem.

Afterward, several researchers used Jeevaraj [5]'s methods to solve IVFFMCDM problems. However, in this paper, it is pointed out that Jeevaraj [5]'s methods fail to find the correct ranking of the alternatives. Hence, it is not appropriate to use Jeevaraj [5]'s methods. To validate this claim, two IVFFMCDM problems are solved using Jeevaraj [5]'s methods, showing that the obtained alternative ranking is not correct. To resolve this, Jeevaraj [5]'s methods are modified. Furthermore, to illustrate the modified methods, the correct ranking of the alternatives of the considered IVFFMCDM problems is obtained by the modified methods. Finally, the ranking of the alternatives of the two real-life IVFFMCDM problems, considered by Jeevaraj [5] to illustrate his proposed approach, is obtained by the modified methods.

This paper is organized as follows: In Section 2, some basic definitions are discussed. In Section 3, Jeevaraj [5]'s ranking method for comparing IVFFNs is discussed. In Section 4, Jeevaraj [5]'s methods for solving IVFFMCDM problems are discussed. In Section 5, two IVFFMCDM problems are solved using Jeevaraj [5]'s methods, and it is shown that the obtained alternative ranking is not correct. In Section 6, the reasons for the inappropriateness of Jeevaraj [5]'s methods are discussed. In Section 7, Jeevaraj [5]'s methods for solving IVFFMCDM problems are modified to resolve their inappropriateness. In Section 8, the appropriateness of the modified methods is discussed. In Section 9, the correct ranking of the alternatives of the considered IVFFMCDM problems is obtained by the modified methods. In Section 10, the real-life IVFFMCDM problems, considered by Jeevaraj [5] to illustrate his proposed method, are solved by the modified methods. Section 11 concludes the paper.

2. Preliminaries

In this section, some basic definitions are discussed.

Definition 2.1. [3] A set $F = \{\langle x, \mu_F(x), \nu_F(x)\rangle : x \in X\}$, defined on the universal set X, is said to be a Fermatean fuzzy set if the condition $0 < \left(\mu_F(x)\right)^3 + \left(\nu_F(x)\right)^3 \leq 1 \ \ \forall x \in X$ is satisfied, where $\mu_F(x) \in [0,1]$ and $\nu_F(x) \in [0,1]$ represent the degree of membership and the degree of non-membership of the element x belonging to the set F, respectively. For any Fermatean fuzzy set F and $x \in X$, $\pi_F(x) = \sqrt{1 - \left(\mu_F(x)\right)^3 + \left(\nu_F(x)\right)^3}$ is called the degree of hesitancy or the degree of indeterminacy of the element x belonging to the set F. Also, the number $F = (\mu_F, \nu_F)$ is called a Fermatean fuzzy number.

Definition 2.2. [5] A set $F = \left\{\langle x, \left[\mu_{F_L}(x), \mu_{F_U}(x)\right], \left[\nu_{F_L}(x), \nu_{F_U}(x)\right]\rangle : x \in X\right\}$, defined on the universal set X, is said to be an IVFFS if the condition $0 < \left(\mu_{F_U}(x)\right)^3 + \left(\nu_{F_U}(x)\right)^3 \leq 1 \ \ \forall x \in X$ is satisfied, where $\left[\mu_{F_L}(x), \mu_{F_U}(x)\right] \subseteq [0,1]$ and $\left[\nu_{F_L}(x), \nu_{F_U}(x)\right] \subseteq [0,1]$ represent the intervals of the degree of membership and the degree of non-membership of the element x belonging to the set F, respectively. For any IVFFS F and $x \in X$, $\pi_F(x) = \left[\pi_{F_L}(x), \pi_{F_U}(x)\right] = \left[\sqrt[3]{1 - \left(\mu_{F_U}(x)\right)^3 - \left(\nu_{F_U}(x)\right)^3}, \sqrt[3]{1 - \left(\mu_{F_L}(x)\right)^3 - \left(\nu_{F_L}(x)\right)^3}\right]$ is called the interval of degree of hesitancy or the interval of degree of indeterminacy of the element x belonging to the set F. Also, the number $F = \left(\left[\mu_{F_L}(x), \mu_{F_U}(x)\right], \left[\nu_{F_L}(x), \nu_{F_U}(x)\right]\right)$ is called an IVFFN.

Definition 2.3. [5] Let $F = \left(\left[\mu_{F_L}, \mu_{F_U}\right], \left[\nu_{F_L}, \nu_{F_U}\right]\right), F_1 = \left(\left[\mu_{F_{1L}}, \mu_{F_{1U}}\right], \left[\nu_{F_{1L}}, \nu_{F_{1U}}\right]\right)$ and $F_2 = \left(\left[\mu_{F_{2L}}, \mu_{F_{2U}}\right], \left[\nu_{F_{2L}}, \nu_{F_{2U}}\right]\right)$ be three IVFFNs. Then

(i) $F_1 \oplus F_2 = \left(\left[\sqrt[3]{\mu^3_{F_{1L}} + \mu^3_{F_{2L}} - \mu^3_{F_{1L}}\mu^3_{F_{2L}}}, \sqrt[3]{\mu^3_{F_{1U}} + \mu^3_{F_{2U}} - \mu^3_{F_{1U}}\mu^3_{F_{2U}}}\right], \left[\nu_{F_{1L}}\nu_{F_{2L}}, \nu_{F_{1U}}\nu_{F_{2U}}\right]\right)$.

(ii) $F_1 \otimes F_2 = \left(\left[\mu_{F_{1L}}\mu_{F_{2L}}, \mu_{F_{1U}}\mu_{F_{2U}}\right], \left[\sqrt[3]{\nu^3_{F_{1L}} + \nu^3_{F_{2L}} - \nu^3_{F_{1L}}\nu^3_{F_{2L}}}, \sqrt[3]{\nu^3_{F_{1U}} + \nu^3_{F_{2U}} - \nu^3_{F_{1U}}\nu^3_{F_{2U}}}\right]\right)$.

(iii) $\lambda F = \left(\left[\sqrt[3]{1 - \left(1 - \mu^3_{F_L}\right)^\lambda}, \sqrt[3]{1 - \left(1 - \mu^3_{F_U}\right)^\lambda}\right], \left[\nu^\lambda_{F_L}, \nu^\lambda_{F_U}\right]\right), \lambda > 0.$

(iv) $\lambda^F = \left(, \left[\mu^\lambda_{F_L}, \mu^\lambda_{F_U}\right] \left[\sqrt[3]{1 - \left(1 - \nu^3_{F_L}\right)^\lambda}, \sqrt[3]{1 - \left(1 - \nu^3_{F_U}\right)^\lambda}\right]\right), \lambda > 0.$

Definition 2.4. [5] Let $F_1 = \left[\mu_{F_{1L}}, \mu_{F_{1U}}\right], \left[\nu_{F_{1L}}, \nu_{F_{1U}}\right]$ and $F_2 = \left[\mu_{F_{2L}}, \mu_{F_{2U}}\right], \left[\nu_{F_{2L}}, \nu_{F_{2U}}\right]$ be any two IVFFNs. Then $D_{GE}(F_1, F_2)$ represents the generalized Euclidean distance between F_1 and F_2 and it can be evaluated by expression (1).

$$D_{GE}(F_1, F_2) =$$

$$\sqrt{\frac{\left(\mu^3_{F_{1L}} - \mu^3_{F_{2L}}\right)^2 + \left(\mu^3_{F_{1U}} - \mu^3_{F_{2U}}\right)^2 + \left(\nu^3_{F_{1L}} - \nu^3_{F_{2L}}\right)^2 + \left(\nu^3_{F_{1U}} - \nu^3_{F_{2U}}\right)^2 + \left(\left(1 - \mu^3_{F_{1U}} - \nu^3_{F_{1U}}\right) - \left(1 - \mu^3_{F_{2U}} - \nu^3_{F_{2U}}\right)\right)^2 + \left(\left(1 - \mu^3_{F_{1L}} - \nu^3_{F_{1L}}\right) - \left(1 - \mu^3_{F_{2L}} - \nu^3_{F_{2L}}\right)\right)^2}{6}}. \tag{1}$$

3. Jeevaraj's ranking method

Jeevaraj [5] proposed the following ranking method to compare two IVFFNs $F_1 = \left(\left[\mu_{F_{1L}}, \mu_{F_{1U}}\right], \left[\nu_{F_{1L}}, \nu_{F_{1U}}\right]\right)$ and $F_2 = \left(\left[\mu_{F_{2L}}, \mu_{F_{2U}}\right], \left[\nu_{F_{2L}}, \nu_{F_{2U}}\right]\right)$.

Step 1: Evaluate $J_M(F_1) = \frac{\mu_{F_{1L}}^3 + \mu_{F_{1U}}^3 - \nu_{F_{1L}}^3 - \nu_{F_{1U}}^3}{2}$, $J_M(F_2) = \frac{\mu_{F_{2L}}^3 + \mu_{F_{2U}}^3 - \nu_{F_{2L}}^3 - \nu_{F_{2U}}^3}{2}$ and check that $J_M(F_1) > J_M(F_2)$ or $J_M(F_1) < J_M(F_2)$ or $J_M(F_1) = J_M(F_2)$.
Case (i): If $J_M(F_1) < J_M(F_2)$ then $F_1 < F_2$. Hence, maximum $\{F_1, F_2\} = F_2$ and minimum $\{F_1, F_2\} = F_1$.
Case (ii): If $J_M(F_1) > J_M(F_2)$ then $F_1 > F_2$. Hence, maximum $\{F_1, F_2\} = F_1$ and minimum $\{F_1, F_2\} = F_2$.
Case (iii): If $J_M(F_1) = J_M(F_2)$ then go to Step 2.
Step 2: Evaluate $J_H(F_1) = \frac{\mu_{F_{1L}}^3 + \mu_{F_{1U}}^3 + \nu_{F_{1L}}^3 + \nu_{F_{1U}}^3}{2}$, $J_H(F_2) = \frac{\mu_{F_{2L}}^3 + \mu_{F_{2U}}^3 + \nu_{F_{2L}}^3 + \nu_{F_{2U}}^3}{2}$ and check that $J_H(F_1) > J_H(F_2)$ or $J_H(F_1) < J_H(F_2)$ or $J_H(F_1) = J_H(F_2)$.
Case (i): If $J_H(F_1) < J_H(F_2)$ then $F_1 < F_2$. Hence, maximum $\{F_1, F_2\} = F_2$ and minimum $\{F_1, F_2\} = F_1$.
Case (ii): If $J_H(F_1) > J_H(F_2)$ then $F_1 > F_2$. Hence, maximum $\{F_1, F_2\} = F_1$ and minimum $\{F_1, F_2\} = F_2$.
Case (iii): If $J_H(F_1) = J_H(F_2)$ then go to Step 3.
Step 3: Evaluate $J_P(F_1) = \frac{-\mu_{F_{1L}}^3 + \mu_{F_{1U}}^3 + \nu_{F_{1L}}^3 - \nu_{F_{1U}}^3}{2}$, $J_P(F_2) = \frac{-\mu_{F_{2L}}^3 + \mu_{F_{2U}}^3 + \nu_{F_{2L}}^3 - \nu_{F_{2U}}^3}{2}$ and check that $J_P(F_1) > J_P(F_2)$ or $J_P(F_1) < J_P(F_2)$ or $J_P(F_1) = J_P(F_2)$.
Case (i): If $J_P(F_1) > J_P(F_2)$ then $F_1 < F_2$. Hence, maximum $\{F_1, F_2\} = F_2$ and minimum $\{F_1, F_2\} = F_1$.
Case (ii): If $J_P(F_1) < J_P(F_2)$ then $F_1 > F_2$. Hence, maximum $\{F_1, F_2\} = F_1$ and minimum $\{F_1, F_2\} = F_2$.
Case (iii): If $J_H(F_1) = J_H(F_2)$ then go to Step 4.
Step 4: Evaluate $J_C(F_1) = \frac{-\mu_{F_{1L}}^3 + \mu_{F_{1U}}^3 - \nu_{F_{1L}}^3 + \nu_{F_{1U}}^3}{2}$, $J_C(F_2) = \frac{-\mu_{F_{2L}}^3 + \mu_{F_{2U}}^3 - \nu_{F_{2L}}^3 + \nu_{F_{2U}}^3}{2}$ and check that $J_C(F_1) < J_C(F_2)$ or $J_C(F_1) > J_C(F_2)$ or $J_C(F_1) = J_C(F_2)$.
Case (i): If $J_C(F_1) < J_C(F_2)$ then $F_1 < F_2$. Hence, maximum $\{F_1, F_2\} = F_2$ and minimum $\{F_1, F_2\} = F_1$.
Case (ii): If $J_C(F_1) > J_C(F_2)$ then $F_1 > F_2$. Hence, maximum $\{F_1, F_2\} = F_1$ and minimum $\{F_1, F_2\} = F_2$.
Case (iii): If $J_C(F_1) = J_C(F_2)$ then $F_1 = F_2$. Hence, maximum $\{F_1, F_2\} =$ minimum $\{F_1, F_2\} = F_1 = F_2$.

4. Jeevaraj's IVFFMCDM methods

Jeevaraj [5] proposed two methods to solve IVFFMCDM problems. In this section, both methods are discussed.

4.1. Jeevaraj's first IVFFMCDM method

Jeevaraj [5] proposed the following method to solve IVFFMCDM problems having an interval-valued Fermatean fuzzy (IVFF) decision matrix $M = (m_{ij})_{r\times s} = \left(\left[\mu_{L_{ij}}, \mu_{U_{ij}}\right], \left[\nu_{L_{ij}}, \nu_{U_{ij}}\right]\right)_{r\times s}$, where the IVFFN m_{ij} represents the rating value of the i^{th} alternative A_i over the j^{th} benefit criteria Cr_j.
Step 1: Transform the IVFF decision matrix M into the weighted IVFF decision matrix $M' =$

$\left(m_{ij}^{'}\right)_{r\times s}$, where $m_{ij}^{'} = m_{ij} \,.\, w_j, i = 1 \; to \; r, j = 1 \; to \; s$, the non-negative real number w_j is the weight of the j^{th} benefit criteria, and $\sum_{j=1}^{s} w_j = 1$.

Step 2: Using the following steps, transform the weighted IVFF decision matrix $M^{'}$ into the weighted crisp decision matrix $S = \left(s_{ij}\right)_{r\times s}$.

Step 2(a): Transform the weighted IVFF decision matrix $M^{'}$ into the weighted crisp decision matrix $S = \left(J_M\left(m_{ij}^{'}\right)\right)_{r\times s}$ and check that all the elements $J_M\left(m_{ij}^{'}\right)$ of the weighted crisp decision matrix S are distinct or not.

Case (i): If all the elements $J_M\left(m_{ij}^{'}\right)$ are distinct, then go to Step 3.

Case (ii): If some elements $J_M\left(m_{ij}^{'}\right)$ are equal, then replace such elements $J_M\left(m_{ij}^{'}\right)$ with $J_H\left(m_{ij}^{'}\right)$ and go to Step 2(b).

Step 2(b): Check if all the elements $J_H\left(m_{ij}^{'}\right)$ are distinct or not.

Case (i): If all the elements $J_H\left(m_{ij}^{'}\right)$ are distinct, then go to Step 3.

Case (ii): If some elements $J_H\left(m_{ij}^{'}\right)$ are equal, then replace such elements $J_H\left(m_{ij}^{'}\right)$ with $J_P\left(m_{ij}^{'}\right)$ and go to Step 2(c).

Step 2(c): Check if all the elements $J_P\left(m_{ij}^{'}\right)$ are distinct or not.

Case (i): If all the elements $J_P\left(m_{ij}^{'}\right)$ are distinct, then go to Step 3.

Case (ii): If some elements $J_P\left(m_{ij}^{'}\right)$ are equal, then replace such elements $J_P\left(m_{ij}^{'}\right)$ with $J_C\left(m_{ij}^{'}\right)$ and go to Step 3.

Step 3: Evaluate the interval-valued positive ideal solution (IVFFPIS) $P_I = \left(p_j\right)_{1\times s}$ and the interval-valued negative ideal solution (IVFFNIS) $N_I = \left(n_j\right)_{1\times s}$ where

$$p_j = maximum_{1\le i\le r}\left(\left[\mu_{Lij}^{'}, \mu_{Uij}^{'}\right], \left[\nu_{Lij}^{'}, \nu_{Uij}^{'}\right]\right)$$
$$= \begin{pmatrix} \left[maximum_{1\le i\le r}\left\{\mu_{Lij}^{'}\right\}, maximum_{1\le i\le r}\left\{\mu_{Uij}^{'}\right\}\right], \\ \left[minimum_{1\le i\le r}\left\{\nu_{Lij}^{'}\right\}, minimum_{1\le i\le r}\left\{\nu_{Uij}^{'}\right\}\right] \end{pmatrix},$$
$$n_j = minimum_{1\le i\le r}\left(\left[\mu_{Lij}^{'}, \mu_{Uij}^{'}\right], \left[\nu_{Lij}^{'}, \nu_{Uij}^{'}\right]\right)$$
$$= \begin{pmatrix} \left[minimum_{1\le i\le r}\left\{\mu_{Lij}^{'}\right\}, minimum_{1\le i\le r}\left\{\mu_{Uij}^{'}\right\}\right], \\ \left[maximum_{1\le i\le r}\left\{\nu_{Lij}^{'}\right\}, maximum_{1\le i\le r}\left\{\nu_{Uij}^{'}\right\}\right] \end{pmatrix}.$$

Step 4: Using the following steps, transform the IVFFPIS $P_I = \left(p_j\right)_{1\times s}$ into the crisp positive ideal solution $CP_I = \left(t_j\right)_{1\times s}$.

Step 4(a): Transform the IVFFPIS $P_I = \left(p_j\right)_{1\times s}$ into the crisp positive ideal solution $CP_I = \left(J_M\left(p_j\right)\right)_{1\times s}$ and check if all the elements $J_M\left(p_j\right)$ of the crisp positive ideal solution CP_I are distinct or not.

Case (i): If all the elements $J_M\left(p_j\right)$ are distinct, then go to Step 5.

Case (ii): If some elements $J_M\left(p_j\right)$ are equal, then replace such elements $J_M\left(p_j\right)$ with $J_H\left(p_j\right)$ and go to Step 4(b).

Step 4(b): Check if all the elements $J_H\left(p_j\right)$ are distinct or not.

Case (i): If all the elements $J_H\left(p_j\right)$ are distinct, then go to Step 5.

Case (ii): If some elements $J_H\left(p_j\right)$ are equal, then replace such elements $J_H\left(p_j\right)$ with $J_P\left(p_j\right)$ and go to Step 4(c).

Step 4(c): Check if all elements $J_P\left(p_j\right)$ are distinct or not.

Case (i): If all the elements $J_P\left(p_j\right)$ are distinct, then go to Step 5.

Case (ii): If some elements $J_P(p_j)$ are equal, then replace such elements $J_P(p_j)$ with $J_C(p_j)$ and go to Step 5.
Step 5: Using the following steps, transform the IVFFNIS $N_I = (n_j)_{1\times s}$ into the crisp negative ideal solution $CN_I = (v_j)_{1\times s}$.
Step 5(a): Transform the IVFFNIS $N_I = (n_j)_{1\times s}$ into the crisp negative ideal solution $CN_I = \left(J_M(n_j)\right)_{1\times s}$ and check if all the elements $J_M(n_j)$ of the crisp negative ideal solution CN_I are distinct or not.
Case (i): If all the elements $J_M(n_j)$ are distinct, then go to Step 6.
Case (ii): If some elements $J_M(n_j)$ are equal, then replace such elements $J_M(n_j)$ with $J_H(n_j)$ and go to Step 5(b).
Step 5(b): Check if all the elements $J_H(n_j)$ are distinct or not.
Case 1: If all the elements $J_H(n_j)$ are distinct, then go to Step 6.
Case 2: If some elements $J_H(n_j)$ are equal, then replace such elements $J_H(n_j)$ with $J_P(n_j)$ and go to Step 5(c).
Step 5(c): Check if all the elements $J_P(n_j)$ are distinct or not.
Case 1: If all the elements $J_P(n_j)$ are distinct, then go to Step 6.
Case 2: If some elements $J_P(n_j)$ are equal, then replace such elements $J_P(n_j)$ with $J_C(n_j)$ and go to Step 6.
Step 6: Using expressions (2) and (3), evaluate the distance between the i^{th} row of weighted crisp decision matrix S with the crisp positive ideal solution $CP_I = (t_j)_{1\times s}$ and the crisp negative ideal solution $CN_I = (v_j)_{1\times s}$.

$$d_i(S, CP_I) = \sqrt{\frac{\sum_{j=1}^{s}\left(s_{ij}-t_j\right)^2}{s}},\ i = 1,2,\ldots,r \tag{2}$$

$$d_i(S, CN_I) = \sqrt{\frac{\sum_{j=1}^{s}\left(s_{ij}-v_j\right)^2}{s}}, i = 1,2,\ldots,r. \tag{3}$$

Step 7: Using expression (4), evaluate the relative closeness $CC(A_i)$ of the i^{th} alternative A_i and check that $CC(A_i) > CC(A_j)$ or $CC(A_i) < CC(A_j)$ or $CC(A_i) = CC(A_j)$.

$$CC(A_i) = \frac{d_i(S,N_I)}{d_i(S,P_I)+d_i(S,N_I)}, i = 1,2,\ldots,r. \tag{4}$$

Case (i): If $CC(A_i) > CC(A_j)$, then $A_i > A_j$.
Case (ii): If $CC(A_i) < CC(A_j)$, then $A_i < A_j$.
Case (iii): If $CC(A_i) = CC(A_j)$, then $A_i = A_j$.

4.2. Jeevaraj's second IVFFMCDM method

Jeevaraj [5] proposed the following method to solve IVFFMCDM problems having the IVFF decision matrix $M = (m_{ij})_{r\times s} = \left(\left(\left[\mu_{L_{ij}}, \mu_{U_{ij}}\right], \left[\nu_{L_{ij}}, \nu_{U_{ij}}\right]\right)\right)_{r\times s}$, where IVFFN m_{ij} represents the ranking of the i^{th} alternative A_i over the j^{th} benefit criteria Cr_j.
Step 1: Transform the IVFF decision matrix M into the weighted IVFF decision matrix $M' = (m'_{ij})_{r\times s}$, where $m'_{ij} = m_{ij}.\, w_j, i = 1\ to\ r, j = 1\ to\ s, w_j$ is the weight of the j^{th} benefit criteria

represented by a real number and $\sum_{j=1}^{s} w_j = 1$.

Step 2: Evaluate the IVFFPIS $P_I = (p_j)_{1\times s}$ and the IVFFNIS $N_I = (n_j)_{1\times s}$ where

$$p_j = maximum_{1\le i\le r}\left([\mu'_{Lij}, \mu'_{Uij}], [\nu'_{Lij}, \nu'_{Uij}]\right)$$
$$= \begin{pmatrix} [maximum_{1\le i\le r}\{\mu'_{Lij}\}, maximum_{1\le i\le r}\{\mu'_{Uij}\}], \\ [minimum_{1\le i\le r}\{\nu'_{Lij}\}, minimum_{1\le i\le r}\{\nu'_{Uij}\}] \end{pmatrix},$$
$$n_j = minimum_{1\le i\le r}\left([\mu'_{Lij}, \mu'_{Uij}], [\nu'_{Lij}, \nu'_{Uij}]\right)$$
$$= \begin{pmatrix} [minimum_{1\le i\le r}\{\mu'_{Lij}\}, minimum_{1\le i\le r}\{\mu'_{Uij}\}], \\ [maximum_{1\le i\le r}\{\nu'_{Lij}\}, maximum_{1\le i\le r}\{\nu'_{Uij}\}] \end{pmatrix}.$$

Step 3: Using expression (1), evaluate the distance $D_{GE}(A_i, P_I)$ between the i^{th} row of weighted IVFF decision matrix M' and the IVFFPIS P_I and the distance $D_{GE}(A_i, N_I)$ between the i^{th} row of weighted IVFF decision matrix M' and the IVFFNIS N_I.

Step 4: Using expression (5), evaluate the relative closeness $CC(A_i)$ of the i^{th} alternative A_i and check that $CC(A_i) > CC(A_j)$ or $CC(A_i) < CC(A_j)$ or $CC(A_i) = CC(A_j)$.

$$CC(A_i) = \frac{D_{GE}(A_i, N_I)}{D_{GE}(A_i, P_I) + D_{GE}(A_i, N_I)}, i = 1,2,\ldots,r. \tag{5}$$

Case (i): If $CC(A_i) > CC(A_j)$, then $A_i > A_j$.

Case (ii): If $CC(A_i) < CC(A_j)$, then $A_i < A_j$.

Case (iii): If $CC(A_i) = CC(A_j)$, then $A_i = A_j$.

5. Inappropriateness of Jeevaraj's IVFFMCDM methods

In this section, two IVFFMCDM problems are solved using Jeevaraj [5]'s IVFFMCDM methods to show that the obtained results are not correct.

5.1. Inappropriateness of Jeevaraj's first IVFFMCDM method

Let $M = (m_{ij})_{2\times 2} = \begin{bmatrix} ([0.2,0.3],[0,0]) & ([0,0.4],[0.3,0.3]) \\ ([0,\sqrt[3]{0.035}],[0,0]) & ([0,0.4],[0.3,0.3]) \end{bmatrix}$ represents the IVFF decision matrix of an IVFFMCDM problem having two alternatives (A_1 and A_2) and two benefit criteria (Cr_1 and Cr_2). Since the rating value of both alternatives corresponding to the second criterion is the same, the ranking of the alternatives is independent from the second criterion. Also, as the rating values of both alternatives corresponding to the first criterion are distinct, the rank of both alternatives should be different. In this section, it is shown that by solving the considered IVFFMCDM problem by Jeevaraj [5]'s first IVFFMCDM method, the ranking of both alternatives is the same. Hence, it is inappropriate to use Jeevaraj [5]'s first IVFFMCDM method to solve IVFFMCDM problems.

If it is assumed that $w_1 = 1$ and $w_2 = 0$, then by using Jeevaraj [5]'s first IVFFMCDM method, the ranking of both alternatives of the considered IVFFMCDM problem can be obtained as follows.

Step 1: According to Step 1 of Jeevaraj [5]'s first IVFFMCDM method, the IVFF decision matrix M can be transformed into the weighted IVFF decision matrix

$$M' = (m'_{ij})_{2\times 2} = \begin{bmatrix} J_M([0.2,0.3],[0,0]) & J_M([0,0],[1,1]) \\ J_M([0,\sqrt[3]{0.035}],[0,0]) & J_M([0,0],[1,1]) \end{bmatrix}.$$

Step 2: According to Step 2(a) of Jeevaraj [5]'s first IVFFMCDM method, the weighted IVFF decision matrix M' can be transformed into the weighted crisp decision matrix

$$S = (s_{ij})_{2\times 2} = \begin{bmatrix} J_M([0.2,0.3],[0,0]) & J_M([0,0],[1,1]) \\ J_M([0,\sqrt[3]{0.035}],[0,0]) & J_M([0,0],[1,1]) \end{bmatrix} = \begin{bmatrix} 0.017 & -1 \\ 0.017 & -1 \end{bmatrix}.$$

Since $J_M(s_{11}) = J_M(s_{21})$ and $J_M(s_{12}) = J_M(s_{22})$. So, according to Case (ii) of Step 2(a), there is a need to replace $J_M(s_{11}), J_M(s_{21}), J_M(s_{12})$ and $J_M(s_{22})$ with $J_H(s_{11}), J_H(s_{21}), J_H(s_{12})$, and $J_H(s_{22})$ respectively.

$$S = (s_{ij})_{2\times 2} = \begin{bmatrix} J_H([0.2,0.3],[0,0]) & J_H([0,0],[1,1]) \\ J_H([0,\sqrt[3]{0.035}],[0,0]) & J_H([0,0],[1,1]) \end{bmatrix} = \begin{bmatrix} 0.017 & 1 \\ 0.017 & 1 \end{bmatrix}.$$

Since $J_H(s_{11}) = J_H(s_{21})$ and $J_H(s_{12}) = J_H(s_{22})$. So, according to Case (ii) of Step 2(b), there is a need to replace $J_H(s_{11}), J_H(s_{21}), J_H(s_{12})$ and $J_H(s_{22})$ with $J_P(s_{11}), J_P(s_{21}), J_P(s_{12})$ and $J_P(s_{22})$ respectively.

$$S = (s_{ij})_{2\times 2} = \begin{bmatrix} J_P([0.2,0.3],[0,0]) & J_P([0,0],[1,1]) \\ J_P([0,\sqrt[3]{0.035}],[0,0]) & J_P([0,0],[1,1]) \end{bmatrix} = \begin{bmatrix} 0.009 & 0 \\ 0.017 & 0 \end{bmatrix}.$$

Since $J_P(s_{12}) = J_P(s_{22})$. So, according to Case (ii) of Step 2(c), there is a need to replace $J_P(s_{12})$ and $J_P(s_{22})$ with $J_C(s_{12})$ and $J_C(s_{22})$ respectively.

$$S = (s_{ij})_{2\times 2} = \begin{bmatrix} J_P([0.2,0.3],[0,0]) & J_C([0,0],[1,1]) \\ J_P([0,\sqrt[3]{0.035}],[0,0]) & J_C([0,0],[1,1]) \end{bmatrix} = \begin{bmatrix} 0.009 & 0 \\ 0.017 & 0 \end{bmatrix}.$$

Step 3: On applying Step 3 of Jeevaraj [5]'s first IVFFMCDM method, the obtained IVFFPIS is
$P_I = [p_1 \quad p_2]_{1\times 2}$
where
$p_1 = \left(\left[maximum\ (0.2,0), maximum\ \left(0.3, \sqrt[3]{0.035}\right)\right], [minimum\ (0,0), minimum\ (0,0)]\right)$
$= ([0.2,0.3],[0,0])$,
$p_2 = ([maximum\ (0,0), maximum\ (0,0)], [minimum\ (1,1), minimum\ (1,1)])$
$= ([0,0],[1,1])$
and the obtained IVFFNIS is
$N_I = [n_1 \quad n_2]_{1\times 2}$
where
$n_1 = \left(\left[minimum\ (0.2,0), minimum\ \left(0.3, \sqrt[3]{0.035}\right)\right], [maximum\ (0,0), maximum\ (0,0)]\right)$
$= \left(\left[0, \sqrt[3]{0.035}\right], [0,0]\right)$,
$n_2 = ([minimum\ (0,0), minimum\ (0,0)], [maximum\ (1,1), maximum\ (1,1)])$
$= ([0,0],[1,1])$.

Step 4: According to Step 4 of Jeevaraj [5]'s first IVFFMCDM method, the IVFFPIS P_I can be transformed into the crisp positive ideal solution CP_I.

$$CP_I = [J_M([0.2,0.3],[0,0]) \quad J_M([0,0],[1,1])] = [0.017 \ -1].$$

Step 5: According to Step 5 of Jeevaraj [5]'s first IVFFMCDM method, the IVFFNIS N_I can be transformed into the crisp negative ideal solution CN_I.

$$CN_I = \left[J_M\left(\left[0,\sqrt[3]{0.035}\right],[0,0]\right) \quad J_M([0,0],[1,1])\right] = [0.017 \quad -1].$$

Step 6: According to Step 6 of Jeevaraj [5]'s first IVFFMCDM method,

$d_1(A_1, P_I) = \sqrt{\frac{(0.009-0.017)^2+(0+1)^2}{2}} = 0.70713,$

$d_2(A_2, P_I) = \sqrt{\frac{(0.017-0.017)^2+(0+1)^2}{2}} = 0.70711,$

$d_1(A_1, N_I) = \sqrt{\frac{(0.009-0.017)^2+(0+1)^2}{2}} = 0.70713,$

$d_2(A_2, N_I) = \sqrt{\frac{(0.017-0.017)^2+(0+1)^2}{2}} = 0.70711.$

Step 7: According to Step 7 of Jeevaraj [5]'s first IVFFMCDM method,

$$CC(A_1) = \frac{0.70713}{0.70713+0.70713} = 0.5, CC(A_2) = \frac{0.70711}{0.70711+0.70711} = 0.5.$$

Since $CC(A_1) = CC(A_2) = 0.5$. So, according to Case (iii) of Step 7 of Jeevaraj [5]'s first IVFFMCDM method, $A_1 = A_2$.

5.2. Inappropriateness of Jeevaraj's second IVFFMCDM method

Let $M = (m_{ij})_{2\times 2} = \begin{bmatrix} ([0.1,0.2],[0.1,0.2]) & ([0.3,0.4],[0.3,0.4]) \\ ([0.2,0.3],[0.2,0.3]) & ([0.3,0.4],[0.3,0.4]) \end{bmatrix}$ represent the IVFF decision matrix of an IVFFMCDM problem having two alternatives and two benefit criteria.

If it is assumed that $w_1 = 1$ and $w_2 = 0$, then using Jeevaraj [5]'s second IVFFMCDM method, the ranking of the alternatives of the considered IVFFMCDM problem can be obtained as follows:
Step 1: According to Step 1 of Jeevaraj [5]'s second IVFFMCDM method, the IVFF matrix M can be transformed into the weighted IVFF decision matrix $M' = (m'_{ij})_{2\times 2} = \begin{bmatrix} ([0.1,0.2],[0.1,0.2]) & ([0,0],[1,1]) \\ ([0.2,0.3],[0.2,0.3]) & ([0,0],[1,1]) \end{bmatrix}$.
Step 2: On applying Step 2 of Jeevaraj [5]'s second IVFFMCDM method, the obtained IVFFPIS is
$P_I = [p_1 \quad p_2]_{1\times 2}$
where
$p_1 = maximum\,\{([0.1,0.2],[0.1,0.2]),([0.2,0.3],[0.2,0.3])\}$
$= ([maximum\,(0.1,0.2), maximum\,(0.2,0.3)],[minimum\,(0.1,0.2), minimum\,(0.2,0.3)])$
$= ([0.2,0.3],[0.1,0.2]),$
$p_2 = maximum\,\{([0,0],[1,1]),([0,0],[1,1])\}$
$= ([maximum\,(0,0), maximum\,(0,0)],[minimum\,(1,1), minimum\,(1,1)])$
$= ([0,0],[1,1])$
and the obtained IVFFNIS is
$N_I = [n_1 \quad n_2]_{1\times 2}$
where
$n_1 = ([minimum\,(0.1,0.2), minimum\,(0.2,0.3)],[maximum\,(0.1,0.2), maximum\,(0.2,0.3)])$
$= ([0.1,0.2],[0.2,0.3]),$
$n_2 = ([minimum\,(0,0), minimum\,(0,0)],[maximum\,(1,1), maximum\,(1,1)])$
$= ([0,0],[1,1]).$
Step 3: According to Step 3 of Jeevaraj [5]'s second IVFFMCDM method,

$$D_{GE}(A_1, P_I) = 0.0116, D_{GE}(A_2, P_I) = 0.0116, D_{GE}(A_1, N_I) = 0.0116 \text{ and } D_{GE}(A_2, N_I) = 0.0116.$$

Step 4: According to Step 4 of Jeevaraj [5]'s second IVFFMCDM method,

$$CC(A_1) = \frac{0.0116}{0.0116+0.0116} = 0.5, CC(A_2) = \frac{0.0116}{0.0116+0.0116} = 0.5.$$

Since $CC(A_1) = CC(A_2) = 0.5$.
So, according to Case (iii) of Step 4 of Jeevaraj [5]'s second IVFFMCDM method, $A_1 = A_2$.

6. Reason for inappropriateness

In Section 5, it was shown that Jeevaraj [5]'s IVFFMCDM methods fail to find the correct ranking of the alternatives. Hence, it is inappropriate to use Jeevaraj [5]'s IVFFMCDM methods. In this section, reasons for this inappropriateness are discussed.

6.1. Reasons for the inappropriateness of Jeevaraj's first IVFFMCDM method

Jeevaraj [5]'s first IVFFMCDM method fails to find the correct ranking of the alternatives due to the following reasons:

In Step 2, the weighted IVFF decision matrix M' is transformed into the weighted crisp decision matrix S; in Step 3, the IVFFPIS $P_I = \left(p_j\right)_{1\times s}$ is transformed into the crisp positive ideal solution CP_I; and in Step 4, IVFFNIS $N_I = \left(n_j\right)_{1\times s}$ is transformed into the crisp negative ideal solution CN_I. Then, the existing crisp TOPSIS [12] is used to find the ranking of the alternatives. However, the transformed weighted crisp decision matrix S will not necessarily be equivalent to the weighted IVFF decision matrix M', the transformed crisp positive ideal solution CP_I will not necessarily be equivalent to IVFFPIS $P_I = \left(p_j\right)_{1\times s}$, and the transformed crisp negative ideal solution CN_I will not necessarily be equivalent to IVFFNIS $N_I = \left(n_j\right)_{1\times s}$ as it may happen that $F_1 \neq F_2$ but $J_i(F_1) = J_i(F_2), i = M$ or H or P or C.

The following validates this claim:

(i) It is obvious from Step 1 of Section 5.1 that the first and second elements of the first column of the weighted IVFF decision matrix M' are distinct. This indicates that the rating value of the first alternative and the second alternative, provided by a decision-maker, is distinct. However, it is obvious from Step 2 of Section 5.1 that the first and second elements of the first column of the transformed weighted crisp decision matrix S are equal. This indicates that the rating value of the first alternative and the second alternative, provided by a decision-maker, is the same.

(ii) It is obvious from Step 4 of Section 5.1 that the first and second elements of the crisp positive ideal solution are 0.017 and -1, respectively. However, according to the weighted crisp decision matrix $S = \begin{bmatrix} 0.009 & 0 \\ 0.017 & 0 \end{bmatrix}$, the first and second elements of the crisp positive ideal solution are maximum $\{0.009, 0.017\} = 0.017$ and maximum $\{0,0\} = 0$, respectively.

(iii) It is obvious from Step 5 of Section 5.1 that the first and second elements of the crisp negative ideal solution are 0.017 and -1, respectively. However, according to the weighted crisp decision matrix $S = \begin{bmatrix} 0.009 & 0 \\ 0.017 & 0 \end{bmatrix}$, the first and second elements of the crisp positive ideal solution are minimum$\{0.009, 0.017\} = 0.009$ and minimum $\{0,0\} = 0$, respectively.

6.2. Reasons for the inappropriateness of Jeevaraj's second IVFFMCDM method

Jeevaraj [5]'s second IVFFMCDM method fails to find the correct ranking of the alternatives due to the following reasons:

In Step 2 of Jeevaraj [5]'s second IVFFMCDM method, it is assumed that $p_j = maximum_{1\le i\le r}\left([\mu'_{Lij},\mu'_{Uij}],[\nu'_{Lij},\nu'_{Uij}]\right) = \begin{pmatrix}[maximum_{1\le i\le r}\{\mu'_{Lij}\}, maximum_{1\le i\le r}\{\mu'_{Uij}\}],\\ [minimum_{1\le i\le r}\{\nu'_{Lij}\}, minimum_{1\le i\le r}\{\nu'_{Uij}\}]\end{pmatrix}$ represents the j^{th} element of the IVFFPIS $P_I = \left(p_j\right)_{1\times s}$ and $n_j = minimum_{1\le i\le r}\left([\mu'_{Lij},\mu'_{Uij}],[\nu'_{Lij},\nu'_{Uij}]\right) = \begin{pmatrix}[minimum_{1\le i\le r}\{\mu'_{Lij}\}, minimum_{1\le i\le r}\{\mu'_{Uij}\}],\\ [maximum_{1\le i\le r}\{\nu'_{Lij}\}, maximum_{1\le i\le r}\{\nu'_{Uij}\}]\end{pmatrix}$ represents the j^{th} element of the IVFFNIS $N_I = \left(n_j\right)_{1\times s}$.

It is a well-known fact that if p_j represents the j^{th} element of the IVFFPIS $P_I = \left(p_j\right)_{1\times s}$ and if n_j represents the j^{th} element of the IVFFNIS $N_I = \left(n_j\right)_{1\times s}$, then p_j should be equal to $\left([\mu'_{Lij},\mu'_{Uij}],[\nu'_{Lij},\nu'_{Uij}]\right)$ for some i as well as n_j should be equal to $\left([\mu'_{Lij},\mu'_{Uij}],[\nu'_{Lij},\nu'_{Uij}]\right)$ for some i. However, it is obvious from Step 2 of Section 5.2 that,

$p_1 = maximum\,\{([0.1,0.2],[0.1,0.2]),([0.2,0.3],[0.2,0.3])\}$
$= ([maximum\,(0.1,0.2), maximum\,(0.2,0.3)],[minimum\,(0.1,0.2), minimum\,(0.2,0.3)])$
$= ([0.2,0.3],[0.1,0.2])$ and
$n_1 = minimum\,\{([0.1,0.2],[0.1,0.2]),([0.2,0.3],[0.2,0.3])\}$
$= ([minimum\,(0.1,0.2), minimum\,(0.2,0.3)],[maximum\,(0.1,0.2), maximum\,(0.2,0.3)])$
$= ([0.1,0.2],[0.2,0.3])$.

It is obvious that $p_1 = ([0.2,0.3],[0.1,0.2])$ is neither $([0.1,0.2],[0.1,0.2])$ nor $([0.2,0.3],[0.2,0.3])$ and $n_1 = ([0.1,0.2],[0.2,0.3])$ is neither $([0.1,0.2],[0.1,0.2])$ nor $([0.2,0.3],[0.2,0.3])$.

7. Modified IVFFMCDM methods

In this section, Jeevaraj [5]'s methods are modified to resolve their inappropriateness.

7.1. First modified IVFFMCDM method

The steps of the first modified IVFFMCDM method are as follows:

Step 1: Transform the IVFF decision matrix $M = \left(m_{ij}\right)_{r\times s}$ into the weighted IVFF decision matrix $M' = \left(m'_{ij}\right)_{r\times s}$, where $m'_{ij} = m_{ij}.\,w_j, i = 1\ to\ r, j = 1\ to\ s$, the non-negative real number w_j is the weight of the j^{th} benefit criteria and $\sum_{j=1}^{s} w_j = 1$.

Step 2: Using the existing method [5], discussed in Section 3, find the ranking of all the elements of the weighted IVFF decision matrix M' in increasing order.

Step 3: Assign the natural number i to the IVFFN having i^{th} position in the ranking.

Step 4: Transform the weighted IVFF decision matrix M' into its equivalent weighted crisp decision matrix $M'' = \left(m''_{ij}\right)_{r\times s}$ by replacing the IVFFN with the assigned natural number.

Step 5: Apply the following steps of the existing crisp TOPSIS (Hwang and Yoon, 1981) to find the ranking of the alternatives.

Step 5(a): Evaluate the crisp positive ideal solution $CP_I = (t_j)_{1\times s}$ and the crisp negative ideal solution $CN_I = (v_j)_{1\times s}$ where
$t_j = \underset{1\le i\le r}{\text{maximum}} \{m''_{ij}\}, j = 1,2,\dots,s.$
$v_j = \underset{1\le i\le r}{\text{minimum}} \{m''_{ij}\}, j = 1,2,\dots,s.$
Step 5(b): Using expressions (6) and (7), evaluate the distance between the i^{th} row of weighted crisp decision matrix M'' with the crisp positive ideal solution $CP_I = (t_j)_{1\times s}$ and the crisp negative ideal solution $CN_I = (v_j)_{1\times s}$, respectively.

$$d_i(M'', CP_I) = \sqrt{\frac{\sum_{j=1}^{s}\left(m''_{ij} - t_j\right)^2}{s}},\ i = 1,2,\dots,r \tag{6}$$

$$d_i(M'', CN_I) = \sqrt{\frac{\sum_{j=1}^{s}\left(m''_{ij} - v_j\right)^2}{s}},\ i = 1,2,\dots,r. \tag{7}$$

Step 5(c): Using expression (8), evaluate the relative closeness $CC(A_i)$ of the i^{th} alternative A_i and check that $CC(A_i) > CC(A_j)$ or $CC(A_i) < CC(A_j)$ or $CC(A_i) = CC(A_j)$.

$$CC(A_i) = \frac{d_i(M'', CN_I)}{d_i(M'', CP_I) + d_i(M'', CN_I)}, i = 1,2,\dots,r. \tag{8}$$

Case (i): If $CC(A_i) > CC(A_j)$ then $A_i > A_j$.
Case (ii): If $CC(A_i) < CC(A_j)$ then $A_i < A_j$.
Case (iii): If $CC(A_i) = CC(A_j)$ then $A_i = A_j$.

7.2. Second modified IVFFMCDM method

The steps of the second modified IVFFMCDM method are as follows:
Step 1: Transform the IVFF decision matrix M into the weighted IVFF decision matrix $M' = (m'_{ij})_{r\times s}$, where $m'_{ij} = m_{ij} . w_j, i = 1\ to\ r, j = 1\ to\ s$, the non-negative real number w_j is the weight of the j^{th} benefit criteria and $\sum_{j=1}^{s} w_j = 1$.
Step 2: Using the existing method [5], discussed in Section 3, evaluate $p_j = \underset{1\le i\le r}{\text{maximum}} \{m'_{ij}\}, j = 1,2,\dots,s$, $n_j = \underset{1\le i\le r}{\text{minimum}}\{m'_{ij}\}, j = 1,2,\dots,s$. Hence, evaluate the IVFFPIS $P_I = (p_j)_{1\times s}$ and the IVFFNIS $N_I = (n_j)_{1\times s}$.
Step 3: Using expression (1), evaluate the distance between the i^{th} row of weighted IVFF decision matrix M' with the IVFFPIS $P_I = (p_j)_{1\times s}$ and the IVFFNIS $N_I = (n_j)_{1\times s}$.
Step 4: Using expression (5), evaluate the relative closeness $CC(A_i)$ of the i^{th} alternative A_i and check that $CC(A_i) > CC(A_j)$ or $CC(A_i) < CC(A_j)$ or $CC(A_i) = CC(A_j)$.
Case (i): If $CC(A_i) > CC(A_j)$, then $A_i > A_j$.
Case (ii): If $CC(A_i) < CC(A_j)$, then $A_i < A_j$.
Case (iii): If $CC(A_i) = CC(A_j)$, then $A_i = A_j$.

8. Appropriateness of the modified methods

In this section, it is pointed out that the inappropriateness occurring in Jeevaraj [5]'s IVFFMCDM methods does not occur in the modified IVFFMCDM methods. Hence, it is appropriate to use these to solve IVFFMCDM problems.

8.1. Appropriateness of the first modified IVFFMCDM method

In Section 6.1, it is pointed out that in Jeevaraj [5]'s first IVFFMCDM method, the transformed weighted crisp decision matrix S is not equivalent to the weighted IVFF decision matrix M', the transformed crisp positive ideal solution CP_I is not equivalent to IVFFPIS $P_I = \left(p_j\right)_{1\times s}$, and the transformed crisp negative ideal solution CN_I is not equivalent to IVFFNIS $N_I = \left(n_j\right)_{1\times s}$.

However, in the first modified IVFFMCDM method, the transformed weighted crisp decision matrix S is equivalent to the weighted IVFF decision matrix M', the transformed crisp positive ideal solution CP_I is equivalent to IVFFPIS $P_I = \left(p_j\right)_{1\times s}$ as two distinct natural numbers are assigned to two distinct IVFFNs, and the transformed crisp negative ideal solution CN_I is equivalent to IVFFNIS $N_I = \left(n_j\right)_{1\times s}$ as two distinct natural numbers are assigned to two distinct IVFFNs.

8.2. Appropriateness of the second modified IVFFMCDM method

In Section 6.2, it is pointed out that if p_j and n_j represent the j^{th} element of the IVFFPIS $P_I = \left(p_j\right)_{1\times s}$ and the IVFFNIS $N_I = \left(n_j\right)_{1\times s}$, respectively. Then, p_j should be equal to $\left([\mu'_{Lij}, \mu'_{Uij}], [\nu'_{Lij}, \nu'_{Uij}]\right)$ for some i and n_j should be equal to $\left([\mu'_{Lij}, \mu'_{Uij}], [\nu'_{Lij}, \nu'_{Uij}]\right)$ for some i. For the values of p_j and n_j obtained by Jeevaraj [5]'s second IVFFMCDM method, this condition is not satisfying, while it can be easily verified that for the values of p_j and n_j obtained by the second modified IVFFMCDM method, this condition is satisfying.

9. Correct results of the IVFFMCDM problems

In Section 5, two IVFFMCDM problems are considered to show that Jeevaraj [5]'s IVFFMCDM methods fail to find the correct ranking of the alternatives. In this section, the correct ranking of the alternatives of the same IVFFMCDM problems is obtained by the modified methods.

9.1. Correct results of the first IVFFMCDM problem

Using the first modified IVFFMCDM method, the correct ranking of the alternatives of the first IVFFMCDM problem, considered in Section 5.1, can be obtained as follows:

Step 1: According to Step 1 of the first modified IVFFMCDM method, the IVFF decision matrix M can be transformed into the weighted IVFF decision matrix M'.

$$M' = \left(m'_{ij}\right)_{2\times 2} = \begin{bmatrix} ([0.2,0.3],[0,0]) & ([0,0],[1,1]) \\ ([0,\sqrt[3]{0.035}],[0,0]) & ([0,0],[1,1]) \end{bmatrix}.$$

Step 2: According to Step 2 of the first modified IVFFMCDM method, there is a need to apply the existing method [5], discussed in Section 3, to evaluate the ranking of all the elements of the weighted

IVFF decision matrix M'.

The following steps of the existing method clearly indicate that $([0,0],[1,1]) < \left([0,\sqrt[3]{0.035}],[0,0]\right) < ([0.2,0.3],[0,0])$.

Step 2(a): It can be easily verified that $J_M([0.2,0.3],[0,0]) = 0.017, J_M\left([0,\sqrt[3]{0.035}],[0,0]\right) = 0.017, J_M([0,0],[1,1]) = -1$.

Since $J_M([0,0],[1,1]) < \left(J_M([0.2,0.3],[0,0]) = J_M\left([0,\sqrt[3]{0.035}],[0,0]\right)\right)$. So, according to Step 1 of the existing method, discussed in Section 3, $([0,0],[1,1]) < ([0.2,0.3],[0,0])$ and $([0,0],[1,1]) < \left([0,\sqrt[3]{0.035}],[0,0]\right)$. But, as $J_M([0.2,0.3],[0,0]) = J_M\left([0,\sqrt[3]{0.035}],[0,0]\right)$, i.e., Case (iii) of Step 1 of the existing method, discussed in Section 3, is satisfying. So, there is a need to go to Step 2 of the existing method to find the ranking of $([0.2,0.3],[0,0])$ and $\left([0,\sqrt[3]{0.035}],[0,0]\right)$.

Step 2(b): It can be easily verified that

$J_H([0.2,0.3],[0,0]) = 0.017, J_H\left([0,\sqrt[3]{0.035}],[0,0]\right) = 0.017$.

Since $J_H([0.2,0.3],[0,0]) = J_H\left([0,\sqrt[3]{0.035}],[0,0]\right)$, i.e., Case (iii) of Step 2 of the existing method, discussed in Section 3, is satisfying. So, there is a need to go to Step 3 of the existing method to find the ranking of $([0.2,0.3],[0,0])$ and $\left([0,\sqrt[3]{0.035}],[0,0]\right)$.

Step 2(c): It can be easily verified that

$J_P([0.2,0.3],[0,0]) = 0.009, J_P\left([0,\sqrt[3]{0.035}],[0,0]\right) = 0.017$.

Since $J_P([0.2,0.3],[0,0]) < J_P\left([0,\sqrt[3]{0.035}],[0,0]\right)$. So, according to Case (ii) of Step 3 of the existing method, discussed in Section 3, $([0.2,0.3],[0,0]) > \left([0,\sqrt[3]{0.035}],[0,0]\right)$.

Hence, $([0,0],[1,1]) < \left([0,\sqrt[3]{0.035}],[0,0]\right) < ([0.2,0.3],[0,0])$.

Step 3: According to Step 3 of the first modified IVFFMCDM method, the natural numbers 1−3 can be assigned to $([0,0],[1,1]), \left([0,\sqrt[3]{0.035}],[0,0]\right)$ and $([0.2,0.3],[0,0])$, respectively.

Step 4: According to Step 4 of the first modified IVFFMCDM method, the weighted IVFF decision matrix M' is transformed into the weighted crisp decision matrix $M'' = \left(m''_{ij}\right)_{3\times 2} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$.

Step 5: According to Step 5 of the first modified IVFFMCDM method, there is a need to apply the following steps of the existing crisp TOPSIS (Hwang and Yoon, 1981) to evaluate the ranking of the alternatives.

Step 5(a): According to Step 5(a) of the first modified IVFFMCDM method,

$CP_I = [\text{maximum}\,\{3,2\} \quad \text{maximum}\,\{1,1\}] = [3 \;\; 1]$.

$CN_I = [\text{minimum}\,\{3,2\} \quad \text{minimum}\,\{1,1\}] = [2 \;\; 1]$.

Step 5(b): According to Step 5(b) of the first modified IVFFMCDM method,

$$d_1(A_1, CP_I) = \sqrt{\frac{(3-3)^2+(1-1)^2}{2}} = 0, d_2(A_2, CP_I) = \sqrt{\frac{(2-3)^2+(1-1)^2}{2}} = 0.707,$$

$$d_1(A_1, CN_I) = \sqrt{\frac{(3-2)^2+(1-1)^2}{2}} = 0.707, d_2(A_2, CN_I) = \sqrt{\frac{(2-2)^2+(1-1)^2}{2}} = 0.$$

Step 5(c): According to Step 5(c) of the first modified IVFFMCDM method,

$$CC(A_1) = \frac{0.707}{0+0.707} = 1, CC(A_2) = \frac{0}{0.707+0} = 0.$$

Since $CC(A_1) > CC(A_2)$. So, according to Case (i) of Step 5(c) of the first modified IVFFMCDM method, $A_1 > A_2$.

9.2. Correct results of the first IVFFMCDM problem

Using the second modified IVFFMCDM method, the correct ranking of the alternatives of the second IVFFMCDM problem, considered in Section 5.2, can be obtained as follows:

Step 1: According to Step 1 of the second modified IVFFMCDM method, the IVFF decision matrix M can be transformed into the weighted IVFF decision matrix M'.

$M' = \left(m'_{ij}\right)_{r\times s} = \begin{bmatrix} ([0.1,0.2],[0.1,0.2]) & ([0,0],[1,1]) \\ ([0.2,0.3],[0.2,0.3]) & ([0,0],[1,1]) \end{bmatrix}.$

Step 2: According to Step 2 of the second modified IVFFMCDM method, there is a need to apply the existing method [5], discussed in Section 3, to evaluate $p_1 = maximum\,\{([0.1,0.2],[0.1,0.2]),([0.2,0.3],[0.2,0.3])\}, p_2 = maximum\,\{([0,0],[1,1]),([0,0],[1,1])\}, n_1 = minimum\,\{([0.1,0.2],[0.1,0.2]),([0.2,0.3],[0.2,0.3])\}, n_2 = minimum\,\{([0,0],[1,1]),([0,0],[1,1])\}$ and hence, $P_I = [p_1 \quad p_2]_{1\times 2}$ and $N_I = [n_1 \quad n_2]_{1\times 2}$.

It can be easily verified that according to the existing method [5], discussed in Section 3, $p_1 = ([0.2,0.3],[0.2,0.3]), p_2 = ([0,0],[1,1])$,
$n_1 = ([0.1,0.2],[0.1,0.2]), n_2 = ([0,0],[1,1])$ and hence,
$P_I = [([0.2,0.3],[0.2,0.3]) \quad ([0,0],[1,1])]_{1\times 2}$,
$N_I = [([0.1,0.2],[0.1,0.2]) \quad ([0,0],[1,1])]_{1\times 2}$.

Step 3: According to Step 3 of the second modified IVFFMCDM method,

$$D_{GE}(A_1, P_I) = 0.0202, D_{GE}(A_2, P_I) = 0, D_{GE}(A_1, N_I) = 0, D_{GE}(A_2, N_I) = 0.0202.$$

Step 4: According to Step 4 of the second modified IVFFMCDM method,

$$CC(A_1) = \frac{0}{0.0202+0} = 0, CC(A_2) = \frac{0.0202}{0+0.0202} = 1.$$

Since $CC(A_2) > CC(A_1)$. So, according to Case (i) of Step 4 of the second modified IVFFMCDM method, $A_2 > A_1$.

10. Ranking of the alternatives of the existing real-life IVFFMCDM problems

Jeevaraj [5] solved two real-life IVFFMCDM problems using his proposed methods.

In this section, the ranking of the alternatives of the same real-life IVFFMCDM problems is obtained by the proposed methods.

10.1. Ranking of the alternatives of the first real-life IVFFMCDM problem

Jeevaraj [5] applied the first IVFFMCDM method to obtain the ranking of the alternatives of the following real-life IVFFMCDM problem.

An investment company wants to invest in a software development project as the best option. The four possible alternatives are:

(i) A_1 is a mail development project.
(ii) A_2 is a game development project.
(iii) A_3 is a browser development project.
(iv) A_4 is a music player.

The development project is to be evaluated for potential investment based on the following four criteria:

(i) Cr_1 is economic feasibility.

(ii) Cr_2 is technological feasibility.
(iii) Cr_3 is staff feasibility.
(iv) Cr_4 is period feasibility.

The opinion of decision-makers, provided in terms of IVFFNs, about the performance of the alternatives with respect to criteria is represented by the IVFF decision matrix M. The weights assigned to the first, second, third, and fourth criteria, provided by the decision-makers, are $w_1 = 0.35, w_2 = 0.20$, $w_3 = 0.15$, and $w_4 = 0.30$, respectively.

$$M = (m_{ij})_{4\times4} = \begin{bmatrix} ([0.3,0.45],[0.1,0.2]) & ([0.4,0.7],[0.15,0.35]) & ([0.45,0.8],[0.3,0.7]) & ([0.1,0.3],[0.3,0.3]) \\ ([0.5,0.7],[0.6,0.8]) & ([0.6,0.8],[0.3.0.3]) & ([0.4,0.7],[0.35,0.4]) & ([0.3,0.6],[0.3,0.35]) \\ ([0.4,0.6],[0.3,0.5]) & ([0.5,0.65],[0.6,0.8]) & ([0.1,0.4],[0.4,0.5]) & ([0.3,0.4],[0.3,0.7]) \\ ([0.6,0.7],[0.2,0.4]) & ([0.5,0.75],[0.1,0.3]) & ([0.4,0.7],[0.5,0.75]) & ([0.2,0.7],[0.2,0.4]) \end{bmatrix}_{4\times4}.$$

Using the first modified IVFFMCDM method, the ranking of the alternatives of the first real-life IVFFMCDM problem can be obtained as follows.

Step 1: According to Step 1 of the first modified IVFFMCDM method, the IVFF decision matrix M can be transformed into the weighted IVFF decision matrix $M' = (m'_{ij})_{4\times4} =$

$$\begin{bmatrix} ([0.2120,0.3204],[0.4467,0.5639]) & ([0.2360,0.4319],[0.6843,0.8106]) & ([0.2423,0.4673],[0.8348,0.9479]) & ([0.0670,0.2015],[0.6968,0.6968]) \\ ([0.3574,0.5152],[0.8383,0.9249]) & ([0.3622,0.5113],[0.7860,0.7860]) & ([0.2145,0.3938],[0.8543,0.8716]) & ([0.2015,0.4129],[0.6968,0.7298]) \\ ([0.2839,0.4338],[0.6561,0.7849]) & ([0.2976,0.3692],[0.9029,0.9564]) & ([0.0531,0.2415],[0.8716,0.9013]) & ([0.2015,0.2698],[0.6968,0.8985]) \\ ([0.4338,0.5152],[0.5693,0.7256]) & ([0.2976,0.4700],[0.6310,0.7860]) & ([0.2145,0.3938],[0.9013,0.9578]) & ([0.1340,0.4910],[0.6170,0.7597]) \end{bmatrix}_{4\times4}.$$

Step 2: According to Step 2 of the first modified IVFFMCDM method, there is a need to apply the existing method [5], discussed in Section 3, to evaluate the ranking of all the elements of the weighted IVFF decision matrix M'.

The following steps of the existing method clearly indicate that
$([0.2145,0.3938],[0.9013,0.9578]) < ([0.2976,0.3692],[0.9029,0.9564]) <$
$([0.0531,0.2415],[0.8716,0.9013]) < ([0.2423,0.4673],[0.8348,0.9479]) <$
$([0.2145,0.3938],[0.8543,0.8716]) < ([0.3574,0.5152],[0.8383,0.9249]) <$
$([0.2015,0.2698],[0.6968,0.8985]) < ([0.3622,0.5113],[0.7860,0.7860]) <$
$([0.2360,0.4319],[0.6843,0.8106]) < ([0.0670,0.2015],[0.6968,0.6968]) <$
$([0.2839,0.4338],[0.6561,0.7849]) < ([0.2015,0.4129],[0.6968,0.7298]) <$
$([0.2976,0.4700],[0.6310,0.7860]) < ([0.1340,0.4910],[0.6170,0.7597]) <$
$([0.4338,0.5152],[0.5693,0.7256]) < ([0.2120,0.3204],[0.4467,0.5639])$.

Step 2(a): It can be easily verified that
$J_M([0.2120,0.3204],[0.4467,0.5639]) = -0.1156,$
$J_M([0.3574,0.5152],[0.8383,0.9249]) = -0.5968,$

$J_M([0.2839,0.4338],[0.6561,0.7849]) = -0.3305,$
$J_M([0.4338,0.5152],[0.5693,0.7256]) = -0.1741,$
$J_M([0.2360,0.4319],[0.6843,0.8106]) = -0.3796,$
$J_M([0.3622,0.5113],[0.7860,0.7860]) = -0.3950,$
$J_M([0.2976,0.3692],[0.9029,0.9564]) = -0.7611,$
$J_M([0.2976,0.4700],[0.6310,0.7860]) = -0.3033,$
$J_M([0.2423,0.4673],[0.8348,0.9479]) = -0.6586,$
$J_M([0.2145,0.3938],[0.8543,0.8716]) = -0.6073,$
$J_M([0.0531,0.2415],[0.8716,0.9013]) = -0.6921,$
$J_M([0.2145,0.3938],[0.9013,0.9578]) = -0.7698,$
$J_M([0.0670,0.2015],[0.6968,0.6968]) = -0.3341,$
$J_M([0.2015,0.4129],[0.6968,0.7298]) = -0.3243,$
$J_M([0.2015,0.2698],[0.6968,0.8985]) = -0.5180$ and
$J_M([0.1340,0.4910],[0.6170,0.7597]) = -0.2762.$

It is obvious that
$J_M([0.2145,0.3938],[0.9013,0.9578]) < J_M([0.2976,0.3692],[0.9029,0.9564]) <$
$J_M([0.0531,0.2415],[0.8716,0.9013]) < J_M([0.2423,0.4673],[0.8348,0.9479]) <$
$J_M([0.2145,0.3938],[0.8543,0.8716]) < J_M([0.3574,0.5152],[0.8383,0.9249]) <$
$J_M([0.2015,0.2698],[0.6968,0.8985]) < J_M([0.3622,0.5113],[0.7860,0.7860]) <$
$J_M([0.2360,0.4319],[0.6843,0.8106]) < J_M([0.0670,0.2015],[0.6968,0.6968]) <$
$J_M([0.2839,0.4338],[0.6561,0.7849]) < J_M([0.2015,0.4129],[0.6968,0.7298]) <$
$J_M([0.2976,0.4700],[0.6310,0.7860]) < J_M([0.1340,0.4910],[0.6170,0.7597]) <$
$J_M([0.4338,0.5152],[0.5693,0.7256]) < J_M([0.2120,0.3204],[0.4467,0.5639]).$

Hence,
$([0.2145,0.3938],[0.9013,0.9578]) < ([0.2976,0.3692],[0.9029,0.9564]) <$
$([0.0531,0.2415],[0.8716,0.9013]) < ([0.2423,0.4673],[0.8348,0.9479]) <$
$([0.2145,0.3938],[0.8543,0.8716]) < ([0.3574,0.5152],[0.8383,0.9249]) <$
$([0.2015,0.2698],[0.6968,0.8985]) < ([0.3622,0.5113],[0.7860,0.7860]) <$
$([0.2360,0.4319],[0.6843,0.8106]) < ([0.0670,0.2015],[0.6968,0.6968]) <$
$([0.2839,0.4338],[0.6561,0.7849]) < ([0.2015,0.4129],[0.6968,0.7298]) <$
$([0.2976,0.4700],[0.6310,0.7860]) < ([0.1340,0.4910],[0.6170,0.7597]) <$
$([0.4338,0.5152],[0.5693,0.7256]) < ([0.2120,0.3204],[0.4467,0.5639]).$

Step 3: According to Step 3 of the first modified IVFFMCDM method, the natural numbers $1,2,\ldots,16$ can be assigned to $([0.2145,0.3938],[0.9013,0.9578]), ([0.2976,0.3692],[0.9029,0.9564]),$ $([0.0531,0.2415],[0.8716,0.9013]), ([0.2423,0.4673],[0.8348,0.9479]),$ $([0.2145,0.3938],[0.8543,0.8716]), ([0.3574,0.5152],[0.8383,0.9249]),$ $([0.2015,0.2698],[0.6968,0.8985]), ([0.3622,0.5113],[0.7860,0.7860]),$ $([0.2360,0.4319],[0.6843,0.8106]), ([0.0670,0.2015],[0.6968,0.6968]),$ $([0.2839,0.4338],[0.6561,0.7849]), ([0.2015,0.4129],[0.6968,0.7298]),$ $([0.2976,0.4700],[0.6310,0.7860]), ([0.1340,0.4910],[0.6170,0.7597]),$ $([0.4338,0.5152],[0.5693,0.7256])$ and $([0.2120,0.3204],[0.4467,0.5639])$, respectively.

Step 4: According to Step 4 of the first modified IVFFMCDM method, the weighted IVFF decision matrix M' is transformed into the weighted crisp decision matrix

$$M'' = (m''_{ij})_{4\times 4} = \begin{bmatrix} 16 & 9 & 4 & 10 \\ 6 & 8 & 5 & 12 \\ 11 & 2 & 3 & 7 \\ 15 & 13 & 1 & 14 \end{bmatrix}_{4\times 4}.$$

Step 5: According to Step 5 of the first modified IVFFMCDM method, there is a need to apply the following steps of the existing crisp TOPSIS [12] to evaluate the ranking of the alternatives.

Step 5(a): According to Step 5(a) of the first modified IVFFMCDM method, $CP_I =$ [maximum $\{16,6,11,15\}$ maximum $\{9,8,2,13\}$ maximum $\{4,5,3,1\}$ maximum $\{10,12,7,14\}$] $= [16\ \ 13\ \ 5\ \ 14]$.

$CN_I =$ [minimum $\{16,6,11,15\}$ minimum $\{9,8,2,13\}$ minimum $\{4,5,3,1\}$ minimum $\{10,12,7,14\}$] $= [6\ \ 2\ \ 1\ \ 7]$.

Step 5(b): According to Step 5(b) of the first modified IVFFMCDM method, $d_1(A_1, CP_I) = 2.872, d_2(A_2, CP_I) = 5.678, d_3(A_3, CP_I) = 7.053, d_4(A_4, CP_I) = 2.06, d_1(A_1, CN_I) = 6.461, d_2(A_2, CN_I) = 4.387, d_3(A_3, CN_I) = 2.692$, $d_4(A_4, CN_I) = 7.921$.

Step 5(c): According to Step 5(c) of the first modified IVFFMCDM method,
$CC(A_1) = 0.69, CC(A_2) = 0.43, CC(A_3) = 0.27, CC(A_4) = 0.79$

Since $CC(A_4) > CC(A_1) > CC(A_2) > CC(A_3)$. So, according to Case (i) of Step 5(c) of the first modified IVFFMCDM method, $A_4 > A_1 > A_2 > A_3$.

10.2. Ranking of the alternatives of the second real-life IVFFMCDM problem

Jeevaraj [5] applied the second IVFFMCDM method to obtain the ranking of the alternatives of the following real-life IVFFMCDM problem:

There is a panel with five possible alternatives to invest money:

(i) A_1 is a car company.
(ii) A_2 is a food company.
(iii) A_3 is a computer company.
(iv) A_4 is a construction company.
(v) A_5 is a textile company.

The investment company must decide for the best alternative based on their performance with respect to the following three criteria:

(i) Cr_1 is the risk analysis.
(ii) Cr_2 is the growth analysis.
(iii) Cr_3 is the environmental impact analysis.

The opinion of decision-makers, provided in terms of IVFFNs, about the performance of the alternatives with respect to criteria is represented by the IVFF decision matrix M. The weights assigned to the first, second, and third criteria, provided by the decision-makers, are $w_1 = 0.35$, $w_2 = 0.20$, and $w_3 = 0.45$, respectively.

$$M = (m_{ij})_{5\times 3} = \begin{bmatrix} ([0.15,0.3],[0.3,0.6]) & ([0.35,0.55],[0.35,0.45]) \\ ([0.5,0.6],[0.35,0.55]) & ([0.15,0.3],[0.45,0.6]) \\ ([0.2,0.35],[0.15,0.6]) & ([0,0.25],[0.2,0.5]) \\ ([0.4,0.7],[0.1,0.35]) & ([0.4,0.6],[0.1,0.3]) \\ ([0.1,0.7],[0.2,0.4]) & ([0.2,0.7],[0,0.2]) \end{bmatrix}$$

$$\begin{matrix} ([0.15,0.3],[0.6,0.6]) \\ ([0.25,0.6],[0.4,0.7]) \\ ([0.2,0.45],[0.3,0.6]) \\ ([0.4,0.35],[0.3,0.55]) \\ ([0.2,0.5],[0.15,0.6]) \end{matrix}\Bigg]_{5\times 3}.$$

Using the second modified IVFFMCDM method, the ranking of the alternatives of the second real-life IVFFMCDM problem can be obtained as follows:

Step 1: According to Step 1 of the second modified method, the IVFF decision matrix M can be transformed into the weighted IVFF decision matrix

$$M' = \left(m'_{ij}\right)_{5\times 3} = \begin{bmatrix} ([0.1057,0.2120],[0.6561,0.8363]) & ([0.2059,0.3294],[0.8106,0.8524]) & ([0.1150,0.2305],[0.5817,0.7941]) \\ ([0.3574,0.4338],[0.6925,0.8112]) & ([0.0878,0.1761],[0.8524,0.9029]) & ([0.1919,0.4698],[0.6621,0.8517]) \\ ([0.1411,0.2478],[0.5148,0.8363]) & ([0,0.1465],[0.7248,0.8706]) & ([0.1534,0.3478],[0.5817,0.7946]) \\ ([0.2839,0.5152],[0.4467,0.6925]) & ([0.2360,0.3622],[0.6310,0.7860]) & ([0.0766,0.2693],[0.5817,0.7641]) \\ ([0.0705,0.5152],[0.5693,0.7256]) & ([0.1171,0.4319],[0,0.7248]) & ([0.1534,0.3878],[0.4258,0.7946]) \end{bmatrix}_{5\times 3}.$$

Step 2: According to Step 2 of the second modified IVFFMCDM method, there is a need to apply the existing method [5], discussed in Section 3, to evaluate $p_1 = maximum \left\{\begin{matrix} ([0.1057,0.2120],[0.6561,0.8363]),([0.3574,0.4338],[0.6925,0.8112]), \\ ([0.1411,0.2478],[0.5148,0.8363]),([0.2839,0.5152],[0.4467,0.6925]), \\ ([0.0705,0.5152],[0.5693,0.7256]) \end{matrix}\right\}$,

$$p_2 = maximum \left\{\begin{matrix} ([0.2059,0.3294],[0.8106,0.8524]),([0.0878,0.1761],[0.8524,0.9029]), \\ ([0,0.1465],[0.7248,0.8706]),([0.2360,0.3622],[0.6310,0.7860]), \\ ([0.1171,0.4319],[0,0.7248]) \end{matrix}\right\},$$

$$p_3 = maximum \left\{\begin{matrix} ([0.1150,0.2305],[0.5817,0.7941]),([0.1919,0.4698],[0.6621,0.8517]), \\ ([0.1534,0.3478],[0.5817,0.7946]),([0.0766,0.2693],[0.5817,0.7641]), \\ ([0.1534,0.3878],[0.4258,0.7946]) \end{matrix}\right\},$$

$$n_1 = minimum \left\{\begin{matrix} ([0.1057,0.2120],[0.6561,0.8363]),([0.3574,0.4338],[0.6925,0.8112]), \\ ([0.1411,0.2478],[0.5148,0.8363]),([0.2839,0.5152],[0.4467,0.6925]), \\ ([0.0705,0.5152],[0.5693,0.7256]) \end{matrix}\right\},$$

$$n_2 = minimum \left\{\begin{matrix} ([0.2059,0.3294],[0.8106,0.8524]),([0.0878,0.1761],[0.8524,0.9029]), \\ ([0,0.1465],[0.7248,0.8706]),([0.2360,0.3622],[0.6310,0.7860]), \\ ([0.1171,0.4319],[0,0.7248]) \end{matrix}\right\},$$

$$n_3 = minimum \left\{\begin{matrix} ([0.1150,0.2305],[0.5817,0.7941]),([0.1919,0.4698],[0.6621,0.8517]), \\ ([0.1534,0.3478],[0.5817,0.7946]),([0.0766,0.2693],[0.5817,0.7641]), \\ ([0.1534,0.3878],[0.4258,0.7946]) \end{matrix}\right\}$$

and hence, $P_I = [p_1 \quad p_2 \quad p_3]_{1\times 3}$ and $N_I = [n_1 \quad n_2 \quad n_3]_{1\times 3}$.

It can be easily verified that according to the existing method [3, Section 4, Definition 4.13], discussed in Section 3, $p_1 = ([0.2839,0.5152],[0.4467,0.6925]), p_2 = ([0.1171,0.4319],[0,0.7248]), p_3 = ([0.1534,0.3878],[0.4258,0.7946]), n_1 = ([0.1057,0.2120],[0.6561,0.8363]), n_2 = ([0.0878,0.1761],[0.8524,0.9029]), n_3 =$

$([0.1919,0.4698],[0.6621,0.8517])$ and hence,

$$P_I = \begin{bmatrix} ([0.2839,0.5152],[0.4467,0.6925]) & ([0.1171,0.4319],[0,0.7248]) & ([0.1534,0.3878],[0.4258,0.7946]) \end{bmatrix}_{1\times 3},$$

$$N_I = \begin{bmatrix} ([0.1057,0.2120],[0.6561,0.8363]) & ([0.0878,0.1761],[0.8524,0.9029]) & ([0.1919,0.4698],[0.6621,0.8517]) \end{bmatrix}_{1\times 3}.$$

Step 3: According to Step 3 of the second modified IVFFMCDM method, $D_{GE}(A_1, P_I) = 0.572$, $D_{GE}(A_2, P_I) = 0.733$, $D_{GE}(A_3, P_I) = 0.425$, $D_{GE}(A_4, P_I) = 0.240$, $D_{GE}(A_5, P_I) = 0.05$, $D_{GE}(A_1, N_I) = 0.195$, $D_{GE}(A_2, N_I) = 0.598$, $D_{GE}(A_3, N_I) = 0.344$, $D_{GE}(A_4, N_I) = 0.555$ and $D_{GE}(A_5, N_I) = 0.669$.

Step 4: According to Step 4 of the second modified IVFFMCDM method, $CC(A_1) = 0.25$, $CC(A_2) = 0.45$, $CC(A_3) = 0.44$, $CC(A_4) = 0.69$, and $CC(A_5) = 0.92$.

Since $CC(A_5) > CC(A_4) > CC(A_2) > CC(A_3) > CC(A_1)$, So, according to Case (i) of Step 4 of the second modified IVFFMCDM method, $A_5 > A_4 > A_2 > A_3 > A_1$.

11. Conclusions

It is shown that Jeevaraj [5]'s IVFFMCDM methods fail to find the correct ranking of the alternatives. Hence, it is inappropriate to use Jeevaraj [5]'s methods to solve IVFFMCDM problems. Modified IVFFMCDM methods are proposed corresponding to Jeevaraj [5]'s methods. Furthermore, the ranking of the alternatives of the existing IVFFMCDM problems [5] is obtained using the proposed methods.

Author contributions

Raina Ahuja: Conceptualization, methodology, writing – original draft; Meraj Ali Khan: Funding acquisition, visualization, writing – review & editing; Parul Tomar: Software, writing – review & editing; Amit Kumar: Visualization, supervision; S. S. Appadoo: Visualization, supervision; Ibrahim Al-Dayel: Funding acquisition, visualization, writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interest.

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