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Research article

Asymptotic theory for QMLE for power-transformed asymmetric double autoregressive models

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Abstract: To better capture asymmetry and heavy-tailedness, we have proposed a power-transformed asymmetric double autoregressive (PTADAR(p,q)) model. First we gave a sufficient condition for the existence of a strict stationarity solution of the PTADAR(p,q) model. Then we studied the quasi-maximum likelihood estimation (QMLE) of the model, and proved the consistency and asymptotic normality for the QMLE estimator. We set the power parameter $\delta > 0$, which includes $\delta = 1, 2$, and in empirical application, the power parameter $\delta > 0$ may be unknown. This could overcome the shortcomings of the double autoregressive (DAR(p,q)) model and asymmetry linear double autoregressive model, where the power parameter is only limited to 2 or 1. Based on QMLE, we proposed Akaike's information criterion (AIC) and Bayesian information criterion (BIC) for model selection. Illustrations and an empirical example show our model's usefulness, and we also compared it with other models.

Keywords: power-transformed asymmetric DAR model; power parameter; estimation; daily exchange rate
Mathematics Subject Classification: 62F10, 62F12

1. Introduction

Volatility is an important foundation for formulating financial derivative product prices, asset allocation, risk management, and quantitative trading strategies. To investigate the temporal variation of volatility, the conditional heteroskedasticity models were innovatively proposed by R. F. Engle [1] and T. Bollerslev [2]. Meanwhile an obvious disadvantage of the two models is their symmetry, which fails to capture positive and negative aspects, specifically, the distinction between good and bad news, and its influence on volatility. However, in practical applications, volatility often exhibits asymmetric effects. Asymmetry features or the leverage effect is well known in financial series. The asymmetry of volatility refers to the difference in volatility caused by price increases and decreases in financial markets. Usually, the volatility caused by price drops is greater than that caused by price increases. This phenomenon is called asymmetry of volatility. The reason for the asymmetry of volatility is that market participants have different attitudes when facing price increases or decreases. In the literature, many models have been proposed to capture asymmetric (leverage) effects, such as how G. W. Schwert [3] considered the relation of stock volatility with financial leverage, and the exponential garch model proposed by D. B. Nelson [4]. Based on Box-Cox transformation, some scholars studied power-transformed autoregressive conditional heteroscedasticity models, see, for examples, Hwang et al. [5], who introduced a power-transformed autoregressive conditional heteroskedasticity (ARCH) model, Pan et al. [6], who considered the power-transformed generalized autoregressive conditional heteroskedasticity (GARCH) model, and Francq et al. [7,8], who studied the non-stationary asymmetric GARCH and inference for volatility models with covariates. Tao et al. [9] suggested a first-order asymmetric GARCH model under symmetric stable innovation. Note that the volatilities are unobservable. S. Ling [10, 11] proposed the double autoregressive (DAR) model, and studied the structure and estimation. In the last twenty years, the DAR model has garnered many scholars' research interest. Similarly to the GARCH model, the DAR model also neglects the asymmetry. To overcome this deficiency, some variants have been studied. Li et al. [12, 13] considered the threshold DAR model, Zhang et al. [14] proposed a threshold AR-ARCH model, and Tan et al. [15, 16] introduced the asymmetric linear double autoregression and dual-asymmetry linear double AR model. The connection function between volatility and observed data is only considered linear or quadratic, which has some limitations. Enlightened by the power-transformed ARCH model and the DAR model, to fill this gap, and to better capture asymmetry and heavy-tailed phenomena, we propose a power-transformed asymmetric double autoregressive PTADAR(p,q) model.

There are three contributions in our paper. First, to our knowledge, currently there is no research literature on power-transformed asymmetric double autoregressive models. The new model is an important extension of the double autoregressive model, which considers power-transformed and asymmetry together. Illustrations and an empirical example show our model's usefulness. The PTADAR(p,q) can deal with heavy-tailed data, and show the asymmetric effects. Second, we give a sufficient condition for a strict stationarity solution of the PTADAR(p,q) model under a mild condition, which only requires a fractional moment of $\{y_t\}$. Third, we study the quasi-maximum likelihood estimation of the model. This model provides a good choice for future research on asymmetric effects and handling heavy-tailed data.

This paper is structured as follows: In Section 2, we obtain a sufficient condition for $\{y_t\}$ to be strictly stationary and the asymptotic behavior of the quasi-maximum likelihood estimation (QMLE) for model (2.1). Simulations and a real data example are given in Sections 3 and 4, respectively. Section 5 provides the conclusions. The proofs of Theorems 2.1 and 2.2 are provided in Appendix A.

2. QMLE with asymptotics

We consider a power-transformed asymmetric double autoregressive (hereafter PTADAR(p,q)) model:

$$\begin{cases} y_t = u + \sum_{i=1}^{p} [\phi_{i+} y_{t-i}^{+} + \phi_{i-} y_{t-i}^{-}] + \epsilon_t, \\ \epsilon_t = \sigma_t \eta_t, \qquad t \in \mathbb{Z}, \\ \sigma_t^{\delta} = \omega + \sum_{j=1}^{q} \left[\alpha_{j+} (y_{t-j}^{+})^{\delta} + \alpha_{j-} (y_{t-j}^{-})^{\delta} \right], \end{cases} \tag{2.1}$$

where $y^+ = \max(y, 0)$, $y^- = \max(-y, 0)$, $\{\eta_t\}$ is a sequence of random variables, which are independent and identically distributed (i.i.d.), and $\{y_s : s < t\}$ is independent of η_t. The parameters are u, ϕ_{i+}, $\phi_{i-} \in (-\infty, +\infty)$; and δ, ω, α_{j+}, $\alpha_{j-} \in (0, +\infty)$; $(i = 1, \cdots, p; j = 1, \cdots, q)$. When $\delta = 2$, $\phi_{i+} = \phi_{i-}$, and $\alpha_{j+} = \alpha_{j+}$, the model (2.1) becomes a classical DAR(p,q). Li et al. [17], Guo et al. [18], Jiang et al. [19], Gong et al. [20], Zhu et al. [21], Zhang et al. [14], and Li et al. [12,13] studied the threshold DAR(p,q) model. When $\delta = 1$ and $\phi_{i+} = \phi_{i-}$, the model (2.1) becomes the asymmetric linear double autoregression model, which has been studied by Tan et al. [15, 16].

Let $\boldsymbol{\theta} = (\boldsymbol{\phi}', \delta, \boldsymbol{\alpha}')'$, where $\boldsymbol{\phi} = (u, \phi_{1+}, \phi_{1-}, \cdots, \phi_{p+}, \phi_{p-})'$, $\boldsymbol{\alpha} = (\omega, \alpha_{1+}, \alpha_{1-}, \cdots, \alpha_{q+}, \alpha_{q-})'$, and $\boldsymbol{\theta}$ is the parameter vector of model (2.1), $\boldsymbol{\theta} \in \Theta$, where the parameter space $\Theta \subset (-\infty, \infty)^{1+2p} \times (0, \infty) \times (0, \infty)^{1+2q}$. Denote $\boldsymbol{\theta}_0$ as the true parameter vector, with $\boldsymbol{\theta}_0 = (\boldsymbol{\phi}_0', \delta_0, \boldsymbol{\alpha}_0')'$, $\boldsymbol{\phi}_0 = (u_0, \phi_{01+}, \phi_{01-}, \cdots, \phi_{0p+}, \phi_{0p-})'$, and $\boldsymbol{\alpha}_0 = (\omega_0, \alpha_{01+}, \alpha_{01-}, \cdots, \alpha_{0q+}, \alpha_{0q-})'$. $\{y_1, \cdots, y_n\}$ be a observed data set from model (2.1).

A QMLE of $\boldsymbol{\theta}_0$ is defined as $\widehat{\boldsymbol{\theta}}_n = \arg\max_{\boldsymbol{\theta} \in \Theta} L_n(\boldsymbol{\theta})$, where the negative conditional log-likelihood function $L_n(\boldsymbol{\theta})$ (ignoring a constant) can be written as

$$L_n(\boldsymbol{\theta}) = \sum_{t=m+1}^{n} l_t(\boldsymbol{\theta}), \quad m = \max(p, q), \quad l_t(\boldsymbol{\theta}) = -\frac{1}{2} \left[\log \sigma_t^2(\boldsymbol{\alpha}) + \frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})} \right], \tag{2.2}$$

with $\epsilon_t(\boldsymbol{\phi}) = y_t - \boldsymbol{\phi}' \boldsymbol{Y_{t-1}}, \sigma_t(\boldsymbol{\alpha}) = [\boldsymbol{\alpha}' \boldsymbol{X_{t-1}}]^{1/\delta}, \sigma_t^2(\boldsymbol{\alpha}) = [\boldsymbol{\alpha}' \boldsymbol{X_{t-1}}]^{2/\delta}, \sigma_t^{\delta}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}' \boldsymbol{X_{t-1}}$, where $\boldsymbol{Y}_t = (1, y_t^+, y_t^-, \cdots, y_{t-p+1}^+, y_{t-p+1}^-)'$, $\boldsymbol{X}_t = (1, (y_t^+)^{\delta}, (y_t^-)^{\delta}, \cdots, (y_{t-q+1}^+)^{\delta}, (y_{t-q+1}^-)^{\delta})'$, and we denote $\boldsymbol{X}_t^0 = (1, (y_t^+)^{\delta_0}, (y_t^-)^{\delta_0}, \cdots, (y_{t-q+1}^+)^{\delta_0}, (y_{t-q+1}^-)^{\delta_0})'$. Next, we give the conditions for the model.

(A1) Let $f(x)$ be the continuous density function of η_t, where $f(x)$ is symmetric, $f(x) > 0$ a.s. in $\mathbb{R}$, and $E|\eta_t|^r < \infty$, for some $r \in (0, \infty)$.

(A2) $\{\eta_t\}$ is i.i.d. with $E(\eta_t) = 0$, $E(\eta_t^2) = 1$.

(A3) $\underline{u} \le u \le \overline{u}$, $\underline{\phi} \le \phi_{i+}, \phi_{i-} \le \overline{\phi}$, $\underline{\delta} \le \delta \le \overline{\delta}$, $\underline{\omega} \le \omega \le \overline{\omega}$, and $\underline{\alpha} \le \alpha_{j+}$, $\alpha_{j-} \le \overline{d}$; where $(i = 1, \cdots, p; j = 1, \cdots, q)$, $\underline{u}$, $\overline{u}$, $\underline{\phi}$, $\overline{\phi}$ are real constants; and $\underline{\delta}$, $\overline{\delta}$, $\underline{\omega}$, $\overline{\omega}$, $\underline{\alpha}, and \overline{\alpha}$ are all positive constants. $\boldsymbol{\theta}_0 \in \mathring{\Theta}$, with $\mathring{\Theta}$ denoting the interior of Θ, and Θ is a compact set.

(A4) Let the sequence $\{y_t\}$ $(t = 0, \pm 1, \cdots)$ be strictly stationary and ergodic, with $E|y_t|^r < \infty$, for some $r \in (0, \infty)$.

(A5) $E\eta_t^4 < \infty$.

Theorem 2.1. *If condition (A1) holds, and the following condition holds:*

(c1) if $0 < r < \delta \le 1$ *or* $0 < r < 1 < \delta$, *then* $\sum_{i=1}^{m} \left[\phi_i^r + \alpha_j^{r/\delta} E(|\eta_1|^r) \right] < 1$,

then there exists a unique strictly stationary geometrically ergodic solution $\{y_t\}$ *for model (2.1), with* $E(|y_t|^r) < \infty$.

Theorem 2.2. *If (A2)–(A4) hold, we obtain*

(i) $\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 \to 0$ *in probability, as* $n \to \infty$.

(ii) Meanwhile, if (A5) holds, then $\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \Rightarrow N\left(0, \Sigma_0^{-1}\Omega_0\Sigma_0^{-1}\right)$ *as* $n \to \infty$, *where we let* $\Rightarrow$ *be the convergence in distribution, the matrix* $\Sigma_0 = -E\left(\frac{\partial^2 l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right)$, *and* $\Omega_0 = E\left\{\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}'}\right\}$.

3. Simulation studies

In this section we show the performance of QMLE, and use the PTADAR(1,1) model to generate data:

$$\begin{cases} y_t = 1 - 0.15y_{t-1}^+ - 0.4y_{t-1}^- + \epsilon_t, \quad \epsilon_t = \eta_t\sigma_t, \\ \sigma_t = \left[0.8 + 0.2(y_{t-1}^+)^{\delta_0} + 0.3(y_{t-1}^-)^{\delta_0}\right]^{1/\delta_0}, \end{cases} \tag{3.1}$$

denote $\boldsymbol{\zeta}_0 = (u, \phi_{1+}, \phi_{1-}, \omega, \alpha_{1+}, \alpha_{1-})' = (1, -0.15, -0.4, 0.8, 0.2, 0.3)'$, where η_t obeys $N(0,1)$ or $t(5)$ distribution. In Tables 1 and 2, the parameter $\delta_0 = 1$ and 2 is known. In Tables 3 and 4, the parameter δ_0 is unknown, and the values of δ_0 are set to 1 and 2, respectively.

Table 1. QMLE for model (3.1), with $\boldsymbol{\zeta}_0 = (1, -0.15, -0.4, 0.8, 0.2, 0.3)'$.

n	η_t		$\widehat{u}$	$\widehat{\phi}_{1+}$	$\widehat{\phi}_{1-}$	$\widehat{\omega}$	$\widehat{\beta}_{1+}$	$\widehat{\beta}_{1-}$
	$N(0,1)$	Bias	0.0011	-0.0017	0.0092	0.0029	-0.0042	-0.0194
500		ESD	0.0706	0.0670	0.1540	0.0543	0.0479	0.1147
		ASD	0.0739	0.0647	0.1570	0.0522	0.0457	0.1111
		Bias	0.0016	-0.0040	-0.0014	0.0002	-0.0007	-0.0123
1000		ESD	0.0520	0.0464	0.1118	0.0382	0.0330	0.0807
		ASD	0.0522	0.0459	0.1108	0.0369	0.0325	0.0783
		Bias	0.0017	-0.0012	-0.0002	-0.0010	0.0004	-0.0082
2000		ESD	0.0372	0.0326	0.0794	0.0257	0.0232	0.0570
		ASD	0.0369	0.0324	0.0788	0.0261	0.0229	0.0557
	$t(5)$	Bias	-0.0023	0.0009	0.0063	0.0000	-0.0079	-0.0300
500		ESD	0.0748	0.0675	0.1570	0.0811	0.0794	0.1760
		ASD	0.0721	0.0660	0.1517	0.1020	0.0933	0.2145
		Bias	-0.0013	-0.0001	0.0093	-0.0008	-0.0024	-0.0196
1000		ESD	0.0507	0.0478	0.1140	0.0634	0.0578	0.1293
		ASD	0.0512	0.0470	0.1086	0.0724	0.0664	0.1535
		Bias	0.0013	-0.0010	0.0001	0.0003	-0.0024	-0.0081
2000		ESD	0.0380	0.0344	0.0806	0.0485	0.0432	0.1007
		ASD	0.0363	0.0333	0.0776	0.0513	0.0471	0.1098

Table 2. QMLE for model (3.1), with $\zeta_0 = (1, -0.15, -0.4, 0.8, 0.2, 0.3)'$.

n	η_t		$\widehat{u}$	$\widehat{\phi}_{1+}$	$\widehat{\phi}_{1-}$	$\widehat{\omega}$	$\widehat{\beta}_{1+}$	$\widehat{\beta}_{1-}$
	$N(0,1)$	Bias	-0.0006	-0.0004	0.0035	-0.0003	-0.0028	-0.0285
500		ESD	0.0783	0.0700	0.1541	0.0781	0.0504	0.1713
		ASD	0.0774	0.0684	0.1469	0.0778	0.0493	0.1577
		Bias	-0.0038	0.0023	0.0058	0.0012	0.0016	-0.0080
1000		ESD	0.0552	0.0484	0.1072	0.0571	0.0359	0.1242
		ASD	0.0550	0.0486	0.1049	0.0554	0.0353	0.1152
		Bias	0.0019	-0.0012	-0.0009	-0.0006	-0.0003	-0.0046
2000		ESD	0.0399	0.0357	0.0763	0.0383	0.0245	0.0869
		ASD	0.0389	0.0343	0.0746	0.0392	0.0248	0.0822
	$t(5)$	Bias	-0.0001	-0.0003	0.0042	-0.0089	0.0013	-0.0268
500		ESD	0.0771	0.0727	0.1526	0.1170	0.0874	0.2417
		ASD	0.0761	0.0705	0.1442	0.1506	0.1028	0.2903
		Bias	0.0009	-0.0017	-0.0005	-0.0006	-0.0006	-0.0148
1000		ESD	0.0542	0.0499	0.1090	0.0891	0.0716	0.1856
		ASD	0.0542	0.0502	0.1049	0.1079	0.0733	0.2155
		Bias	-0.0012	0.0010	0.0034	-0.0022	0.0015	-0.0084
2000		ESD	0.0390	0.0359	0.0781	0.0682	0.0495	0.1366
		ASD	0.0384	0.0356	0.0751	0.0766	0.0523	0.1557

Table 3. Simulations for model (3.1), with $\boldsymbol{\theta}_0 = (1, -0.15, -0.4, 1, 0.8, 0.2, 0.3)'$.

n	η_t		$\widehat{u}$	$\widehat{\phi}_{1+}$	$\widehat{\phi}_{1-}$	$\widehat{\delta}$	$\widehat{\omega}$	$\widehat{\beta}_{1+}$	$\widehat{\beta}_{1-}$
	$N(0,1)$	Bias	0.0018	-0.0022	0.0075	0.1467	-0.0030	-0.0181	-0.0401
500		ESD	0.0757	0.0675	0.1555	0.9118	0.0701	0.0533	0.1420
		ASD	0.0735	0.0645	0.1548	0.7858	0.0679	0.0542	0.1330
		Bias	0.0006	-0.0010	0.0074	0.0166	0.0003	-0.0101	-0.0222
1000		ESD	0.0511	0.0452	0.1095	0.5299	0.0446	0.0357	0.0926
		ASD	0.0520	0.0456	0.1111	0.4716	0.0434	0.0353	0.0928
		Bias	0.0004	-0.0001	0.0038	-0.0017	-0.0011	-0.0034	-0.0113
2000		ESD	0.0364	0.0320	0.0779	0.3310	0.0296	0.0245	0.0653
		ASD	0.0368	0.0323	0.0789	0.3173	0.0297	0.0240	0.0653
		Bias	0.0099	-0.0024	0.0021	0.0784	0.0007	-0.0040	-0.0007
5000		ESD	0.0205	0.0135	0.0519	0.2873	0.0154	0.0169	0.0393
		ASD	0.0234	0.0205	0.0504	0.2116	0.0190	0.0152	0.0431
		Bias	-0.0025	0.0027	0.0104	-0.0095	0.0009	-0.0024	-0.0068
10^4		ESD	0.0191	0.0132	0.0317	0.0979	0.0116	0.0102	0.0312
		ASD	0.0165	0.0145	0.0354	0.1408	0.0131	0.0103	0.0291
		Bias	0.0008	-0.0003	-0.0011	0.0125	-0.0006	0.0005	-0.0000
10^5		ESD	0.0037	0.0027	0.0106	0.0575	0.0035	0.0034	0.0092
		ASD	0.0052	0.0046	0.0112	0.0446	0.0041	0.0033	0.0093
	$t(5)$	Bias	0.0053	-0.0055	0.0054	-0.0513	-0.0087	-0.0097	-0.0630
500		ESD	0.0723	0.0658	0.1543	0.7037	0.0887	0.0796	0.1727
		ASD	0.0707	0.0653	0.1471	1.3582	0.1276	0.1049	0.2283
		Bias	-0.0006	0.0002	-0.0010	0.0266	-0.0056	-0.0104	-0.0333
1000		ESD	0.0524	0.0481	0.1111	0.6205	0.0717	0.0609	0.1532
		ASD	0.0508	0.0467	0.1068	0.8819	0.0873	0.0739	0.1735
		Bias	-0.0001	0.0008	-0.0001	0.0036	-0.0016	-0.0075	-0.0203
2000		ESD	0.0370	0.0347	0.0767	0.4781	0.0498	0.0426	0.1098
		ASD	0.0361	0.0331	0.0769	0.5979	0.0581	0.0504	0.1218
		Bias	-0.0039	0.0041	0.0092	0.0584	0.0033	-0.0078	-0.0185
5000		ESD	0.0225	0.0216	0.0516	0.4719	0.0443	0.0353	0.0683
		ASD	0.0230	0.0210	0.0490	0.3971	0.0371	0.0315	0.0750
		Bias	0.0014	0.0014	-0.0146	-0.1317	0.0012	0.0025	-0.0179
10^4		ESD	0.0182	0.0179	0.0254	0.2035	0.0240	0.0234	0.0370
		ASD	0.0161	0.0148	0.0345	0.2297	0.0234	0.0209	0.0494
		Bias	0.0009	-0.0001	-0.0046	-0.0155	0.0009	0.0004	0.0070
10^5		ESD	0.0054	0.0047	0.0088	0.0834	0.0057	0.0062	0.0124
		ASD	0.0051	0.0047	0.0110	0.0790	0.0077	0.0067	0.0163

Table 4. Simulations for model (3.1), with $\boldsymbol{\theta}_0 = (1, -0.15, -0.4, 2, 0.8, 0.2, 0.3)'$.

n	η_t		$\widehat{u}$	$\widehat{\phi}_{1+}$	$\widehat{\phi}_{1-}$	$\widehat{\delta}$	$\widehat{\omega}$	$\widehat{\beta}_{1+}$	$\widehat{\beta}_{1-}$
	$N(0,1)$	Bias	0.0017	-0.0042	0.0061	0.1439	-0.0084	-0.0162	-0.0449
500		ESD	0.0786	0.0698	0.15296	1.1068	0.0895	0.0655	0.1704
		ASD	0.0769	0.0680	0.1462	1.3002	0.0926	0.0722	0.1591
		Bias	0.0047	-0.0040	-0.0069	0.0610	-0.0023	-0.0092	-0.0341
1000		ESD	0.0552	0.0492	0.1056	0.8235	0.0608	0.0477	0.1194
		ASD	0.0547	0.0483	0.1039	0.8337	0.0607	0.0495	0.1116
		Bias	0.0001	-0.0013	-0.0009	0.0272	-0.0025	-0.0040	-0.0127
2000		ESD	0.0392	0.0347	0.0751	0.5818	0.0405	0.0350	0.0827
		ASD	0.0388	0.0343	0.0745	0.5536	0.0411	0.0345	0.0825
		Bias	0.0020	-0.0010	-0.0032	0.1501	0.0038	-0.0074	-0.0345
5000		ESD	0.0231	0.0213	0.0499	0.3060	0.0321	0.0185	0.0549
		ASD	0.0247	0.0218	0.0465	0.3651	0.0267	0.0230	0.0525
		Bias	-0.0015	0.0024	-0.0057	0.0276	-0.0010	0.0023	0.0046
10^4		ESD	0.0164	0.0196	0.0242	0.2925	0.0118	0.0161	0.0323
		ASD	0.0174	0.0154	0.0336	0.2364	0.0180	0.0155	0.0376
		Bias	0.0019	-0.0014	-0.0040	0.0091	-0.0002	-0.0011	0.0002
10^5		ESD	0.0049	0.0041	0.0071	0.0503	0.0049	0.0045	0.0110
		ASD	0.0055	0.0049	0.0106	0.0744	0.0056	0.0049	0.0118
	$t(5)$	Bias	0.0019	-0.0030	0.0107	-0.1658	-0.0120	-0.0014	-0.0449
500		ESD	0.0728	0.0710	0.1461	1.0000	0.1182	0.0934	0.2102
		ASD	0.0749	0.0695	0.1420	2.0663	0.1741	0.1369	0.2866
		Bias	-0.0016	0.0005	0.0066	-0.0649	-0.0129	-0.0018	-0.0262
1000		ESD	0.0563	0.0517	0.1075	0.8322	0.0902	0.0654	0.1731
		ASD	0.0535	0.0497	0.1042	1.4579	0.1185	0.0974	0.2165
		Bias	0.0014	-0.0016	0.0033	-0.0017	-0.0039	-0.0046	-0.0186
2000		ESD	0.0395	0.0351	0.0744	0.7012	0.0719	0.0524	0.1413
		ASD	0.0382	0.0354	0.0746	1.0443	0.0827	0.0693	0.1588
		Bias	0.0038	-0.0025	-0.0027	-0.0478	0.0066	-0.0012	-0.0041
5000		ESD	0.0257	0.0221	0.0491	0.5113	0.0472	0.0412	0.0999
		ASD	0.0243	0.0225	0.0479	0.6227	0.0499	0.0429	0.1010
		Bias	0.0022	-0.0033	0.0086	0.0881	-0.0143	0.0005	-0.0125
10^4		ESD	0.0177	0.0171	0.0369	0.3945	0.0364	0.0290	0.0580
		ASD	0.0172	0.0160	0.0336	0.4597	0.0360	0.0318	0.0732
		Bias	-0.0008	0.0009	-0.0003	0.0027	-0.0013	0.0012	-0.0018
10^5		ESD	0.0050	0.0041	0.0112	0.1438	0.0095	0.0155	0.0179
		ASD	0.0055	0.0051	0.0108	0.1375	0.0110	0.0097	0.0231

In the four tables, we used 1000 iterations, and set the sample size $n = 500, 1000, 2000$ in Tables 1 and 2, and sample size $n = 500, 1000, 2000, 5000, 10,000, 100,000$ in Tables 3 and 4. We can see that the values of Biases, ESDs, and ASDs decrease when the sample size increases. Furthermore, the ESDs (empirical standard deviations) are similar to the ASDs (asymptotic standard deviations).

We used the R program to calculate the quasi-maximum likelihood estimation of the parameter vector $\boldsymbol{\theta}_0$. For the PTADAR(1,1) model, the QMLE of $\boldsymbol{\theta}_0$ is defined as $\widehat{\boldsymbol{\theta}}_n = \arg\max_{\theta\in\Theta} L_n(\boldsymbol{\theta})$, where the negative conditional log-likelihood function $L_n(\boldsymbol{\theta})$ (ignoring a constant) can be written as

$$L_n(\boldsymbol{\theta}) = \sum_{t=2}^{n} l_t(\boldsymbol{\theta}), \quad l_t(\boldsymbol{\theta}) = -\frac{1}{2}\left[\log\sigma_t^2(\boldsymbol{\alpha}) + \frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})}\right]. \tag{3.2}$$

Next, we estimate the ASDs (asymptotic standard deviations). From Theorem 2.2, we have the stationary case $\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 \to 0$ in probability, as $n \to \infty$. Meanwhile, $\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \Rightarrow N\left(0, \Sigma_0^{-1}\Omega_0\Sigma_0^{-1}\right)$ as $n \to \infty$, where we let $\Rightarrow$ be the convergence in distribution, the matrix $\Sigma_0 = -E\left(\frac{\partial^2 l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right)$, and $\Omega_0 = E\left\{\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}'}\right\}$. $\Omega_0 = E\begin{pmatrix}\Omega_0^{11} & \Omega_0^{12} & \Omega_0^{13}\\ \Omega_0^{21} & \Omega_0^{22} & \Omega_0^{23}\\ \Omega_0^{31} & \Omega_0^{32} & \Omega_0^{33}\end{pmatrix} = \begin{pmatrix}E\Omega_0^{11} & E\Omega_0^{12} & E\Omega_0^{13}\\ E\Omega_0^{21} & E\Omega_0^{22} & E\Omega_0^{23}\\ E\Omega_0^{31} & E\Omega_0^{32} & E\Omega_0^{33}\end{pmatrix}$, where $(\Omega_0^{ij})' = \Omega_0^{ji}, i, j = 1, 2, 3.$
$E\left[\frac{\epsilon_t(\boldsymbol{\phi})}{\sigma_t(\boldsymbol{\alpha})}\right] = E(\eta_t) = 0$, $E\left[\frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})}\right] = E(\eta_t^2) = 1$, $k_1 = E(\eta_t^3)$, $k_2 = E(\eta_t^4) - 1$, with

$$\begin{aligned}
E\Omega_0^{11} &= E\left[\frac{\epsilon_t^2(\boldsymbol{\phi})\boldsymbol{Y}_{t-1}\boldsymbol{Y}_{t-1}'}{\sigma_t^4(\boldsymbol{\alpha})}\right] = E\left[\frac{\boldsymbol{Y}_{t-1}\boldsymbol{Y}_{t-1}'}{\sigma_t^2(\boldsymbol{\alpha})}\right], E\Omega_0^{12} = 0, E\Omega_0^{13} = 0,\\
E\Omega_0^{22} &= E\left\{\frac{k_2}{\delta^2}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]^2\right\},\\
E\Omega_0^{23} &= E\left\{\frac{k_2\boldsymbol{X}_{t-1}'}{\delta^2\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]\right\},\\
E\Omega_0^{33} &= E\left[\frac{\left(\eta_t^4 - 1\right)\boldsymbol{X}_{t-1}\boldsymbol{X}_{t-1}'}{\delta^2\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)^2}\right] = E\left[\frac{k_2\boldsymbol{X}_{t-1}\boldsymbol{X}_{t-1}'}{\delta^2\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)^2}\right].
\end{aligned}$$

The matrix Ω_0 can be estimated by $\widehat{\Omega}_{0n} = \begin{pmatrix}\widehat{\Omega}_{0n}^{11} & \widehat{\Omega}_{0n}^{12} & \widehat{\Omega}_{0n}^{13}\\ \widehat{\Omega}_{0n}^{21} & \widehat{\Omega}_{0n}^{22} & \widehat{\Omega}_{0n}^{23}\\ \widehat{\Omega}_{0n}^{31} & \widehat{\Omega}_{0n}^{32} & \widehat{\Omega}_{0n}^{33}\end{pmatrix}$,when η_t obeys $N(0,1)$, $k_1 = 0$, $k_2 = 2$, $\widehat{\Omega}_{0n}^{ij} = \frac{1}{n}\sum_{t=2}^{n}\Omega_{0n}^{ij}(\widehat{\theta}_n), i, j = 1, 2, 3.$ When η_t obeys $t(5)$, let $\eta_t = \frac{\eta_t}{(E(\eta_t^2))^{0.5}}$, $k_1 = 0$, $k_2 = 8$, $\Sigma_0 = E\left(-\frac{\partial^2 l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right) = E\begin{pmatrix}\Sigma_0^{11} & \Sigma_0^{12} & \Sigma_0^{13}\\ \Sigma_0^{21} & \Sigma_0^{22} & \Sigma_0^{23}\\ \Sigma_0^{31} & \Sigma_0^{32} & \Sigma_0^{33}\end{pmatrix} = \begin{pmatrix}E\Sigma_0^{11} & E\Sigma_0^{12} & E\Sigma_0^{13}\\ E\Sigma_0^{21} & E\Sigma_0^{22} & E\Sigma_0^{23}\\ E\Sigma_0^{31} & E\Sigma_0^{32} & E\Sigma_0^{33}\end{pmatrix}$, where $(\Sigma_0^{ij})' = \Sigma_0^{ji}, i, j = 1, 2, 3$. Then we have

$$\begin{aligned}
E\Sigma_0^{11} &= E\left[\frac{\boldsymbol{Y}_{t-1}\boldsymbol{Y}_{t-1}'}{\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)^{2/\delta}}\right], E\Sigma_0^{12} = \mathbf{0}_{(2p+1)\times 1}, E\Sigma_0^{13} = \mathbf{0}_{(2p+1)\times(2q+1)},\\
E\Sigma_0^{22} &= \frac{2}{\delta^2}E\left[\frac{1}{\delta}\log\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\right]^2,
\end{aligned}$$

$$E\Sigma_0^{23} = E\left\{\frac{-2X'_{t-1}}{\delta^2\alpha' X_{t-1}}\left[\frac{1}{\delta}\log\left(\alpha' X_{t-1}\right) - \frac{M_t^1}{\alpha' X_{t-1}}\right]\right\},$$

$$E\Sigma_0^{33} = E\left\{-\frac{\left[1-\left(1+\frac{2}{\delta}\right)\frac{\epsilon_t^2(\phi)}{\sigma_t^2(\alpha)}\right] X_{t-1}X'_{t-1}}{\delta\left(\alpha' X_{t-1}\right)^2}\right\} = E\left[\frac{2X_{t-1}X'_{t-1}}{\delta^2\left(\alpha' X_{t-1}\right)^2}\right].$$

The matrix Σ_0 can be estimated by $\widehat{\Sigma}_{0n} = \begin{pmatrix} \widehat{\Sigma}_{0n}^{11} & \widehat{\Sigma}_{0n}^{12} & \widehat{\Sigma}_{0n}^{13} \\ \widehat{\Sigma}_{0n}^{21} & \widehat{\Sigma}_{0n}^{22} & \widehat{\Sigma}_{0n}^{23} \\ \widehat{\Sigma}_{0n}^{31} & \widehat{\Sigma}_{0n}^{32} & \widehat{\Sigma}_{0n}^{33} \end{pmatrix}$, $\widehat{\Sigma}_{0n}^{ij} = \frac{1}{n}\sum_{t=2}^{n}\Sigma_{0n}^{ij}(\widehat{\theta}_n), i, j = 1, 2, 3.$

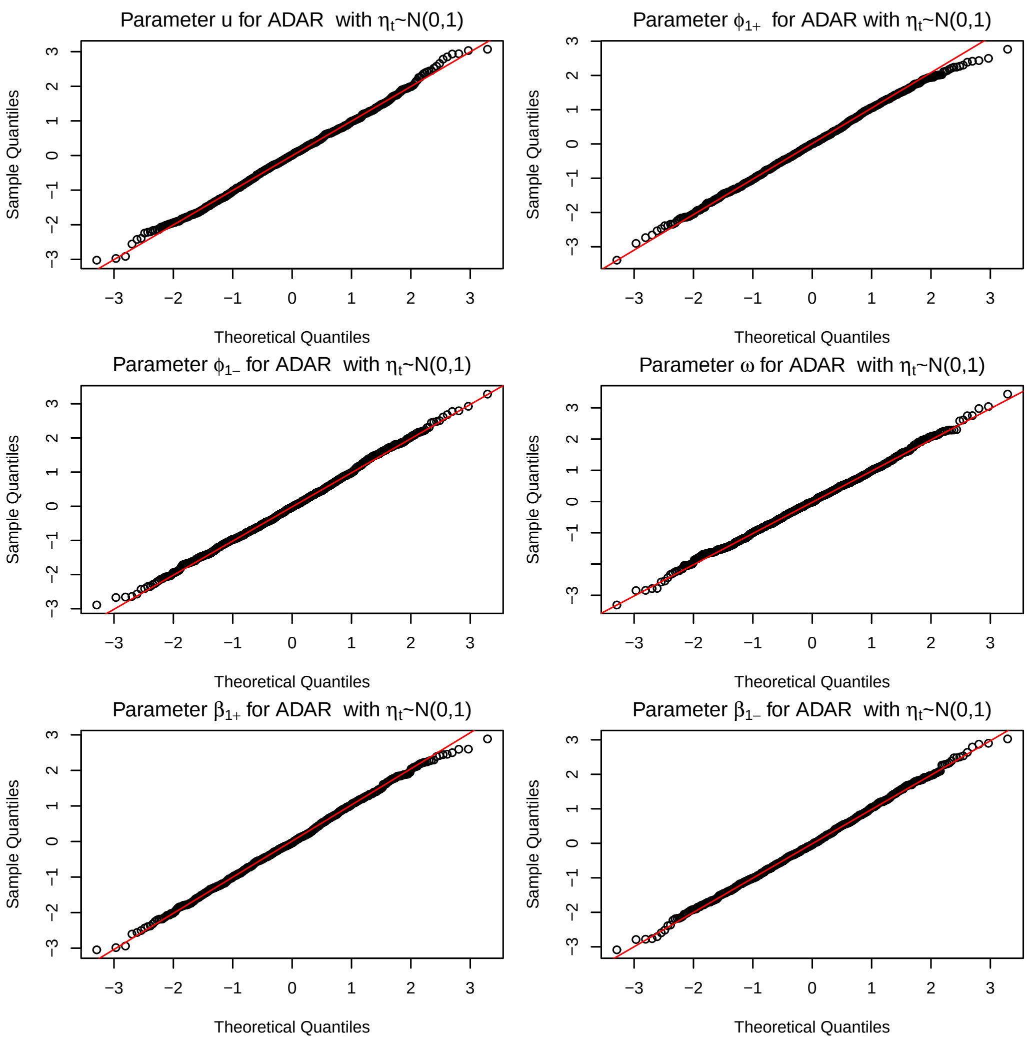

Figure 1. The figure displays the QQ-plot of the QMLE estimator for each element of the parameter vector from the ADAR model when the power parameter is known and the value is 1, where the error series obey N(0,1). The sample size in the simulations is $n = 2000$.

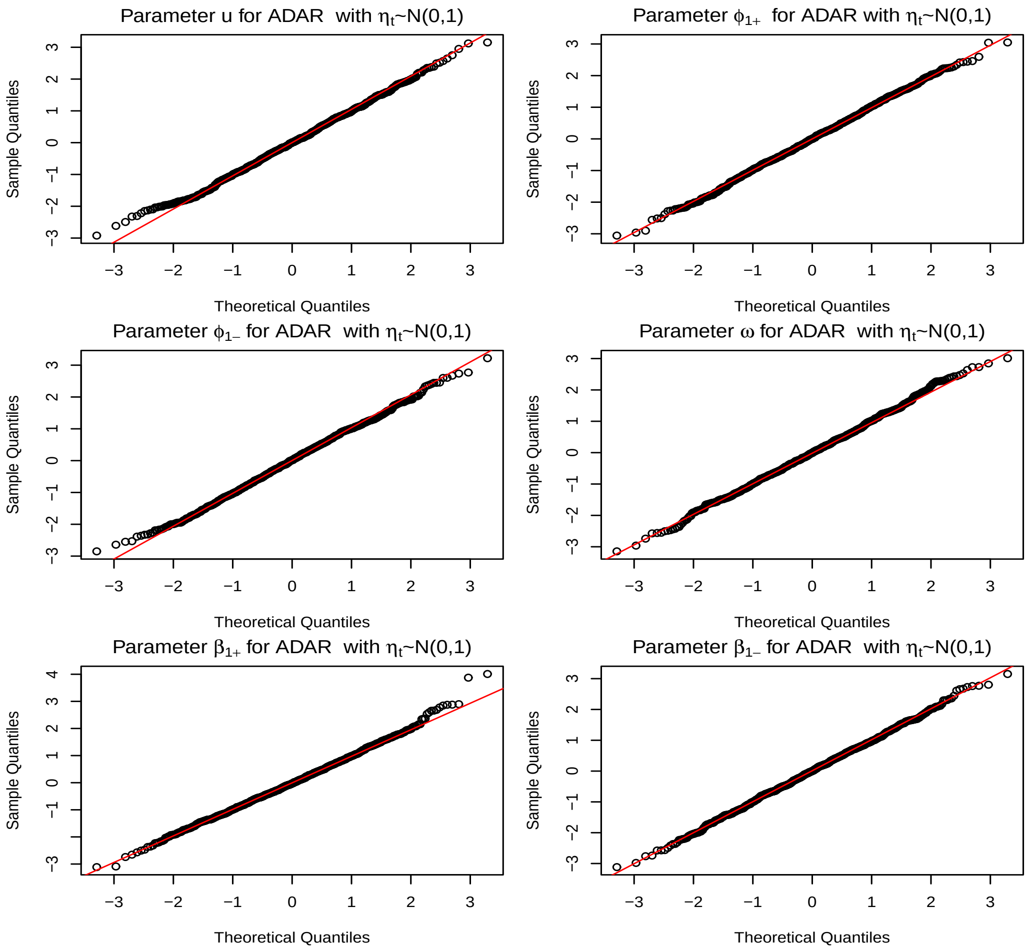

Figure 2. The figure displays the QQ-plot of the QMLE estimator for each element of the parameter vector from the ADAR model when the power parameter is known and the value is 2, where the error series obey N(0,1). The sample size in the simulations is $n = 2000$.

In Tables 1 and 2, the power parameter is known, and simulation results are presented and suggest that biases and standard errors generally decrease as the sample size n increases, suggesting consistency. In addition, from Figures 1 to 4, we also present QQ-plots for the QMLE estimates of the parameter vector, which suggest that the distribution of the estimates looks reasonably close to the normal distribution. In Tables 3 and 4, the power parameter is unknown, and simulation results are presented and suggest that biases and standard errors generally decrease as the sample size n increases, suggesting consistency. Note that the power parameter may have coupling effects with other parameters, making it difficult to effectively decouple in small samples, resulting in estimation results showing multi-modal distribution or local optimal solutions. Maximum likelihood estimation is susceptible to extreme values in small samples, and the estimation of δ may deviate from the true value, exhibiting high variance characteristics. We found it interesting that when the sample size is large enough such as the sample size exceeds 100,000, the maximum likelihood estimation of δ performs well, and the estimation results show better stability. We also present QQ-plots (in Figures 5 to 8) for the QMLE estimates of the parameter vector, which suggest that the distribution of the estimates looks reasonably close to the normal distribution.

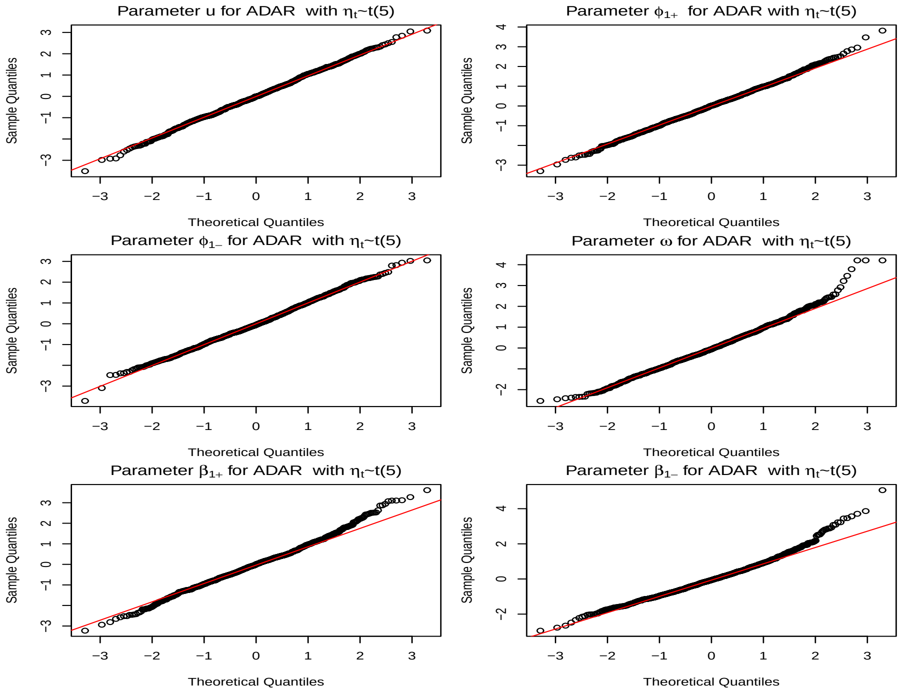

Figure 3. The figure displays the QQ-plot of the QMLE estimator for each element of the parameter vector from the ADAR model when the power parameter is known and the value is 1, where the error series obey t(5). The sample size in the simulations is $n = 2000$.

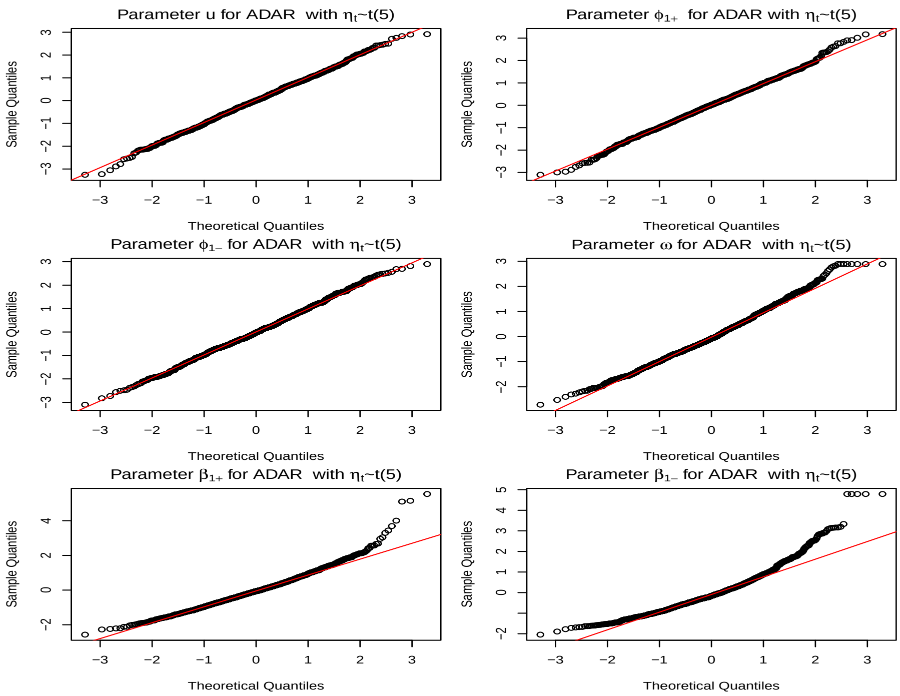

Figure 4. The figure displays the QQ-plot of the QMLE estimator for each element of the parameter vector from the ADAR model when the power parameter is known and the value is 2, where the error series obey t(5). The sample size in the simulations is $n = 2000$.

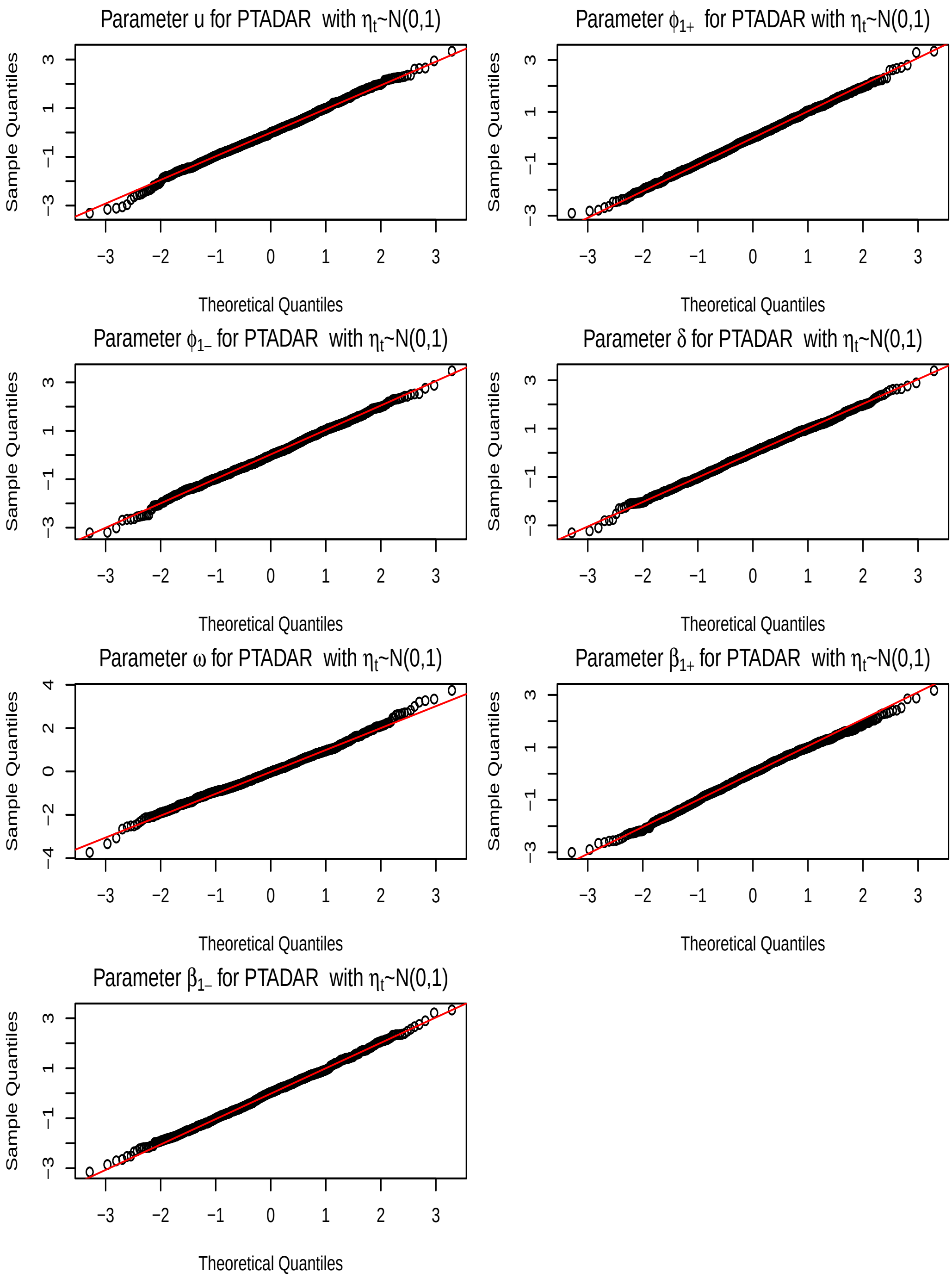

Figure 5. The figure displays the QQ-plot of the QMLE estimator for each element of the parameter vector from the PTADAR model when the power parameter is unknown and the true value is 1, where the error series obey N(0,1). The sample size in the simulations is n = 100,000.

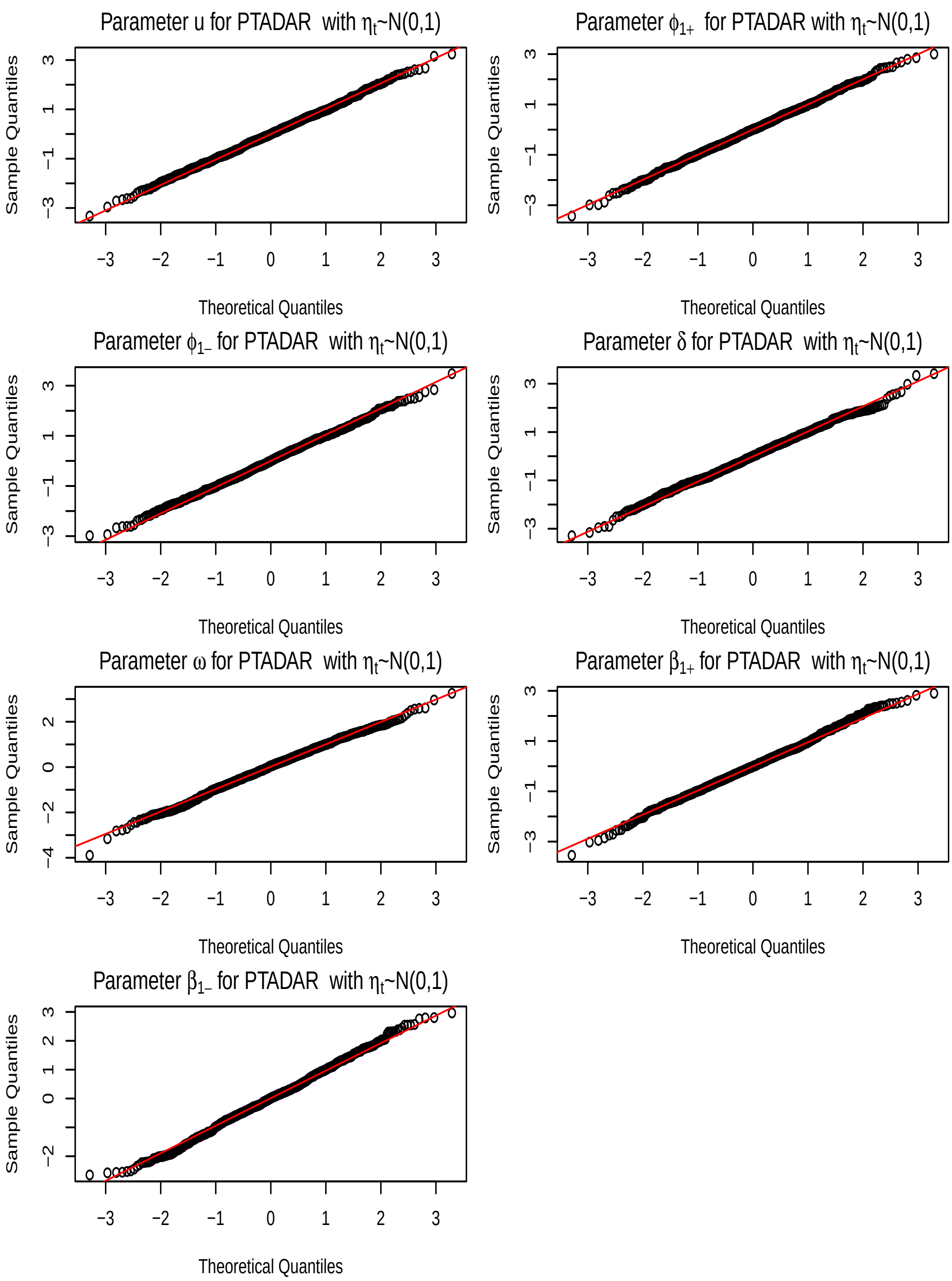

Figure 6. The figure displays the QQ-plot of the QMLE estimator for each element of the parameter vector from the PTADAR model when the power parameter is unknown and the true value is 2, where the error series obey N(0,1). The sample size in the simulations is n = 100,000.

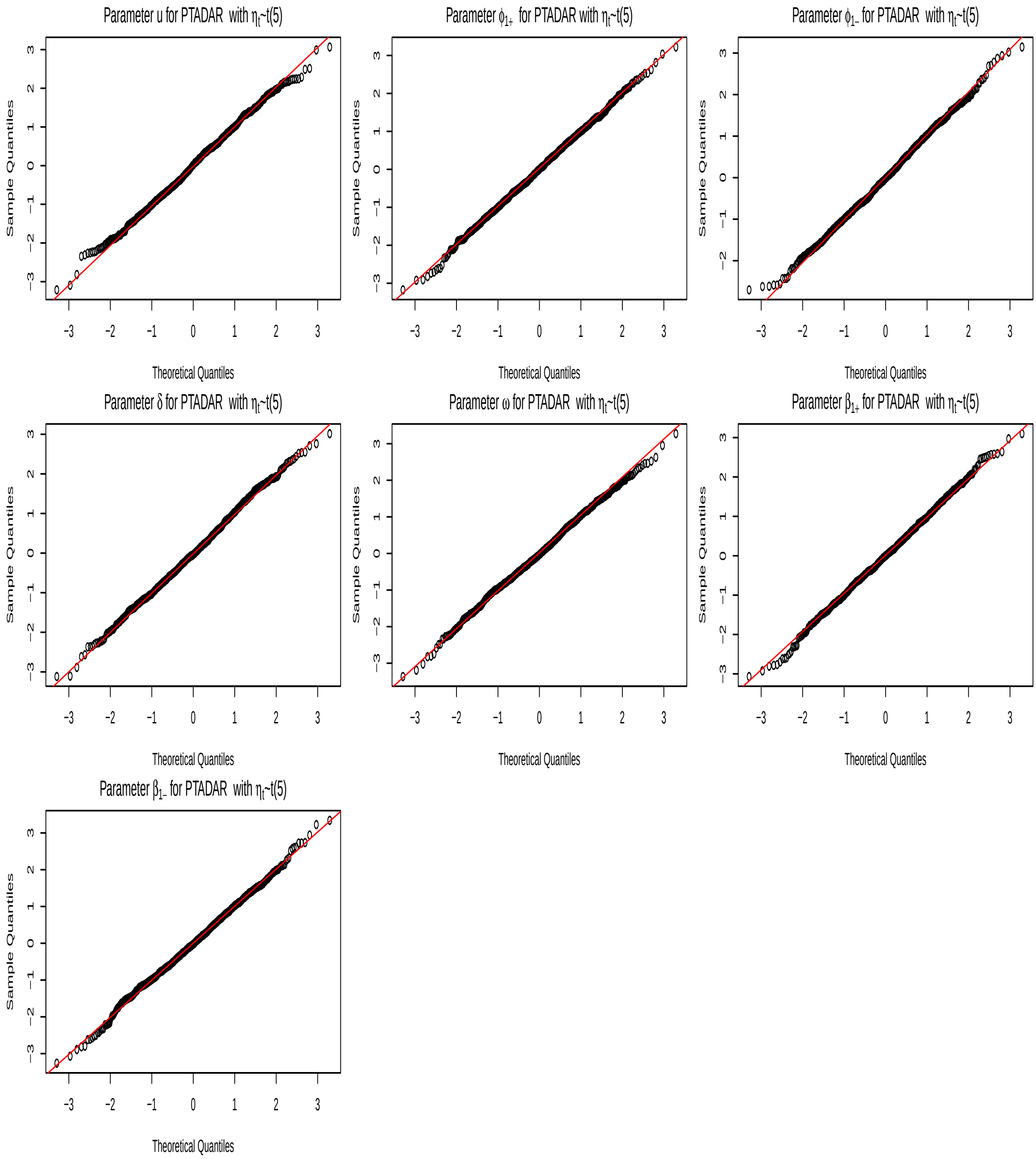

Figure 7. The figure displays the QQ-plot of the QMLE estimator for each element of the parameter vector from the PTADAR model when the power parameter is unknown and the true value is 1, where the error series obey t(5). The sample size in the simulations is n = 100,000.

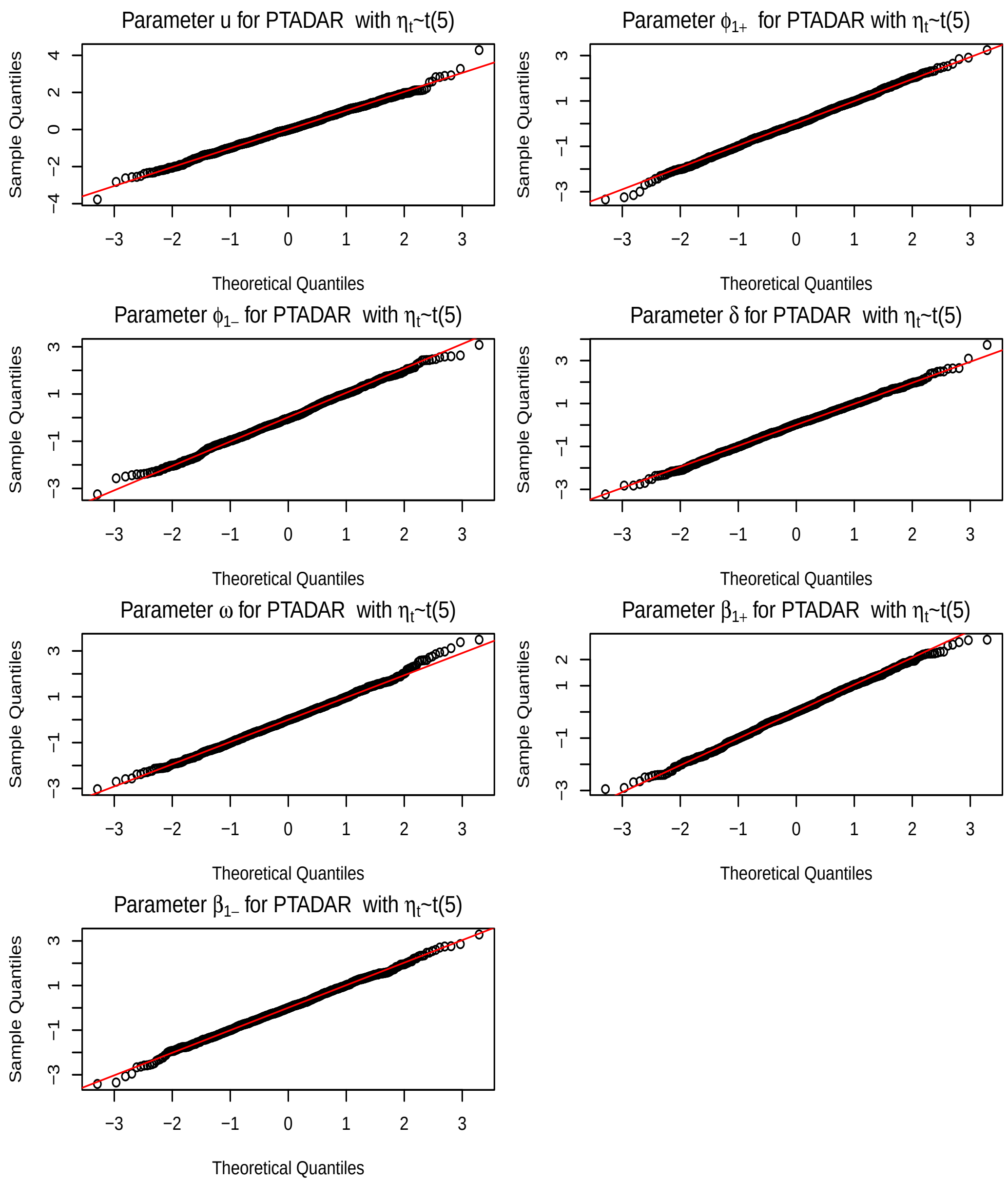

Figure 8. The figure displays the QQ-plot of the QMLE estimator for each element of the parameter vector from the PTADAR model when the power parameter is unknown and the true value is 2, where the error series obey t(5). The sample size in the simulations is n = 100,000.

4. Application

The data set $\{x_t\}_{t=1}^{802}$ is the daily exchange rate of USD to RMB from August 12, 2020, to April 26, 2024, which can be found in the website of China Money (https://www.chinamoney.com.cn). Let $\{y_t\}_{t=1}^{801}$ be the log return percentage sequence of $\{x_t\}_{t=1}^{802}$, with $y_t = 100(\log x_t - \log x_{t-1})$, see Figure 9. Denote $H_{n,k}^L = \left\{\frac{1}{k}\sum_{i=1}^k \left(\log \frac{y_{(i)}}{y_{(k+1)}}\right)\right\}^{-1}$ and $H_{n,k}^R = \left\{\frac{1}{k}\sum_{i=1}^k \left(\log \frac{y_{(n-i+1)}}{y_{(n-k)}}\right)\right\}^{-1}$ to be the left-tail and right-tail Hill estimators , where $\{y_{(t)}\}_{t=1}^{801}$ are decresing order statistics of $\{y_t\}_{t=1}^{801}$, see Figure 10. The Hill estimators are mostly less than 2, which shows that the data set $\{y_t\}_{t=1}^{801}$ is heavy-tailed.

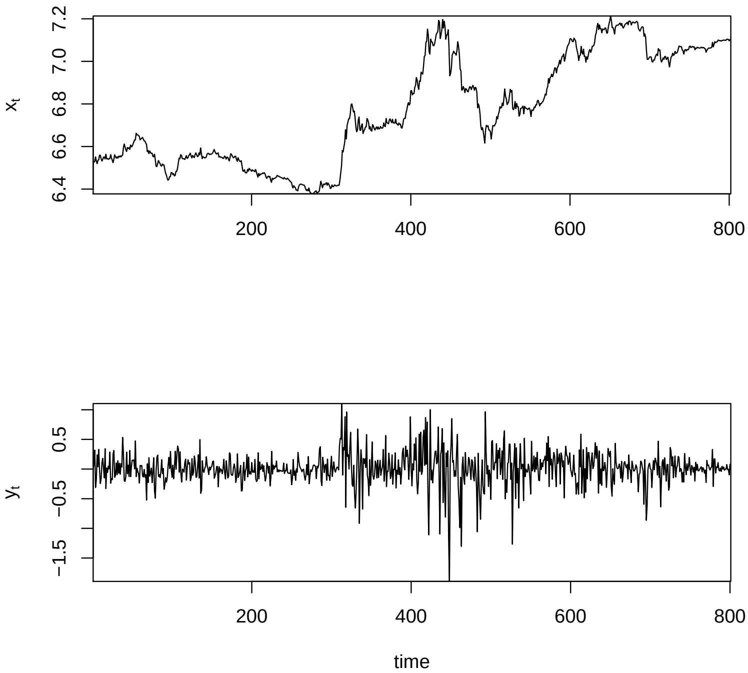

Figure 9. Daily exchange rate of USD to RMB from August 12, 2020, to April 26, 2024.

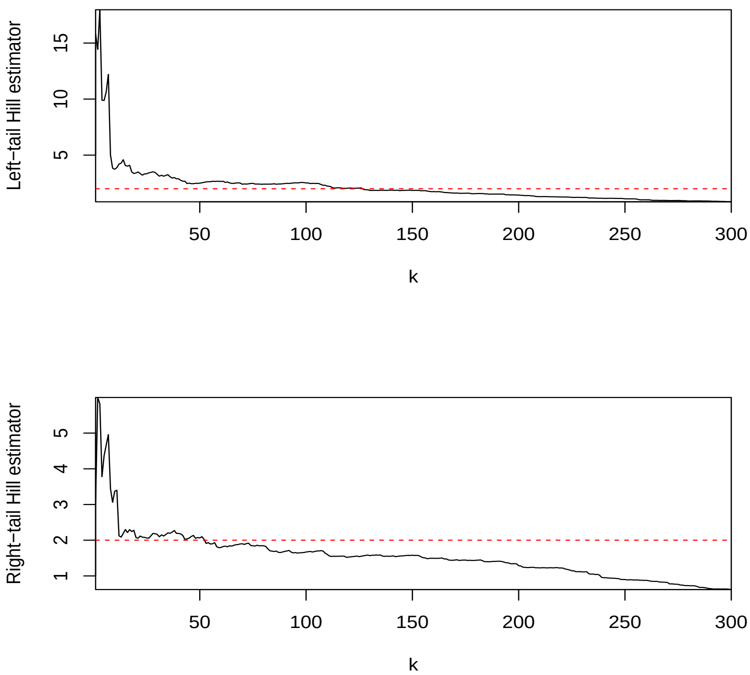

Figure 10. Left-tail Hill estimator and right-tail Hill estimator.

By the QMLE methold, we set $p_{max} = q_{max} = 10$, and compare the values of the AIC and BIC. The data $\{y_t\}_{t=1}^{801}$ is found from the following PTADAR(1,1) fitted model: AIC = −1007.018, BIC = -967.217. AIC stands for the Akaike information criterion, where AIC = $2(2p + 2q + 3) - 2ln(L)$, BIC stands for the Bayesian Information Criterion, where BIC = $(2p + 2q + 3)ln(n) - 2ln(L)$, and L is the value of $L_n(\boldsymbol{\theta})$ defined in (2.1). Numbers in the parentheses are standard errors of the parameters.

$$\begin{cases} y_t = 0.0207 + 0.0310 y_{t-1}^{+} - 0.0981 y_{t-1}^{-} + \epsilon_t, \ \epsilon_t = \sigma_t \eta_t \\ \quad (0.0118)(0.0780) \qquad (0.0886) \\ \sigma_t^{1.1471} = 0.1568 + 0.3584 (y_{t-1}^{+})^{1.1471} + 0.4385 (y_{t-1}^{-})^{1.1471} \\ \qquad (0.0703)(0.0653) \qquad\qquad (0.0815) \quad (0.3110) \end{cases} \tag{4.1}$$

Alternatively, we can also develop other fitted models: for instance, the ADAR(1,1) ($\delta = 1$) model, where AIC = −1008.0169, BIC = −973.9017.

$$\begin{cases} y_t = 0.0258 - 0.0051 y_{t-1}^{+} - 0.1665 y_{t-1}^{-} + \epsilon_t, \ \epsilon_t = \sigma_t \eta_t \\ \quad (0.0120) \ (0.0767) \qquad (0.0891) \\ \sigma_t = 0.1994 + 0.3185 y_{t-1}^{+} + 0.4202 y_{t-1}^{-} \\ \quad (0.0085) \ (0.0542) \qquad (0.0630) \end{cases} \tag{4.2}$$

The ADAR(1,1) ($\delta = 2$) model shows AIC = −1008.0184, BIC = −973.9032.

$$\begin{cases} y_t = 0.0262 - 0.0017 y_{t-1}^{+} - 0.1744 y_{t-1}^{-} + \epsilon_t, \ \epsilon_t = \sigma_t \eta_t \\ \quad (0.0126) \ (0.0786) \qquad (0.0951) \\ \sigma_t^2 = 0.0492 + 0.3230 (y_{t-1}^{+})^2 + 0.5586 (y_{t-1}^{-})^2 \\ \quad (0.0033) \ (0.0808) \qquad (0.1212) \end{cases} \tag{4.3}$$

Obviously the coefficients of y_{t-1}^{+} , y_{t-1}^{-}, $(y_{t-1}^{-})^{1.1471}$, $(y_{t-1}^{+})^{1.1471}$, $(y_{t-1}^{-})^2$ and $(y_{t-1}^{+})^2$ are significantly different. It is appropriate to consider asymmetric effects between volatility and y_t, where η_t is i.i.d $N(0,1)$. The above three models have their own advantages. The parameter coefficients of model (4.1) have smaller standard errors compared to model (4.3). The standard error of the parameter coefficients in the first equation of model (4.1) is smaller than that in model (4.2). By the AIC and BIC, we select the model (4.3) to fit the data $\{y_t\}_{t=1}^{801}$.

5. Conclusions

Motivated by the power transformed ARCH model and the double autoregressive, to better capture asymmetry and heavy-tailed phenomena, this paper introduces a power-transformed asymmetric double autoregressive (PTADAR(p,q)) model. The new model includes diverse nonlinear and asymmetric double autoregressive models as special cases. We set the power parameter $\delta > 0$. When $\delta = 1$, it becomes an asymmetric linear double autoregressive model (Tan et al. [15, 16]). When $\delta = 2$, it becomes an asymmetric double autoregressive model (Zhang et al. [14]). When $\delta = 2$, $\phi_{i+} = \phi_{i-}$, and $\alpha_{j+} = \alpha_{j+}$, $(i = 1, \cdots, p; j = 1, \cdots, q)$, it becomes a double autoregressive model (Ling, S. [10, 11], Zhu et al. [21], Guo et al. [18], Jiang et al. [19], Gong et al. [20]). Moreover in empirical application, the power parameter $\delta > 0$ may be unknown. This could overcome the constraint of power parameter values of variants of the double autoregressive (DAR(p,q)) model. We give a sufficient condition for a strict stationarity solution of the PTADAR(p,q) model, and then study the quasi-maximum likelihood estimation of the model, obtaining the strong consistency and asymptotic normality of the QMLE. Illustrations and an empirical example show our model's usefulness. The PTADAR(p,q) can better deal with heavy-tailed data and capture the asymmetric effects.

For further research, we can consider the following: (1) The sufficient and necessary conditions for the existence of strictly stationary solutions, which may pose challenges. (2) Robust estimation: when the error sequence has an infinite fourth moment, we can consider robust estimations, such as least absolute deviation estimation. (3) The problem of model inference under non-stationary conditions. We will attempt to complete these issues in the future.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflicts of interest in this paper.

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A. Proof for theorems

A.1. Proof for Theorem 2.1

Note that

$$\eta_t\sigma_t(\boldsymbol{\alpha}) = \epsilon_t(\boldsymbol{\phi}) = y_t - \boldsymbol{\phi}'\boldsymbol{Y}_{t-1}, \sigma_t(\boldsymbol{\alpha}) = (\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{1/\delta},$$
$$\sigma_t^2(\boldsymbol{\alpha}) = (\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta} = \exp\left\{\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}\right\} = \exp\left\{\frac{2}{\delta}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})\right\}.$$

Let

$$\boldsymbol{M}_t^1 = \sum_{j=1}^{q} \left[\alpha_{j+} \left(y_{t-j}^+\right)^{\delta} \log\left(y_{t-j}^+\right) + \alpha_{j-} \left(y_{t-j}^-\right)^{\delta} \log\left(y_{t-j}^-\right)\right],$$

$$\boldsymbol{M}_t^2 = \sum_{j=1}^{q} \left[\alpha_{j+} \left(y_{t-j}^+\right)^{\delta} \left(\log\left(y_{t-j}^+\right)\right)^2 + \alpha_{j-} \left(y_{t-j}^-\right)^{\delta} \left(\log\left(y_{t-j}^-\right)\right)^2\right],$$

$$\boldsymbol{X}_t^1 = \left(0, (y_t^+)^{\delta} \log(y_t^+), (y_t^-)^{\delta} \log(y_t^-), \cdots, \left(y_{t-q+1}^+\right)^{\delta} \log\left(y_{t-q+1}^+\right), \left(y_{t-q+1}^-\right)^{\delta} \log\left(y_{t-q+1}^-\right)\right)'.$$

Then we can obtain the first and second derivatives of $l_t(\boldsymbol{\theta})$:

$$\begin{aligned}
\frac{\partial l_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} &= \left(-\frac{1}{2}\right) \frac{\partial}{\partial \boldsymbol{\theta}} \left[\frac{2}{\delta} \log \sigma_t^{\delta}(\boldsymbol{\alpha}) + \frac{\epsilon_t^2(\boldsymbol{\phi})}{(\sigma_t^{\delta}(\boldsymbol{\alpha}))^{2/\delta}}\right], \\
\frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} &= \left(-\frac{1}{2}\right) \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \left[\frac{2}{\delta} \log \sigma_t^{\delta}(\boldsymbol{\alpha}) + \frac{\epsilon_t^2(\boldsymbol{\phi})}{(\sigma_t^{\delta}(\boldsymbol{\alpha}))^{2/\delta}}\right], \\
\frac{\partial l_t(\boldsymbol{\theta})}{\partial \boldsymbol{\phi}} &= \frac{\epsilon_t(\boldsymbol{\phi}) \boldsymbol{Y}_{t-1}}{\sigma_t^2(\boldsymbol{\alpha})}, \frac{\partial l_t(\boldsymbol{\theta})}{\partial \delta} = \frac{1}{\delta} \left[\frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})} - 1\right] \left[\frac{\boldsymbol{M}_t^1}{\sigma_t^{\delta}(\boldsymbol{\alpha})} - \log(\sigma_t(\boldsymbol{\alpha}))\right], \\
\frac{\partial l_t(\boldsymbol{\theta})}{\partial \boldsymbol{\alpha}} &= \frac{\boldsymbol{X}_{t-1}}{\delta \sigma_t^{\delta}(\boldsymbol{\alpha})} \left[\frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})} - 1\right], \\
\frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial \boldsymbol{\phi} \partial \boldsymbol{\phi}'} &= \frac{-\boldsymbol{Y}_{t-1} \boldsymbol{Y}_{t-1}'}{\sigma_t^2(\boldsymbol{\alpha})}, \frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial \boldsymbol{\phi} \partial \delta} = \frac{2 \epsilon_t(\boldsymbol{\phi}) \boldsymbol{Y}_{t-1}}{\delta \sigma_t^2(\boldsymbol{\alpha})} \left[\log(\sigma_t(\boldsymbol{\alpha})) - \frac{\boldsymbol{M}_t^1}{\sigma_t^{\delta}(\boldsymbol{\alpha})}\right], \\
\frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial \boldsymbol{\phi} \partial \boldsymbol{\alpha}'} &= \frac{-2 \epsilon_t(\boldsymbol{\phi}) \boldsymbol{Y}_{t-1} \boldsymbol{X}_{t-1}'}{\delta \sigma_t^2(\boldsymbol{\alpha}) \sigma_t^{\delta}(\boldsymbol{\alpha})}, \\
\frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial \delta^2} &= \frac{-2}{\delta^2} \frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})} \left[\frac{1}{\delta} \log(\boldsymbol{\alpha}' \boldsymbol{X}_{t-1}) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}' \boldsymbol{X}_{t-1}}\right]^2 \\
&\quad + \frac{1}{\delta} \left[\frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})} - 1\right] \left[\frac{\boldsymbol{M}_t^2}{\boldsymbol{\alpha}' \boldsymbol{X}_{t-1}} - \frac{\left(\boldsymbol{M}_t^1\right)^2}{(\boldsymbol{\alpha}' \boldsymbol{X}_{t-1})^2} + \frac{2}{\delta^2} \log(\boldsymbol{\alpha}' \boldsymbol{X}_{t-1}) - \frac{2}{\delta} \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}' \boldsymbol{X}_{t-1}}\right], \\
\frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial \delta \partial \boldsymbol{\alpha}'} &= \frac{2 \epsilon_t^2(\boldsymbol{\phi}) \boldsymbol{X}_{t-1}'}{\delta^2 \sigma_t^2(\boldsymbol{\alpha}) \boldsymbol{\alpha}' \boldsymbol{X}_{t-1}} \left[\frac{1}{\delta} \log(\boldsymbol{\alpha}' \boldsymbol{X}_{t-1}) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}' \boldsymbol{X}_{t-1}}\right] \\
&\quad + \frac{1}{\delta \boldsymbol{\alpha}' \boldsymbol{X}_{t-1}} \left[\frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})} - 1\right] \left[\boldsymbol{X}_{t-1}^{1\prime} - \left(\boldsymbol{M}_t^1 + \frac{1}{\delta}\right) \boldsymbol{X}_{t-1}'\right], \\
\frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}'} &= \frac{\left[1 - \left(1 + \frac{2}{\delta}\right) \frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})}\right] \boldsymbol{X}_{t-1} \boldsymbol{X}_{t-1}'}{\delta (\boldsymbol{\alpha}' \boldsymbol{X}_{t-1})^2}.
\end{aligned}$$

First we prove the conclusion holds under the condition of (c1). Note that

$$\begin{aligned}
\epsilon_t(\boldsymbol{\phi}) &= y_t - \boldsymbol{\phi}' \boldsymbol{Y}_{t-1}, \sigma_t(\boldsymbol{\alpha}) = [\boldsymbol{\alpha}' \boldsymbol{X}_{t-1}]^{1/\delta}, \\
\sigma_t^2(\boldsymbol{\alpha}) &= [\boldsymbol{\alpha}' \boldsymbol{X}_{t-1}]^{2/\delta} \\
&= \exp\left\{\log[\boldsymbol{\alpha}' \boldsymbol{X}_{t-1}]^{2/\delta}\right\} \\
&= \exp\left\{\frac{2}{\delta} \log[\boldsymbol{\alpha}' \boldsymbol{X}_{t-1}]\right\}.
\end{aligned}$$

Let $\phi_i = \max(|\phi_{i+}|, |\phi_{i-}|)$, $\alpha_j = \max(|\alpha_{j+}|, |\alpha_{j-}|)$, $(i = 1, \cdots, p; j = 1, \cdots, q)$. For convenience, let $p = q = m$, with $m = \max(p, q)$. In fact if $m > p$, then $\phi_i = 0\,(i = p+1, \cdots, m)$, if $m > q$, then $\alpha_j = 0\,(j = q+1, \cdots, m)$. Hence $y_t = \boldsymbol{\phi}'\boldsymbol{Y}_{t-1} + \sigma_t(\boldsymbol{\alpha})\eta_t$, $y^+ + y^- = |y|$. When $0 < r \le \delta$, we obtain

$$
\begin{aligned}
[\sigma_t(\boldsymbol{\alpha})]^r &\le \left[\omega + \sum_{j=1}^{m} \alpha_j |y_{t-j}|^\delta\right]^{r/\delta} \\
&\le \omega^{r/\delta} + \sum_{j=1}^{m} \alpha_j^{r/\delta} |y_{t-j}|^r, \qquad \text{(A.1)}
\end{aligned}
$$

when $0 < \delta < r$, and by the c_r inequality, we obtain

$$
\begin{aligned}
[\sigma_t(\boldsymbol{\alpha})]^r &\le \left[\omega + \sum_{j=1}^{m} \alpha_j |y_{t-j}|^\delta\right]^{r/\delta} \\
&\le (m+1)^{r-1} \left[\omega^{r/\delta} + \sum_{j=1}^{m} \alpha_j^{r/\delta} |y_{t-j}|^r\right]. \qquad \text{(A.2)}
\end{aligned}
$$

When $0 < r \le 1$, we can get

$$
|\boldsymbol{\phi}'\boldsymbol{Y}_{t-1}|^r \le u^r + \sum_{i=1}^{m} \phi_i^r |y_{t-i}|^r. \qquad \text{(A.3)}
$$

Then if $0 < r \le 1 < \delta$ or $0 < r \le \delta \le 1$, by inequalities (6.1) and (6.3), we obtain

$$
\begin{aligned}
E(|y_1|^r \mid Y_{0m} = y_{0m}) &\le E|\boldsymbol{\phi}'\boldsymbol{Y}_0|^r + E\,[\sigma_1(\boldsymbol{\alpha})]^r\, E(|\eta_1|^r) \\
&\le u^r + \sum_{i=1}^{m} \phi_i^r |y_{1-i}|^r + \omega^{r/\delta} + \sum_{j=1}^{m} \alpha_j^{r/\delta} |y_{1-j}|^r E(|\eta_1|^r),
\end{aligned}
$$

with $y_{0m} = (y_0, \cdots, y_{1-m})' \in R^m$, and we denote $c_i = \phi_i^r + \alpha_i^{r/\delta} E(|\eta_1|^r)$, where $\sum_{i=1}^m c_i < 1$. By Theorem 2.1 and Corollary 2.2 in Daren et al. [22], then there exists a unique strictly stationary geometrically ergodic solution $\{y_t\}$ for model (2.1), with $E(|y_t|^r) < \infty$. Hence under the condition of (c1), the conclusion of Theorem 2.1 holds. Thus we complete the proof of Theorem 2.1.

Remark A.1. *The conditions of Theorem 1 are mild, and only require $E(|y_t|^r) < \infty$ under a fractional moment, where $r \in (0, 1)$.*

A.2. Proof for Theorem 2.2

To prove Theorem 2.2, we need a lemma. Let $\|\cdot\|$ be the Euclidean norm.

Lemma A.1. *If the conditions (A1)–(A3) hold, then*

(i) $E \sup\limits_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} |l_t(\boldsymbol{\theta})| < \infty$, $E \sup\limits_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left\|\frac{\partial l_t(\boldsymbol{\theta})}{\partial \theta}\right\| < \infty$, $E \sup\limits_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left\|\frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial\theta\partial\theta'}\right\| < \infty$;

(ii) $\sup\limits_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left|\frac{1}{n} L_n(\boldsymbol{\theta}) - E l_t(\boldsymbol{\theta})\right| = o_p(1)$, $E \sup\limits_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left\|\frac{1}{n}\sum_{t=m+1}^{n} \left\{\left[\frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial\theta\partial\theta'}\right] - E\left[\frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial\theta\partial\theta'}\right]\right\}\right\| = o_p(1)$;

(iii) when $\boldsymbol{\theta} = \boldsymbol{\theta}_0$, $E l_t(\boldsymbol{\theta})$ has the unique maximum value;

(iv) when $k_1 = E\eta_t^3 < \infty, E\eta_t^4 < \infty, k_2 = E\eta_t^4 - 1 < \infty$, then $\mathbf{\Sigma}_0$ and $\mathbf{\Omega}_0$ are all positive definite and finite;

(v) $\frac{1}{\sqrt{n}}\sum_{t=m+1}^{n}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}} \Rightarrow N(0,\Omega_0)$ *as* $n\to\infty$.

Proof. (i) There exists a constant $r \in (0,1)$, such that $0 < r < \delta$ and $E|y_t|^r < \infty$, and we denote $\overline{\omega}^* = \max(1,\overline{\omega})$. By the c_r inequality and Jensen's inequality, we obtain

$$
\begin{aligned}
&E\sup_{\theta\in\Theta}|l_t(\boldsymbol{\theta})|\\
&= E\sup_{\theta\in\Theta}\left|\left(-\frac{1}{2}\right)\left[\frac{2}{\delta}\log\sigma_t^{\delta}(\boldsymbol{\alpha}) + \frac{\epsilon_t^2(\boldsymbol{\phi})}{(\sigma_t^{\delta}(\boldsymbol{\alpha}))^{2/\delta}}\right]\right|\\
&\leqslant E\sup_{\theta\in\Theta}\left|\left(-\frac{1}{2}\right)\frac{2}{\delta}\log\sigma_t^{\delta}(\boldsymbol{\phi})\right| + E\sup_{\theta\in\Theta}\left|\left(-\frac{1}{2}\right)\frac{\epsilon_t^2(\boldsymbol{\phi})}{(\sigma_t^{\delta}(\boldsymbol{\phi}))^{2/\delta}}\right|\\
&= E\sup_{\theta\in\Theta}\left|\frac{1}{\delta}\log\left[\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right]\right| + E\sup_{\theta\in\Theta}\left|\frac{\left[y_t-\boldsymbol{\phi}'\boldsymbol{Y}_{t-1}\right]^2}{2\left[\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right]^{2/\delta}}\right|\\
&\leqslant E\sup_{\theta\in\Theta}\left\{I\left[\frac{1}{\delta}\log\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)\geqslant 1\right]\times\frac{1}{\delta}\log\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)\right\}\\
&\quad + E\sup_{\theta\in\Theta}\left\{I\left[\frac{1}{\delta}\log\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)<1\right]\times\frac{1}{\delta}\log\left[\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right]\right\} + E\sup_{\theta\in\Theta}\left|\frac{\left[y_t-\boldsymbol{\phi}'\boldsymbol{Y}_{t-1}\right]^2}{2\left[\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right]^{2/\delta}}\right|\\
&\leqslant \frac{1}{r}\log\left[(\overline{\omega}^*)^{\frac{r}{\delta}} + \overline{\alpha}^{\frac{r}{\delta}}\sum_{j=1}^{q}E\left|y_{t-j}\right|^r\right] - I\left(\underline{\omega}<1\right)\left[\frac{1}{\delta}\log\left(\underline{\omega}\right)\right]\\
&\quad + E\sup_{\theta\in\Theta}\left|\frac{\epsilon_t^2(\boldsymbol{\phi_0})}{2\left[\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right]^{2/\delta}}\right| + E\sup_{\theta\in\Theta}\left|\frac{\epsilon_t(\boldsymbol{\phi_0})\left[(\boldsymbol{\phi}'-\boldsymbol{\phi}_{\mathbf{0}}')\boldsymbol{Y}_{t-1}\right]}{\left[\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right]^{2/\delta}}\right| + E\sup_{\theta\in\Theta}\left|\frac{\left[(\boldsymbol{\phi}'-\boldsymbol{\phi}_{\mathbf{0}}')\boldsymbol{Y}_{t-1}\right]^2}{2\left[\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right]^{2/\delta}}\right|\\
&= \frac{1}{r}\log\left[(\overline{\omega}^*)^{\frac{r}{\delta}} + \overline{\alpha}^{\frac{r}{\delta}}\sum_{j=1}^{q}E\left|y_{t-j}\right|^r\right] - I\left(\underline{\omega}<1\right)\left[\frac{1}{\delta}\log\left(\underline{\omega}\right)\right]\\
&\quad + E\sup_{\theta\in\Theta}\left|\frac{\eta_t^2\left[\boldsymbol{\alpha}_{\mathbf{0}}'\boldsymbol{X}_{t-1}^0\right]^{2/\delta_0}}{2\left[\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right]^{2/\delta}}\right| + E\sup_{\theta\in\Theta}\left|\frac{\left[(\boldsymbol{\phi}'-\boldsymbol{\phi}_{\mathbf{0}}')\boldsymbol{Y}_{t-1}\right]^2}{2\left[\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right]^{2/\delta}}\right|\\
&= \frac{1}{r}\log\left[(\overline{\omega}^*)^{\frac{r}{\delta}} + \overline{\alpha}^{\frac{r}{\delta}}\sum_{j=1}^{q}E(y_{t-j}^{+})^r + E(y_{t-j}^{-})^r\right] - I\left(\underline{\omega}<1\right)\left[\frac{1}{\delta}\log\left(\underline{\omega}\right)\right]\\
&\quad + E\sup_{\theta\in\Theta}\left|\frac{\left[\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right]^{2/\delta_0}}{2\left[\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right]^{2/\delta}}\right| + E\sup_{\theta\in\Theta}\left|\frac{\left[(\boldsymbol{\phi}'-\boldsymbol{\phi}_{\mathbf{0}}')\boldsymbol{Y}_{t-1}\right]^2}{2\left[\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right]^{2/\delta}}\right| < \infty.
\end{aligned}
$$

Note that

$$
\epsilon_t(\boldsymbol{\phi_0}) = \eta_t\left[\boldsymbol{\alpha}_{\mathbf{0}}'\boldsymbol{X}_{t-1}^0\right]^{1/\delta_0},
$$
$$
y_t - \boldsymbol{\phi}'\boldsymbol{Y}_{t-1} = \epsilon_t(\boldsymbol{\phi_0}) - (\boldsymbol{\phi}'-\boldsymbol{\phi}_{\mathbf{0}}')\boldsymbol{Y}_{t-1},
$$

and the third inequality holds. Hence, we obtain $E\sup_{\theta\in\Theta}|l_t(\boldsymbol{\theta})| < \infty$. By the same way, we can obtain $E\sup_{\theta\in\Theta}\left\|\frac{\partial l_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right\| < \infty$, $E\sup_{\theta\in\Theta}\left\|\frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right\| < \infty$.

(ii) By Theorem 3.1 in Ling [23] and (i), we obtain $\sup_{\theta\in\Theta}\left|\frac{1}{n}L_n(\boldsymbol{\theta}) - El_t(\boldsymbol{\theta})\right| = o_p(1)$, and

$$E\sup_{\theta\in\Theta}\left\|\frac{1}{n}\sum_{t=m+1}^{n}\left\{\left[\frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right] - E\left[\frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right]\right\}\right\| = o_p(1).$$

(iii) Let $\boldsymbol{c}_1$ be a $(2p+1)\times 1$ constant vector, where $\boldsymbol{c}_1 = (c_0, c_{1+}, c_{1-}, \cdots, c_{p+}, c_{p-})'$. First we give the proof showing that $\boldsymbol{c}_1 = 0$ if $\boldsymbol{c}_1'\boldsymbol{Y}_t = 0$ a.s. If $\boldsymbol{c}_1'\boldsymbol{Y}_t = 0$ a.s., and $\boldsymbol{c}_1' \neq 0$, for convenience, let $c_{1+} = 1$, and hence $y_t^+ = -c_0 - c_{1-}y_t^- - \sum_{i=2}^{p}\left[c_{i+}y_{t-i+1}^+ + c_{i-}y_{t-i+1}^-\right]$ a.s. Consider $y_t = y_t^+ - y_t^-$, if $c_{1-} = -c_{1+} = -1$, and hence $y_t = y_t^+ - y_t^- = -c_0 - \sum_{i=2}^{p}\left[c_{i+}y_{t-i+1}^+ + c_{i-}y_{t-i+1}^-\right]$ for $\eta_t = \frac{\epsilon_t(\boldsymbol{\phi_0})}{\sigma_t(\boldsymbol{\alpha_0})} = \frac{y_t - \boldsymbol{\phi}_0'\boldsymbol{Y}_{t-1}}{\left[\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right]^{1/\delta_0}}$, and $\{y_s : s < t\}$ is independent of η_t. Thus, we obtain $E\left(\eta_t^2\right) = E(\eta_t)E\left(\frac{-c_0-\sum_{i=2}^{p}\left[c_{i+}y_{t-i+1}^+ + c_{i-}y_{t-i+1}^-\right]-\boldsymbol{\phi}_0'\boldsymbol{Y}_{t-1}}{\left[\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right]^{1/\delta_0}}\right) = 0$. In fact $E\left(\eta_t^2\right) = 1$, which is a contradiction. Hence $\boldsymbol{c}_1 = 0$. On the other side, if $c_{1-} \neq -1$, consider $y_t = y_t^+ - y_t^-$, $y_t = \boldsymbol{\phi}_0'\boldsymbol{Y}_{t-1} + \epsilon_t(\boldsymbol{\phi_0})$. Thus $-(1+c_{1-})y_t^- = c_0 + \sum_{i=2}^{p}\left[c_{i+}y_{t-i+1}^+ + c_{i-}y_{t-i+1}^-\right] + \boldsymbol{\phi}_0'\boldsymbol{Y}_{t-1} + \eta_t\left[\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right]^{1/\delta_0}$. For convenience, let $1 + c_{1-} > 0$, and then $\frac{-(1+c_{1-})y_t^-}{\left[\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right]^{1/\delta_0}} = \frac{c_0+\sum_{i=2}^{p}\left[c_{i+}y_{t-i+1}^+ + c_{i-}y_{t-i+1}^-\right]+\boldsymbol{\phi}_0'\boldsymbol{Y}_{t-1}}{\left[\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right]^{1/\delta_0}} + \eta_t$, denote $M = -\frac{c_0+\sum_{i=2}^{p}\left[c_{i+}y_{t-i+1}^+ + c_{i-}y_{t-i+1}^-\right]+\boldsymbol{\phi}_0'\boldsymbol{Y}_{t-1}}{\left[\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right]^{1/\delta_0}}$, and obviously $P(y_t^- < 0) = P(\eta_t > M) > 0$, which is a contradiction with $P(y_t^- < 0) = 0$. Hence, $\boldsymbol{c}_1 = 0$ if $\boldsymbol{c}_1'\boldsymbol{Y}_t = 0$ a.s. Similarly we can prove $(\delta, \alpha) = (\delta_0, \alpha_0)$ if $\sigma_t^\delta(\boldsymbol{\alpha}) = \sigma_t^{\delta_0}(\boldsymbol{\alpha_0})$ a.s.

Recall that

$$\begin{aligned}
El_t(\boldsymbol{\theta}) &= E\left(-\frac{1}{2}\right)\left[\frac{2}{\delta}\log\sigma_t^\delta(\boldsymbol{\alpha}) + \frac{\epsilon_t^2(\boldsymbol{\phi})}{(\sigma_t^\delta(\boldsymbol{\alpha}))^{2/\delta}}\right] \\
&= -E\left[\frac{1}{\delta}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}) + \frac{(y_t - \boldsymbol{\phi}'\boldsymbol{Y}_{t-1})^2}{2(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] \\
&= -E\left[\frac{1}{\delta}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})\right] - E\left[\frac{\epsilon_t^2(\boldsymbol{\phi_0})}{2(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] - E\left[\frac{\epsilon_t(\boldsymbol{\phi_0})\left[(\boldsymbol{\phi}' - \boldsymbol{\phi}_0')\boldsymbol{Y}_{t-1}\right]}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] \\
&\quad - E\left[\frac{\left[(\boldsymbol{\phi}' - \boldsymbol{\phi}_0')\boldsymbol{Y}_{t-1}\right]^2}{2(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] \\
&= -E\left[\frac{1}{\delta}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})\right] - E\left[\frac{\eta_t^2\left(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right)^{2/\delta_0}}{2(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] - E\left[\frac{\left[(\boldsymbol{\phi}' - \boldsymbol{\phi}_0')\boldsymbol{Y}_{t-1}\right]^2}{2(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] \\
&= -E\left[\frac{1}{2}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}\right] - E\left[\frac{\left(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right)^{2/\delta_0}}{2(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] - E\left[\frac{\left[(\boldsymbol{\phi}' - \boldsymbol{\phi}_0')\boldsymbol{Y}_{t-1}\right]^2}{2(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right]
\end{aligned}$$

$$= -\frac{1}{2}E\left[-\log\frac{(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0)^{2/\delta_0}}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}} + \frac{\left(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right)^{2/\delta_0}}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] - E\left[\frac{1}{2}\log(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0)^{2/\delta_0}\right]$$
$$- E\left[\frac{\left[(\boldsymbol{\phi}' - \boldsymbol{\phi}_0')\boldsymbol{Y}_{t-1}\right]^2}{2\,(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right].$$

Consider for any $x > 0$, the function $x - \log(x) \geqslant 1$, if and only if $x = 1$. Then $1 - \log(1) = 1$. Thus

$$E\left[-\log\frac{(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0)^{2/\delta_0}}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}} + \frac{\left(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right)^{2/\delta_0}}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] \geqslant 1,$$

if and only if

$$P\left[\frac{\left(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right)^{2/\delta_0}}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}} = 1\right] = 1,$$

which means

$$\frac{\left(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right)^{2/\delta_0}}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}} = 1,$$

a.s. and

$$(\delta, \alpha) = (\delta_0, \alpha_0).$$

Then

$$E\left[-\log\frac{(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0)^{2/\delta_0}}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}} + \frac{\left(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right)^{2/\delta_0}}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] = 1.$$

Hence,

$$-\frac{1}{2}E\left[-\log\frac{(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0)^{2/\delta_0}}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}} + \frac{\left(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right)^{2/\delta_0}}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] \leqslant -\frac{1}{2},$$

if and only if

$$(\delta, \alpha) = (\delta_0, \alpha_0),$$

and then

$$-\frac{1}{2}E\left[-\log\frac{(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0)^{2/\delta_0}}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}} + \frac{\left(\boldsymbol{\alpha}_0'\boldsymbol{X}_{t-1}^0\right)^{2/\delta_0}}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] = -\frac{1}{2}.$$

Note that

$$-E\left\{\frac{\left[(\boldsymbol{\phi}' - \boldsymbol{\phi}_0')\boldsymbol{Y}_{t-1}\right]^2}{2\,(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right\} \leqslant 0,$$

if and only if

$$\phi = \phi_0,$$

and then

$$-E\left\{\frac{\left[(\boldsymbol{\phi}'-\boldsymbol{\phi}_{\mathbf{0}}')\boldsymbol{Y}_{t-1}\right]^2}{2\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)^{2/\delta}}\right\}=0.$$

Hence, $El_t(\boldsymbol{\theta})$ reaches its unique maximum at $\boldsymbol{\theta}_{\mathbf{0}}$.

(iv) Note that

$$\Omega_0=E\left\{\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}}\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}'}\right\}=E\begin{pmatrix}\Omega_0^{11}&\Omega_0^{12}&\Omega_0^{13}\\\Omega_0^{21}&\Omega_0^{22}&\Omega_0^{23}\\\Omega_0^{31}&\Omega_0^{32}&\Omega_0^{33}\end{pmatrix}=\begin{pmatrix}E\Omega_0^{11}&E\Omega_0^{12}&E\Omega_0^{13}\\E\Omega_0^{21}&E\Omega_0^{22}&E\Omega_0^{23}\\E\Omega_0^{31}&E\Omega_0^{32}&E\Omega_0^{33}\end{pmatrix},$$

where $(\Omega_0^{ij})'=\Omega_0^{ji}, i,j=1,2,3$. $E\left[\frac{\epsilon_t(\boldsymbol{\phi})}{\sigma_t(\boldsymbol{\alpha})}\right]=E(\eta_t)=0$, $E\left[\frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})}\right]=E(\eta_t^2)=1$, $k_1=E(\eta_t^3)$, $k_2=E(\eta_t^4)-1$, and

$$\begin{aligned}
E\Omega_0^{11}&=E\left[\frac{\epsilon_t^2(\boldsymbol{\phi})\boldsymbol{Y}_{t-1}\boldsymbol{Y}_{t-1}'}{\sigma_t^4(\boldsymbol{\alpha})}\right]=E\left[\frac{\boldsymbol{Y}_{t-1}\boldsymbol{Y}_{t-1}'}{\sigma_t^2(\boldsymbol{\alpha})}\right],\\
E\Omega_0^{12}&=E\left\{\frac{\eta_t^3\boldsymbol{Y}_{t-1}}{\delta\sigma_t(\boldsymbol{\alpha})}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})}-\log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]\right\}=E\left\{\frac{k_1\boldsymbol{Y}_{t-1}}{\delta\sigma_t(\boldsymbol{\alpha})}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})}-\log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]\right\},\\
E\Omega_0^{13}&=E\left[\frac{\eta_t^3\boldsymbol{Y}_{t-1}}{\sigma_t(\boldsymbol{\alpha})}\frac{\boldsymbol{X}_{t-1}'}{\delta\sigma_t^\delta(\boldsymbol{\alpha})}\right]=E\left[\frac{k_1\boldsymbol{Y}_{t-1}}{\sigma_t(\boldsymbol{\alpha})}\frac{\boldsymbol{X}_{t-1}'}{\delta\sigma_t^\delta(\boldsymbol{\alpha})}\right],\\
E\Omega_0^{22}&=E\left\{\frac{1}{\delta}\left[\frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})}-1\right]\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})}-\log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]\right\}^2\\
&=E\left\{\frac{1}{\delta^2}\left[\eta_t^4-1\right]\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})}-\log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]^2\right\}\\
&=E\left\{\frac{k_2}{\delta^2}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})}-\log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]^2\right\},\\
E\Omega_0^{23}&=E\left\{\frac{1}{\delta}\left[\frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})}-1\right]\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})}-\log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]\left[\frac{\eta_t^2\boldsymbol{X}_{t-1}'}{\delta\sigma_t^\delta(\boldsymbol{\alpha})}\right]\right\}\\
&=E\left\{\frac{\left(\eta_t^4-1\right)\boldsymbol{X}_{t-1}'}{\delta^2\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})}-\log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]\right\}\\
&=E\left\{\frac{k_2\boldsymbol{X}_{t-1}'}{\delta^2\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})}-\log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]\right\},\\
E\Omega_0^{33}&=E\left[\frac{\left(\eta_t^4-1\right)\boldsymbol{X}_{t-1}\boldsymbol{X}_{t-1}'}{\delta^2\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)^2}\right]=E\left[\frac{k_2\boldsymbol{X}_{t-1}\boldsymbol{X}_{t-1}'}{\delta^2\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)^2}\right].
\end{aligned}$$

$$\Sigma_0=E\left(-\frac{\partial^2 l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right)=E\begin{pmatrix}\Sigma_0^{11}&\Sigma_0^{12}&\Sigma_0^{13}\\\Sigma_0^{21}&\Sigma_0^{22}&\Sigma_0^{23}\\\Sigma_0^{31}&\Sigma_0^{32}&\Sigma_0^{33}\end{pmatrix}=\begin{pmatrix}E\Sigma_0^{11}&E\Sigma_0^{12}&E\Sigma_0^{13}\\E\Sigma_0^{21}&E\Sigma_0^{22}&E\Sigma_0^{23}\\E\Sigma_0^{31}&E\Sigma_0^{32}&E\Sigma_0^{33}\end{pmatrix},$$

where $(\Sigma_0^{ij})' = \Sigma_0^{ji}$, $i, j = 1, 2, 3$. Where we have

$$
\begin{aligned}
E\Sigma_0^{11} &= E\left[\frac{\boldsymbol{Y}_{t-1}\boldsymbol{Y}'_{t-1}}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right], E\Sigma_0^{12} = E\left\{-\frac{2\eta_t\boldsymbol{Y}_{t-1}}{\delta\sigma_t(\boldsymbol{\alpha})}\left[\log(\sigma_t(\boldsymbol{\alpha})) - \frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})}\right]\right\} = \boldsymbol{0}_{(2p+1)\times 1},\\
E\Sigma_0^{13} &= E\left[\frac{2\epsilon_t(\boldsymbol{\phi})\boldsymbol{Y}_{t-1}\boldsymbol{X}'_{t-1}}{\delta\sigma_t^2(\boldsymbol{\alpha})\sigma_t^\delta(\boldsymbol{\alpha})}\right] = \boldsymbol{0}_{(2p+1)\times(2q+1)},\\
E\Sigma_0^{22} &= E\left\{\frac{2}{\delta^2}\frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})}\left[\frac{1}{\delta}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\right]^2\right\}\\
&\quad - E\left\{\frac{1}{\delta}\left[\frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})} - 1\right]\left[\frac{\boldsymbol{M}_t^2}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}} - \frac{\left(\boldsymbol{M}_t^1\right)^2}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^2} + \frac{2}{\delta^2}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}) - \frac{2}{\delta}\frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\right]\right\}\\
&= \frac{2}{\delta^2}E\left[\frac{1}{\delta}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\right]^2,\\
E\Sigma_0^{23} &= E\left\{\frac{-2\boldsymbol{X}'_{t-1}}{\delta^2\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\left[\frac{1}{\delta}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\right]\right\},\\
E\Sigma_0^{33} &= E\left\{-\frac{\left[1 - \left(1 + \frac{2}{\delta}\right)\frac{\epsilon_t^2(\boldsymbol{\phi})}{\sigma_t^2(\boldsymbol{\alpha})}\right]\boldsymbol{X}_{t-1}\boldsymbol{X}'_{t-1}}{\delta(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^2}\right\} = E\left[\frac{2\boldsymbol{X}_{t-1}\boldsymbol{X}'_{t-1}}{\delta^2(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^2}\right].
\end{aligned}
$$

By (i) in Lemma A.1 and $E\eta_t^3 < \infty, E\eta_t^4 < \infty$, for some positive constant λ, we obtain $\|\Omega_0(i, j)\| < \lambda$, $\|\Sigma_0(i, j)\| < \lambda$, $i, j = 1, 2, 3$. Hence, Ω_0 and Σ_0 are all finite. Denote $\boldsymbol{x} = (\boldsymbol{x}'_1, \boldsymbol{x}_2, \boldsymbol{x}'_3)'$, where $\boldsymbol{x}'_1 \in \mathbb{R}^{2p+1}$, $\boldsymbol{x}_2 \in \mathbb{R}$, $\boldsymbol{x}'_3 \in \mathbb{R}^{2q+1}$ are arbitrary nonzero constant vectors. We calculate $\boldsymbol{x}'\Omega_0\boldsymbol{x}$, and then can obtain $\boldsymbol{x}'\Omega_0\boldsymbol{x} = (\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3)\boldsymbol{x} = \boldsymbol{T}_1\boldsymbol{x}_1 + \boldsymbol{T}_2\boldsymbol{x}_2 + \boldsymbol{T}_3\boldsymbol{x}_3$, where

$$
\begin{aligned}
\boldsymbol{T}_1 &= E\left[\frac{\boldsymbol{x}'_1\boldsymbol{Y}_{t-1}\boldsymbol{Y}'_{t-1}}{\sigma_t^2(\boldsymbol{\alpha})}\right] + \boldsymbol{x}_2 E\left\{\frac{k_1\boldsymbol{Y}'_{t-1}}{\delta\sigma_t(\boldsymbol{\alpha})}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log(\sigma_t(\boldsymbol{\alpha}))\right]\right\} + E\left[\frac{\boldsymbol{x}'_3k_1\boldsymbol{X}_{t-1}\boldsymbol{Y}'_{t-1}}{\sigma_t(\boldsymbol{\alpha})\delta\sigma_t^\delta(\boldsymbol{\alpha})}\right],\\
\boldsymbol{T}_2 &= E\left\{\frac{\boldsymbol{x}'_1k_1\boldsymbol{Y}_{t-1}}{\delta\sigma_t(\boldsymbol{\alpha})}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log(\sigma_t(\boldsymbol{\alpha}))\right]\right\} + E\left\{\frac{\boldsymbol{x}_2k_2}{\delta^2}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log(\sigma_t(\boldsymbol{\alpha}))\right]^2\right\}\\
&\quad + E\left\{\frac{\boldsymbol{x}'_3k_2\boldsymbol{X}_{t-1}}{\delta^2(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log(\sigma_t(\boldsymbol{\alpha}))\right]\right\},\\
\boldsymbol{T}_3 &= E\left[\frac{\boldsymbol{x}'_1k_1\boldsymbol{Y}_{t-1}\boldsymbol{X}'_{t-1}}{\sigma_t(\boldsymbol{\alpha})\delta\sigma_t^\delta(\boldsymbol{\alpha})}\right] + E\left\{\frac{\boldsymbol{x}_2k_2\boldsymbol{X}'_{t-1}}{\delta^2(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log\sigma_t(\boldsymbol{\alpha})\right]\right\} + E\left[\frac{\boldsymbol{x}'_3k_2\boldsymbol{X}_{t-1}\boldsymbol{X}'_{t-1}}{\delta^2(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^2}\right]
\end{aligned}
$$

and then

$$
\begin{aligned}
\boldsymbol{x}'\Omega_0\boldsymbol{x} &= \boldsymbol{T}_1\boldsymbol{x}_1 + \boldsymbol{T}_2\boldsymbol{x}_2 + \boldsymbol{T}_3\boldsymbol{x}_3 \\
&= E\left[\frac{\boldsymbol{x}_1'\boldsymbol{Y}_{t-1}\boldsymbol{Y}_{t-1}'\boldsymbol{x}_1}{\sigma_t^2(\boldsymbol{\alpha})}\right] + E\left\{\frac{\boldsymbol{x}_2 k_1 \boldsymbol{Y}_{t-1}'\boldsymbol{x}_1}{\delta\sigma_t(\boldsymbol{\alpha})}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]\right\} + E\left[\frac{\boldsymbol{x}_3' k_1 \boldsymbol{X}_{t-1}\boldsymbol{Y}_{t-1}'\boldsymbol{x}_1}{\sigma_t(\boldsymbol{\alpha})\delta\sigma_t^\delta(\boldsymbol{\alpha})}\right] \\
&\quad + E\left\{\frac{\boldsymbol{x}_1' k_1 \boldsymbol{Y}_{t-1}\boldsymbol{x}_2}{\delta\sigma_t(\boldsymbol{\alpha})}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]\right\} + E\left\{\frac{\boldsymbol{x}_2 k_2 \boldsymbol{x}_2}{\delta^2}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]^2\right\} \\
&\quad + E\left\{\frac{\boldsymbol{x}_3' k_2 \boldsymbol{X}_{t-1}\boldsymbol{x}_2}{\delta^2\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]\right\} + E\left[\frac{\boldsymbol{x}_1' k_1 \boldsymbol{Y}_{t-1}\boldsymbol{X}_{t-1}'\boldsymbol{x}_3}{\sigma_t(\boldsymbol{\alpha})\delta\sigma_t^\delta(\boldsymbol{\alpha})}\right] \\
&\quad + E\left\{\frac{\boldsymbol{x}_2 k_2 \boldsymbol{X}_{t-1}'\boldsymbol{x}_3}{\delta^2\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log\sigma_t(\boldsymbol{\alpha})\right]\right\} + E\left[\frac{\boldsymbol{x}_3' k_2 \boldsymbol{X}_{t-1}\boldsymbol{X}_{t-1}'\boldsymbol{x}_3}{\delta^2\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)^2}\right] \\
&= E\left[\frac{\left(\boldsymbol{x}_1'\boldsymbol{Y}_{t-1}\right)^2}{\sigma_t^2(\boldsymbol{\alpha})}\right] + E\left\{\frac{2\boldsymbol{x}_2 k_1 \boldsymbol{Y}_{t-1}'\boldsymbol{x}_1}{\delta\sigma_t(\boldsymbol{\alpha})}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]\right\} + E\left[\frac{2\boldsymbol{x}_3' k_1 \boldsymbol{X}_{t-1}\boldsymbol{Y}_{t-1}'\boldsymbol{x}_1}{\sigma_t(\boldsymbol{\alpha})\delta\sigma_t^\delta(\boldsymbol{\alpha})}\right] \\
&\quad + E\left\{\frac{k_2\boldsymbol{x}_2^2}{\delta^2}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]^2\right\} + E\left\{\frac{2k_2\boldsymbol{x}_2\boldsymbol{x}_3'\boldsymbol{X}_{t-1}}{\delta^2\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log\left(\sigma_t(\boldsymbol{\alpha})\right)\right]\right\} \\
&\quad + E\left[\frac{k_2\left(\boldsymbol{x}_3'\boldsymbol{X}_{t-1}\right)^2}{\delta^2\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)^2}\right].
\end{aligned}
$$

Consider that the density function of η_t is symmetric, and then $k_1 = E\eta_t^3 = 0$. Thus

$$
\boldsymbol{x}'\Omega_0\boldsymbol{x} = E\left[\frac{\left(\boldsymbol{x}_1'\boldsymbol{Y}_{t-1}\right)^2}{\sigma_t^2(\boldsymbol{\alpha})}\right] + k_2 E\left\{\frac{\boldsymbol{x}_2}{\delta}\left[\frac{\boldsymbol{M}_t^1}{\sigma_t^\delta(\boldsymbol{\alpha})} - \log\left(\sigma_t(\boldsymbol{\alpha})\right)\right] + \frac{\boldsymbol{x}_3'\boldsymbol{X}_{t-1}}{\delta\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)}\right\}^2 > 0,
$$

hence Ω_0 is positive definite.

Similarly calculate $\boldsymbol{x}'\Sigma_0\boldsymbol{x}$, and then we can obtain $\boldsymbol{x}'\Sigma_0\boldsymbol{x} = (\boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3)\boldsymbol{x}$, where

$$
\begin{aligned}
\boldsymbol{D}_1 &= \boldsymbol{x}_1'E\Sigma_0^{11} + \boldsymbol{x}_2E\Sigma_0^{21} + \boldsymbol{x}_3'E\Sigma_0^{31} = E\left[\frac{\boldsymbol{x}_1'\boldsymbol{Y}_{t-1}\boldsymbol{Y}_{t-1}'}{\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)^{2/\delta}}\right], \\
\boldsymbol{D}_2 &= \boldsymbol{x}_1'E\Sigma_0^{12} + \boldsymbol{x}_2E\Sigma_0^{22} + \boldsymbol{x}_3'E\Sigma_0^{32} \\
&= \frac{2\boldsymbol{x}_2}{\delta^2}E\left[\frac{1}{\delta}\log\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\right]^2 - E\left\{\frac{2\boldsymbol{x}_3'\boldsymbol{X}_{t-1}}{\delta^2\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\left[\frac{1}{\delta}\log\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\right]\right\}, \\
\boldsymbol{D}_3 &= \boldsymbol{x}_1'E\Sigma_0^{13} + \boldsymbol{x}_2E\Sigma_0^{23} + \boldsymbol{x}_3'E\Sigma_0^{33} \\
&= E\left\{\frac{-2\boldsymbol{x}_2\boldsymbol{X}_{t-1}'}{\delta^2\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\left[\frac{1}{\delta}\log\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\right]\right\} - E\left[\frac{-2\boldsymbol{x}_3'\boldsymbol{X}_{t-1}\boldsymbol{X}_{t-1}'}{\delta^2\left(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}\right)^2}\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
\boldsymbol{x}'\Sigma_0\boldsymbol{x} &= \boldsymbol{D}_1\boldsymbol{x}_1 + \boldsymbol{D}_2\boldsymbol{x}_2 + \boldsymbol{D}_3\boldsymbol{x}_3 \\
&= E\left[\frac{\boldsymbol{x}_1'\boldsymbol{Y}_{t-1}\boldsymbol{Y}_{t-1}'\boldsymbol{x}_1}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] + \frac{2\boldsymbol{x}_2^2}{\delta^2}E\left[\frac{1}{\delta}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\right]^2 \\
&\quad + E\left\{\frac{-2\boldsymbol{x}_3'\boldsymbol{X}_{t-1}\boldsymbol{x}_2}{\delta^2\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\left[\frac{1}{\delta}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\right]\right\} + E\left[\frac{2\boldsymbol{x}_3'\boldsymbol{X}_{t-1}\boldsymbol{X}_{t-1}'\boldsymbol{x}_3}{\delta^2(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^2}\right] \\
&\quad - E\left\{\frac{2\boldsymbol{x}_2\boldsymbol{X}_{t-1}'\boldsymbol{x}_3}{\delta^2\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\left[\frac{1}{\delta}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\right]\right\} \\
&= E\left[\frac{\left(\boldsymbol{x}_1'\boldsymbol{Y}_{t-1}\right)^2}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] + \frac{2\boldsymbol{x}_2^2}{\delta^2}E\left[\frac{1}{\delta}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\right]^2 \\
&\quad - E\left\{\frac{4\boldsymbol{x}_2\boldsymbol{x}_3'\boldsymbol{X}_{t-1}}{\delta^2\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\left[\frac{1}{\delta}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\right]\right\} + E\left[\frac{2\left(\boldsymbol{x}_3'\boldsymbol{X}_{t-1}\right)^2}{\delta^2(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^2}\right] \\
&= E\left[\frac{\left(\boldsymbol{x}_1'\boldsymbol{Y}_{t-1}\right)^2}{(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})^{2/\delta}}\right] + 2\left\{\frac{\boldsymbol{x}_2}{\delta}E\left[\frac{1}{\delta}\log(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}) - \frac{\boldsymbol{M}_t^1}{\boldsymbol{\alpha}'\boldsymbol{X}_{t-1}}\right] - E\left[\frac{\left(\boldsymbol{x}_3'\boldsymbol{X}_{t-1}\right)}{\delta(\boldsymbol{\alpha}'\boldsymbol{X}_{t-1})}\right]\right\}^2 \\
&> 0,
\end{aligned}
$$

and then Σ_0 is positive definite, and (iv) holds.

(v) By the Crámer-Wold device and martingale central limit theorem, we can obtain that (v) holds. Thus we complete the proof of Lemma A.1.

Using Lemma A.1 (i), (ii) $\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}}\left|\frac{1}{n}L_n(\boldsymbol{\theta}) - El_t(\boldsymbol{\theta})\right| = o_p(1)$, (iii), and (iv), we have established all the conditions for consistency in Theorem 4.1.1 in Amemiya, T. [24], then and Theorem 2.2 (i) holds. Using Lemma A.1 (ii) $E\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}}\left\|\frac{1}{n}\sum_{t=m+1}^n\left\{\left[\frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right] - E\left[\frac{\partial^2 l_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right]\right\}\right\| = o_p(1)$, we obtain $\frac{1}{n}\sum_{t=m+1}^n\left[\frac{\partial^2 l_t(\widehat{\boldsymbol{\theta}}_n)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right] \to -\Sigma_0$ in probability, for any sequence $\widehat{\boldsymbol{\theta}}_n = \boldsymbol{\theta}_0 + o_p(1)$. Using the Taylor expansion, we obtain $\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) = \Sigma_0^{-1}\frac{1}{\sqrt{n}}\sum_{t=m+1}^n\frac{\partial l_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}} + o_p(1)$. By Lemma A.1 (v) and Theorem 4.1.3 in T. Amemiya [24], we have established all the conditions for the asymptotic normality, hence Theorem 2.2 (ii) holds. Thus we complete the proof of Theorem 2.2. □