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Research article

A fresh look at Volterra integral equations: a fixed point approach in $\mathfrak{F}$-bipolar metric spaces

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Abstract: Our aim of the research article was to obtain fixed point results for generalized contractions equipped with precisely defined control functions in the framework of $\mathfrak{F}$-bipolar metric spaces. As a consequence, some well-known results from the literature were derived as special cases of our leading theorems. To validate the theoretical findings and demonstrate their practical relevance, a non-trivial illustrative example is included. Furthermore, we explored the applicability of our major results by addressing the existence and uniqueness of solutions to Volterra integral equations, a class of problems frequently arising in mathematical modeling of real-world systems. In addition, we extended the utility of our fixed point theorems to homotopy theory, where they served as a foundational tool in analyzing the existence of continuous deformations between mappings, thereby contributing to the study of topological invariants and the structural properties of function spaces.

Keywords: fixed point; $\mathfrak{F}$-bipolar metric space; rational contraction; Volterra integral equation; homotopy theory
Mathematics Subject Classification: 46S40, 47H10, 54H25

1. Introduction

One of the most principal and primary part in fixed point theory is the underlying space as well as the contractive condition and the contractive mapping. In 1905, a French mathematician M. Fréchet [1] instigated the idea of metric space (MS), which performs a constructive role in the first result in this theory. Since then, there have been plenty of generalizations of MSs by eliminating or relaxing some axioms, altering the metric function or extracting the approach. Czerwik [2] replaced a number $s \geq 1$ on the right- side of triangular inequality of MS and presented the thought of b-metric spaces (b-MSs) in 1993. Branciari [3] introduced the groundbreaking concept of generalized metric spaces (GMSs) by modifying the triangle inequality of MS and replacing it with a weaker "quadrilateral inequality" involving three terms on the right-hand side. By relaxing the strict triangle inequality, Branciari encompassed a much broader range of "distances" than traditional metric spaces could handle. This enabled researchers to model and analyze relationships in various contexts that would not fit neatly into the classical metric framework. Jleli et al. [4] satisfied the triangle inequality in a function, which is continuous and established a new MS called $\mathfrak{F}$-metric space ($\mathfrak{F}$-MS). In the above generalizations of MSs, we identify the distance connecting the members of only one set. Hence, an inquiry occurs regarding distance joining the members of two disparate sets can be debated? Similar issues of determining distance can be encountered in distinct domains of the analysis. Moreover, Mutlu et al. [5] initiated a new MS named the bipolar metric space (bip MS) to encounter such problems. Rawat et al. [6] consolidated the above mentioned two creative ideas, especially bip MSs and $\mathfrak{F}$-MSs to initiate the conception of $\mathfrak{F}$-bipolar metric spaces ($\mathfrak{F}$-bip MSs). Alamri [7] harnessed the concept of this new space to secure further significant results in this context.

On the other hand, the Banach Contraction Principle (BCP) [8] stands as a cornerstone of fixed point theory. It asserts that a self-mapping on a complete MS that satisfies a contraction condition possesses a unique fixed point. However, the mapping in BCP is inherently assumed to be continuous. A critical limitation of this principle lies in its inability to accommodate discontinuous self-mappings within its framework. This shortcoming was addressed by Kannan [9], who established a fixed point theorem that does not require continuity of the mapping. Building upon these foundational results, Reich [10] unified the approaches of Banach and Kannan, proving a more general theorem in 1971. Around the same time, Ćirić [11] introduced the concept of generalized contractions, offering further flexibility in the contractive conditions and significantly broadening the scope of fixed point theory. Eventually, Fisher [12] introduced a rational expression into the theory, further enriching the landscape of fixed point results. In the context of modern generalized MSs, Al-Mazrooei et al. [13] extended these ideas by proving fixed point theorems for rational contractions in $\mathfrak{F}$-MSs, thereby highlighting the adaptability of rational contractive conditions in more abstract settings. Within the framework of bip MSs, several researchers have significantly enriched the theory. Gürdal et al. [14] studied fixed points of α-ψ-contractive mappings, while Gaba et al. [15] introduced the concept of (α, BK)-contractions, expanding the control function approach. For more details in this direction, we recommend researchers to consult [16–18].

The interplay between fixed point theory and integral equations extends far beyond pure mathematics, finding substantial applications in modern inverse problems, partial differential equations (PDEs), and image processing. Researchers such as Dong et al. [19] and Lin et al. [20], have applied nonlinear PDE models and level-set methods to imaging and inverse problems, where fixed point theory plays a crucial role in ensuring convergence and solution stability. Zhang and Hofmann [21] developed non-negativity preserving iterative regularization techniques for ill-posed problems, which align naturally with contractive-type mappings. Baravdish et al. [22] explored damped second-order flow models for image denoising, which can be linked to Volterra-type integral formulations analyzed via fixed point methods. Collectively, these studies highlight the synergy between fixed point theory, nonlinear integral equations, and modern applications in PDEs and computational imaging, emphasizing the relevance of generalized MSs such as $\mathfrak{F}$-bip MSs.

In this work, we establish fixed point results for generalized contractions equipped with precisely defined control functions in the setting of $\mathfrak{F}$-bip MSs. Our approach not only broadens the scope of fixed point theory but also encapsulates several well-known results from the literature, which appear as particular instances of the theorems presented herein. To reinforce the significance and applicability of the theoretical framework, we provide a meaningful and non-trivial example. Beyond this, we apply our principal results to discuss the existence and uniqueness of solutions to Volterra integral equations, which are integral to modeling dynamic processes in various scientific domains. Moreover, the developed fixed point techniques are extended to homotopy theory, where they offer a powerful tool for studying continuous transformations between mappings, thereby contributing to a deeper understanding of topological invariants and the geometric structure of function spaces.

2. Preliminaries

The well-known Banach Contraction Principle [8] on complete metric space (CMS) is given as follows.

Theorem 1. *([8]) Let* $T\colon (X,d) \to (X,d)$ *and* (X,d) *be a CMS. If there exists* $\theta \in [0,1)$ *such that*

$$d(T\mathfrak{a}, T\mathfrak{b}) \leq \theta d(\mathfrak{a}, \mathfrak{b}),$$

for all $\mathfrak{a}, \mathfrak{b} \in X$*, then there exists* $\mathfrak{a}^* \in X$ *such that* $T\mathfrak{a}^* = \mathfrak{a}^*$ *that is unique.*

In [9], Kannan proved a result for the mapping, which is not certainly continuous in such way.

Theorem 2. *([9]) Let* $T\colon (X,d) \to (X,d)$ *and* (X,d) *be a CMS. If there exists* $\theta \in [0,\frac{1}{2})$ *such that*

$$d(T\mathfrak{a}, T\mathfrak{b}) \leq \theta\left(d(\mathfrak{a}, T\mathfrak{a}) + d(\mathfrak{b}, T\mathfrak{b})\right),$$

for all $\mathfrak{a}, \mathfrak{b} \in X$*, then there exists* $\mathfrak{a}^* \in X$ *such that* $T\mathfrak{a}^* = \mathfrak{a}^*$ *that is unique.*

Reich [10] merged the above two fixed point results in this way.

Theorem 3. *Let* $T\colon (X,d) \to (X,d)$ *and* (X,d) *be a CMS. If there exist* $\theta_1, \theta_2 \in [0,1)$ *such that* $\theta_1 + 2\theta_2 < 1$ *and*

$$d(T\mathfrak{a}, T\mathfrak{b}) \leq \theta_1 d(\mathfrak{a}, \mathfrak{b}) + \theta_2\left(d(\mathfrak{a}, T\mathfrak{a}) + d(\mathfrak{b}, T\mathfrak{b})\right),$$

for all $\mathfrak{a}, \mathfrak{b} \in X$*, then there exists* $\mathfrak{a}^* \in X$ *such that* $T\mathfrak{a}^* = \mathfrak{a}^*$ *that is unique.*

In [12], Fisher proved the following result in this way.

Theorem 4. *([12]) Let* $T\colon (X,d) \to (X,d)$ *and* (X,d) *be a CMS. If there exist* $\theta_1, \theta_2 \in [0,1)$ *such that* $\theta_1 + \theta_2 < 1$ *and*

$$d(T\mathfrak{a}, T\mathfrak{b}) \leq \theta_1 d(\mathfrak{a}, \mathfrak{b}) + \theta_2 \frac{d(\mathfrak{a}, T\mathfrak{a})\, d(T\mathfrak{b}, \mathfrak{b})}{1 + d(\mathfrak{a}, \mathfrak{b})},$$

for all $\mathfrak{a}, \mathfrak{b} \in X$*, then there exists* $\mathfrak{a}^* \in X$ *such that* $T\mathfrak{a}^* = \mathfrak{a}^*$ *that is unique.*

Jleli et al. [4] presented a compulsive expansion of a MS in this fashion.

Let $\mathcal{F}$ be the set of all continuous mappings $\curlywedge : \mathbb{R}^+ \to \mathbb{R}$. fulfilling these axioms:

($\mathfrak{F}_1$) $\curlywedge(t_1) < \curlywedge(t_2)$, for $t_1 < t_2$,
($\mathfrak{F}_2$) for $\{t_\iota\} \subseteq \mathbb{R}^+$, $\lim_{\iota\to\infty} t_\iota = 0 \Leftrightarrow \lim_{\iota\to\infty} \curlywedge(t_\iota) = -\infty$.

Definition 1. *([4]) Let* $X \neq \emptyset$ *and let* $d : X \times X \to \mathbb{R}^+$*. Suppose that there exists* $(\curlywedge, \beta) \in \mathcal{F} \times \mathbb{R}^+$ *such that for all* $(\mathfrak{a}, \mathfrak{b}) \in X \times X$, *(i)* $d(\mathfrak{a},\mathfrak{b})=0 \Longleftrightarrow \mathfrak{a}=\mathfrak{b}$,

(ii) $d(\mathfrak{a},\mathfrak{b})=d(\mathfrak{b},\mathfrak{a})$,

(iii) for all $(\mathfrak{w}_\iota)_{\iota=1}^{p} \subset X$ *with* $(\mathfrak{w}_1,\mathfrak{w}_p)=(\mathfrak{a},\mathfrak{b})$*, we have*

$$d(\mathfrak{a},\mathfrak{b})>0 \Rightarrow \curlywedge(d(\mathfrak{a},\mathfrak{b})) \leq \curlywedge\left(\sum_{\iota=1}^{p-1} d\left(\mathfrak{w}_\iota,\mathfrak{w}_{\iota+1}\right)\right)+\beta,$$

where $p \geq 2$*. Then* (X,d) *is professed to be an* $\mathfrak{F}$*-MS.*

Mutlu et al. [5] pioneered the bip MS concept with this perspective.

Definition 2. *([5]) Consider* $X \neq \emptyset$ *and* $Y \neq \emptyset$ *and* $d: X \times Y \to \mathbb{R}^+$ *be a mapping. If the mapping* d *verifies*

(bi$_1$*)* $d(\mathfrak{a},\mathfrak{b})=0 \Longleftrightarrow \mathfrak{a}=\mathfrak{b}$,

(bi$_2$*)* $d(\mathfrak{a},\mathfrak{b})=d(\mathfrak{b},\mathfrak{a})$*, if* $\mathfrak{a},\mathfrak{b} \in X \cap Y$,

(bi$_3$*)* $d(\mathfrak{a},\mathfrak{b}) \leq d(\mathfrak{a},\mathfrak{b}^/)+d(\mathfrak{a}^/,\mathfrak{b}^/)+d(\mathfrak{a}^/,\mathfrak{b})$,

for all $(\mathfrak{a},\mathfrak{b}),(\mathfrak{a}^/,\mathfrak{b}^/) \in X \times Y$*. Then* (X,Y,d) *is recognized as bip MS.*

Definition 3. *([5]) Consider* (X_1,Y_1,d_1) *and* (X_2,Y_2,d_2) *be two bip MS. A mapping* $T:(X_1,Y_1,d_1) \rightrightarrows (X_2,Y_2,d_2)$ *is called a covariant, if* $T(X_1) \subseteq X_2$ *and* $T(Y_1) \subseteq Y_2$*. Likewise, a mapping* $T:(X_1,Y_1,d_1) \rightleftarrows (X_2,Y_2,d_2)$ *is contravariant, if* $T(X_1) \subseteq Y_2$ *and* $T(X_2) \subseteq Y_1$.

Rawat et al. [6] synthesized the aforementioned inventive ideas, especially bip MS and $\mathfrak{F}$-MS and initiated the thought of $\mathfrak{F}$-bip MS as follows:

Definition 4. *([6]) Let* $X \neq \emptyset$ *and* $Y \neq \emptyset$ *and let* $d: X \times Y \to [0,+\infty)$ *and there exists* $(\curlywedge,\beta) \in \mathcal{F} \times \mathbb{R}^+$ *such that for all* $(\mathfrak{a},\mathfrak{b}) \in X \times Y$,

(D$_1$*)* $d(\mathfrak{a},\mathfrak{b})=0 \Longleftrightarrow \mathfrak{a}=\mathfrak{b}$,

(D$_2$*)* $d(\mathfrak{a},\mathfrak{b})=d(\mathfrak{b},\mathfrak{a})$*, if* $\mathfrak{a},\mathfrak{b} \in X \cap Y$,

(D$_3$*) for all* $(\mathfrak{w}_\iota)_{\iota=1}^{p} \subset X$ *and* $(\varkappa_\iota)_{\iota=1}^{p} \subset Y$ *with* $(\mathfrak{w}_1,\varkappa_p)=(\mathfrak{a},\mathfrak{b})$*, we have*

$$d(\mathfrak{a},\mathfrak{b})>0 \text{ implies } \curlywedge(d(\mathfrak{a},\mathfrak{b})) \leq \curlywedge\left(\sum_{\iota=1}^{p-1} d\left(\mathfrak{w}_{\iota+1},\varkappa_\iota\right)+\sum_{\iota=1}^{p} d\left(\mathfrak{w}_\iota,\varkappa_\iota\right)\right)+\beta,$$

where $p \geq 2$*, then* (X,Y,d) *is professed to be an* $\mathfrak{F}$*-bip MS.*

Remark 1. *([6]) Considering* $Y=X$, $p=2\iota$, $\mathfrak{w}_\jmath=\mathfrak{w}_{2\jmath-1}$ *and* $\varkappa_\jmath=\mathfrak{w}_{2\jmath}$ *in Definition 4, we obtain a sequence* $\left(\mathfrak{w}_\jmath\right)_{\jmath=1}^{2\iota} \in X$ *with* $(\mathfrak{w}_1,\mathfrak{w}_{2\iota})=(\mathfrak{a},\mathfrak{b})$ *such that axiom (iii) of Definition 1 hold. Therefore, each* $\mathfrak{F}$*-MS qualifies as an* $\mathfrak{F}$*-bipolar MS, though the reverse is not applicable.*

Definition 5. *([6]) Let* (X,Y,d) *be an* $\mathfrak{F}$*-bip MS.*

(i) A sequence $(\mathfrak{a}_\iota,\mathfrak{b}_\iota)$ *on* $X \times Y$ *is called a bisequence.*

(ii) If $(\mathfrak{a}_\iota)$ *and* $(\mathfrak{b}_\iota)$ *are convergent, then* $(\mathfrak{a}_\iota,\mathfrak{b}_\iota)$ *is also convergent, and if* $(\mathfrak{a}_\iota)$ *and* $(\mathfrak{b}_\iota)$ *converge to a same point, then* $(\mathfrak{a}_\iota,\mathfrak{b}_\iota)$ *is said to be biconvergent.*

(iii) A bisequence $(\mathfrak{a}_\iota,\mathfrak{b}_\iota)$ *in* (X,Y,d) *is termed as Cauchy bisequence, if for any given* $\epsilon>0$*, there exists* $\iota_0 \in \mathbb{N}$ *in order to* $d\left(\mathfrak{a}_\iota,\mathfrak{b}_p\right)<\epsilon$*, for* $\iota,p \geq \iota_0$.

Definition 6. *([6]) An* $\mathfrak{F}$*-bip MS* (X,Y,d) *is denoted as complete when each Cauchy bisequence within* (X,Y,d) *converges.*

3. Major results

Throughout this section, we depict a complete $\mathfrak{F}$-bip MS by (X, Y, d).

Theorem 5. *Let* $T : (X, Y, d) \rightleftarrows (X, Y, d)$ *be contravariant and continuous mapping. If there exist the mappings* $\theta_1, \theta_2, \theta_3 : (X \cup Y) \times (X \cup Y) \to [0, 1)$ *such that*

(a) $\theta_1 (T\mathfrak{a}, \mathfrak{b}) \leq \theta_1 (\mathfrak{a}, \mathfrak{b})$ *and* $\theta_1 (\mathfrak{a}, T\mathfrak{b}) \leq \theta_1 (\mathfrak{a}, \mathfrak{b})$,
$\theta_2 (T\mathfrak{a}, \mathfrak{b}) \leq \theta_2 (\mathfrak{a}, \mathfrak{b})$ *and* $\theta_2 (\mathfrak{a}, T\mathfrak{b}) \leq \theta_2 (\mathfrak{a}, \mathfrak{b})$,
$\theta_3 (T\mathfrak{a}, \mathfrak{b}) \leq \theta_3 (\mathfrak{a}, \mathfrak{b})$ *and* $\theta_3 (\mathfrak{a}, T\mathfrak{b}) \leq \theta_3 (\mathfrak{a}, \mathfrak{b})$,
(b) $\theta_1 (\mathfrak{a}, \mathfrak{b}) + \theta_2 (\mathfrak{a}, \mathfrak{b}) + \theta_3 (\mathfrak{a}, \mathfrak{b}) < 1$,
(c)

$$d (T\mathfrak{b}, T\mathfrak{a}) \leq \theta_1 (\mathfrak{a}, \mathfrak{b}) d (\mathfrak{a}, \mathfrak{b}) + \theta_2 (\mathfrak{a}, \mathfrak{b}) d (\mathfrak{a}, T\mathfrak{a}) + \theta_3 (\mathfrak{a}, \mathfrak{b}) d (T\mathfrak{b}, \mathfrak{b}), \tag{3.1}$$

then there exists a unique point $\mathfrak{a}^* \in X \cup Y$ *such that* $T\mathfrak{a}^* = \mathfrak{a}^*$.

Proof. Let $\mathfrak{a}_0 \in X$. Define the bisequence $(\mathfrak{a}_\iota, \mathfrak{b}_\iota)$ in (X, Y, d) by

$$\mathfrak{b}_\iota = T\mathfrak{a}_\iota \text{ and } \mathfrak{a}_{\iota+1} = T\mathfrak{b}_\iota,$$

for all $\iota \in \mathbb{N}$. By the inequality (3.1), we have

$$\begin{aligned}
d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) &= d (T\mathfrak{b}_{\iota-1}, T\mathfrak{a}_\iota) \\
&\leq \theta_1 (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2 (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, T\mathfrak{a}_\iota) + \theta_3 (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) d (T\mathfrak{b}_{\iota-1}, \mathfrak{b}_{\iota-1}) \\
&= \theta_1 (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2 (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_3 (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) \\
&= \theta_1 (T\mathfrak{b}_{\iota-1}, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2 (T\mathfrak{b}_{\iota-1}, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_3 (T\mathfrak{b}_{\iota-1}, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) \\
&\leq \theta_1 (\mathfrak{b}_{\iota-1}, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2 (\mathfrak{b}_{\iota-1}, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_3 (\mathfrak{b}_{\iota-1}, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) \\
&= \theta_1 (T\mathfrak{a}_{\iota-1}, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2 (T\mathfrak{a}_{\iota-1}, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_3 (T\mathfrak{a}_{\iota-1}, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}),
\end{aligned}$$

which yields

$$\begin{aligned}
d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) &\leq \theta_1 (\mathfrak{a}_{\iota-1}, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2 (\mathfrak{a}_{\iota-1}, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_3 (\mathfrak{a}_{\iota-1}, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) \\
&\leq \cdots \leq \theta_1 (\mathfrak{a}_0, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2 (\mathfrak{a}_0, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_3 (\mathfrak{a}_0, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}).
\end{aligned}$$

This further entails that

$$\begin{aligned}
d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) &\leq \theta_1 (\mathfrak{a}_0, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2 (\mathfrak{a}_0, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_3 (\mathfrak{a}_0, \mathfrak{b}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) \\
&= \theta_1 (\mathfrak{a}_0, T\mathfrak{a}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2 (\mathfrak{a}_0, T\mathfrak{a}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_3 (\mathfrak{a}_0, T\mathfrak{a}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}),
\end{aligned}$$

that is,

$$\begin{aligned}
d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) &\leq \theta_1 (\mathfrak{a}_0, \mathfrak{a}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2 (\mathfrak{a}_0, \mathfrak{a}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_3 (\mathfrak{a}_0, \mathfrak{a}_{\iota-1}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) \\
&= \theta_1 (\mathfrak{a}_0, T\mathfrak{b}_{\iota-2}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2 (\mathfrak{a}_0, T\mathfrak{b}_{\iota-2}) d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_3 (\mathfrak{a}_0, T\mathfrak{b}_{\iota-2}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) \\
&\leq \theta_1 (\mathfrak{a}_0, \mathfrak{b}_{\iota-2}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2 (\mathfrak{a}_0, \mathfrak{b}_{\iota-2}) d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_3 (\mathfrak{a}_0, \mathfrak{b}_{\iota-2}) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) \\
&\leq \cdots \leq \theta_1 (\mathfrak{a}_0, \mathfrak{b}_0) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2 (\mathfrak{a}_0, \mathfrak{b}_0) d (\mathfrak{a}_\iota, \mathfrak{b}_\iota) \theta_3 (\mathfrak{a}_0, \mathfrak{b}_0) d (\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}).
\end{aligned}$$

It yields that

$$d(\mathfrak{a}_\iota,\mathfrak{b}_\iota)\leq\frac{\theta_1(\mathfrak{a}_0,\mathfrak{b}_0)+\theta_3(\mathfrak{a}_0,\mathfrak{b}_0)}{1-\theta_2(\mathfrak{a}_0,\mathfrak{b}_0)}d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}). \tag{3.2}$$

Moreover,

$$\begin{aligned}
d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) &= d(T\mathfrak{b}_{\iota-1},T\mathfrak{a}_{\iota-1})\\
&\leq \theta_1(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_2(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},T\mathfrak{a}_{\iota-1})+\theta_3(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})d(T\mathfrak{b}_{\iota-1},\mathfrak{b}_{\iota-1})\\
&= \theta_1(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_2(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_3(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})\\
&= \theta_1(T\mathfrak{b}_{\iota-2},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_2(T\mathfrak{b}_{\iota-2},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_3(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}),
\end{aligned}$$

which implies

$$\begin{aligned}
d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) &\leq \theta_1(\mathfrak{b}_{\iota-2},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_2(\mathfrak{b}_{\iota-2},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_3(\mathfrak{b}_{\iota-2},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})\\
&= \theta_1(T\mathfrak{a}_{\iota-2},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_2(T\mathfrak{a}_{\iota-2},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_3(T\mathfrak{a}_{\iota-2},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})\\
&\leq \theta_1(\mathfrak{a}_{\iota-2},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_2(\mathfrak{a}_{\iota-2},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_3(T\mathfrak{a}_{\iota-2},\mathfrak{b}_{\iota-1})d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})\\
&\leq \cdots\leq\theta_1(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_2(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_3(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}),
\end{aligned}$$

signifying that

$$\begin{aligned}
d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) &\leq \theta_1(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_2(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_3(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})\\
&= \theta_1(\mathfrak{a}_0,T\mathfrak{a}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_2(\mathfrak{a}_0,T\mathfrak{a}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_3(\mathfrak{a}_0,T\mathfrak{a}_{\iota-1})d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})\\
&\leq \theta_1(\mathfrak{a}_0,\mathfrak{a}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_2(\mathfrak{a}_0,\mathfrak{a}_{\iota-1})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_3(\mathfrak{a}_0,\mathfrak{a}_{\iota-1})d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})\\
&= \theta_1(\mathfrak{a}_0,T\mathfrak{b}_{\iota-2})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_2(\mathfrak{a}_0,T\mathfrak{b}_{\iota-2})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_3(\mathfrak{a}_0,T\mathfrak{b}_{\iota-2})d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}),
\end{aligned}$$

which implies

$$\begin{aligned}
d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) &\leq \theta_1(\mathfrak{a}_0,\mathfrak{b}_{\iota-2})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_2(\mathfrak{a}_0,\mathfrak{b}_{\iota-2})d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_3(\mathfrak{a}_0,\mathfrak{b}_{\iota-2})d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})\\
&\leq \cdots\leq\theta_1(\mathfrak{a}_0,\mathfrak{b}_0)d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_2(\mathfrak{a}_0,\mathfrak{b}_0)d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})+\theta_3(\mathfrak{a}_0,\mathfrak{b}_0)d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}),
\end{aligned}$$

implying definitively that

$$d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})\leq\frac{\theta_1(\mathfrak{a}_0,\mathfrak{b}_0)+\theta_2(\mathfrak{a}_0,\mathfrak{b}_0)}{1-\theta_3(\mathfrak{a}_0,\mathfrak{b}_0)}d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}). \tag{3.3}$$

Taking $\mu=\max\left\{\frac{\theta_1(\mathfrak{a}_0,\mathfrak{b}_0)+\theta_3(\mathfrak{a}_0,\mathfrak{b}_0)}{1-\theta_2(\mathfrak{a}_0,\mathfrak{b}_0)},\frac{\theta_1(\mathfrak{a}_0,\mathfrak{b}_0)+\theta_2(\mathfrak{a}_0,\mathfrak{b}_0)}{1-\theta_3(\mathfrak{a}_0,\mathfrak{b}_0)}\right\}<1$. Therefore by (3.2) and (3.3), we have

$$d(\mathfrak{a}_\iota,\mathfrak{b}_\iota)\leq\mu^{2\iota}d(\mathfrak{a}_0,\mathfrak{b}_0). \tag{3.4}$$

Likewise, we get

$$d(\mathfrak{a}_{\iota+1},\mathfrak{b}_\iota)\leq\mu^{2\iota+1}d(\mathfrak{a}_0,\mathfrak{b}_0), \tag{3.5}$$

for all $\iota\in\mathbb{N}$. Under condition (D_3) for $(\curlywedge,\beta)\in\mathcal{F}\times\mathbb{R}^+$ and a chosen $\epsilon>0$, a $\delta>0$ exists with the property that

$$0<t<\delta\implies\curlywedge(t)<\curlywedge(\epsilon)-\beta. \tag{3.6}$$

By the inequalities (3.4) and (3.5), we get

$$\begin{aligned}
&\sum_{\iota=\iota}^{p-1} d(\mathfrak{a}_{\iota+1}, \mathfrak{b}_\iota) + \sum_{\iota=\iota}^{p} d(\mathfrak{a}_\iota, \mathfrak{b}_\iota) \\
\leq\ & \left(\mu^{2\iota} + \mu^{2\iota+2} + ... + \mu^{2p}\right) d(\mathfrak{a}_0, \mathfrak{b}_0) + \left(\mu^{2\iota+1} + \mu^{2\iota+3} + ... + \mu^{2p-1}\right) d(\mathfrak{a}_0, \mathfrak{b}_0) \\
\leq\ & \mu^{2\iota} \sum_{\iota=0}^{\infty} \mu^\iota d(\mathfrak{a}_0, \mathfrak{b}_0) = \frac{\mu^{2\iota}}{1-\mu} d(\mathfrak{a}_0, \mathfrak{b}_0),
\end{aligned}$$

for $p > \iota$. Since $\lim_{\iota\to\infty} \frac{\mu^{2\iota}}{1-\mu} d(\mathfrak{a}_0, \mathfrak{b}_0) = 0$, so there exists $\iota_0 \in \mathbb{N}$, such that

$$0 < \frac{\mu^{2\iota}}{1-\mu} d(\mathfrak{a}_0, \mathfrak{b}_0) < \delta,$$

for $\iota \geq \iota_0$. Combining $(\mathfrak{F}_1)$ and (3.6) for $p > \iota \geq \iota_0$ gives us

$$\curlywedge\left(\sum_{\iota=\iota}^{p-1} d(\mathfrak{a}_{\iota+1}, \mathfrak{b}_\iota) + \sum_{\iota=\iota}^{p} d(\mathfrak{a}_\iota, \mathfrak{b}_\iota)\right) \leq \curlywedge\left(\frac{\mu^{2\iota}}{1-\mu} d(\mathfrak{a}_0, \mathfrak{b}_0)\right) < \curlywedge(\epsilon) - \beta. \tag{3.7}$$

By (D_3) and Eq (3.7), we get that $d\left(\mathfrak{a}_\iota, \mathfrak{b}_p\right) > 0$, yields

$$\curlywedge\left(d\left(\mathfrak{a}_\iota, \mathfrak{b}_p\right)\right) \leq \curlywedge\left(\sum_{\iota=\iota}^{p-1} d(\mathfrak{a}_{\iota+1}, \mathfrak{b}_\iota) + \sum_{\iota=\iota}^{p} d(\mathfrak{a}_\iota, \mathfrak{b}_\iota)\right) + \beta < \curlywedge(\epsilon).$$

Likewise, for $\iota > p \geq \iota_0$, $d\left(\mathfrak{a}_\iota, \mathfrak{b}_p\right) > 0$ yields

$$\curlywedge\left(d\left(\mathfrak{a}_\iota, \mathfrak{b}_p\right)\right) \leq \curlywedge\left(\sum_{\iota=p}^{\iota-1} d(\mathfrak{a}_{\iota+1}, \mathfrak{b}_\iota) + \sum_{\iota=p}^{\iota} d(\mathfrak{a}_\iota, \mathfrak{b}_\iota)\right) + \beta < \curlywedge(\epsilon).$$

Hence, by $(\mathfrak{F}_1)$, $d\left(\mathfrak{a}_\iota, \mathfrak{b}_p\right) < \epsilon$, for all $p, \iota \geq \iota_0$. Therefore, $(\mathfrak{a}_\iota, \mathfrak{b}_\iota)$ is a Cauchy bisequence in (X, Y, d). By the completeness of (X, Y, d), we have $(\mathfrak{a}_\iota, \mathfrak{b}_\iota) \to \mathfrak{a}^* \in X \cap Y$. Thus, $(\mathfrak{a}_\iota) \to \mathfrak{a}^*$, $(\mathfrak{b}_\iota) \to \mathfrak{a}^*$. Also, since T is continuous, we have

$$(\mathfrak{a}_\iota) \to \mathfrak{a}^* \implies (\mathfrak{b}_\iota) = (T\mathfrak{a}_\iota) \to T\mathfrak{a}^*.$$

Also, given that $(\mathfrak{b}_\iota)$ converges to a unique limit $\mathfrak{a}^*$ within the intersection of X and Y, then $T\mathfrak{a}^* = \mathfrak{a}^*$. Consequently, T possesses a fixed point. □

Now, assuming T possesses an additional fixed point σ, then $T\sigma = \sigma$ implies that $\sigma \in X \cup Y$. Thus,

$$d(\mathfrak{a}^*, \sigma) = d(T\mathfrak{a}^*, T\sigma) \leq \theta_1(\sigma, \mathfrak{a}^*)\, d(\sigma, \mathfrak{a}^*) + \theta_2(\sigma, \mathfrak{a}^*)\, d(\sigma, T\sigma) + \theta_3(\sigma, \mathfrak{a}^*)\, d(T\mathfrak{a}^*, \mathfrak{a}^*) = \theta_1(\sigma, \mathfrak{a}^*)\, d(\sigma, \mathfrak{a}^*)$$

is contradiction, except $\mathfrak{a}^* = \sigma$.

Example 1. *Let* $X = \mathbb{N} \cup \{0\}$ *and* $Y = \frac{1}{n} \cup \{0\}$. *Define* $d : X \times Y \to [0, \infty)$ *as follows:*

$$d(\mathfrak{a}, \mathfrak{b}) = \begin{cases} 0, \ \textit{if} \ (\mathfrak{a}, \mathfrak{b}) = \left(1, \frac{1}{2}\right), \\ |\mathfrak{a} - \mathfrak{b}|, \ \textit{otherwise.} \end{cases}$$

Then, (X, Y, d) *is complete* $\mathfrak{F}$*-bip MS for* $\curlywedge(t) = \ln t$, $t > 0$ *and* $\beta > 3$. *Define* $T : X \cup Y \rightleftarrows X \cup Y$ *by*

$$T(\mathfrak{a}) = \begin{cases} \frac{1}{1+\mathfrak{a}}, \ \textit{if} \ \mathfrak{a} \in X, \\ \frac{1-\mathfrak{a}}{\mathfrak{a}}, \ \textit{if} \ \mathfrak{a} \in Y \backslash \{0\}, \\ 0, \quad \textit{if} \ \mathfrak{a} = 0. \end{cases}$$

Then, T is contravariant mapping. Define $\theta_1, \theta_2, \theta_3 : (X \cup Y) \times (X \cup Y) \to [0, 1)$ *as follows:*

$$\theta_1(\mathfrak{a}, \mathfrak{b}) = \frac{5}{6},$$

$$\theta_2(\mathfrak{a}, \mathfrak{b}) = \frac{1}{9 + \max\{|\mathfrak{a}|, |\mathfrak{b}|\}},$$

and

$$\theta_3(\mathfrak{a}, \mathfrak{b}) = \frac{1}{40 + \max\{|\mathfrak{a}|, |\mathfrak{b}|\}}.$$

It is very simple to satisfy all the assumptions of Theorem 5. Hence, by Theorem 5, $T(0) = 0$.

Corollary 1. *Let* $T : (X, Y, d) \rightleftarrows (X, Y, d)$ *be contravariant and continuous mapping. If there exist* $\theta_1, \theta_2, \theta_3 : (X \cup Y) \to [0, 1)$ *such that*

(a) $\theta_1(T\mathfrak{a}) \leq \theta_1(\mathfrak{a})$ *and* $\theta_1(T\mathfrak{b}) \leq \theta_1(\mathfrak{b})$, $\theta_2(T\mathfrak{a}) \leq \theta_2(\mathfrak{a})$ *and* $\theta_2(T\mathfrak{b}) \leq \theta_2(\mathfrak{b})$, $\theta_3(T\mathfrak{a}) \leq \theta_3(\mathfrak{a})$ *and* $\theta_3(T\mathfrak{b}) \leq \theta_3(\mathfrak{b})$,

(b) $\theta_1(\mathfrak{a}) + \theta_2(\mathfrak{a}) + \theta_3(\mathfrak{a}) < 1$,

(c)

$$d(T\mathfrak{b}, T\mathfrak{a}) \leq \theta_1(\mathfrak{a}) d(\mathfrak{a}, \mathfrak{b}) + \theta_2(\mathfrak{a}) d(\mathfrak{a}, T\mathfrak{a}) + \theta_3(\mathfrak{a}) d(T\mathfrak{b}, \mathfrak{b}),$$

then there exists a unique point $\mathfrak{a}^* \in X \cup Y$ *such that* $T\mathfrak{a}^* = \mathfrak{a}^*$.

Proof. Define $\theta_1, \theta_2, \theta_3 : (X \cup Y) \times (X \cup Y) \to [0, 1)$ be

$$\theta_1(\mathfrak{a}, \mathfrak{b}) = \theta_1(\mathfrak{a}), \ \theta_2(\mathfrak{a}, \mathfrak{b}) = \theta_2(\mathfrak{a}) \ \text{and} \ \theta_3(\mathfrak{a}, \mathfrak{b}) = \theta_3(\mathfrak{a}).$$

Then, for all $(\mathfrak{a}, \mathfrak{b}) \in (X \cup Y) \times (X \cup Y)$, we have

(a)

$$\theta_1(T\mathfrak{a}, \mathfrak{b}) = \theta_1(T\mathfrak{a}) \leq \theta_1(\mathfrak{a}) = \theta_1(\mathfrak{a}, \mathfrak{b})$$

$$\theta_2(T\mathfrak{a}, \mathfrak{b}) = \theta_2(T\mathfrak{a}) \leq \theta_2(\mathfrak{a}) = \theta_2(\mathfrak{a}, \mathfrak{b})$$

$$\theta_3(T\mathfrak{a}, \mathfrak{b}) = \theta_3(T\mathfrak{a}) \leq \theta_3(\mathfrak{a}) = \theta_3(\mathfrak{a}, \mathfrak{b}).$$

(b) $\theta_1(\mathfrak{a}, \mathfrak{b}) + \theta_2(\mathfrak{a}, \mathfrak{b}) + \theta_3(\mathfrak{a}, \mathfrak{b}) = \theta_1(\mathfrak{a}) + \theta_2(\mathfrak{a}) + \theta_3(\mathfrak{a}) < 1$,

(c)

$$\begin{aligned} d(T\mathfrak{b}, T\mathfrak{a}) &\leq \theta_1(\mathfrak{a}) d(\mathfrak{a}, \mathfrak{b}) + \theta_2(\mathfrak{a}) d(\mathfrak{a}, T\mathfrak{a}) + \theta_3(\mathfrak{a}) d(T\mathfrak{b}, \mathfrak{b}) \\ &= \theta_1(\mathfrak{a}, \mathfrak{b}) d(\mathfrak{a}, \mathfrak{b}) + \theta_2(\mathfrak{a}, \mathfrak{b}) d(\mathfrak{a}, T\mathfrak{a}) + \theta_3(\mathfrak{a}, \mathfrak{b}) d(T\mathfrak{b}, \mathfrak{b}). \end{aligned}$$

By Theorem 5, T has a unique fixed point. □

Corollary 2. *Let* $T : (X, Y, d) \rightleftarrows (X, Y, d)$ *be contravariant and continuous mapping. If there exist* $\theta_1, \theta_2, \theta_3 \in [0, 1)$ *such that* $\theta_1 + \theta_2 + \theta_3 < 1$ *and*

$$d(T\mathfrak{b}, T\mathfrak{a}) \leq \theta_1 d(\mathfrak{a}, \mathfrak{b}) + \theta_2 d(\mathfrak{a}, T\mathfrak{a}) + \theta_3 d(T\mathfrak{b}, \mathfrak{b}),$$

for all $(\mathfrak{a}, \mathfrak{b}) \in (X \cup Y) \times (X \cup Y)$*, then there exists a unique point* $\mathfrak{a}^* \in X \cup Y$ *such that* $T\mathfrak{a}^* = \mathfrak{a}^*$.

Proof. Take $\theta_1(\cdot) = \theta_1$, $\theta_2(\cdot) = \theta_2$ and $\theta_3(\cdot) = \theta_3$ in Corollary 1. □

Corollary 3. *([6]) Let* $T : (X, Y, d) \rightleftarrows (X, Y, d)$ *be a contravariant and continuous mapping. If there exists* $\theta \in [0, 1)$ *such that*

$$d(T\mathfrak{b}, T\mathfrak{a}) \leq \theta d(\mathfrak{a}, \mathfrak{b}),$$

for all $(\mathfrak{a}, \mathfrak{b}) \in (X \cup Y) \times (X \cup Y)$*, then there exists a unique point* $\mathfrak{a}^* \in X \cup Y$ *such that* $T\mathfrak{a}^* = \mathfrak{a}^*$.

Proof. Take $\theta_1 = \theta$, $\theta_2 = \theta_3 = 0$ in Corollary 2. □

Remark 2. *If we put* $\curlywedge(t) = \ln(t)$ *and* $\beta = 0$ *in Corollary 3, then the notion of* $\mathfrak{F}$*-bip MS is restricted to notion of bip MS, and we acquire the prime theorem of Mutlu et al. [5] as an immediate conclusion.*

Remark 3. *If we take* $X = Y$ *in the result Corollary 3, then the idea of* $\mathfrak{F}$*-bip MS is decreased to the thought of* $\mathfrak{F}$*-MS and we deduce the principal theorem of Jleli et al. [4].*

Theorem 6. *Let* $T : (X, Y, d) \rightleftarrows (X, Y, d)$ *be a contravariant and continuous mapping. If there exist the mappings* $\theta_1, \theta_2 : (X \cup Y) \times (X \cup Y) \rightarrow [0, 1)$ *such that*

(a) $\theta_1(T\mathfrak{a}, \mathfrak{b}) \leq \theta_1(\mathfrak{a}, \mathfrak{b})$ *and* $\theta_1(\mathfrak{a}, T\mathfrak{b}) \leq \theta_1(\mathfrak{a}, \mathfrak{b})$, $\theta_2(T\mathfrak{a}, \mathfrak{b}) \leq \theta_2(\mathfrak{a}, \mathfrak{b})$ *and* $\theta_2(\mathfrak{a}, T\mathfrak{b}) \leq \theta_2(\mathfrak{a}, \mathfrak{b})$,

(b) $\theta_1(\mathfrak{a}, \mathfrak{b}) + \theta_2(\mathfrak{a}, \mathfrak{b}) < 1$,

(c)

$$d(T\mathfrak{b}, T\mathfrak{a}) \leq \theta_1(\mathfrak{a}, \mathfrak{b}) d(\mathfrak{a}, \mathfrak{b}) + \theta_2(\mathfrak{a}, \mathfrak{b}) \frac{d(\mathfrak{a}, T\mathfrak{a}) d(T\mathfrak{b}, \mathfrak{b})}{1 + d(\mathfrak{a}, \mathfrak{b})}, \tag{3.8}$$

for all $(\mathfrak{a}, \mathfrak{b}) \in (X \cup Y) \times (X \cup Y)$*, then there exists a unique point* $\mathfrak{a}^* \in X \cup Y$ *such that* $T\mathfrak{a}^* = \mathfrak{a}^*$.

Proof. Let $\mathfrak{a}_0 \in X$ and $\mathfrak{b}_0 \in Y$. Define the bisequence $(\mathfrak{a}_\iota, \mathfrak{b}_\iota)$ in (X, Y, d) by

$$\mathfrak{b}_\iota = T\mathfrak{a}_\iota \text{ and } \mathfrak{a}_{\iota+1} = T\mathfrak{b}_\iota,$$

for all $\iota \in \mathbb{N}$. From (3.8), we get

$$\begin{aligned}
d(\mathfrak{a}_\iota, \mathfrak{b}_\iota) &= d(T\mathfrak{b}_{\iota-1}, T\mathfrak{a}_\iota) \\
&\leq \theta_1(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) d(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) \frac{d(\mathfrak{a}_\iota, T\mathfrak{a}_\iota) d(T\mathfrak{b}_{\iota-1}, \mathfrak{b}_{\iota-1})}{1 + d(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1})} \\
&= \theta_1(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) d(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) \frac{d(\mathfrak{a}_\iota, \mathfrak{b}_\iota) d(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1})}{1 + d(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1})} \\
&\leq \theta_1(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) d(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) d(\mathfrak{a}_\iota, \mathfrak{b}_\iota) \\
&= \theta_1(T\mathfrak{b}_{\iota-1}, \mathfrak{b}_{\iota-1}) d(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2(T\mathfrak{b}_{\iota-1}, \mathfrak{b}_{\iota-1}) d(\mathfrak{a}_\iota, \mathfrak{b}_\iota) \\
&\leq \theta_1(\mathfrak{b}_{\iota-1}, \mathfrak{b}_{\iota-1}) d(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{b}_{\iota-1}, \mathfrak{b}_{\iota-1}) d(\mathfrak{a}_\iota, \mathfrak{b}_\iota) \\
&= \theta_1(T\mathfrak{a}_{\iota-1}, \mathfrak{b}_{\iota-1}) d(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) + \theta_2(T\mathfrak{a}_{\iota-1}, \mathfrak{b}_{\iota-1}) d(\mathfrak{a}_\iota, \mathfrak{b}_\iota)
\end{aligned}$$

$$\begin{aligned}
&\leq \theta_1(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) \\
&\leq \cdots \leq \theta_1(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota).
\end{aligned}$$

This further strengthens the conclusion that

$$\begin{aligned}
d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) &\leq \theta_1(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) \\
&= \theta_1(\mathfrak{a}_0,T\mathfrak{a}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,T\mathfrak{a}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) \\
&\leq \theta_1(\mathfrak{a}_0,\mathfrak{a}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{a}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) \\
&= \theta_1(\mathfrak{a}_0,T\mathfrak{b}_{\iota-2})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,T\mathfrak{b}_{\iota-2})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) \\
&\leq \theta_1(\mathfrak{a}_0,\mathfrak{b}_{\iota-2})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_{\iota-2})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) \\
&\leq \cdots \leq \theta_1(\mathfrak{a}_0,\mathfrak{b}_0)\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_0)\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota),
\end{aligned}$$

implying definitively that

$$d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) \leq \frac{\theta_1(\mathfrak{a}_0,\mathfrak{b}_0)}{1-\theta_2(\mathfrak{a}_0,\mathfrak{b}_0)} d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}). \tag{3.9}$$

Moreover,

$$\begin{aligned}
d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) &= d(T\mathfrak{b}_{\iota-1},T\mathfrak{a}_{\iota-1}) \\
&\leq \theta_1(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) \frac{d(\mathfrak{a}_{\iota-1},T\mathfrak{a}_{\iota-1})\, d(T\mathfrak{b}_{\iota-1},\mathfrak{b}_{\iota-1},)}{1+d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})} \\
&= \theta_1(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) \frac{d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})}{1+d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})},
\end{aligned}$$

which yields that

$$\begin{aligned}
d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) &\leq \theta_1(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})\, d(\mathfrak{b}_{\iota-1},\mathfrak{a}_\iota) \\
&= \theta_1(T\mathfrak{b}_{\iota-2},\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) + \theta_2(T\mathfrak{b}_{\iota-2},\mathfrak{b}_{\iota-1})\, d(\mathfrak{b}_{\iota-1},\mathfrak{a}_\iota) \\
&\leq \theta_1(\mathfrak{b}_{\iota-2},\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{b}_{\iota-2},\mathfrak{b}_{\iota-1})\, d(\mathfrak{b}_{\iota-1},\mathfrak{a}_\iota) \\
&= \theta_1(T\mathfrak{a}_{\iota-2},\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) + \theta_2(T\mathfrak{a}_{\iota-2},\mathfrak{b}_{\iota-1})\, d(\mathfrak{b}_{\iota-1},\mathfrak{a}_\iota) \\
&\leq \theta_1(\mathfrak{a}_{\iota-2},\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_{\iota-2},\mathfrak{b}_{\iota-1})\, d(\mathfrak{b}_{\iota-1},\mathfrak{a}_\iota) \\
&\leq \cdots \leq \theta_1(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})\, d(\mathfrak{b}_{\iota-1},\mathfrak{a}_\iota),
\end{aligned}$$

which further contributes to the evidence that

$$\begin{aligned}
d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) &\leq \theta_1(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})\, d(\mathfrak{b}_{\iota-1},\mathfrak{a}_\iota) \\
&= \theta_1(\mathfrak{a}_0,T\mathfrak{a}_{\iota-1})\, d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,T\mathfrak{a}_{\iota-1})\, d(\mathfrak{b}_{\iota-1},\mathfrak{a}_\iota) \\
&\leq \theta_1(\mathfrak{a}_0,\mathfrak{a}_{\iota-1})\, d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{a}_{\iota-1})\, d(\mathfrak{b}_{\iota-1},\mathfrak{a}_\iota) \\
&= \theta_1(\mathfrak{a}_0,T\mathfrak{b}_{\iota-2})\, d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,T\mathfrak{b}_{\iota-2})\, d(\mathfrak{b}_{\iota-1},\mathfrak{a}_\iota) \\
&\leq \theta_1(\mathfrak{a}_0,\mathfrak{b}_{\iota-2})\, d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_{\iota-2})\, d(\mathfrak{b}_{\iota-1},\mathfrak{a}_\iota) \\
&\leq \cdots \leq \theta_1(\mathfrak{a}_0,\mathfrak{b}_0)\, d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_0)\, d(\mathfrak{b}_{\iota-1},\mathfrak{a}_\iota).
\end{aligned}$$

Hence,

$$d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) \leq \frac{\theta_1(\mathfrak{a}_0,\mathfrak{b}_0)}{1-\theta_2(\mathfrak{a}_0,\mathfrak{b}_0)} d(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1}). \tag{3.10}$$

Now, if we take $\frac{\theta_1(\mathfrak{a}_0,\mathfrak{b}_0)}{1-\theta_2(\mathfrak{a}_0,\mathfrak{b}_0)} = \mu$, then (3.9) and (3.10) becomes

$$d(\mathfrak{a}_\iota, \mathfrak{b}_\iota) \leq \mu d(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}), \tag{3.11}$$

and

$$d(\mathfrak{a}_\iota, \mathfrak{b}_{\iota-1}) \leq \mu d(\mathfrak{a}_{\iota-1}, \mathfrak{b}_{\iota-1}). \tag{3.12}$$

Applying both (3.11) and (3.12), we can infer that

$$d(\mathfrak{a}_\iota, \mathfrak{b}_\iota) \leq \mu^{2\iota} d(\mathfrak{a}_0, \mathfrak{b}_0). \tag{3.13}$$

Likewise,

$$d(\mathfrak{a}_{\iota+1}, \mathfrak{b}_\iota) \leq \mu^{2\iota+1} d(\mathfrak{a}_0, \mathfrak{b}_0), \tag{3.14}$$

for all $\iota \in \mathbb{N}$. For any fixed $\epsilon > 0$ and a pair $(\curlywedge, \beta) \in \mathcal{F} \times \mathbb{R}^+$ satisfying (D_3), we can find a $\delta > 0$ such that

$$0 < t < \delta \implies \curlywedge(t) < \curlywedge(\epsilon) - \beta. \tag{3.15}$$

From (3.13) and (3.14), we get

$$\begin{aligned} & \sum_{\iota=\iota}^{p-1} d(\mathfrak{a}_{\iota+1}, \mathfrak{b}_\iota) + \sum_{\iota=\iota}^{p} d(\mathfrak{a}_\iota, \mathfrak{b}_\iota) \\ \leq\ & \left(\mu^{2\iota} + \mu^{2\iota+2} + ... + \mu^{2p}\right) d(\mathfrak{a}_0, \mathfrak{b}_0) + \left(\mu^{2\iota+1} + \mu^{2\iota+3} + ... + \mu^{2p-1}\right) d(\mathfrak{a}_0, \mathfrak{b}_0) \\ \leq\ & \mu^{2\iota} \sum_{\iota=0}^{\infty} \mu^\iota d(\mathfrak{a}_0, \mathfrak{b}_0) = \frac{\mu^{2\iota}}{1-\mu} d(\mathfrak{a}_0, \mathfrak{b}_0), \end{aligned}$$

for $p > \iota$. Since $\lim_{\iota\to\infty} \frac{\mu^{2\iota}}{1-\mu} d(\mathfrak{a}_0, \mathfrak{b}_0) = 0$, so there exists $\iota_0 \in \mathbb{N}$, such that

$$0 < \frac{\mu^{2\iota}}{1-\mu} d(\mathfrak{a}_0, \mathfrak{b}_0) < \delta,$$

for $\iota \geq \iota_0$. Therefore, for $p > \iota \geq \iota_0$, employing $(\mathfrak{F}_1)$ and inequality (3.15), we obtain

$$\curlywedge\left(\sum_{\iota=\iota}^{p-1} d(\mathfrak{a}_{\iota+1}, \mathfrak{b}_\iota) + \sum_{\iota=\iota}^{p} d(\mathfrak{a}_\iota, \mathfrak{b}_\iota)\right) \leq \curlywedge\left(\frac{\mu^{2\iota}}{1-\mu} d(\mathfrak{a}_0, \mathfrak{b}_0)\right) < \curlywedge(\epsilon) - \beta. \tag{3.16}$$

By (D_3) and Eq (3.16), we get that $d\left(\mathfrak{a}_\iota, \mathfrak{b}_p\right) > 0$ implies

$$\curlywedge\left(d\left(\mathfrak{a}_\iota, \mathfrak{b}_p\right)\right) \leq \curlywedge\left(\sum_{\iota=\iota}^{p-1} d(\mathfrak{a}_{\iota+1}, \mathfrak{b}_\iota) + \sum_{\iota=\iota}^{p} d(\mathfrak{a}_\iota, \mathfrak{b}_\iota)\right) + \beta < \curlywedge(\epsilon).$$

Likewise, for $\iota > p \geq \iota_0$, $d\left(\mathfrak{a}_\iota, \mathfrak{b}_p\right) > 0$ yields

$$\curlywedge\left(d\left(\mathfrak{a}_\iota, \mathfrak{b}_p\right)\right) \leq \curlywedge\left(\sum_{\iota=p}^{\iota-1} d(\mathfrak{a}_{\iota+1}, \mathfrak{b}_\iota) + \sum_{\iota=p}^{\iota} d(\mathfrak{a}_\iota, \mathfrak{b}_\iota)\right) + \beta < \curlywedge(\epsilon).$$

Subsequently, according to $(\mathfrak{F}_1)$, $d\left(\mathfrak{a}_\iota, \mathfrak{b}_p\right) < \epsilon$, for all $p, \iota \geq \iota_0$. Therefore, $(\mathfrak{a}_\iota, \mathfrak{b}_\iota)$ constitutes a Cauchy bisequence in (X, Y, d). By the completeness of (X, Y, d), so $(\mathfrak{a}_\iota, \mathfrak{b}_\iota) \longrightarrow \mathfrak{a}^* \in X \cap Y$. So $(\mathfrak{a}_\iota) \to \mathfrak{a}^*, (\mathfrak{b}_\iota) \to \mathfrak{a}^*$. Also, since T is continuous, we get

$$(\mathfrak{a}_\iota) \to \mathfrak{a}^* \implies (\mathfrak{b}_\iota) = (T\mathfrak{a}_\iota) \to T\mathfrak{a}^*.$$

Also, since $(\mathfrak{b}_\iota)$ has a limit $\mathfrak{a}^*$ in $X \cap Y$. Hence, $T\mathfrak{a}^* = \mathfrak{a}^*$. So T has a fixed point. □

Let $T\mathfrak{a}' = \mathfrak{a}' \neq \mathfrak{a}^* = T\mathfrak{a}^*$. Then,

$$d\left(\mathfrak{a}^*, \mathfrak{a}'\right) = d\left(T\mathfrak{a}^*, T\mathfrak{a}'\right) \leq \theta_1\left(\mathfrak{a}', \mathfrak{a}^*\right) d\left(\mathfrak{a}', \mathfrak{a}^*\right) + \theta_2\left(\mathfrak{a}', \mathfrak{a}^*\right) \frac{d\left(\mathfrak{a}', T\mathfrak{a}'\right) d\left(T\mathfrak{a}^*, \mathfrak{a}^*\right)}{1 + d\left(\mathfrak{a}', \mathfrak{a}^*\right)} = \theta_1\left(\mathfrak{a}', \mathfrak{a}^*\right) d\left(\mathfrak{a}', \mathfrak{a}^*\right),$$

which is a contradiction, except $\mathfrak{a}^* = \mathfrak{a}'$.

Corollary 4. *Let $T : (X, Y, d) \rightleftarrows (X, Y, d)$ be contravariant and continuous mapping. If there exist the mappings $\theta_1, \theta_2 : (X \cup Y) \to [0, 1)$ such that*

(a) $\theta_1(T\mathfrak{a}) \leq \theta_1(\mathfrak{a})$ *and* $\theta_1(T\mathfrak{b}) \leq \theta_1(\mathfrak{b})$,
$\theta_2(T\mathfrak{a}) \leq \theta_2(\mathfrak{a})$ *and* $\theta_2(T\mathfrak{b}) \leq \theta_2(\mathfrak{b})$,
(b) $\theta_1(\mathfrak{a}) + \theta_2(\mathfrak{a}) < 1$,
(c)

$$d(T\mathfrak{b}, T\mathfrak{a}) \leq \theta_1(\mathfrak{a}) d(\mathfrak{a}, \mathfrak{b}) + \theta_2(\mathfrak{a}) \frac{d(\mathfrak{a}, T\mathfrak{a}) d(T\mathfrak{b}, \mathfrak{b})}{1 + d(\mathfrak{a}, \mathfrak{b})},$$

for all $(\mathfrak{a}, \mathfrak{b}) \in (X \cup Y) \times (X \cup Y)$, then there exists a unique point $\mathfrak{a}^* \in X \cup Y$ such that $T\mathfrak{a}^* = \mathfrak{a}^*$.

Proof. Define $\theta_1, \theta_2 : (X \cup Y) \times (X \cup Y) \to [0, 1)$ be

$$\theta_1(\mathfrak{a}, \mathfrak{b}) = \theta_1(\mathfrak{a}) \text{ and } \theta_2(\mathfrak{a}, \mathfrak{b}) = \theta_2(\mathfrak{a}).$$

Then, for all $(\mathfrak{a}, \mathfrak{b}) \in (X \cup Y) \times (X \cup Y)$, we have

(a)

$$\theta_1(T\mathfrak{a}, \mathfrak{b}) = \theta_1(T\mathfrak{a}) \leq \theta_1(\mathfrak{a}) = \theta_1(\mathfrak{a}, \mathfrak{b}),$$

$$\theta_2(T\mathfrak{a}, \mathfrak{b}) = \theta_2(T\mathfrak{a}) \leq \theta_2(\mathfrak{a}) = \theta_2(\mathfrak{a}, \mathfrak{b}),$$

(b) $\theta_1(\mathfrak{a}, \mathfrak{b}) + \theta_2(\mathfrak{a}, \mathfrak{b}) = \theta_1(\mathfrak{a}) + \theta_2(\mathfrak{a}) < 1$,
(c)

$$d(T\mathfrak{b}, T\mathfrak{a}) \leq \theta_1(\mathfrak{a}) d(\mathfrak{a}, \mathfrak{b}) + \theta_2(\mathfrak{a}) \frac{d(\mathfrak{a}, T\mathfrak{a}) d(T\mathfrak{b}, \mathfrak{b})}{1 + d(\mathfrak{a}, \mathfrak{b})} = \theta_1(\mathfrak{a}, \mathfrak{b}) d(\mathfrak{a}, \mathfrak{b}) + \theta_2(\mathfrak{a}, \mathfrak{b}) \frac{d(\mathfrak{a}, T\mathfrak{a}) d(T\mathfrak{b}, \mathfrak{b})}{1 + d(\mathfrak{a}, \mathfrak{b})}.$$

By Theorem 6, T has a unique fixed point. □

Corollary 5. *([7]) Let $T : (X, Y, d) \rightleftarrows (X, Y, d)$ be contravariant and continuous mapping. If there exist the constants $\theta_1, \theta_2 \in [0, 1)$ such that $\theta_1 + \theta_2 < 1$ and*

$$d(T\mathfrak{b}, T\mathfrak{a}) \leq \theta_1 d(\mathfrak{a}, \mathfrak{b}) + \theta_2 \frac{d(\mathfrak{a}, T\mathfrak{a}) d(T\mathfrak{b}, \mathfrak{b})}{1 + d(\mathfrak{a}, \mathfrak{b})},$$

for all $(\mathfrak{a}, \mathfrak{b}) \in (X \cup Y) \times (X \cup Y)$, then there exists a unique point $\mathfrak{a}^* \in X \cup Y$ such that $T\mathfrak{a}^* = \mathfrak{a}^*$.

Proof. Take $\theta_1(\cdot) = \theta_1$ and $\theta_2(\cdot) = \theta_2$ in Corollary 4. □

Remark 4. *If $X = Y$ in above Corollary, then $\mathfrak{F}$-bip MS is converted to $\mathfrak{F}$-MS, and we obtain a key theorem that extends the results of Ahmad et al. [13] as an outcome of our main result.*

4. Application

4.1. Integral equations

Integral equations, particularly those with Volterra-type kernels, have long presented intriguing challenges in various fields. In this subsection, we embark on a journey to unlock solutions for such an equation, wielding the powerful tool of rational contractions within the elegant realm of $\mathfrak{F}$-bip MS. By navigating the unique properties of contravariant mappings in this setting, we not only demonstrate the existence of a solution but potentially also its uniqueness. This approach offers both theoretical rigor and practical applicability, promising valuable insights into the equation's behavior and solution.

Beyond mathematical intrigue, Volterra integral equations are intricately woven into the fabric of signal processing, a discipline that breathes life into our technological world. Imagine the bustling cityscape of information, filled with the murmurs of digital signals carrying voices, music, and countless other data streams. Volterra integral equations act as meticulous tailors in this vibrant space, shaping and reshaping these signals to our needs. Moreover, their applicability extends to inverse problems in partial differential equations, as demonstrated by Shcheglov et al. [23], and to ecological modeling, such as predator-prey dynamics, through nonlinear mixed Volterra-Fredholm integral equations in complex-valued suprametric spaces, as investigated by Abdou [24].

Consider a Volterra integral equation

$$\varphi(\mathfrak{t}) = g(\mathfrak{t}) + \int_{X\cup Y} K(\mathfrak{t}, \mathfrak{b}, \varphi(\mathfrak{t}))d\mathfrak{b}, \tag{4.1}$$

represents a mathematical model used to describe the behavior of a system when processing signals. Let us break down the components of this equation:

- $\varphi(\mathfrak{t})$ is the output signal, representing the response of the system at time t,
- $g(\mathfrak{t})$ is the input signal or excitation applied to the system at time t,
- $\int_{X\cup Y} K(\mathfrak{t}, \mathfrak{b}, \varphi(\mathfrak{t}))d\mathfrak{b}$ is the integral term, which involves a kernel function $K(\mathfrak{t}, \mathfrak{b}, \varphi(\mathfrak{t}))$ and is integrated over a Lebesgue measurable set $X \cup Y$, $g(\mathfrak{t})$ is a given function and $\int_{X\cup Y} K(\mathfrak{a}, \mathfrak{b}, \varphi(\mathfrak{t}))d\mathfrak{b}$ is an integral term involving a kernel function $K(\mathfrak{a}, \mathfrak{b}, \varphi(\mathfrak{t}))$. The domain of integration is $X \cup Y$. This integral captures the memory or history-dependent effects of the system.

In signal processing, the Volterra integral equation is often used to model nonlinear systems. The kernel function $K(\mathfrak{a}, \mathfrak{b}, \varphi(\mathfrak{t}))$ describes how the current output at time t depends not only on the current input $g(\mathfrak{t})$ but also on the past values of the output $\varphi(\mathfrak{t})$. This memory effect is particularly useful for capturing the behavior of systems that exhibit nonlinear and memory-dependent responses.

Theorem 7. *Consider the Volterra integral Eq (4.1). If the underlying conditions are met:*

(i) There is a continuous function $\Upsilon : X^2 \cup Y^2 \to [0,\infty)$ *and control functions* $\theta_1, \theta_2, \theta_3 : \mathfrak{L}^\infty(Y) \times \mathfrak{L}^\infty(Y) \to [0, 1)$ *such that*

$$|K(\mathfrak{t}, \mathfrak{b}, \varphi(\mathfrak{b})) - K(\mathfrak{t}, \mathfrak{b}, \phi(\mathfrak{b}))| \leq \Upsilon(\mathfrak{t}, \mathfrak{b}) \left\{ \begin{array}{c} \theta_1(\phi(\mathfrak{b}), \varphi(\mathfrak{b})) |\phi(\mathfrak{b}) - \varphi(\mathfrak{b})| \\ +\theta_2(\phi(\mathfrak{b}), \varphi(\mathfrak{b})) |\phi(\mathfrak{b}) - \mathfrak{A}\phi(\mathfrak{b})| \\ +\theta_3(\phi(\mathfrak{b}), \varphi(\mathfrak{b})) |\mathfrak{A}\varphi(\mathfrak{b}) - \varphi(\mathfrak{b})| \end{array} \right\},$$

for all $\mathfrak{t}, \mathfrak{b} \in \left(X^2 \cup Y^2\right)$, $\mathfrak{A} : \mathfrak{L}^\infty(X) \cup \mathfrak{L}^\infty(Y) \to \mathfrak{L}^\infty(X) \cup \mathfrak{L}^\infty(Y)$,

(ii) $\left\|\int_{X\cup Y}\Upsilon(\mathfrak{t},\mathfrak{b})\,d\mathfrak{b}\right\|\leq 1$, *i.e.* $\sup_{\mathfrak{t}\in X\cup Y}\int_{X\cup Y}|\Upsilon(\mathfrak{t},\mathfrak{b})|\,d\mathfrak{b}\leq 1$,
(iii) $\theta_1(\mathfrak{A}\phi(\mathfrak{b}),\varphi(\mathfrak{b}))\leq\theta_1(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))$ *and* $\theta_1(\phi(\mathfrak{b}),\mathfrak{A}\varphi(\mathfrak{b}))\leq\theta_1(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))$,
$\theta_2(\mathfrak{A}\phi(\mathfrak{b}),\varphi(\mathfrak{b}))\leq\theta_2(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))$ *and* $\theta_2(\phi(\mathfrak{b}),\mathfrak{A}\varphi(\mathfrak{b}))\leq\theta_2(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))$,
$\theta_3(\mathfrak{A}\phi(\mathfrak{b}),\varphi(\mathfrak{b}))\leq\theta_3(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))$ *and* $\theta_3(\phi(\mathfrak{b}),\mathfrak{A}\varphi(\mathfrak{b}))\leq\theta_3(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))$,
(iv) $\theta_1(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))+\theta_2(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))+\theta_3(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))<1$.

Under these conditions, the integral Eq (4.1) possesses a unique solution within the combined space $\mathfrak{L}^\infty(X)\cup\mathfrak{L}^\infty(Y)$.

Proof. Consider $\mathfrak{X}$ as $\mathfrak{L}^\infty(X)$ and $\mathfrak{Z}$ as $\mathfrak{L}^\infty(Y)$, both being normed linear spaces, with X and Y representing Lebesgue measureable sets and $m(X\cup Y)<\infty$. Define $d:\mathfrak{X}\times\mathfrak{Z}\to\mathbb{R}^+$ by

$$d(\mathfrak{b},\mathfrak{a})=\|\mathfrak{b}-\mathfrak{a}\|_\infty,$$

for all $\mathfrak{b},\mathfrak{a}\in\mathfrak{X}\times\mathfrak{Z}$. Then $(\mathfrak{X},\mathfrak{Z},d)$ is complete $\mathfrak{F}$-bip MS and $\mathfrak{J}:\mathfrak{X}\cup\mathfrak{Z}\to\mathfrak{X}\cup\mathfrak{Z}$ is defined by

$$\mathfrak{J}(\varphi(\mathfrak{t}))=g(\mathfrak{t})+\int_{X\cup Y}K(\mathfrak{t},\mathfrak{b},\varphi(\mathfrak{t}))d\mathfrak{b},$$

for $\mathfrak{t}\in X\cup Y$. Now

$$\begin{aligned}
d(\mathfrak{J}(\varphi(\mathfrak{t})),\mathfrak{J}(\phi(\mathfrak{t}))) &= \|\mathfrak{J}(\varphi(\mathfrak{t}))-\mathfrak{J}(\phi(\mathfrak{t}))\|\\
&= \left|\int_{X\cup Y}K(\mathfrak{t},\mathfrak{b},\varphi(\mathfrak{t}))d\mathfrak{b}-\int_{X\cup Y}K(\mathfrak{t},\mathfrak{b},\phi(\mathfrak{t}))d\mathfrak{b}\right|\\
&\leq \int_{X\cup Y}|K(\mathfrak{t},\mathfrak{b},\varphi(\mathfrak{t}))-K(\mathfrak{t},\mathfrak{b},\phi(\mathfrak{t}))|\,d\mathfrak{b}\\
&\leq \int_{X\cup Y}\Upsilon(\mathfrak{t},\mathfrak{b})\left\{\begin{array}{c}\theta_1(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))\|\phi(\mathfrak{b})-\varphi(\mathfrak{b})\|\\+\theta_2(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))\|\phi(\mathfrak{b})-\mathfrak{J}\phi(\mathfrak{b})\|\\+\theta_3(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))\|\mathfrak{J}\varphi(\mathfrak{b})-\varphi(\mathfrak{b})\|\end{array}\right\}d\mathfrak{b}\\
&\leq \left\{\begin{array}{c}\theta_1(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))\|\phi(\mathfrak{b})-\varphi(\mathfrak{b})\|\\+\theta_2(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))\|\phi(\mathfrak{b})-\mathfrak{J}\phi(\mathfrak{b})\|\\+\theta_3(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))\|\mathfrak{J}\varphi(\mathfrak{b})-\varphi(\mathfrak{b})\|\end{array}\right\}\int_{X\cup Y}|\Upsilon(\mathfrak{t},\mathfrak{b})|\,d\mathfrak{b}\\
&\leq \left\{\begin{array}{c}\theta_1(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))\|\phi(\mathfrak{b})-\varphi(\mathfrak{b})\|\\+\theta_2(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))\|\phi(\mathfrak{b})-\mathfrak{J}\phi(\mathfrak{b})\|\\+\theta_3(\phi(\mathfrak{b}),\varphi(\mathfrak{b}))\|\mathfrak{J}\varphi(\mathfrak{b})-\varphi(\mathfrak{b})\|\end{array}\right\}\sup_{\mathfrak{t}\in X\cup Y}\int_{X\cup Y}|\Upsilon(\mathfrak{t},\mathfrak{b})|\,d\mathfrak{b}\\
&\leq \left\{\begin{array}{c}\theta_1(\phi,\varphi)\|\phi-\varphi\|\\+\theta_2(\phi,\varphi)\|\phi-\mathfrak{J}\phi\|\\+\theta_3(\phi,\varphi)\|\mathfrak{J}\varphi-\varphi\|\end{array}\right\}\\
&= \theta_1(\phi,\varphi)\,d(\phi,\varphi)+\theta_2(\phi,\varphi)\,d(\phi,\mathfrak{J}(\phi))+\theta_3(\phi,\varphi)\,d(\mathfrak{J}(\varphi),\varphi).
\end{aligned}$$

Hence, by Theorem 5, the operator $\mathfrak{J}$ has a unique solution in $\mathfrak{X}\cup\mathfrak{Z}$. □

4.2. Homotopy

The concept of fixed points in homotopy theory extends the classical notion from algebraic topology to a broader and more abstract setting. Fixed points arise when considering continuous mappings or transformations between spaces that have a degree of symmetry or self-mapping properties. In the context of homotopy theory, fixed points become particularly intriguing when exploring homotopy classes of mappings, and they serve as key elements in characterizing the structure and behavior of topological spaces.

Theorem 8. *Let* $(\mathfrak{X}, \mathfrak{Z})$ *is a pair of open subset of* (X, Y) *and* $\left(\overline{\mathfrak{X}}, \overline{\mathfrak{Z}}\right)$ *is a pair of closed subset of* (X, Y) *and* $(\mathfrak{X}, \mathfrak{Z}) \subseteq \left(\overline{\mathfrak{X}}, \overline{\mathfrak{Z}}\right)$. *Suppose* $\mathcal{E} : \left(\overline{\mathfrak{X}} \cup \overline{\mathfrak{Z}}\right) \times [0, 1] \to X \cup Y$ *is a function fulfiling the conditions:*

(h1) $\mathfrak{a} \neq \mathcal{E}(\mathfrak{a}, q)$ *for each* $\mathfrak{a} \in \partial\mathfrak{X} \cup \partial\mathfrak{Z}$ *and* $q \in [0, 1]$, *where* $\partial\mathfrak{X}$ *and* $\partial\mathfrak{Z}$ *represents the differential of* $\mathfrak{X}$ *and* $\mathfrak{Z}$ *likewise,*

(h2) for all $\mathfrak{a} \in \overline{\mathfrak{X}}$, $\mathfrak{b} \in \overline{\mathfrak{Z}}$ *and* $q \in [0, 1]$

$$d(\mathcal{E}(\mathfrak{b}, q), \mathcal{E}(\mathfrak{a}, q)) \leq \theta_1(\mathfrak{a}, \mathfrak{b})\, d(\mathfrak{a}, \mathfrak{b}) + \theta_2(\mathfrak{a}, \mathfrak{b}) \frac{d(\mathfrak{a}, \mathcal{E}(\mathfrak{a}, q))\, d(\mathcal{E}(\mathfrak{b}, q), \mathfrak{b})}{1 + d(\mathfrak{a}, \mathfrak{b})},$$

where $\theta_1, \theta_2 : \left(\overline{\mathfrak{X}} \cup \overline{\mathfrak{Z}}\right) \times \left(\overline{\mathfrak{X}} \cup \overline{\mathfrak{Z}}\right) \to [0, 1)$,

(h3) there exists $\aleph \geq 0$ *such that*

$$d(\mathcal{E}(\mathfrak{a}, r), \mathcal{E}(\mathfrak{b}, \tau)) \leq \aleph\, |r - \tau|,$$

for all $\mathfrak{a}, \mathfrak{b} \in \left(\overline{\mathfrak{X}} \cup \overline{\mathfrak{Z}}\right)$ *and* $r, \tau \in [0, 1]$,

(h4) $\theta_1(\mathcal{E}(\mathfrak{a}, r), \mathfrak{b}) \leq \theta_1(\mathfrak{a}, \mathfrak{b})$ *and* $\theta_1(\mathfrak{a}, \mathcal{E}(\mathfrak{b}, \tau)) \leq \theta_1(\mathfrak{a}, \mathfrak{b})$, $\theta_2(\mathcal{E}(\mathfrak{a}, r), \mathfrak{b}) \leq \theta_2(\mathfrak{a}, \mathfrak{b})$ *and* $\theta_2(\mathfrak{a}, \mathcal{E}(\mathfrak{b}, \tau)) \leq \theta_2(\mathfrak{a}, \mathfrak{b})$, *and* $\theta_1(\mathfrak{a}, \mathfrak{b}) + \theta_2(\mathfrak{a}, \mathfrak{b}) < 1$.

A fixed point for $\mathcal{E}(\cdot, 0)$ is a necessary and sufficient condition for $\mathcal{E}(\cdot, 1)$ to have a fixed point as well.

Proof. Let

$$\Theta_1 = \{\pi \in [0, 1] : \mathfrak{a} = \mathcal{E}(\mathfrak{a}, \pi), \mathfrak{a} \in \mathfrak{X}\},$$

and

$$\Theta_2 = \{\tau \in [0, 1] : \mathfrak{b} = \mathcal{E}(\mathfrak{b}, \tau), \mathfrak{b} \in \mathfrak{Z}\}.$$

As $\mathcal{E}(\cdot, 0)$ has a fixed point in $\mathfrak{X} \cup \mathfrak{Z}$, then we obtain $0 \in \Theta_1 \cap \Theta_2$. Thus, $\Theta_1 \cap \Theta_2 \neq \emptyset$. Now, we prove that $\Theta_1 \cap \Theta_2$ is both open and closed in $[0, 1]$. Thus, $\Theta_1 = \Theta_2 = [0, 1]$ by connectedness. Let $(\{\pi_\iota\}_{\iota=1}^{\infty}), (\{\tau_\iota\}_{\iota=1}^{\infty}) \subseteq (\Theta_1, \Theta_2)$ with $(\pi_\iota, \tau_\iota) \to (\rho, \rho) \in [0, 1]$ as $\iota \to \infty$. Now we assert that $\rho \in \Theta_1 \cap \Theta_2$. As $(\pi_\iota, \tau_\iota) \in \Theta_1 \cap \Theta_2$, for $\iota \in \mathbb{N} \cup \{0\}$. Thus, there is $(\mathfrak{a}_\iota, \mathfrak{b}_\iota) \in (\mathfrak{X}, \mathfrak{Z})$ such that $\mathfrak{b}_\iota = \mathcal{E}(\mathfrak{a}_\iota, \pi_\iota)$ and $\mathfrak{a}_{\iota+1} = \mathcal{E}(\mathfrak{b}_\iota, \tau_\iota)$. Also, we get

$$\begin{aligned}
d(\mathfrak{a}_{\iota+1}, \mathfrak{b}_\iota) &= d(\mathcal{E}(\mathfrak{b}_\iota, \tau_\iota), \mathcal{E}(\mathfrak{a}_\iota, \pi_\iota)) \\
&\leq \theta_1(\mathfrak{a}_\iota, \mathfrak{b}_\iota)\, d(\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_2(\mathfrak{a}_\iota, \mathfrak{b}_\iota) \frac{d(\mathfrak{a}_\iota, \mathcal{E}(\mathfrak{a}_\iota, \pi_\iota))\, d(\mathcal{E}(\mathfrak{b}_\iota, \tau_\iota), \mathfrak{b}_\iota)}{1 + d(\mathfrak{a}_\iota, \mathfrak{b}_\iota)} \\
&= \theta_1(\mathfrak{a}_\iota, \mathfrak{b}_\iota)\, d(\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_2(\mathfrak{a}_\iota, \mathfrak{b}_\iota) \frac{d(\mathfrak{a}_\iota, \mathfrak{b}_\iota)\, d(\mathfrak{a}_{\iota+1}, \mathfrak{b}_\iota)}{1 + d(\mathfrak{a}_\iota, \mathfrak{b}_\iota)} \\
&\leq \theta_1(\mathfrak{a}_\iota, \mathfrak{b}_\iota)\, d(\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_2(\mathfrak{a}_\iota, \mathfrak{b}_\iota)\, d(\mathfrak{a}_{\iota+1}, \mathfrak{b}_\iota) \\
&= \theta_1(\mathcal{E}(\mathfrak{b}_{\iota-1}, \tau_{\iota-1}), \mathfrak{b}_\iota)\, d(\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_2(\mathcal{E}(\mathfrak{b}_{\iota-1}, \tau_{\iota-1}), \mathfrak{b}_\iota)\, d(\mathfrak{a}_{\iota+1}, \mathfrak{b}_\iota) \\
&\leq \theta_1(\mathfrak{b}_{\iota-1}, \mathfrak{b}_\iota)\, d(\mathfrak{a}_\iota, \mathfrak{b}_\iota) + \theta_2(\mathfrak{b}_{\iota-1}, \mathfrak{b}_\iota)\, d(\mathfrak{a}_{\iota+1}, \mathfrak{b}_\iota)
\end{aligned}$$

$$\begin{aligned}
&= \theta_1(\mathcal{E}(\mathfrak{a}_{\iota-1},\pi_{\iota-1}),\mathfrak{b}_\iota)\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) + \theta_2(\mathcal{E}(\mathfrak{a}_{\iota-1},\pi_{\iota-1}),\mathfrak{b}_\iota)\, d(\mathfrak{a}_{\iota+1},\mathfrak{b}_\iota)\\
&\le \theta_1(\mathfrak{a}_{\iota-1},\mathfrak{b}_\iota)\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) + \theta_2(\mathfrak{a}_{\iota-1},\mathfrak{b}_\iota)\, d(\mathfrak{a}_{\iota+1},\mathfrak{b}_\iota)\\
&\le \cdots \le \theta_1(\mathfrak{a}_0,\mathfrak{b}_\iota)\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_\iota)\, d(\mathfrak{a}_{\iota+1},\mathfrak{b}_\iota),
\end{aligned}$$

signifying that

$$\begin{aligned}
d(\mathfrak{a}_{\iota+1},\mathfrak{b}_\iota) &\le \theta_1(\mathfrak{a}_0,\mathfrak{b}_\iota)\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_\iota)\, d(\mathfrak{a}_{\iota+1},\mathfrak{b}_\iota)\\
&= \theta_1(\mathfrak{a}_0,\mathcal{E}(\mathfrak{a}_\iota,\pi_\iota))\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) + \theta_2(\mathfrak{a}_0,\mathcal{E}(\mathfrak{a}_\iota,\pi_\iota))\, d(\mathfrak{a}_{\iota+1},\mathfrak{b}_\iota)\\
&\le \theta_1(\mathfrak{a}_0,\mathfrak{a}_\iota)\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) + \theta_2(\mathfrak{a}_0,\mathfrak{a}_\iota)\, d(\mathfrak{a}_{\iota+1},\mathfrak{b}_\iota)\\
&= \theta_1(\mathfrak{a}_0,\mathcal{E}(\mathfrak{b}_{\iota-1},\tau_{\iota-1}))\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) + \theta_2(\mathfrak{a}_0,\mathcal{E}(\mathfrak{b}_{\iota-1},\tau_{\iota-1}))\, d(\mathfrak{a}_{\iota+1},\mathfrak{b}_\iota)\\
&\le \theta_1(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_{\iota+1},\mathfrak{b}_\iota)\\
&\le \cdots \le \theta_1(\mathfrak{a}_0,\mathfrak{b}_0)\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_0)\, d(\mathfrak{a}_{\iota+1},\mathfrak{b}_\iota),
\end{aligned}$$

that is

$$d(\mathfrak{a}_{\iota+1},\mathfrak{b}_\iota) \le \frac{\theta_1(\mathfrak{a}_0,\mathfrak{b}_0)}{1-\theta_2(\mathfrak{a}_0,\mathfrak{b}_0)} d(\mathfrak{a}_\iota,\mathfrak{b}_\iota).$$

Moreover,

$$\begin{aligned}
d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) &= d(\mathcal{E}(\mathfrak{b}_{\iota-1},\tau_{\iota-1}),\mathcal{E}(\mathfrak{a}_\iota,\pi_\iota))\\
&\le \theta_1(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) \frac{d(\mathfrak{a}_\iota,\mathcal{E}(\mathfrak{a}_\iota,\pi_\iota))\, d(\mathcal{E}(\mathfrak{b}_{\iota-1},\tau_{\iota-1}),\mathfrak{b}_{\iota-1})}{1+d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})}\\
&= \theta_1(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) \frac{d(\mathfrak{a}_\iota,\mathfrak{b}_\iota)\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})}{1+d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})}\\
&\le \theta_1(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota)\\
&= \theta_1(\mathcal{E}(\mathfrak{b}_{\iota-1},\tau_{\iota-1}),\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathcal{E}(\mathfrak{b}_{\iota-1},\tau_{\iota-1}),\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota)\\
&\le \theta_1(\mathfrak{b}_{\iota-1},\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{b}_{\iota-1},\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota)\\
&= \theta_1(\mathcal{E}(\mathfrak{a}_{\iota-1},\pi_{\iota-1}),\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathcal{E}(\mathfrak{a}_{\iota-1},\pi_{\iota-1}),\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota)\\
&\le \theta_1(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_{\iota-1},\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota)\\
&\le \cdots \le \theta_1(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota),
\end{aligned}$$

which further implies that

$$\begin{aligned}
d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) &\le \theta_1(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota)\\
&= \theta_1(\mathfrak{a}_0,\mathcal{E}(\mathfrak{a}_{\iota-1},\pi_{\iota-1}))\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathcal{E}(\mathfrak{a}_{\iota-1},\pi_{\iota-1}))\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota)\\
&\le \theta_1(\mathfrak{a}_0,\mathfrak{a}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{a}_{\iota-1})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota)\\
&= \theta_1(\mathfrak{a}_0,\mathcal{E}(\mathfrak{b}_{\iota-2},\tau_{\iota-2}))\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathcal{E}(\mathfrak{b}_{\iota-2},\tau_{\iota-2}))\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota)\\
&\le \theta_1(\mathfrak{a}_0,\mathfrak{b}_{\iota-2})\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_{\iota-2})\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota)\\
&\le \cdots \le \theta_1(\mathfrak{a}_0,\mathfrak{b}_0)\, d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}) + \theta_2(\mathfrak{a}_0,\mathfrak{b}_0)\, d(\mathfrak{a}_\iota,\mathfrak{b}_\iota).
\end{aligned}$$

It yields

$$d(\mathfrak{a}_\iota,\mathfrak{b}_\iota) \le \frac{\theta_1(\mathfrak{a}_0,\mathfrak{b}_0)}{1-\theta_2(\mathfrak{a}_0,\mathfrak{b}_0)} d(\mathfrak{a}_\iota,\mathfrak{b}_{\iota-1}).$$

As in the proof of Theorem 6, we can esaily prove that $(\mathfrak{a}_\iota, \mathfrak{b}_\iota)$ is a Cauchy bisequence in $(\mathfrak{X}, \mathfrak{Z})$. As $(\mathfrak{X}, \mathfrak{Z})$ is complete, so there exists $\rho_1 \in \mathfrak{X} \cap \mathfrak{Z}$ such that $\lim_{\iota\to\infty} (\mathfrak{a}_\iota) = \lim_{\iota\to\infty} (\mathfrak{b}_\iota) = \rho_1$. Now, consider

$$\begin{aligned} d(\mathcal{E}(\rho_1,\tau),\mathfrak{b}_\iota) &= d(\mathcal{E}(\rho_1,\tau),\mathcal{E}(\mathfrak{a}_\iota,\pi_\iota)) \\ &\le \theta_1(\mathfrak{a}_\iota,\rho_1)\, d(\mathfrak{a}_\iota,\rho_1) + \theta_2(\mathfrak{a}_\iota,\rho_1)\,\frac{d(\mathfrak{a}_\iota,\mathcal{E}(\mathfrak{a}_\iota,\pi_\iota))\, d(\mathcal{E}(\rho_1,\tau),\rho_1)}{1+d(\mathfrak{a}_\iota,\theta_1)} \\ &= \theta_1(\mathfrak{a}_\iota,\rho_1)\, d(\mathfrak{a}_\iota,\rho_1) + \theta_2(\mathfrak{a}_\iota,\rho_1)\,\frac{d(\mathfrak{a}_\iota,\mathfrak{b}_\iota)\, d(\mathcal{E}(\rho_1,\tau),\rho_1)}{1+d(\mathfrak{a}_\iota,\rho_1)}. \end{aligned}$$

Applying limit as $\iota \to \infty$, we get $d(\mathcal{E}(\rho_1,\tau),\rho_1) = 0$, which implies that $\mathcal{E}(\rho_1,\tau) = \rho_1$. Similarly, $\mathcal{E}(\rho_1,\pi) = \rho_1$. Thus, $\pi = \tau \in \Theta_1 \cap \Theta_2$, and evidently $\Theta_1 \cap \Theta_2$ is closed set in $[0,1]$. Now we show that $\Theta_1 \cap \Theta_2$ is open in $[0,1]$. Let $(\pi_0,\tau_0) \in (\Theta_1,\Theta_2)$, then there is a bisequence $(\mathfrak{a}_0,\mathfrak{b}_0)$ so that $\mathfrak{a}_0 = \mathcal{E}(\mathfrak{a}_0,\pi_0)$, $\mathfrak{b}_0 = \mathcal{E}(\mathfrak{b}_0,\tau_0)$. Since $\mathfrak{X}\cup\mathfrak{Z}$ is open, so there exists $r > 0$ so that $B_d(\mathfrak{a}_0, r) \subseteq \mathfrak{X}\cup\mathfrak{Z}$ and $B_d(r,\mathfrak{b}_0) \subseteq \mathfrak{X}\cup\mathfrak{Z}$, in which $B_d(\mathfrak{a}_0,r)$ and $B_d(r,\mathfrak{b}_0)$ denote the open balls with centres $\mathfrak{a}_0$ and $\mathfrak{b}_0$ independently and radius r. We select $\pi \in (\tau_0 - \epsilon, \tau_0 + \epsilon)$ and $\tau \in (\pi_0 - \epsilon, \pi_0 + \epsilon)$ such that

$$|\pi - \tau_0| \le \frac{1}{\aleph^\iota} < \frac{\epsilon}{2},$$

$$|\tau - \pi_0| \le \frac{1}{\aleph^\iota} < \frac{\epsilon}{2},$$

and

$$|\pi_0 - \tau_0| \le \frac{1}{\aleph^\iota} < \frac{\epsilon}{2}.$$

Hence, we have

$$\mathfrak{b} \in \overline{B_{\Theta_1\cup\Theta_2}(\mathfrak{a}_0,r)} = \{\mathfrak{b} : \mathfrak{b}_0 \in \mathfrak{Z} : d(\mathfrak{a}_0,\mathfrak{b}) \le r + d(\mathfrak{a}_0,\mathfrak{b}_0)\},$$

and

$$\mathfrak{a} \in \overline{B_{\Theta_1\cup\Theta_2}(r,\mathfrak{b}_0)} = \{\mathfrak{a} : \mathfrak{a}_0 \in \mathfrak{X} : d(\mathfrak{a},\mathfrak{b}_0) \le r + d(\mathfrak{a}_0,\mathfrak{b}_0)\}.$$

Furthermore, we have

$$\begin{aligned} d(\mathcal{E}(\mathfrak{a},\pi),\mathfrak{b}_0) &= d(\mathcal{E}(\mathfrak{a},\pi),\mathcal{E}(\mathfrak{b}_0,\tau_0)) \\ &\le d(\mathcal{E}(\mathfrak{a},\pi),\mathcal{E}(\mathfrak{b},\tau_0)) + d(\mathcal{E}(\mathfrak{a}_0,\pi),\mathcal{E}(\mathfrak{b},\tau_0)) + d(\mathcal{E}(\mathfrak{a}_0,\pi),\mathcal{E}(\mathfrak{b}_0,\tau_0)) \\ &\le 2\aleph|\pi-\tau_0| + d(\mathcal{E}(\mathfrak{a}_0,\pi),\mathcal{E}(\mathfrak{b},\tau_0)) \\ &\le \frac{2}{\aleph^\iota - 1} + \theta_1(\mathfrak{a}_0,\mathfrak{b})\, d(\mathfrak{a}_0,\mathfrak{b}) + \theta_2(\mathfrak{a}_0,\mathfrak{b})\,\frac{d(\mathfrak{a}_0,\mathcal{E}(\mathfrak{a}_0,\pi))\, d(\mathcal{E}(\mathfrak{b},\tau_0),\mathfrak{b})}{1+d(\mathfrak{a}_0,\mathfrak{b})} \\ &= \frac{2}{\aleph^\iota - 1} + \theta_1(\mathfrak{a}_0,\mathfrak{b})\, d(\mathfrak{a}_0,\mathfrak{b}) + \theta_2(\mathfrak{a}_0,\mathfrak{b})\,\frac{d(\mathfrak{a}_0,\mathfrak{a}_0)\, d(\mathfrak{b},\mathfrak{b})}{1+d(\mathfrak{a}_0,\mathfrak{b})} \\ &= \frac{2}{\aleph^\iota - 1} + \theta_1(\mathfrak{a}_0,\mathfrak{b})\, d(\mathfrak{a}_0,\mathfrak{b}) \\ &\le \frac{2}{\aleph^\iota - 1} + d(\mathfrak{a}_0,\mathfrak{b}). \end{aligned}$$

Taking the limit as $\iota \to \infty$, we get

$$d(\mathcal{E}(\mathfrak{a},\pi),\mathfrak{b}_0) \le d(\mathfrak{a}_0,\mathfrak{b}) \le r + d(\mathfrak{a}_0,\mathfrak{b}_0).$$

By corresponding fashion, we get

$$d(\mathfrak{a}_0, \mathcal{E}(\mathfrak{b}, \tau)) \leq d(\mathfrak{a}, \mathfrak{b}_0) \leq r + d(\mathfrak{a}_0, \mathfrak{b}_0).$$

However,

$$d(\mathfrak{a}_0, \mathfrak{b}_0) = d(\mathcal{E}(\mathfrak{a}_0, \pi_0), \mathcal{E}(\mathfrak{b}_0, \tau_0)) \leq \aleph |\pi_0 - \tau_0| \leq \frac{1}{\aleph^{\iota-1}} \to 0$$

as $\iota \to \infty$. This yields that $\mathfrak{a}_0 = \mathfrak{b}_0$. Hence, for $\tau = \pi \in (\tau_0 - \epsilon, \tau_0 + \epsilon)$ and $\mathcal{E}(\cdot, \pi) : \overline{B_{\Theta_1 \cup \Theta_2}(\mathfrak{a}_0, r)} \to \overline{B_{\Theta_1 \cup \Theta_2}(\mathfrak{a}_0, r)}$. Given that all the conditions of Theorem 6 are met, a fixed point in in $\overline{\mathfrak{X}} \cup \overline{\mathfrak{Z}}$ for $\mathcal{E}(\cdot, \pi)$ is assured, and this point is inherently within $\mathfrak{X} \cup \mathfrak{Z}$. Then, $\pi = \tau \in \Theta_1 \cap \Theta_2$ for each $\tau \in (\tau_0 - \epsilon, \tau_0 + \epsilon)$. Hence, $(\tau_0 - \epsilon, \tau_0 + \epsilon) \in \Theta_1 \cap \Theta_2$ which gives $\Theta_1 \cap \Theta_2$ is open in the closed interval [0, 1]. The converse of it can be obtained by the similar procedure. □

5. Conclusions and future work

In this manuscript, we introduced some precise control functions in the contractive inequalities and proved fixed points of contravariant mappings in the situation of this new space. As consequences of our major results, we derived some well-known results of literature. An illustrative example is also flourished to verify the authenticity of accomplished theorem. As an application, we solved a Volterra integral equation, which was predominantly highlighted in signal processing and electrical engineering by employing our results.

Analyzing the convergence rate and stability properties of our iterative scheme under different control functions for rational contractions could offer significant theoretical advancements and optimize convergence speed in practical applications. Applying our fixed point results to model real-world engineering problems in areas like heat transfer, vibration analysis, and circuit design could demonstrate their practical potential and contribute to solving critical engineering challenges.

Author contributions

Mohammed H. Alharbi: conceptualization, formal analysis, investigation, visualization, writing-original draft, writing-review and editing; Jamshaid Ahmad: conceptualization, formal analysis, methodology, validation, writing-original draft, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

References

1. M. Frechet, Sur quelques points du calcul fonctionnel, *Rend. Circ. Matem. Palermo*, **22** (1906), 1–72. https://doi.org/10.1007/BF03018603
2. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis*, **1** (1993), 5–11.
3. A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, *Publ. Math. Debrecen*, **57** (2000), 31–37. https://doi.org/10.5486/PMD.2000.2133
4. M. Jleli, B. Samet, On a new generalization of metric spaces, *J. Fixed Point Theory Appl.*, **20** (2018), 128. https://doi.org/10.1007/s11784-018-0606-6
5. A. Mutlu, U. Gürdal, Bipolar metric spaces and some fixed point theorems, *J. Nonlinear Sci. Appl.*, **9** (2016), 5362–5373. https://doi.org/10.22436/jnsa.009.09.05
6. S. Rawat, R. Dimri, A. Bartwal, $\mathfrak{F}$-bipolar metric spaces and fixed point theorems with applications, *J. Math. Computer Sci.*, **26** (2022), 184–195. https://doi.org/10.22436/jmcs.026.02.08
7. B. Alamri, Fixed point results in $\mathfrak{F}$-bipolar metric spaces with applications, *Mathematics*, **11** (2023), 2399. https://doi.org/10.3390/math11102399
8. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.*, **3** (1922), 133–181.
9. R. Kannan, Some results on fixed points–II, *American Mathematical Monthly*, **76** (1969), 405–408. https://doi.org/10.1080/00029890.1969.12000228
10. S. Reich, A fixed point theorem, *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti*, **51** (1971), 26–28.
11. L. Ćirć, Generalized contractions and fixed point theorems, *Publ. Inst. Math,*, **12** (1971), 19–26.
12. B. Fisher, Mappings satisfying a rational inequality, *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, **24** (1980), 247–251.
13. A. Al-Mazrooei, J. Ahmad, Fixed point theorems for rational contractions in $\mathfrak{F}$-metric spaces, *J. Math. Anal.*, **10** (2019), 79–86.
14. U. Gürdal, A. Mutlu, K. Ozkan, Fixed point results for α-ψ-contractive mappings in bipolar metric spaces, *J. Inequal. Spec. Funct.*, **11** (2020), 64–75.
15. Y. Gaba, M. Aphane, H. Aydi, (α, BK)-contractions in bipolar metric spaces, *J. Math*, **2021** (2021), 5562651. https://doi.org/10.1155/2021/5562651
16. G. Kishore, K. Rao, A. Sombabu, R. Rao, Related results to hybrid pair of mappings and applications in bipolar metric spaces, *J. Math.*, **2019** (2019), 8485412. https://doi.org/10.1155/2019/8485412
17. G. Kishore, K. Rao, H. Isik, B. Rao, A. Sombabu, Covariant mappings and coupled fixed point results in bipolar metric spaces, *Int. J. Nonlinear Anal.*, **12** (2021), 1–15. https://doi.org/10.22075/IJNAA.2021.4650
18. G. Kishore, H. Isik, H. Aydi, B. Rao, D. Prasad, On new types of contraction mappings in bipolar metric spaces and applications, *J. Linear Topological Algebra*, **9** (2020), 253–266.

19. G. Dong, M. Hintermueller, Y. Zhang, A class of second-order geometric quasilinear hyperbolic PDEs and their application in imaging, *SIAM J. Imaging Sci.*, **14** (2021), 645–688. https://doi.org/10.1137/20M1366277

20. G. Lin, X. Cheng, Y. Zhang, A parametric level set based collage method for an inverse problem in elliptic partial differential equations, *J. Comput. Appl. Math.*, **340** (2018), 101–121. https://doi.org/10.1016/j.cam.2018.02.008

21. Y. Zhang, B. Hofmann, Two new non-negativity preserving iterative regularization methods for ill-posed inverse problems, *Inverse Probl. Imag.*, **15** (2021), 229–256. https://doi.org/10.3934/ipi.2020062

22. G. Baravdish, O. Svensson, M. Gulliksson, Y. Zhang, Damped second order flow applied to image denoising, *IMA J. Appl. Math.*, **84** (2019), 1082–1111. https://doi.org/10.1093/imamat/hxz027

23. A. Shcheglov, J. Li, C. Wang, A. Ilin, Y. Zhang, Reconstructing the absorption function in a quasi-linear sorption dynamic model via an iterative regularizing algorithm, *Adv. Appl. Math. Mech.*, **16** (2024), 237–252. https://doi.org/10.4208/aamm.OA-2023-0020

24. A. Abdou, A fixed point approach to predator-prey dynamics via nonlinear mixed Volterra-Fredholm integral equations in complex-valued suprametric spaces, *AIMS Mathematics*, **10** (2025), 6002–6024. https://doi.org/10.3934/math.2025274