AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(4): 8615–8646.
DOI: 10.3934/math.2025395
Received: 23 January 2025
Revised: 29 March 2025
Accepted: 02 April 2025
Published: 14 April 2025

Research article

q-Rung simplified neutrosophic set: A generalization of intuitionistic, Pythagorean and Fermatean neutrosophic sets

Ashraf Al-Quran[1], **Faisal Al-Sharqi**[2,*] **and Abdelhamid Mohammed Djaouti**[1]

[1] Department of Mathematics and Statistics, College of Science, King Faisal University, Al Ahsa 31982, Saudi Arabia

[2] Department of Mathematics, Faculty of Education for Pure Sciences, University of Anbar, Ramadi, Anbar

* **Correspondence:** Email: faisal.ghazi@uoanbar.edu.iq.

Abstract: This research explored a significant and specific case in the definition of neutrosophic sets (N sets), where truth and falsehood degrees were dependent, representing q-Rung orthopair fuzzy (q-ROF) values, while indeterminacy acted independently. Based on this assertion, a new hybrid model called the q-Rung simplified neutrosophic set (q-RSN) set was proposed. This model extended the capabilities of existing frameworks such as the simplified intuitionistic neutrosophic set (Simplified IN) set and Pythagorean neutrosophic set (PyN) set, while simultaneously harnessing the advantages of both N sets and q-ROF sets. To lay the foundation, we established the formal definition of the q-RSN set and outlined its algebraic operations. The properties of these operations were provided, accompanied by their respective proofs. Moving forward, we introduced a method for comparing q-RSN numbers, employing the score function (SF) and accuracy function (AF) as effective tools. To enable the aggregation of q-RSN numbers, we proposed two types of operators: q-RSN weighted averaging operators (q-RSNWAOs) and q-RSN weighted geometric operators (q-RSNWGOs). We established and validated the desirable properties of these operators, including idempotency, boundedness, and monotonicity. Expanding on the proposed aggregation operators (AOs), we presented an algorithmic approach for multi-criteria decision making (MCDM) and provide a practical implementation to showcase the effectiveness of the algorithm. Within the given MCDM problem context, we employed the proposed q-RSNWAOs, q-RSNWGOs, and SFs to ensure a comprehensive evaluation. Additionally, we conducted a sensitivity analysis of the problem parameters to examine the behavior of the optimal solution. Furthermore, we conducted a comparative examination, pitting the proposed method against other related methods. The results obtained from this comparative examination demonstrated a clear and significant superiority of the proposed method over the alternatives.

Keywords: aggregation operators; multi-criteria decision making; neutrosophic set; q-Rung orthopair

fuzzy set; score function
Mathematics Subject Classification: 03E72, 90B50, 68T35

Abbreviations

N Set	Neutrosophic Set
q-ROF	q-Rung Orthopair Fuzzy
q-RSN Set	q-Rung Simplified Neutrosophic Set
IN Set	Intuitionistic Neutrosophic Set
INS Set	Intuitionistic Neutrosophic Soft Set
PyN Set	Pythagorean Neutrosophic Set
AF	Accuracy Function
SF	Score Function
ATCNs	Archimedean t-conorms
ATNs	Archimedean t-norms
q-ROFWAOs	q-Rung Orthopair Fuzzy Weighted Averaging Operators
q-ROFWGOs	q-Rung Orthopair Fuzzy Weighted Geometric Operators
q-RSNWAOs	q-Rung Simplified Neutrosophic Weighted Averaging Operators
q-RSNWGOs	q-Rung Simplified Neutrosophic Weighted Geometric Operators
MCDM	Multi-Criteria Decision Making
S-VN Sets	Single-Valued Neutrosophic Sets
SN Set	Simplified Neutrosophic Set
Simplified NWAOs	Simplified Neutrosophic Weighted Averaging Operators
Simplified NWGOs	Simplified Neutrosophic Weighted Geometric Operators
Soft NS	Soft Neutrosophic Set
SSs	Soft Sets
Vague NS	Vague Neutrosophic Set
D-M	Decision-Making
IF Set	Intuitionistic Fuzzy Set
PyF Set	Pythagorean Fuzzy Set
q-ROF Set	q-Rung Orthopair Fuzzy Set
TM	Truth-Membership
FM	Falsity-Membership
$\mathfrak{X}$	Universal Set
$\mathbb{I}$	The unit interval $[0, 1]$
$\mathcal{T}$	Truth-Membership Degree
$\mathcal{I}$	Indeterminacy- Membership Degree
$\mathcal{F}$	Falsity-Membership Degree
HAOs	Hamacher Aggregation Operators
DAAOs	Dombi–Archimedean aggregation operators
BM	Bonferroni mean

SESs	Soft Expert Sets
Cq-ROF Set	Complex q-Rung Orthopair Fuzzy Set
TOPSIS	Technique for Order Preference by Similarity to Ideal Solution
VIKOR	VlseKriterijumska Optimizacija I Kompromisno Resenje
CODAS	COmbinative Distance-based ASsessment
HSS	Hyper soft set

1. Introduction

Indeterminacy is a state of uncertainty or ambiguity where it becomes challenging to determine or assign a precise value or degree to a specific notion or phenomenon. This term stems from the recognition that our environment contains imprecise aspects that cannot be fully captured by conventional conceptions of uncertainty or ambiguity. To address this more subtle kind of imprecision, Smarandache [1] proposed neutrosphic sets (N-sets), which are designed to handle Multi-Criteria Decision Making (MCDM) situations with ambiguous, partial, and indeterminate information. Within the framework of N-sets, indeterminacy refers to the degree of ambiguity or lack of clarity in the truth, falsity, or both degrees associated with the elements of the set . It is essential for handling partial or inaccurate data, in addition to representing and analyzing uncertain or ambiguous data. Consequently, more advanced variants of N-sets have emerged. Notable examples of these variants include [2–9]. In their work [2], Wang et al. put forth the framework of single-valued neutrosphic sets (S-VN-sets) as a specific case derived from the broader N-set notion. They achieved this by considering the real standard interval $\mathbb{I}$ as the range for the membership values of S-VN-sets. Additionally, they proposed new operators tailored to the specific characteristics of S-VN-sets, aiming to facilitate their application in science and engineering fields. In [3], Ye established the simplified neutrosophic set (SN set), a subclass of the N-set. The SN set is characterized using three real values confined to the interval $\mathbb{I}$. Further, Ye proposed the aggregation operators (AOs) of SN numbers as practical tools for effectively utilizing the SN set in situations involving MCDM. But still, Peng et al. [4] confirmed that the operations of SN set by Ye [3] might not be feasible in certain cases. As a solution, they redefined the operations and AOs for SN numbers, and subsequently established an MCDM method based on these revised operators. Mishra et al. [5] introduced MCDM in the context of the SN set environment. Ali and Smarandache [6] expanded the three SN set memberships from a real-number framework to a complex one. Building on their findings, Al-Quran et al. [7] advanced the concept of SN by incorporating Q-complex neutrosophic sets within the same context. Qiu et al. [8] proposed a dynamic nonlinear SN set for capturing the real-time changing expert partiality information. Ye [9] presented cosine and sine function-based entropy measures of SNSs. Ye et al. [10] proposed a new similarity measure of refined SNSs.

Maji, in his study [11], developed the notion of soft NS by integrating SSs into the N set framework. The aim was to enhance the parameterization capabilities by providing a more comprehensive and complete representation of problem parameters.

In a similar vein, Alkhazaleh strengthened the notion of the N-set and created the framework of vague NS by incorporating vague sets into the N set framework, as outlined in his work [12]. Bhowmik and Pal [13] proposed truth-value N set and its operations. They also defined the intuitionistic neutrosophic sets (IN-sets) based on significance elements within the N set and established their basic operations. As a development of the mathematical structure of the NS to suit the data associated with decision-making (D-M) problems, the scholars work to incorporate the NS with other fuzziness structures. For example, Chinnadurai and Bobin [14] introduced a redefinition of IN sets by incorporating an intuitionistic condition between $\mathcal{T}$ and $\mathcal{F}$, ensuring that their sum does not exceed one. In a related context, Jansi et al. [15] introduced the PyN set, a different variation of the N set. In this variation, the components $\mathcal{T}$ and $\mathcal{F}$ are dependent neutrosophic components. The researchers also put forth a similarity measure specifically designed for PyN set and provided evidence supporting some of its fundamental properties. Radha et al. [16] further the redefined the PyN set, where the components $\mathcal{T}$ and $\mathcal{F}$ are dependent neutrosophic Pythagorean components.

As a consequence, several researchers have developed different AOs for the aforementioned variations of the N set, which have proven to be valuable tools in tackling MCDM challenges [17–19]. Imran et al. [17] proposed a new hybrid structure of the Bonferroni mean and Aczel-Alsina operators for interval-valued intuitionistic fuzzy (IVIF) information to show the interrelationship between the multiple criteria and assist the expertise in the D-M process. In [18], Riaz et al. pioneered the notion of fairly AOs specifically designed for S-VN sets. Farid and Riaz expanded on this by proposing dynamic Einstein AOs for S-VN sets [19]. Garg and Nancy [20] contributed power AOs for linguistic S-VN sets. Liu and Luo elaborated a series of power AOs for SN sets and presented an approach to MCDM that utilizes these operators [21]. Yong et al. [22] proposed Aczel-Alsina AOs for SN sets in MCDM problems. Furthermore, a plethora of other types of AOs for SN sets can be found in references [23–25]. Within the domain of soft NS, Jana and Pal [26] established the concept of soft single-valued neutrosophic weighted averaging operators (SS-VNWAOs) and neutrosophic weighted geometric operators (NWGOs). In a separate study [27], Al-Sharqi et al. employed matrices to represent Q-soft NS and effectively addressed an MCDM problem by employing score functions technique in conjunction with AOs of Q-neutrosophic soft matrices. Building upon their earlier work, the same authors [28] extended their methodology to encompass the realm of bipolar neutrosophic hypersoft sets, employing the same techniques to tackle relevant challenges. The realm of vague NS has been explored by Hashim et al. [29], who employed a sophisticated methodology involving the fusion of Heronian mean operators and Shapley fuzzy measure to address MCDM problems within the domain of interval vague NS. Expanding upon their prior research, the same authors [30] further advanced their studies by addressing MCDM problems in the same domain, this time employing the entropy measure of vague NS in its interval-based form. Additionally, the authors [31] have taken their exploration of vague NS a step further by expanding it to encompass linguistic variables. They have amalgamated this extension with the Decision making trial and evaluation laboratory (DEMATEL) method, effectively integrating it into the D-M process. Within the framework of the IN set, Unver et al. [32] utilized of AOs within the context of IN multi-sets, employing the framework of intuitionistic fuzzy t-norms and t-conorms. Simultaneously, Broumi and Smarandache [33] amalgamated soft sets (SSs) into the IN framework, thereby introducing the notion of INS sets, which has proven instrumental in the realm of D-M. Furthermore, the realm of graph theory has witnessed a notable expansion through the endeavors of diverse researchers, who have effectively extended the principles and methodologies of the IN set [34–36]. Numerous scholars have undertaken the study of the AOs of the PyN set, further expanding the existing body of knowledge in this field. Chellamani and Ajay [37] introduced the integration of the Dombi operator with PyN fuzzy graphs, resulting in the development of diverse fundamental graphical concepts through this amalgamation. Ismail et al. [38] investigated the algebraic operations of PyN sets, thereby broadening their range of applicability. Additionally, they [39] further improved the precision of D-M in DEMATEL by enhancing accuracy through the utilization of Bonferroni mean aggregation within the context of PyN environment. Palanikumar and Arulmozhi [40] proposed Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) and Vlse Kriterijumska Optimizacija I Kompromisno Resenje (VIKOR) methods to process the MCDM problems with the PyNS-information. Asif et al. [41] introduced the structure of PyF Hamacher interactive weighted averaging (PyFHIWA), and they used it in dealing with MCDM. Palanikumar et al. [42] integrated PyN sets with SSs by utilizing the possibility concept and employed the resulting model in D-M processes. Petchimuthu et al. [43] worked in handling uncertain environments when they introduced novel aggregation techniques tailored for complex q-rung picture fuzzy numbers (Cq-RPFNs). Ali [44] builds AOs techniques based on complex p, q-rung orthopair fuzzy sets (Cp,q-ROFSs).

In a separate work, M. Palanikumar and K. Arulmozhi [45] discuss the TOPSIS and VIKOR methods for MCGDM based on PyNS sets. Razak et al. [46] combined PyN sets with interval-valued neutrosophic sets and developed several AOs based on this combination.

Simultaneously, significant progress has been made in the development of two-dimensional uncertainty sets. The intuitionistic fuzzy sets (IF) set was formulated by Atanassov [47], where $\mathcal{T}$ and $\mathcal{F}$ are constructed to satisfy the constraint that their combined value is limited to one. To address this constraint, Yager [48] introduced the principle of the PyF set, in which the sum of the squares of $\mathcal{T}$ and $\mathcal{F}$ is restricted to one. Yager [49] further proposed that the PyF set structure may not always adhere to this constraint in certain cases. Therefore, he presented an alternative solution, the $\mathfrak{q}$-ROF set, characterized by $\mathcal{T}$ and $\mathcal{F}$, with the requirement that the sum of the $\mathfrak{q}$-th power of $\mathcal{T}$ and $\mathcal{F}$ does not surpass one.

As a result, numerous researchers have formulated various AOs for the $\mathfrak{q}$-ROF set, which shown to be effective an tool for overcoming MCDM challenges [50–54]. In [50], Liu and Wang formulated $\mathfrak{q}$-ROFWAOs and $\mathfrak{q}$-ROFWGOs to address D-M information. Seikh and Mandal [51] improved the Archimedean t-conorm (ATCN) and Archimedean t-norm (ATN) operations in the q-ROF context and employed these operations to introduce a series of Archimedean AOs. In the realm of complex numbers, Du et al. [52] expanded the Frank operations to the complex $\mathfrak{q}$-ROF environment and applied these operations to create several complex $\mathfrak{q}$-ROF Frank AOs. Farid et al. [53] developed dynamic $\mathfrak{q}$-ROFAOs to tackle multi-period D-M problems, where decision makers present all information as $\mathfrak{q}$-ROFNs over different time intervals. In a different study, Farid and Riaz [54] created various types of $\mathfrak{q}$-ROF Aczel-Alsina AOs and applied these operators to develop an algorithm for MCDM involving multiple evaluations by decision makers and partial weight information within $\mathfrak{q}$-ROF sets. Additionally, numerous other types of AOs for $\mathfrak{q}$-ROF sets are discussed in references [55–57].

This paper aims to expand the scope of previous works and benefit from the dynamic role of $\mathfrak{q}$ in $\mathfrak{q}$-ROF by introducing the $\mathfrak{q}$-RSN set, thereby tackling more intricate MCDM problems that both IN and PyN are incapable of handling. Therefore, this paper aims to present several important contributions.

(1) The concept of the $\mathfrak{q}$-RSN set is introduced, expanding on and integrating principles from existing neutrosophic set literature.

(2) Evolve some extended aggregation operators based on $\mathfrak{q}$-Rung simplified neutrosophic numbers ($\mathfrak{q}$-RSNNs), such as $\mathfrak{q}$-RSN-weighted averaging operator ($\mathfrak{q}$-RSNWAO) and $\mathfrak{q}$-RSN-weighted geometric operator ($\mathfrak{q}$-RSNWGO), then check their initial properties.

(3) A technique for ranking q-RSN numbers is proposed, employing the accuracy function (SF) and score function (AF) as useful instruments.

(4) An algorithmic method for MCDM is introduced, utilizing the proposed AOs and SFs, along with a practical implementation to demonstrate the algorithm's effectiveness.

(5) A sensitivity and comparison analysis is conducted to highlight the significant advantages of the proposed method over the alternatives.

1.1. Research gap and motivation behind q*-RSN Set*

Based on the in-depth analysis of the previous literary studies, this subsection supplies a mathematical breakdown of the research gap and the motivation for giving out this work. Researchers have presented a multitude of N set variations as solutions to D-M problems. Among these inventive adaptations is the SN set, a simplified variant intended to expedite the dealings of indeterminate data current in the daily life situations. The SN set has three membership degrees, denoted as $\mathcal{T}$, $\mathcal{I}$, and $\mathcal{F}$. Their sum must satisfy the inequality $0 \leq \mathcal{T} + \mathcal{I} + \mathcal{F} \leq 3$.

Subsequently, numerous researchers defined the IN set to address specific scenarios under certain circumstances. In this academic discourse, we shall explicate the definition of the simplified IN set, as proposed by Chinnadurai and Bobin [14]. In their assertion, the simplified IN set is distinguished by three membership degrees, namely $\mathcal{T}$, $\mathcal{I}$, and $\mathcal{F}$, which must conform to the conditions $0 \leq \mathcal{T} + \mathcal{F} \leq 1$ and $0 \leq \mathcal{T} + \mathcal{I} + \mathcal{F} \leq 2$. Within this framework, the elements $\mathcal{T}$ and $\mathcal{F}$ exhibit a dependent relationship, while the selection of $\mathcal{I}$ remains unconstrained. This model is constructed to cater to specific cases requiring meticulous consideration. Under certain circumstances, it has been observed that the total value of $\mathcal{T}$ and $\mathcal{F}$ may surpass 1. Consider the following decision-maker's judgment value $(0.6, 0.3, 0.7)$, where we see that $0 \leq 0.6 + 0.3 + 0.7 \leq 2$, while $0.6 + 0.7 \geq 1$. To address this case, Radha et al. [16] introduced the theory of PyN sets, proposing the requirements $\mathcal{T}^2 + \mathcal{F}^2 \leq 1$ and $0 \leq \mathcal{T}^2 + \mathcal{I}^2 + \mathcal{F}^2 \leq 2$. Using this condition, we can observe that $0.6^2 + 0.7^2 = 0.36 + 0.49 = 0.85 \leq 1$, guaranteeing that the sum of the squares of $\mathcal{T}$, $\mathcal{I}$, and $\mathcal{F}$ remains within the specified range of 2.

Indeed, throughout the process of solving D-M problems, there may be instances where the opinions or preferences of decision-makers cannot adhere to the constraints imposed by the simplified IN and PyN sets. Consider a case in which the decision-maker allocates $\mathcal{T} = 0.7$ and $\mathcal{F} = 0.9$. It is evident that this particular pair $(0.7, 0.9)$ cannot be accommodated within the simplified IN and PyN sets frameworks since $0.7^3 + 0.9^3 \geq 1$. Consequently, in such situations, the theories of simplified IN and PyN sets are insufficient to successfully handle the issues of solving D-M problems. To overcome these limitations, a further extension of the set called the q-RSN set is introduced. This extension is motivated by the progression from the simplified IN set to the PyN set. In the q-RSN set, the idea of raising $\mathcal{T}$ and $\mathcal{F}$ to the power of q is introduced, hence expanding the range of membership grades covered. Incorporating the q-RSN set addresses previously identified difficulties. This extension enables a more validated representation of the decision-maker's thoughts and preferences, as well as a larger variety of membership grades.

The theories of simplified IN and PyN sets can be regarded as specific instances of the q-RSN set. In the context of the q-RSN set, the value of q determines which theory is being employed. For example, setting $\mathfrak{q} = 1$, the q-RSN set corresponds to the simplified IN set and $\mathfrak{q} = 2$ converts it to the PyN set. Thus, the q-RSN set is a complete framework that covers and generalizes these preceding theories, offering flexibility based on the chosen value of q. In particular, we expand on the q-RSN set based on the definitions of the N set, simplified IN set, PyN set and q-ROF set. In depth, a detailed examination of the reasons for introducing these sets serves as a significant impetus for the development of the q-RSN concept. The motivation for this paper stems from the benefits of the suggested methodology as well as the acknowledged shortcomings of previously described methodologies earlier. Thus, this proposed approach offers several advantages, such as:

(1) By utilizing the q-RSN set to represent decision information, it acts as an extension of the simplified IN and PyN sets, incorporating evaluative information that is lacking in the first two models, such as the data point (0.9, 0.2, 0.7). This specific type of data can be represented through our method that cannot be captured by the simplified IN or PyN sets.

(2) The q-RSNAOs are used to combine q-RSN data, ensuring that the original data remains intact without any loss or distortion.

(3) The suggested SFs enable D-M through a straightforward computational process.

(4) The q-RSN set used in our approach is adept at managing imprecise, unclear, inconsistent, and incomplete information. Consequently, this method can effectively address more complex uncertain data in a manner comparable to the IF set, PyF set, and q-ROF set.

1.2. Capture data via fuzzy environments

Currently, there are numerous effective methods and approaches to constructing data for describing daily life situations. Based on this data that describes daily life situations, appropriate fuzzy tools are modeled to handle these situations. Because we live in an era full of daily fluctuations and surprises, the process of constructing data for any problem can be subject to one of the methods shown in Table 1. Accordingly, in the subsection, we present a state-of-the-art review of relevant exploratory methods and approaches for effectively obtaining such data in practical decision-making processes as well as the mechanism of processing it using current fuzzy models and their extensions, highlighting the importance of our proposed concept and how the data presented in Section 6 was captured.

In Section 6 of this article, we talk about how our idea can help fix one of the D-M problems in the field of roads in cities. This is based on made-up data that fits the structure of the q-RSN set. As for the way to deal with this problem, it is by presenting a multistep algorithm that depends on q-RSNWAO or q-RSNWGO of the q-RSN set as described in Section 6.

Table 1. Review of existing papers on data capture techniques and handling mechanisms.

Method	Data capture techniques	D-M approach
IN set [13]	Hypothetical data based on IN Logic Integration	Multi-step algorithm based on similarity measures between two IN sets
PyF set [55]	Hypothetical data based on PyF Logic Integration	Multi-step algorithm based on HAOs for PyF sets
PyN set [45]	Hypothetical data based on PyN Logic Integration	MCGDM based on TOPSIS and VIKOR using AOs of PyN sets
q-ROF set [51]	Hypothetical data based on q-ROF Logic Integration	Multi-step algorithm based on q-ROFWA operator
q-ROF set [53]	Real data based on Expert opinion	Multi-step algorithm based on DAAOs of q-ROF sets
q-ROF set [58]	Hypothetical data based on Expert opinion	Multi-step algorithm based on DAAOs of q-ROF sets
Cq-ROF set [52]	Real data based on Expert opinion	Multi-step algorithm based on FAOs of Cq-ROF sets

1.3. Paper organization

This paper is organized and divided into 8 sections distributed as follows: Section 2 revisits several foundational notions from previously published works that are related to our work. Section 3 discusses the notion of the q-RSN set with algebraic operations and results. Section 4 delves into the aggregation operators utilized in q-RSN numbers. Section 5 introduces a methodology for solving MCDM issues using AOs and SFs tailored for q-RSN sets. Section 6 harnesses the power of the algorithm given in Section 5 to conquer the formidable DM dilemma roads in metropolitan areas. Section 7: This section examines the effect of parameters on ranking outcomes to confirm the stability of the suggested MCDM approach. Section 8 summarizes the paper's conclusions with future directions, which is expected to benefit researchers.

2. Basic knowledge

In this section, we revisit several foundational notions from previously published works.

Definition 2.1. [49] The q-ROF set denoted by $\hat{\omega}$ on $\mathfrak{X}$ is represented as

$$\hat{\omega} = \{(\mathfrak{b}, \mathcal{T}_{\hat{\omega}}(\mathfrak{b}), \mathcal{F}_{\hat{\omega}}(\mathfrak{b})) : \mathfrak{b} \in \mathfrak{X}\},$$

where $\mathcal{T}_{\hat{\omega}}(\mathfrak{b})$ and $\mathcal{F}_{\hat{\omega}}(\mathfrak{b})$ are values in the range $\mathbb{I}$. The following condition must hold: $0 \leq (\mathcal{T}_{\hat{\omega}}(\mathfrak{b}))^q + (\mathcal{F}_{\hat{\omega}}(\mathfrak{b}))^q \leq 1$ $(q \geq 1)$.

Liu and Wang [50] proposed the AOs of the q-ROF numbers in the following way.

Definition 2.2. [50] Consider ${}^{\Lambda}\beth = \langle \mathcal{T}_{{}^{\Lambda}\beth}, \mathcal{F}_{{}^{\Lambda}\beth} : \Lambda = 1, ..., m \rangle$, to be a family of q-ROF numbers and ${}^{\Lambda}\varpi : \{\Lambda = 1, ..., m\}$, to be the weights of ${}^{\Lambda}\beth$. Then,

(1) The q-ROF weighted averaging operator(q-ROFWAO) is defined as:

$$\text{q-ROFWAO}({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) = \left\langle \left(1 - \prod_{\Lambda=1}^{m} \left(1 - (\mathcal{T}_{{}^{\Lambda}\beth})^q\right)^{{}^{\Lambda}\varpi}\right)^{\frac{1}{q}}, \prod_{\Lambda=1}^{m} \left(\mathcal{F}_{{}^{\Lambda}\beth}\right)^{{}^{\Lambda}\varpi} \right\rangle, \ q \geq 1. \tag{1}$$

(2) The $\mathfrak{q}$-ROF weighted geometric operator ($\mathfrak{q}$-ROFWGO) is defined as:

$$\mathfrak{q}\text{-ROFWGO}({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) = \left\langle \prod_{\Lambda=1}^{m} \left(\mathcal{T}_{{}^{\Lambda}\beth}\right)^{{}^{\Lambda}\varpi}, \left(1-\prod_{\Lambda=1}^{m}\left(1-(\mathcal{F}_{{}^{\Lambda}\beth})^{\mathfrak{q}}\right)^{{}^{\Lambda}\varpi}\right)^{\frac{1}{q}} \right\rangle, \ q \geq 1. \tag{2}$$

Definition 2.3. [3] A simplified NS $\hat{\lambda}$ in $\mathfrak{X}$ with a representative element $\flat$ in $\mathfrak{X}$ is expressed as:

$$\hat{\lambda} = \{< \flat; \mathcal{T}_{\hat{\lambda}}(\flat), \mathcal{I}_{\hat{\lambda}}(\flat), \mathcal{F}_{\hat{\lambda}}(\flat) >: \flat \in \mathfrak{X}\},$$

where $\mathcal{T}_{\hat{\lambda}}; \mathcal{I}_{\hat{\lambda}}; \mathcal{F}_{\hat{\lambda}} : \mathfrak{X} \to \mathbb{I}$, satisfies the condition $0 \leq \mathcal{T}_{\hat{\lambda}} + \mathcal{I}_{\hat{\lambda}} + \mathcal{F}_{\hat{\lambda}} \leq 3$.

In this research, we will utilize SNS in which $\mathcal{T}$ and $\mathcal{F}$ values are represented as $\mathfrak{q}$-ROF-values within $\mathbb{I}$.

Bhowmik and Pal [13] provided the following definition for the IN set:

Definition 2.4. [13] The IN set $\hat{\psi}$ over $\mathfrak{X}$ is determined as:

$$\hat{\psi} = \{(\flat, \langle \mathcal{T}_{\hat{\psi}}(\flat), \mathcal{I}_{\hat{\psi}}(\flat), \mathcal{F}_{\hat{\psi}}(\flat)\rangle) : \flat \in \mathfrak{X}\},$$

where $\mathcal{T}_{\hat{\psi}}; \mathcal{I}_{\hat{\psi}}; \mathcal{F}_{\hat{\psi}} : \mathfrak{X} \to \mathbb{I}$, satisfies the conditions $0 \leq \mathcal{T}_{\hat{\psi}}(\flat) + \mathcal{F}_{\hat{\psi}}(\flat) \leq 1$ and $0 \leq \mathcal{T}_{\hat{\psi}}(\flat) + \mathcal{I}_{\hat{\psi}}(\flat) + \mathcal{F}_{\hat{\psi}}(\flat) \leq 2$, for the element $\flat$ in $\mathfrak{X}$.

Radha et al. [16] developed the PyN set as follows:

Definition 2.5. [16] The PyN set $\hat{\phi}$ over $\mathfrak{X}$ is stated as:

$$\hat{\phi} = \{(\flat, \langle \mathcal{T}_{\hat{\phi}}(\flat), \mathcal{I}_{\hat{\phi}}(\flat), \mathcal{F}_{\hat{\phi}}(\flat)\rangle) : \flat \in \mathfrak{X}\},$$

where $\mathcal{T}_{\hat{\phi}}; \mathcal{I}_{\hat{\phi}}; \mathcal{F}_{\hat{\phi}} : \mathfrak{X} \to \mathbb{I}$, fulfills the prerequisites $0 \leq (\mathcal{T}_{\hat{\phi}}(\flat))^2 + (\mathcal{F}_{\hat{\phi}}(\flat))^2 \leq 1$ and $0 \leq (\mathcal{T}_{\hat{\phi}}(\flat))^2 + (\mathcal{I}_{\hat{\phi}}(\flat))^2 + (\mathcal{F}_{\hat{\phi}}(\flat))^2 \leq 2$, for the element $\flat$ in $\mathfrak{X}$.

3. $\mathfrak{q}$-Rung simplified neutrosophic set

In this section, we propose the notion of $\mathfrak{q}$-RSN set.

Definition 3.1. Let $\mathfrak{X}$ denote the universal set. We can define the $\mathfrak{q}$-RSN set $\mathcal{K}$ over $\mathfrak{X}$ as the collection of tuples:

$$\mathcal{K} = \{(\flat, \langle \mathcal{T}_{\mathcal{K}}(\flat), \mathcal{I}_{\mathcal{K}}(\flat), \mathcal{F}_{\mathcal{K}}(\flat)\rangle) : \flat \in \mathfrak{X}\}.$$

In this context, $\mathcal{T}_{\mathcal{K}}(\flat)$, $\mathcal{I}_{\mathcal{K}}(\flat)$, and $\mathcal{F}_{\mathcal{K}}(\flat) \in \mathbb{I}$, represent the membership values associated with the concepts of truth, indeterminacy, and falsity, respectively, for the element $\flat$ in $\mathfrak{X}$, with the condition that: $0 \leq (\mathcal{T}_{\mathcal{K}}(\flat))^{\mathfrak{q}} + (\mathcal{F}_{\mathcal{K}}(\flat))^{\mathfrak{q}} \leq 1$, $\forall \flat \in \mathfrak{X}$, and $\mathfrak{q} \geq 1$.

Remark 3.2. As specified in Definition 3.1, it becomes evident that there exists a dependency between $\mathcal{T}_{\mathcal{K}}(\flat)$ and $\mathcal{F}_{\mathcal{K}}(\flat)$, while $\mathcal{I}_{\mathcal{K}}(\flat)$ remains independent. Consequently, this tuple adheres to restriction: $0 \leq (\mathcal{T}_{\mathcal{K}}(\flat))^{\mathfrak{q}} + \mathcal{I}_{\mathcal{K}}(\flat) + (\mathcal{F}_{\mathcal{K}}(\flat))^{\mathfrak{q}} \leq 2$ and naturally the restriction $0 \leq (\mathcal{T}_{\mathcal{K}}(\flat))^{\mathfrak{q}} + (\mathcal{I}_{\mathcal{K}}(\flat))^{\mathfrak{q}} + (\mathcal{F}_{\mathcal{K}}(\flat))^{\mathfrak{q}} \leq 2$.

Remark 3.3. Based on Definition 3.1, we can observe the following:

(1) When $\mathfrak{q} = 1$, the$\mathfrak{q}$-RSN set is transformed into simplified IN set.

(2) When $\mathfrak{q} = 2$, the $\mathfrak{q}$-RSN set is reduced to PyN set.

Example 3.4. Let's consider the universe $\mathfrak{X} = \{\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3\}$. where,
$\mathcal{K} = \{(\mathfrak{b}_1, \langle 0.9, 0.6, 0.7\rangle); (\mathfrak{b}_2, \langle 0.4, 0.2, 0.7\rangle); (\mathfrak{b}_3, \langle 0.2, 0.9, 0.7\rangle)\}$. Here, $\mathcal{K}$ represents a $\mathfrak{q}$-RSN set with $\mathfrak{q} = 4$.

We now present the definitions of absolute $\mathfrak{q}$-RSN set and null $\mathfrak{q}$-RSN set.

Definition 3.5. Let $\mathcal{K} = \{(\mathfrak{b}, \langle \mathcal{T}_{\mathcal{K}}(\mathfrak{b}), \mathcal{I}_{\mathcal{K}}(\mathfrak{b}), \mathcal{F}_{\mathcal{K}}(\mathfrak{b})\rangle) : \mathfrak{b} \in \mathfrak{X}\}$ be a $\mathfrak{q}$-RSN set over $\mathfrak{X}$. Then, $\mathcal{K}$ is referred to as an absolute $\mathfrak{q}$-RSN set, denoted by $\mathcal{K}_{\tilde{\Psi}}$, if $\mathcal{T}_{\mathcal{K}}(\mathfrak{b}) = 1$, and $\mathcal{I}_{\mathcal{K}}(\mathfrak{b}) = \mathcal{F}_{\mathcal{K}}(\mathfrak{b}) = 0$, i.e., $\mathcal{K}_{\tilde{\Psi}} = \langle 1, 0, 0\rangle$.

Definition 3.6. Let $\mathcal{K} = \{(\mathfrak{b}, \langle \mathcal{T}_{\mathcal{K}}(\mathfrak{b}), \mathcal{I}_{\mathcal{K}}(\mathfrak{b}), \mathcal{F}_{\mathcal{K}}(\mathfrak{b})\rangle) : \mathfrak{b} \in \mathfrak{X}\}$ be a $\mathfrak{q}$-RSN set over $\mathfrak{X}$. Then, $\mathcal{K}$ is referred to as a null $\mathfrak{q}$-RSN set, denoted by $\mathcal{K}_{\tilde{\Phi}}$, if $\mathcal{T}_{\mathcal{K}}(\mathfrak{b}) = 0$, and $\mathcal{I}_{\mathcal{K}}(\mathfrak{b}) = \mathcal{F}_{\mathcal{K}}(\mathfrak{b}) = 1$, i.e., $\mathcal{K}_{\tilde{\Psi}} = \langle 0, 1, 1\rangle$.

Definition 3.7. A collection of $\beth = \langle \mathcal{T}_{\beth}, \mathcal{I}_{\beth}, \mathcal{F}_{\beth}\rangle$ is referred to as a $\mathfrak{q}$-RSN number with $0 \leq (\mathcal{T}_{\beth})^{\mathfrak{q}} + (\mathcal{F}_{\beth})^{\mathfrak{q}} \leq 1,\ \forall \mathfrak{q} \geq 1$.

3.1. The operations for q-RSN numbers

In this space, we elucidate a repertoire of algebraic operations tailored specifically for $\mathfrak{q}$-RSN numbers. These operations are defined by leveraging the operations of simplified NSs as proposed by Peng et al. [4] and the operations of $\mathfrak{q}$-ROF sets as proposed by Liu and Wang [50].

Definition 3.8. Let $\beth = \langle \mathcal{T}_{\beth}, \mathcal{I}_{\beth}, \mathcal{F}_{\beth}\rangle$ and ${}^{*}\beth = \langle \mathcal{T}_{{}^*\beth},\ \mathcal{I}_{{}^*\beth},\ \mathcal{F}_{{}^*\beth}\rangle$ represent two $\mathfrak{q}$-RSN numbers defined on $\mathfrak{X}$ and $\eta > 0$. In this context, the following holds true:

(1) $$\beth \oplus {}^{*}\beth = \Big\langle \big((\mathcal{T}_{\beth})^{\mathfrak{q}} + (\mathcal{T}_{{}^*\beth})^{\mathfrak{q}} - (\mathcal{T}_{\beth})^{\mathfrak{q}} (\mathcal{T}_{{}^*\beth})^{\mathfrak{q}} \big)^{\frac{1}{\mathfrak{q}}}, (\mathcal{I}_{\beth}\, \mathcal{I}_{{}^*\beth}), (\mathcal{F}_{\beth}\, \mathcal{F}_{{}^*\beth}) \Big\rangle, \tag{3}$$

(2) $$\beth \otimes {}^{*}\beth = \Big\langle (\mathcal{T}_{\beth}\, \mathcal{T}_{{}^*\beth}), (\mathcal{I}_{\beth} + \mathcal{I}_{{}^*\beth} - \mathcal{I}_{\beth}\, \mathcal{I}_{{}^*\beth}), \big((\mathcal{F}_{\beth})^{\mathfrak{q}} + (\mathcal{F}_{{}^*\beth})^{\mathfrak{q}} - (\mathcal{F}_{\beth})^{\mathfrak{q}} (\mathcal{F}_{{}^*\beth})^{\mathfrak{q}} \big)^{\frac{1}{\mathfrak{q}}} \Big\rangle, \tag{4}$$

(3) $$\eta\beth = \Big\langle \Big(1 - (1 - (\mathcal{T}_{\beth})^{\mathfrak{q}})^{\eta}\Big)^{\frac{1}{\mathfrak{q}}}, (\mathcal{I}_{\beth})^{\eta},\ (\mathcal{F}_{\beth})^{\eta} \Big\rangle, \tag{5}$$

(4) $$(\beth)^{\eta} = \Big\langle (\mathcal{T}_{\beth})^{\eta}, 1 - (1 - \mathcal{I}_{\beth})^{\eta}, \Big(1 - (1 - (\mathcal{F}_{\beth})^{\mathfrak{q}})^{\eta}\Big)^{\frac{1}{\mathfrak{q}}} \Big\rangle. \tag{6}$$

Example 3.9. Consider $\beth = \langle 0.92, 0.21, 0.85\rangle$ and ${}^{*}\beth = \langle 0.78, 0.54, 0.76\rangle$, which are two 6-RSN numbers. Assuming $\eta = 4$, then,

(1) $\beth \oplus {}^{*}\beth = \langle 0.94, 0.11, 0.65\rangle$,

(2) $\beth \otimes {}^{*}\beth = \langle 0.72, 0.64, 0.89\rangle$,

(3) $\eta\beth = \langle 0.72, 0, 0.52\rangle$,

(4) $(\beth)^{\eta} = \langle 0.72, 0.61, 0.97\rangle$.

Proposition 3.10. *Let* $\beth = \langle \mathcal{T}_{\beth}, \mathcal{I}_{\beth}, \mathcal{F}_{\beth}\rangle$, ${}^{*}\beth = \langle \mathcal{T}_{{}^*\beth}, \mathcal{I}_{{}^*\beth}, \mathcal{F}_{{}^*\beth}\rangle$ *and* ${}^{\diamond}\beth = \langle \mathcal{T}_{{}^\diamond\beth},\ \mathcal{I}_{{}^\diamond\beth}, \mathcal{F}_{{}^\diamond\beth}\rangle$ *denote three* $\mathfrak{q}$*-RSN numbers defined over* $\mathfrak{X}$ *and* $\eta > 0$. *In this context, the following statements are valid:*

(1) $\beth \oplus {}^{*}\beth = {}^{*}\beth \oplus \beth$,

(2) $(\beth \oplus {}^{*}\beth) \oplus {}^{\diamond}\beth = \beth \oplus ({}^{*}\beth \oplus {}^{\diamond}\beth)$,

(3) $\eta\,(\beth \oplus {}^{*}\beth) = \eta\,\beth \oplus \eta\,{}^{*}\beth$,

(4) $\beth \otimes {}^{*}\beth = {}^{*}\beth \otimes \beth$,

(5) $(\beth \otimes {}^{*}\beth) \otimes {}^{\diamond}\beth = \beth \otimes ({}^{*}\beth \otimes {}^{\diamond}\beth)$,

(6) $(\beth \otimes {}^{*}\beth)^{\eta} = (\beth)^{\eta} \otimes ({}^{*}\beth)^{\eta}$.

Proof. The proofs of (3) and (6) will be presented, while the proofs of the remaining items are straightforward and can be omitted.

(3) By applying (3) and (5) from Definition 3.8, we can derive the following expression for the left-hand side of the equation:

$$\eta\,(\beth \oplus {}^{*}\beth) = \eta.\left(\left\langle \left((\mathcal{T}_{\beth})^{q} + (\mathcal{T}_{*\beth})^{q} - (\mathcal{T}_{\beth})^{q}(\mathcal{T}_{*\beth})^{q}\right)^{\frac{1}{q}}, (\mathcal{I}_{\beth}\,\mathcal{I}_{*\beth}), (\mathcal{F}_{\beth}\,\mathcal{F}_{*\beth}) \right\rangle\right),$$

$$= \left\langle \left(1-\left(1-\left(\left((\mathcal{T}_{\beth})^{q} + (\mathcal{T}_{*\beth})^{q} - (\mathcal{T}_{\beth})^{q}(\mathcal{T}_{*\beth})^{q}\right)^{\frac{1}{q}}\right)^{q}\right)^{\eta}\right)^{\frac{1}{q}}, (\mathcal{I}_{\beth}\,\mathcal{I}_{*\beth})^{\eta}, (\mathcal{F}_{\beth}\,\mathcal{F}_{*\beth})^{\eta} \right\rangle,$$

$$= \left\langle \left(1-\left(1-\left((\mathcal{T}_{\beth})^{q} + (\mathcal{T}_{*\beth})^{q} - (\mathcal{T}_{\beth})^{q}(\mathcal{T}_{*\beth})^{q}\right)\right)^{\eta}\right)^{\frac{1}{q}}, (\mathcal{I}_{\beth})^{\eta}(\mathcal{I}_{*\beth})^{\eta}, (\mathcal{F}_{\beth})^{\eta}(\mathcal{F}_{*\beth})^{\eta} \right\rangle,$$

$$= \left\langle \left(1-\left(1-(\mathcal{T}_{\beth})^{q}\right)^{\eta}\left(1-(\mathcal{T}_{*\beth})^{q}\right)^{\eta}\right)^{\frac{1}{q}}, (\mathcal{I}_{\beth})^{\eta}(\mathcal{I}_{*\beth})^{\eta}, (\mathcal{F}_{\beth})^{\eta}(\mathcal{F}_{*\beth})^{\eta} \right\rangle.$$

Similarly, considering the righthand side of the equation, we can express it as follows:

$\eta\beth = \left\langle \left(1-(1-(\mathcal{T}_{\beth})^{q})^{\eta}\right)^{\frac{1}{q}}, (\mathcal{I}_{\beth})^{\eta}, (\mathcal{F}_{\beth})^{\eta} \right\rangle$, $\eta^{*}\beth = \left\langle \left(1-(1-(\mathcal{T}_{*\beth})^{q})^{\eta}\right)^{\frac{1}{q}}, (\mathcal{I}_{*\beth})^{\eta}, (\mathcal{F}_{*\beth})^{\eta} \right\rangle$, then, $\eta\beth \oplus \eta^{*}\beth = \left\langle \left(\left(\left(1-(1-(\mathcal{T}_{\beth})^{q})^{\eta}\right)^{\frac{1}{q}}\right)^{q} + \left(\left(1-(1-(\mathcal{T}_{*\beth})^{q})^{\eta}\right)^{\frac{1}{q}}\right)^{q} - \left(\left(1-(1-(\mathcal{T}_{\beth})^{q})^{\eta}\right)^{\frac{1}{q}}\right)^{q}\left(\left(1-(1-(\mathcal{T}_{*\beth})^{q})^{\eta}\right)^{\frac{1}{q}}\right)^{q}\right)^{\frac{1}{q}}, (\mathcal{I}_{\beth})^{\eta}(\mathcal{I}_{*\beth})^{\eta}, (\mathcal{F}_{\beth})^{\eta}(\mathcal{F}_{*\beth})^{\eta} \right\rangle$,

$$= \left\langle \left(1-(1-(\mathcal{T}_{\beth})^{q})^{\eta} + 1-(1-(\mathcal{T}_{*\beth})^{q})^{\eta} - \left(1-(1-(\mathcal{T}_{\beth})^{q})^{\eta}\right)\left(1-(1-(\mathcal{T}_{*\beth})^{q})^{\eta}\right)\right)^{\frac{1}{q}}, (\mathcal{I}_{\beth})^{\eta}(\mathcal{I}_{*\beth})^{\eta}, (\mathcal{F}_{\beth})^{\eta}(\mathcal{F}_{*\beth})^{\eta} \right\rangle,$$

$$= \left\langle \left(1-\left(1-(\mathcal{T}_{\beth})^{q}\right)^{\eta}\left(1-(\mathcal{T}_{*\beth})^{q}\right)^{\eta}\right)^{\frac{1}{q}}, (\mathcal{I}_{\beth})^{\eta}(\mathcal{I}_{*\beth})^{\eta}, (\mathcal{F}_{\beth})^{\eta}(\mathcal{F}_{*\beth})^{\eta} \right\rangle.$$

Hence, it can be concluded that the right side of the equation is equal to the left side, thereby proving that: $\eta\,(\beth \oplus {}^{*}\beth) = \eta\,\beth \oplus \eta\,{}^{*}\beth$.

(6) By applying (4) and (6) from Definition 3.8, we can derive the following expression for the left-hand side of the equation:

$$(\beth \otimes {}^{*}\beth)^{\eta} =$$

$$\left\langle (\mathcal{T}_{\beth}\,\mathcal{T}_{*\beth}), (\mathcal{I}_{\beth} + \mathcal{I}_{*\beth} - \mathcal{I}_{\beth}\,\mathcal{I}_{*\beth}), \left((\mathcal{F}_{\beth})^{q} + (\mathcal{F}_{*\beth})^{q} - (\mathcal{F}_{\beth})^{q}({}^{*}\mathcal{F}_{*\beth})^{q}\right)^{\frac{1}{q}} \right\rangle^{\eta}$$

$$= \left\langle \left(\mathcal{T}_{\beth}\,\mathcal{T}_{*\beth}\right)^{\eta}, 1-\left(1-[\mathcal{I}_{\beth} + \mathcal{I}_{*\beth} - \mathcal{I}_{\beth}\,\mathcal{I}_{*\beth}]\right)^{\eta}, \left(1-\left(1-\left(\left[(\mathcal{F}_{\beth})^{q} + (\mathcal{F}_{*\beth})^{q} - (\mathcal{F}_{\beth})^{q}(\mathcal{F}_{*\beth})^{q}\right]^{\frac{1}{q}}\right)^{q}\right)^{\eta}\right)^{\frac{1}{q}} \right\rangle$$

$$= \left\langle \left(\mathcal{T}_{\beth}\,\mathcal{T}_{*\beth}\right)^{\eta}, 1-\left(1-[\mathcal{I}_{\beth} + \mathcal{I}_{*\beth} - \mathcal{I}_{\beth}\,\mathcal{I}_{*\beth}]\right)^{\eta}, \left(1-\left(1-\left((\mathcal{F}_{\beth})^{q} + (\mathcal{F}_{*\beth})^{q} - (\mathcal{F}_{\beth})^{q}(\mathcal{F}_{*\beth})^{q}\right)\right)^{\eta}\right)^{\frac{1}{q}} \right\rangle$$

$$= \left\langle \left(\mathcal{T}_{\beth}\right)^{\eta}\left(\mathcal{T}_{*\beth}\right)^{\eta}, 1-\left(1-\mathcal{I}_{\beth}\right)^{\eta}\left(1-\mathcal{I}_{*\beth}\right)^{\eta}, \left(1-\left(1-(\mathcal{F}_{\beth})^{q}\right)^{\eta}\left(1-(\mathcal{F}_{*\beth})^{q}\right)^{\eta}\right)^{\frac{1}{q}} \right\rangle.$$

As for the righthand side of the equation, we can express it as follows:

$(\beth)^{\eta} = \left\langle (\mathcal{T}_{\beth})^{\eta}, 1-(1-\mathcal{I}_{\beth})^{\eta}, \left(1-(1-(\mathcal{F}_{\beth})^{q})^{\eta}\right)^{\frac{1}{q}} \right\rangle,$

$({}^{*}\beth)^{\eta} = \left\langle (\mathcal{T}_{{}^{*}\beth})^{\eta}, 1-(1-\mathcal{I}_{{}^{*}\beth})^{\eta}, \left(1-(1-(\mathcal{F}_{{}^{*}\beth})^{q})^{\eta}\right)^{\frac{1}{q}} \right\rangle,$

$(\beth)^{\eta}\otimes({}^{*}\beth)^{\eta} = \left\langle \left(\mathcal{T}_{\beth}\right)^{\eta}\left(\mathcal{T}_{{}^{*}\beth}\right)^{\eta}, \left(\left(1-(1-\mathcal{I}_{\beth})^{\eta}\right)+\left(1-(1-\mathcal{I}_{{}^{*}\beth})^{\eta}\right)-\left(1-(1-\mathcal{I}_{\beth})^{\eta}\right)\left(1-(1-\mathcal{I}_{{}^{*}\beth})^{\eta}\right)\right), \left(\left(\left(1-(1-(\mathcal{F}_{\beth})^{q})^{\eta}\right)^{\frac{1}{q}}\right)^{q}+\left(\left(1-(1-(\mathcal{F}_{{}^{*}\beth})^{q})^{\eta}\right)^{\frac{1}{q}}\right)^{q}-\left(\left(1-(1-(\mathcal{F}_{\beth})^{q})^{\eta}\right)^{\frac{1}{q}}\right)^{q}\left(\left(1-(1-(\mathcal{F}_{{}^{*}\beth})^{q})^{\eta}\right)^{\frac{1}{q}}\right)^{q}\right)^{\frac{1}{q}} \right\rangle,$

$= \left\langle \left(\mathcal{T}_{\beth}\right)^{\eta}\left(\mathcal{T}_{{}^{*}\beth}\right)^{\eta}, \left(\left(1-(1-\mathcal{I}_{\beth})^{\eta}\right)+\left(1-(1-\mathcal{I}_{{}^{*}\beth})^{\eta}\right)-\left(1-(1-\mathcal{I}_{\beth})^{\eta}\right)\left(1-(1-\mathcal{I}_{{}^{*}\beth})^{\eta}\right)\right), \left(1-(1-(\mathcal{F}_{\beth})^{q})^{\eta}+1-(1-(\mathcal{F}_{{}^{*}\beth})^{q})^{\eta}-\left(1-(1-(\mathcal{F}_{\beth})^{q})^{\eta}\right)\left(1-(1-(\mathcal{F}_{{}^{*}\beth})^{q})^{\eta}\right)\right)^{\frac{1}{q}} \right\rangle,$

$= \left\langle \left(\mathcal{T}_{\beth}\right)^{\eta}\left(\mathcal{T}_{{}^{*}\beth}\right)^{\eta}, 1-\left(1-\mathcal{I}_{\beth}\right)^{\eta}\left(1-\mathcal{I}_{{}^{*}\beth}\right)^{\eta}, \left(1-\left(1-(\mathcal{F}_{\beth})^{q}\right)^{\eta}\left(1-(\mathcal{F}_{{}^{*}\beth})^{q}\right)^{\eta}\right)^{\frac{1}{q}} \right\rangle.$

This proves that $(\beth\otimes{}^{*}\beth)^{\eta} = (\beth)^{\eta}\otimes({}^{*}\beth)^{\eta}$. □

3.2. *The comparison method of* q*-RSN numbers*

Considering the SF and AF of SN numbers and q-ROF numbers as proposed by Peng et al. [4] and Liu and Wang [50], the SF and AF for a q-RSN number are defined as follows.

Definition 3.11. Let $\beth = \langle \mathcal{T}_{\beth}, \mathcal{I}_{\beth}, \mathcal{F}_{\beth} \rangle$ be a q-RSN number. An SF on $\beth$ is denoted by the mapping Σ : q-RSNN($\mathfrak{X}$)$\longrightarrow \mathbb{I}$ and defined as

$$\Sigma_{\beth} = \Sigma(\beth) = \frac{1}{3}[(\mathcal{T}_{\beth})^{q} + (1-\mathcal{I}_{\beth}) + (1-(\mathcal{F}_{\beth})^{q})],\ q \geq 1. \tag{7}$$

q-RSNN($\mathfrak{X}$) is the collection of q-RSN numbers on $\mathfrak{X}$.

Definition 3.12. The AF $\daleth$ of a q-RSN number $\beth = \langle \mathcal{T}_{\beth}, \mathcal{I}_{\beth}, \mathcal{F}_{\beth} \rangle$ is represented by the mapping $\daleth$: q-RSNN($\mathfrak{X}$) $\longrightarrow [-1, 1]$ and described as

$$\daleth_{\beth} = \daleth(\beth) = [(\mathcal{T}_{\beth})^{q} - (\mathcal{F}_{\beth})^{q})],\ q \geq 1. \tag{8}$$

q-RSNN($\mathfrak{X}$) is the collection of q-RSN numbers on $\mathfrak{X}$.

Based on Definitions 3.11 and 3.12, we can outline the procedure for comparing q-RSN numbers in the following manner.

Definition 3.13. Given two q-RSN numbers, ${}^{*}\beth = \langle \mathcal{T}_{{}^{*}\beth}, \mathcal{I}_{{}^{*}\beth}, \mathcal{F}_{{}^{*}\beth} \rangle$ and ${}^{\diamond}\beth = \langle \mathcal{T}_{{}^{\diamond}\beth}, \mathcal{I}_{{}^{\diamond}\beth}, \mathcal{F}_{{}^{\diamond}\beth} \rangle$. The method for comparison can be described as follows.

(1) If $\Sigma_{{}^{*}\beth} < \Sigma_{{}^{\diamond}\beth}$, then ${}^{*}\beth <^{\diamond} \beth$.

(2) If $\Sigma_{{}^{*}\beth} > \Sigma_{{}^{\diamond}\beth}$, then ${}^{*}\beth >^{\diamond} \beth$.

(3) If $\Sigma_{{}^{*}\beth} = \Sigma_{{}^{\diamond}\beth}$ and $\daleth_{{}^{*}\beth} < \daleth_{{}^{\diamond}\beth}$, then ${}^{*}\beth <^{\diamond} \beth$.

(4) If $\Sigma_{^*\beth} = \Sigma_{^\diamond\beth}$ and $\daleth_{^*\beth} > \daleth_{^\diamond\beth}$, then $^*\beth >^\diamond \beth$.

Example 3.14. Given two 4-RSN numbers, $^*\beth = \langle 0.85,\ 0.3,\ 0.78\rangle$ and $^\diamond\beth = \langle 0.9,\ 0.4,\ 0.75\rangle$. Based on Definition 3.11, $\Sigma(^*\beth) = \frac{1}{3}[(\mathcal{T}_{^*\beth})^q + (1-(\mathcal{F}_{^*\beth})^q) + (1-\mathcal{I}_{^*\beth})] = \frac{1}{3}[(0.85)^4 + (1-(0.78)^4) + (1-0.3)]$ $= 0.617$ and $\Sigma(^\diamond\beth) = \frac{1}{3}[(\mathcal{T}_{^\diamond\beth})^q + (1-(\mathcal{F}_{^\diamond\beth})^q) + (1-\mathcal{I}_{^\diamond\beth})] = \frac{1}{3}[(0.9)^4 + (1-(0.75)^4) + (1-0.4)] = 0.647$. According to Definition 3.13, $\Sigma(^\diamond\beth) > \Sigma(^*\beth)$. This implies that $^\diamond\beth$ is greater than $^*\beth$.

4. Aggregation operators for q-RSN numbers

Continuing from the algebraic operations of q-RSN numbers, we delve into the aggregation operators utilized in q-RSN numbers.

4.1. q-RSNWA operator

In this context, we establish the definition of the q-RSNWAO and examine its inherent characteristics.

Definition 4.1. Let $^\Lambda\beth = \langle \mathcal{T}_{^\Lambda\beth},\ \mathcal{I}_{^\Lambda\beth},\ \mathcal{F}_{^\Lambda\beth} : \Lambda = 1, ..., m\rangle$ be a family of q-RSN numbers defined on $\mathfrak{X}$. The q-RSNWAO is a transformation denoted as
q-RSNWA: q-RSNN($\mathfrak{X}$) $\longrightarrow$ q-RSNN($\mathfrak{X}$), defined by:

$$\text{q-RSNWA}(^1\beth, ^2\beth, ..., ^m\beth) = {}^1\varpi\ ^1\beth \oplus {}^2\varpi\ ^2\beth \oplus ... {}^m\varpi\ ^m\beth,$$

where $^\Lambda\varpi \in [0\ ,\ 1]$ represents the weight associated with $^\Lambda\beth$, $\forall\Lambda = 1, ..., m$ and $\sum\limits_{\Lambda=1}^{m} {}^\Lambda\varpi = 1$.

Theorem 4.2. *Consider $^\Lambda\beth = \langle \mathcal{T}_{^\Lambda\beth},\ \mathcal{I}_{^\Lambda\beth},\ \mathcal{F}_{^\Lambda\beth} : \Lambda = 1, ..., m\rangle$, to be a family of q-RSN numbers and $^\Lambda\varpi : \{\Lambda = 1, ..., m\}$, to be the weights of $^\Lambda\beth$. Then,*

$$\text{q-RSNWA}(^1\beth, ^2\beth, ..., ^m\beth) = \left\langle \left(1 - \prod_{\Lambda=1}^{m}\left(1 - (\mathcal{T}_{^\Lambda\beth})^q\right)^{^\Lambda\varpi}\right)^{\frac{1}{q}}, \prod_{\Lambda=1}^{m}\left(\mathcal{I}_{^\Lambda\beth}\right)^{^\Lambda\varpi}, \prod_{\Lambda=1}^{m}\left(\mathcal{F}_{^\Lambda\beth}\right)^{^\Lambda\varpi}\right\rangle,$$

$$q \geq 1. \tag{9}$$

Proof. To establish the validity of this theorem, we will employ the mathematical induction technique.

1. For $m = 2$, since $^1\varpi\ ^1\beth = \left\langle \left(1 - (1 - (\mathcal{T}_{^1\beth})^q)^{^1\varpi}\right)^{\frac{1}{q}},\ (\mathcal{I}_{^1\beth})^{^1\varpi},\ (\mathcal{F}_{^1\beth})^{^1\varpi}\right\rangle$,

$^2\varpi\ ^2\beth = \left\langle \left(1 - (1 - (\mathcal{T}_{^2\beth})^q)^{^2\varpi}\right)^{\frac{1}{q}},\ (\mathcal{I}_{^2\beth})^{^2\varpi},\ (\mathcal{F}_{^2\beth})^{^2\varpi}\right\rangle$,

$$\text{q-RSNWA}(^1\beth, ^2\beth) = \left\langle \left(\left(\left(1 - (1 - (\mathcal{T}_{^1\beth})^q)^{^1\varpi}\right)^{\frac{1}{q}}\right)^q + \left(\left(1 - (1 - (\mathcal{T}_{^2\beth})^q)^{^2\varpi}\right)^{\frac{1}{q}}\right)^q - \left(\left(1 - (1 - (\mathcal{T}_{^1\beth})^q)^{^1\varpi}\right)^{\frac{1}{q}}\right)^q \left(\left(1 - (1 - (\mathcal{T}_{^2\beth})^q)^{^2\varpi}\right)^{\frac{1}{q}}\right)^q \right)^{\frac{1}{q}}, (\mathcal{I}_{^1\beth})^{^1\varpi}(\mathcal{I}_{^2\beth})^{^2\varpi}, (\mathcal{F}_{^1\beth})^{^1\varpi}(\mathcal{F}_{^2\beth})^{^2\varpi} \right\rangle,$$

$$= \left\langle \left(1 - (1 - (\mathcal{T}_{^1\beth})^q)^{^1\varpi} + 1 - (1 - (\mathcal{T}_{^2\beth})^q)^{^2\varpi} - \left(1 - (1 - (\mathcal{T}_{^1\beth})^q)^{^1\varpi}\right)\left(1 - (1 - (\mathcal{T}_{^2\beth})^q)^{^2\varpi}\right) \right)^{\frac{1}{q}}, (\mathcal{I}_{^1\beth})^{^1\varpi}(\mathcal{I}_{^2\beth})^{^2\varpi}, (\mathcal{F}_{^1\beth})^{^1\varpi}(\mathcal{F}_{^2\beth})^{^2\varpi} \right\rangle,$$

$$= \left\langle \left(1 - \left(1 - (\mathcal{T}_{^1\beth})^q\right)^{^1\varpi}\left(1 - (\mathcal{T}_{^2\beth})^q\right)^{^2\varpi}\right)^{\frac{1}{q}},\ (\mathcal{I}_{^1\beth})^{^1\varpi}\ (\mathcal{I}_{^2\beth})^{^2\varpi},\ (\mathcal{F}_{^1\beth})^{^1\varpi}\ (\mathcal{F}_{^2\beth})^{^2\varpi} \right\rangle.$$

$$= \left\langle \left(1 - \prod_{\Lambda=1}^{2}\left(1 - (\mathcal{T}_{^\Lambda\beth})^q\right)^{^\Lambda\varpi}\right)^{\frac{1}{q}}, \prod_{\Lambda=1}^{2}\left(\mathcal{I}_{^\Lambda\beth}\right)^{^\Lambda\varpi}, \prod_{\Lambda=1}^{2}\left(\mathcal{F}_{^\Lambda\beth}\right)^{^\Lambda\varpi}\right\rangle.$$

It is evident that Eq (1) is valid for the case of $m = 2$.

2. Assuming that Eq (1) is valid for $\Lambda = m$. Then,

$\mathfrak{q}$-RSNWA$({}^{1}\gimel, {}^{2}\gimel, ..., {}^{m}\gimel) =$

$$\left\langle \left(1 - \prod_{\Lambda=1}^{m} \left(1 - ({\mathcal{T}}_{{}^{\Lambda}\gimel})^{\mathfrak{q}}\right)^{{}^{\Lambda}\varpi}\right)^{\frac{1}{q}}, \prod_{\Lambda=1}^{m} \left({\mathcal{I}}_{{}^{\Lambda}\gimel}\right)^{{}^{\Lambda}\varpi}, \prod_{\Lambda=1}^{m} \left({\mathcal{F}}_{{}^{\Lambda}\gimel}\right)^{{}^{\Lambda}\varpi} \right\rangle.$$

When $\Lambda = m+1$, utilizing the operational rules inherent to $\mathfrak{q}$-RSN numbers, we obtain the following:

$\mathfrak{q}$-RSNWA$({}^{1}\gimel, {}^{2}\gimel, ..., {}^{m+1}\gimel) = \mathfrak{q}$-RSNWA$({}^{1}\gimel, {}^{2}\gimel, ..., {}^{m}\gimel) \oplus {}^{m+1}\varpi\, {}^{m+1}\gimel$,

$$= \left\langle \left(1 - \prod_{\Lambda=1}^{m} \left(1 - ({\mathcal{T}}_{{}^{\Lambda}\gimel})^{\mathfrak{q}}\right)^{{}^{\Lambda}\varpi}\right)^{\frac{1}{q}}, \prod_{\Lambda=1}^{m} \left({\mathcal{I}}_{{}^{\Lambda}\gimel}\right)^{{}^{\Lambda}\varpi}, \prod_{\Lambda=1}^{m} \left({\mathcal{F}}_{{}^{\Lambda}\gimel}\right)^{{}^{\Lambda}\varpi} \right\rangle \oplus$$

$$\left\langle \left(1 - (1 - ({\mathcal{T}}_{{}^{m+1}\gimel})^{\mathfrak{q}})^{{}^{m+1}\varpi}\right)^{\frac{1}{\mathfrak{q}}}, ({\mathcal{I}}_{{}^{m+1}\gimel})^{{}^{m+1}\varpi}, ({\mathcal{F}}_{{}^{m+1}\gimel})^{{}^{m+1}\varpi} \right\rangle,$$

$$= \left[\left\langle \left(1 - \prod_{\Lambda=1}^{m} (1 - ({\mathcal{T}}_{\Lambda_\gimel})^{q})^{\Lambda_\varpi}\right) + \left(1 - (1 - ({\mathcal{T}}_{m+1_\gimel})^{q})^{m+1_\varpi}\right) - \left(1 - \prod_{\Lambda=1}^{m} (1 - ({\mathcal{T}}_{\Lambda_\gimel})^{q})^{\Lambda_\varpi}\right) \right.\right.$$

$$\left.\left. (1 - (1 - ({\mathcal{T}}_{m+1_\gimel})^{q}))^{m+1_\varpi}, \left(\prod_{\Lambda=1}^{m} ({\mathcal{I}}_{\Lambda_\gimel})^{\Lambda_\varpi}\right)\left({\mathcal{I}}_{m+1_\gimel}^{m+1_\varpi}\right)\left(\prod_{\Lambda=1}^{m} ({\mathcal{F}}_{\Lambda_\gimel})^{\Lambda_\varpi}\right)\left({\mathcal{F}}_{m+1_\gimel}^{m+1_\varpi}\right), \right\rangle \right]$$

for $\Lambda = 1, 2, ..., m$.

$$= \left[\left\langle \left(1 - \prod_{\Lambda=1}^{m} (1 - ({\mathcal{T}}_{\Lambda_\gimel})^{q})^{\Lambda_\varpi}(1 - ({\mathcal{T}}_{m+1_\gimel})^{q})^{m+1_\varpi}\right)^{\frac{1}{q}} \right\rangle, \left\langle \prod_{\Lambda=1}^{m+1} ({\mathcal{I}}_{\Lambda_\gimel})^{\Lambda_\varpi} \right\rangle, \left\langle \prod_{\Lambda=1}^{m+1} (I_{\Lambda_\gimel})^{\Lambda_\varpi} \right\rangle \right]$$

$$= \left\langle \left(1 - \prod_{\Lambda=1}^{m+1} \left(1 - ({\mathcal{T}}_{{}^{\Lambda}\gimel})^{\mathfrak{q}}\right)^{{}^{\Lambda}\varpi}\right)^{\frac{1}{q}}, \prod_{\Lambda=1}^{m+1} \left({\mathcal{I}}_{{}^{\Lambda}\gimel}\right)^{{}^{\Lambda}\varpi}, \prod_{\Lambda=1}^{m+1} \left({\mathcal{F}}_{{}^{\Lambda}\gimel}\right)^{{}^{\Lambda}\varpi} \right\rangle, \ \Lambda = 1, 2, ..., m+1.$$

Therefore, we can conclude that when $\Lambda = m + 1$, Eq (1) remains valid. Based on the preceding Steps (1) and (2), it is evident that Eq (1) holds for all values of Λ.

We can establish the following properties for the $\mathfrak{q}$-RSNWAO.

Proposition 4.3. *Idempotency Property: Consider a collection of* $\mathfrak{q}$*-RSN numbers denoted by* ${}^{\Lambda}\gimel = \langle {\mathcal{T}}_{{}^{\Lambda}\gimel}, {\mathcal{I}}_{{}^{\Lambda}\gimel}, {\mathcal{F}}_{{}^{\Lambda}\gimel} : \Lambda = 1, ..., m\rangle$. *If* ${}^{\Lambda}\gimel = \gimel = \langle {\mathcal{T}}_{\gimel}, {\mathcal{I}}_{\gimel}, {\mathcal{F}}_{\gimel}\rangle$, $\forall \Lambda = 1, ..., m$. *Then,* $\mathfrak{q}$*-RSNWA*$({}^{1}\gimel, {}^{2}\gimel, ..., {}^{m}\gimel) = \gimel = \langle {\mathcal{T}}_{\gimel}, {\mathcal{I}}_{\gimel}, {\mathcal{F}}_{\gimel}\rangle$.

Proof. Since ${}^{\Lambda}\gimel = \gimel = \langle {\mathcal{T}}_{\gimel}, {\mathcal{I}}_{\gimel}, {\mathcal{F}}_{\gimel}\rangle$, $\forall \Lambda = 1, ..., m$. Then, based on Theorem 4.2,

$\mathfrak{q}$-RSNWA$({}^{1}\gimel, {}^{2}\gimel, ..., {}^{m}\gimel) =$

$$\left\langle \left(1 - \prod_{\Lambda=1}^{m} \left(1 - ({\mathcal{T}}_{\gimel})^{\mathfrak{q}}\right)^{{}^{\Lambda}\varpi}\right)^{\frac{1}{q}}, \prod_{\Lambda=1}^{m} \left({\mathcal{I}}_{\gimel}\right)^{{}^{\Lambda}\varpi}, \prod_{\Lambda=1}^{m} \left({\mathcal{F}}_{\gimel}\right)^{{}^{\Lambda}\varpi} \right\rangle,$$

$$= \left\langle \left(1 - (1 - ({\mathcal{T}}_{\gimel})^{q})^{\sum_{\Lambda=1}^{m} \Lambda_\varpi}\right)^{\frac{1}{q}}, ({\mathcal{I}}_{\gimel})^{\sum_{\Lambda=1}^{m} \Lambda_\varpi}, ({\mathcal{F}}_{\gimel})^{\sum_{\Lambda=1}^{m} \Lambda_\varpi} \right\rangle,$$

$$= \left\langle \left(1 - (1 - ({\mathcal{T}}_{\gimel})^{q})\right)^{\frac{1}{q}}, ({\mathcal{I}}_{\gimel}), ({\mathcal{F}}_{\gimel}) \right\rangle = \langle {\mathcal{T}}_{\gimel}, {\mathcal{I}}_{\gimel}, {\mathcal{F}}_{\gimel}\rangle = \gimel.$$

□

Proposition 4.4. *Boundedness Property: Consider a collection of* $\mathfrak{q}$*-RSN numbers denoted by* ${}^{\Lambda}\gimel = \langle {\mathcal{T}}_{{}^{\Lambda}\gimel}, {\mathcal{I}}_{{}^{\Lambda}\gimel}, {\mathcal{F}}_{{}^{\Lambda}\gimel} : \Lambda = 1, ..., m\rangle$. *If* ${}^{-}\gimel = \langle {\mathcal{T}}_{{}^{-}\gimel}, {\mathcal{I}}_{{}^{+}\gimel}, {\mathcal{F}}_{{}^{+}\gimel}\rangle$ *and* ${}^{+}\gimel = \langle {\mathcal{T}}_{{}^{+}\gimel}, {\mathcal{I}}_{{}^{-}\gimel}, {\mathcal{F}}_{{}^{-}\gimel}\rangle$, *where,* ${\mathcal{T}}_{{}^{-}\gimel} = \min_{\Lambda}\{{\mathcal{T}}_{{}^{\Lambda}\gimel}\}$, ${\mathcal{T}}_{{}^{+}\gimel} = \max_{\Lambda}\{{\mathcal{T}}_{{}^{\Lambda}\gimel}\}$, ${\mathcal{I}}_{{}^{-}\gimel} = \min_{\Lambda}\{{\mathcal{I}}_{{}^{\Lambda}\gimel}\}$, ${\mathcal{I}}_{{}^{+}\gimel} = \max_{\Lambda}\{{\mathcal{I}}_{{}^{\Lambda}\gimel}\}$, ${\mathcal{F}}_{{}^{-}\gimel} = \min_{\Lambda}\{{\mathcal{F}}_{{}^{\Lambda}\gimel}\}$ *and* ${\mathcal{F}}_{{}^{+}\gimel} = \max_{\Lambda}\{{\mathcal{F}}_{{}^{\Lambda}\gimel}\}$ *then,* ${}^{-}\gimel \leq \mathfrak{q} - RSNWA({}^{1}\gimel, {}^{2}\gimel, ..., {}^{m}\gimel) \leq^{+} \gimel$.

Proof. Since ${\mathcal{T}}_{{}^{-}\gimel} \leq {\mathcal{T}}_{{}^{\Lambda}\gimel} \leq {\mathcal{T}}_{{}^{+}\gimel}$ then, $\forall \mathfrak{q} \geq 1$, and we obtain:

$(\mathcal{T}_{-\beth})^{q} \leq (\mathcal{T}_{{}^{\Lambda}\beth})^{q} \leq (\mathcal{T}_{+\beth})^{q} \Rightarrow 1-(\mathcal{T}_{-\beth})^{q} \geq 1-(\mathcal{T}_{{}^{\Lambda}\beth})^{q} \geq 1-(\mathcal{T}_{+\beth})^{q} \Rightarrow \left(1-(\mathcal{T}_{-\beth})^{q}\right)^{\Lambda\varpi} \geq \left(1-(\mathcal{T}_{{}^{\Lambda}\beth})^{q}\right)^{\Lambda\varpi} \geq \left(1-(\mathcal{T}_{+\beth})^{q}\right)^{\Lambda\varpi} \Rightarrow \prod_{\Lambda=1}^{m}\left(1-(\mathcal{T}_{-\beth})^{q}\right)^{\Lambda\varpi} \geq \prod_{\Lambda=1}^{m}\left(1-(\mathcal{T}_{{}^{\Lambda}\beth})^{q}\right)^{\Lambda\varpi} \geq \prod_{\Lambda=1}^{m}\left(1-(\mathcal{T}_{+\beth})^{q}\right)^{\Lambda\varpi} \Rightarrow 1-\prod_{\Lambda=1}^{m}\left(1-(\mathcal{T}_{-\beth})^{q}\right)^{\Lambda\varpi} \leq 1-\prod_{\Lambda=1}^{m}\left(1-(\mathcal{T}_{{}^{\Lambda}\beth})^{q}\right)^{\Lambda\varpi} \leq 1-\prod_{\Lambda=1}^{m}\left(1-(\mathcal{T}_{+\beth})^{q}\right)^{\Lambda\varpi} \Rightarrow \left(1-\prod_{\Lambda=1}^{m}\left(1-(\mathcal{T}_{-\beth})^{q}\right)^{\Lambda\varpi}\right)^{\frac{1}{q}} \leq \left(1-\prod_{\Lambda=1}^{m}\left(1-(\mathcal{T}_{{}^{\Lambda}\beth})^{q}\right)^{\Lambda\varpi}\right)^{\frac{1}{q}} \leq \left(1-\prod_{\Lambda=1}^{m}\left(1-(\mathcal{T}_{+\beth})^{q}\right)^{\Lambda\varpi}\right)^{\frac{1}{q}}$. Since, $\left(1-\prod_{\Lambda=1}^{m}\left(1-(\mathcal{T}_{-\beth})^{q}\right)^{\Lambda\varpi}\right)^{\frac{1}{q}} = \mathcal{T}_{-\beth}$, and $\left(1-\prod_{\Lambda=1}^{m}\left(1-(\mathcal{T}_{+\beth})^{q}\right)^{\Lambda\varpi}\right)^{\frac{1}{q}} = \mathcal{T}_{+\beth}$, then,

$\mathcal{T}_{-\beth} \leq \left(1-\prod_{\Lambda=1}^{m}\left(1-(\mathcal{T}_{{}^{\Lambda}\beth})^{q}\right)^{\Lambda\varpi}\right)^{\frac{1}{q}} \leq \mathcal{T}_{+\beth}$. Likewise, as $\mathcal{I}_{-\beth} \leq \mathcal{I}_{{}^{\Lambda}\beth} \leq \mathcal{I}_{+\beth}$, then, $(\mathcal{I}_{-\beth})^{\Lambda\varpi} \leq (\mathcal{I}_{{}^{\Lambda}\beth})^{\Lambda\varpi} \leq (\mathcal{I}_{+\beth})^{\Lambda\varpi} \Rightarrow \prod_{\Lambda=1}^{m}(\mathcal{I}_{-\beth})^{\Lambda\varpi} \leq \prod_{\Lambda=1}^{m}(\mathcal{I}_{{}^{\Lambda}\beth})^{\Lambda\varpi} \leq \prod_{\Lambda=1}^{m}(\mathcal{I}_{+\beth})^{\Lambda\varpi}$. Since $\prod_{\Lambda=1}^{m}(\mathcal{I}_{-\beth})^{\Lambda\varpi} = \mathcal{I}_{-\beth}$ and $\prod_{\Lambda=1}^{m}(\mathcal{I}_{+\beth})^{\Lambda\varpi} = \mathcal{I}_{+\beth}$ then, $\mathcal{I}_{-\beth} \leq \prod_{\Lambda=1}^{m}(\mathcal{I}_{{}^{\Lambda}\beth})^{\Lambda\varpi} \leq \mathcal{I}_{+\beth}$.

In the same way, as $\mathcal{F}_{-\beth} \leq \mathcal{F}_{{}^{\Lambda}\beth} \leq \mathcal{F}_{+\beth}$, then $(\mathcal{F}_{-\beth})^{\Lambda\varpi} \leq (\mathcal{F}_{{}^{\Lambda}\beth})^{\Lambda\varpi} \leq (\mathcal{F}_{+\beth})^{\Lambda\varpi} \Rightarrow \prod_{\Lambda=1}^{m}(\mathcal{F}_{-\beth})^{\Lambda\varpi} \leq \prod_{\Lambda=1}^{m}(\mathcal{F}_{{}^{\Lambda}\beth})^{\Lambda\varpi} \leq \prod_{\Lambda=1}^{m}(\mathcal{F}_{+\beth})^{\Lambda\varpi}$. Since $\prod_{\Lambda=1}^{m}(\mathcal{F}_{-\beth})^{\Lambda\varpi} = \mathcal{F}_{-\beth}$ and $\prod_{\Lambda=1}^{m}(\mathcal{F}_{+\beth})^{\Lambda\varpi} = \mathcal{F}_{+\beth}$ then $\mathcal{F}_{-\beth} \leq \prod_{\Lambda=1}^{m}(\mathcal{F}_{{}^{\Lambda}\beth})^{\Lambda\varpi} \leq \mathcal{F}_{+\beth}$.

Now, let q-RSNWA$({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) = \beth = \langle \mathcal{T}_{\beth}, \mathcal{I}_{\beth}, \mathcal{F}_{\beth}\rangle$. Then,

$\Sigma(\beth) = \frac{1}{3}\left[(\mathcal{T}_{\beth})^{q} + (1-(\mathcal{F}_{\beth})^{q}) + (1-\mathcal{I}_{\beth})\right] \geq \frac{1}{3}\left[(\mathcal{T}_{-\beth})^{q} + (1-(\mathcal{F}_{+\beth})^{q}) + (1-\mathcal{I}_{+\beth})\right] = \Sigma({}^{-}\beth)$ and $\Sigma(\beth) = \frac{1}{3}\left[(\mathcal{T}_{\beth})^{q} + (1-(\mathcal{F}_{\beth})^{q}) + (1-\mathcal{I}_{\beth})\right] \leq \frac{1}{3}\left[(\mathcal{T}_{+\beth})^{q} + (1-(\mathcal{F}_{-\beth})^{q}) + (1-\mathcal{I}_{-\beth})\right] = \Sigma({}^{+}\beth)$.

This implies ${}^{-}\beth \leq q-RNWA({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) \leq {}^{+}\beth$. □

Proposition 4.5. *Monotonicity Property: Consider two collections of* q*-RSN numbers, denoted by* ${}^{\Lambda}\beth = \langle \mathcal{T}_{{}^{\Lambda}\beth}, \mathcal{I}_{{}^{\Lambda}\beth}, \mathcal{F}_{{}^{\Lambda}\beth} : \Lambda = 1, ..., m\rangle$ *and* ${}^{\Lambda}\beth^{*} = \langle \mathcal{T}_{{}^{\Lambda}\beth^{*}}, \mathcal{I}_{{}^{\Lambda}\beth^{*}}, \mathcal{F}_{{}^{\Lambda}\beth^{*}} : \Lambda = 1, ..., m\rangle$. *If,* $\mathcal{T}_{{}^{\Lambda}\beth} \leq \mathcal{T}_{{}^{\Lambda}\beth^{*}}$, $\mathcal{I}_{{}^{\Lambda}\beth} \geq \mathcal{I}_{{}^{\Lambda}\beth^{*}}$ *and* $\mathcal{F}_{{}^{\Lambda}\beth} \geq \mathcal{F}_{{}^{\Lambda}\beth^{*}}$, $\forall \Lambda = 1, ..., m$, *then,* $q-RSNWA({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) \leq q-RSNWA({}^{1}\beth^{*}, {}^{2}\beth^{*}, ..., {}^{m}\beth^{*})$.

Proof. Since $\mathcal{T}_{{}^{\Lambda}\beth} \leq \mathcal{T}_{{}^{\Lambda}\beth^{*}}$, for $q \geq 1$, we obtain

$(\mathcal{T}_{{}^{\Lambda}\beth})^{q} \leq (\mathcal{T}_{{}^{\Lambda}\beth^{*}})^{q} \Rightarrow 1-(\mathcal{T}_{{}^{\Lambda}\beth})^{q} \geq 1-(\mathcal{T}_{{}^{\Lambda}\beth^{*}})^{q} \Rightarrow (1-(\mathcal{T}_{{}^{\Lambda}\beth})^{q})^{\Lambda\varpi} \geq (1-(\mathcal{T}_{{}^{\Lambda}\beth^{*}})^{q})^{\Lambda\varpi} \Rightarrow \prod_{\Lambda=1}^{m}(1-(\mathcal{T}_{{}^{\Lambda}\beth})^{q})^{\Lambda\varpi} \geq \prod_{\Lambda=1}^{m}(1-(\mathcal{T}_{{}^{\Lambda}\beth^{*}})^{q})^{\Lambda\varpi} \Rightarrow 1-\prod_{\Lambda=1}^{m}(1-(\mathcal{T}_{{}^{\Lambda}\beth})^{q})^{\Lambda\varpi} \leq 1-\prod_{\Lambda=1}^{m}(1-(\mathcal{T}_{{}^{\Lambda}\beth^{*}})^{q})^{\Lambda\varpi} \Rightarrow \left(1-\prod_{\Lambda=1}^{m}\left(1-(\mathcal{T}_{{}^{\Lambda}\beth})^{q}\right)^{\Lambda\varpi}\right)^{\frac{1}{q}} \leq \left(1-\prod_{\Lambda=1}^{m}\left(1-(\mathcal{T}_{{}^{\Lambda}\beth^{*}})^{q}\right)^{\Lambda\varpi}\right)^{\frac{1}{q}}$.

Similarly, since $\mathcal{I}_{{}^{\Lambda}\beth} \geq \mathcal{I}_{{}^{\Lambda}\beth^{*}}$, then $(\mathcal{I}_{{}^{\Lambda}\beth})^{\Lambda\varpi} \geq (\mathcal{I}_{{}^{\Lambda}\beth^{*}})^{\Lambda\varpi} \Rightarrow \prod_{\Lambda=1}^{m}(\mathcal{I}_{{}^{\Lambda}\beth})^{\Lambda\varpi} \geq \prod_{\Lambda=1}^{m}(\mathcal{I}_{{}^{\Lambda}\beth^{*}})^{\Lambda\varpi}$. In a similar vein, as $\mathcal{F}_{{}^{\Lambda}\beth} \geq \mathcal{F}_{{}^{\Lambda}\beth^{*}}$, then $(\mathcal{F}_{{}^{\Lambda}\beth})^{\Lambda\varpi} \geq (\mathcal{F}_{{}^{\Lambda}\beth^{*}})^{\Lambda\varpi} \Rightarrow \prod_{\Lambda=1}^{m}(\mathcal{F}_{{}^{\Lambda}\beth})^{\Lambda\varpi} \geq \prod_{\Lambda=1}^{m}(\mathcal{F}_{{}^{\Lambda}\beth^{*}})^{\Lambda\varpi}$.

Now, let $\mathfrak{q}$-RSNWA$({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) = \beth = \langle \mathcal{T}_{\beth},\ \mathcal{I}_{\beth},\ \mathcal{F}_{\beth} \rangle$ and $\mathfrak{q}$-RSNWA$({}^{1}\beth^{*}, {}^{2}\beth^{*}, ..., {}^{m}\beth^{*}) = \beth^{*} = \langle \mathcal{T}_{\beth^{*}},\ \mathcal{I}_{\beth^{*}},\ \mathcal{F}_{\beth^{*}} \rangle$. Then,

$\Sigma(\beth) = \frac{1}{3}\big[(\mathcal{T}_{\beth})^{q} + (1-(\mathcal{F}_{\beth})^{q}) + (1-\mathcal{I}_{\beth})\big] \leq \frac{1}{3}\big[(\mathcal{T}_{\beth^{*}})^{q} + (1-(\mathcal{F}_{\beth^{*}})^{q}) + (1-\mathcal{I}_{\beth^{*}})\big] = \Sigma(\beth^{*})$. This indicates that $\mathfrak{q} - RSNWA({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) \leq \mathfrak{q} - RSNWA({}^{1}\beth^{*}, {}^{2}\beth^{*}, ..., {}^{m}\beth^{*})$. □

4.2. $\mathfrak{q}$-*RSNWG operator*

In this area, we define and investigate the essential characteristics of the $\mathfrak{q}$-RSNWGO.

Definition 4.6. Let ${}^{\Lambda}\beth = \langle \mathcal{T}_{{}^{\Lambda}\beth},\ \mathcal{I}_{{}^{\Lambda}\beth},\ \mathcal{F}_{{}^{\Lambda}\beth} : \Lambda = 1, ..., m \rangle$ be a family of $\mathfrak{q}$-RSN numbers defined on $\mathfrak{X}$. The $\mathfrak{q}$-RSNWGO is a transformation denoted as $\mathfrak{q}$-RSNWG: $\mathfrak{q}$-RSNN$(\mathfrak{X}) \longrightarrow \mathfrak{q}$-RSNN$(\mathfrak{X})$, defined by:

$$\mathfrak{q}\text{-RSNWG}({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) = {}^{1}\varpi\ {}^{1}\beth \otimes {}^{2}\varpi\ {}^{2}\beth \otimes ... {}^{m}\varpi\ {}^{m}\beth,$$

where ${}^{\Lambda}\varpi \in \mathbb{I}$ represents the weight associated with ${}^{\Lambda}\beth, \forall \Lambda = 1, ..., m$ and $\sum\limits_{\Lambda=1}^{m} {}^{\Lambda}\varpi = 1$.

Theorem 4.7. *Consider* ${}^{\Lambda}\beth = \langle \mathcal{T}_{{}^{\Lambda}\beth},\ \mathcal{I}_{{}^{\Lambda}\beth},\ \mathcal{F}_{{}^{\Lambda}\beth} : \Lambda = 1, ..., m \rangle$*, to be a family of* $\mathfrak{q}$*-RSN numbers and* ${}^{\Lambda}\varpi : \{\Lambda = 1, ..., m\}$*, to be the weights of* ${}^{\Lambda}\beth$*. Then,*

$$\mathfrak{q}\text{-}RSNWG({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) = \left\langle \prod_{\Lambda=1}^{m}\left(\mathcal{T}_{{}^{\Lambda}\beth}\right)^{{}^{\Lambda}\varpi}, 1 - \prod_{\Lambda=1}^{m}\left(1 - (\mathcal{I}_{{}^{\Lambda}\beth})\right)^{{}^{\Lambda}\varpi}, \left(1 - \prod_{\Lambda=1}^{m}\left(1 - (\mathcal{F}_{{}^{\Lambda}\beth})^{\mathfrak{q}}\right)^{{}^{\Lambda}\varpi}\right)^{\frac{1}{q}} \right\rangle,$$
$$q \geq 1. \tag{10}$$

Proof. To establish the validity of this theorem, we will employ the mathematical induction technique.

(1) For $m = 2$, since ${}^{1}\varpi\ {}^{1}\beth = \left\langle (\mathcal{T}_{{}^{1}\beth})^{{}^{1}\varpi}\ 1 - (1 - \mathcal{I}_{{}^{1}\beth})^{{}^{1}\varpi}, \left(1 - (1 - (\mathcal{F}_{{}^{1}\beth})^{\mathfrak{q}})^{{}^{1}\varpi}\right)^{\frac{1}{\mathfrak{q}}} \right\rangle$,

${}^{2}\varpi\ {}^{2}\beth = \left\langle (\mathcal{T}_{{}^{2}\beth})^{{}^{2}\varpi}\ 1 - (1 - \mathcal{I}_{{}^{2}\beth})^{{}^{2}\varpi}, \left(1 - (1 - (\mathcal{F}_{{}^{2}\beth})^{\mathfrak{q}})^{{}^{2}\varpi}\right)^{\frac{1}{\mathfrak{q}}} \right\rangle$,

$$\mathfrak{q}\text{-RSNWG}({}^{1}\beth, {}^{2}\beth) = \left\langle (\mathcal{T}_{{}^{1}\beth})^{{}^{1}\varpi} (\mathcal{T}_{{}^{2}\beth})^{{}^{2}\varpi}, \left(\left(1-(1-\mathcal{I}_{{}^{1}\beth})^{{}^{1}\varpi}\right) + \left(1-(1-\mathcal{I}_{{}^{2}\beth})^{{}^{2}\varpi}\right) - \left(1-(1-\mathcal{I}_{{}^{1}\beth})^{{}^{1}\varpi}\right)\left(1 - (1-\mathcal{I}_{{}^{2}\beth})^{{}^{2}\varpi}\right)\right), \left(\left(\left(1-(1-(\mathcal{F}_{{}^{1}\beth})^{\mathfrak{q}})^{{}^{1}\varpi}\right)^{\frac{1}{\mathfrak{q}}}\right)^{\mathfrak{q}} + \left(\left(1-(1-(\mathcal{F}_{{}^{2}\beth})^{\mathfrak{q}})^{{}^{2}\varpi}\right)^{\frac{1}{\mathfrak{q}}}\right)^{\mathfrak{q}} - \left(\left(1-(1-(\mathcal{F}_{{}^{1}\beth})^{\mathfrak{q}})^{{}^{1}\varpi}\right)^{\frac{1}{\mathfrak{q}}}\right)^{\mathfrak{q}}\left(\left(1-(1-(\mathcal{F}_{{}^{2}\beth})^{\mathfrak{q}})^{{}^{2}\varpi}\right)^{\frac{1}{\mathfrak{q}}}\right)^{\mathfrak{q}}\right)^{\frac{1}{\mathfrak{q}}} \right\rangle,$$

$$= \left\langle (\mathcal{T}_{{}^{1}\beth})^{{}^{1}\varpi} (\mathcal{T}_{{}^{2}\beth})^{{}^{2}\varpi}, \left(\left(1-(1-\mathcal{I}_{{}^{1}\beth})^{{}^{1}\varpi}\right) + \left(1-(1-\mathcal{I}_{{}^{2}\beth})^{{}^{2}\varpi}\right) - \left(1-(1-\mathcal{I}_{{}^{1}\beth})^{{}^{1}\varpi}\right)\left(1-(1-\mathcal{I}_{{}^{2}\beth})^{{}^{2}\varpi}\right)\right), \left(1-(1-(\mathcal{F}_{{}^{1}\beth})^{\mathfrak{q}})^{{}^{1}\varpi} + 1-(1-(\mathcal{F}_{{}^{2}\beth})^{\mathfrak{q}})^{{}^{2}\varpi} - \left(1-(1-(\mathcal{F}_{{}^{1}\beth})^{\mathfrak{q}})^{{}^{1}\varpi}\right)\left(1-(1-(\mathcal{F}_{{}^{2}\beth})^{\mathfrak{q}})^{{}^{2}\varpi}\right)\right)^{\frac{1}{\mathfrak{q}}} \right\rangle,$$

$$= \left\langle (\mathcal{T}_{{}^{1}\beth})^{{}^{1}\varpi} (\mathcal{T}_{{}^{2}\beth})^{{}^{2}\varpi}, 1 - \left(1-\mathcal{I}_{{}^{1}\beth}\right)^{{}^{1}\varpi}\left(1-\mathcal{I}_{{}^{2}\beth}\right)^{{}^{2}\varpi}, \left(1-\left(1-(\mathcal{F}_{{}^{1}\beth})^{\mathfrak{q}}\right)^{{}^{1}\varpi}\left(1-(\mathcal{F}_{{}^{2}\beth})^{\mathfrak{q}}\right)^{{}^{2}\varpi}\right)^{\frac{1}{\mathfrak{q}}} \right\rangle.$$

$$= \left\langle \prod_{\Lambda=1}^{2}\left(\mathcal{T}_{{}^{\Lambda}\beth}\right)^{{}^{\Lambda}\varpi}, 1 - \prod_{\Lambda=1}^{2}\left(1 - (\mathcal{I}_{{}^{\Lambda}\beth})\right)^{{}^{\Lambda}\varpi}, \left(1 - \prod_{\Lambda=1}^{2}\left(1 - (\mathcal{F}_{{}^{\Lambda}\beth})^{\mathfrak{q}}\right)^{{}^{\Lambda}\varpi}\right)^{\frac{1}{q}} \right\rangle.$$

It is evident that Eq (10) is valid for the case of $m = 2$.

(2) Assume that Eq (10) is valid for $\Lambda = m$. Then,

$$\mathfrak{q}\text{-RSNWG}({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) = \left\langle \prod_{\Lambda=1}^{m} \left(\mathcal{T}_{^{\Lambda}\beth}\right)^{^{\Lambda}\varpi},\ 1 - \prod_{\Lambda=1}^{m} \left(1 - (\mathcal{I}_{^{\Lambda}\beth})\right)^{^{\Lambda}\varpi},\ \left(1 - \prod_{\Lambda=1}^{m} \left(1 - (\mathcal{F}_{^{\Lambda}\beth})^{q}\right)^{^{\Lambda}\varpi}\right)^{\frac{1}{q}} \right\rangle.$$

When $\Lambda = m+1$, utilizing the operational rules inherent to $\mathfrak{q}$-RSN numbers, we obtain the following:

$$\mathfrak{q}\text{-RSNWG}({}^{1}\beth, {}^{2}\beth, ..., {}^{m+1}\beth) = \mathfrak{q}\text{-RSNWG}({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) \otimes {}^{m+1}\varpi\ {}^{m+1}\beth,$$

$$= \left\langle \prod_{\Lambda=1}^{m} (\mathcal{T}_{\Lambda_\beth})^{\Lambda_\varpi}, 1 - \prod_{\Lambda=1}^{m} (1 - (\mathcal{I}_{\Lambda_\beth}))^{\Lambda_\varpi}, \left(1 - \prod_{\Lambda=1}^{m} (1 - (\mathcal{F}_{\Lambda_\beth})^{q})^{\Lambda_\varpi}\right)^{\frac{1}{q}} \right\rangle \otimes$$

$$= \left[\left\langle \left(\prod_{\Lambda=1}^{m} \left(\mathcal{T}_{\Lambda_\beta}\right)^{\Lambda_\varpi}\right) \left(\mathcal{T}^{m+1_\varpi}_{m+1_\beth}\right), \left(1 - \prod_{\Lambda=1}^{m} (1 - (\mathcal{I}_{\Lambda_\beth}))^{\Lambda_\varpi} + (1 - (1 - \mathcal{I}_{m+1_\beth})^{m+1_\varpi}\right) - \right.\right.$$

$$\left(1 - \prod_{\Lambda=1}^{m} (1 - (\mathcal{I}_{\Lambda_\beth}))^{\Lambda_\varpi}\right) (1 - (1 - \mathcal{I}_{m+1_\beth})^{m+1_\varpi})\Big), \left(\left(1 - \prod_{\Lambda=1}^{m} (1 - (\mathcal{F}_{\Lambda_\beth})^{q})^{\Lambda_\varpi}\right) \left((1 - (1 - (\mathcal{F}_{m+1_\beth})^{q})^{m+1_\varpi}\right)\right.$$

$$\left.\left.- \left(1 - \prod_{\Lambda=1}^{m} (1 - (\mathcal{F}_{\Lambda_\beth})^{q})^{\Lambda_\varpi}\right) (1 - (1 - (\mathcal{F}_{m+1_\beth})^{q})^{m+1_\varpi}\right)^{\frac{1}{q}}\right\rangle\right], \text{ for } \Lambda = 1, 2, ..., m.$$

$$= \left[\left\langle \prod_{\Lambda=1}^{m+1} (\mathcal{T}_{\Lambda_\beth})^{\Lambda_\varpi} \right\rangle, \left\langle 1 - \prod_{\Lambda=1}^{m} (1 - (\mathcal{I}_{\Lambda_\beth}))^{\Lambda_\varpi} (1 - (\mathcal{I}_{m+1_\beth}))^{m+1_\varpi} \right\rangle,\right.$$

$$\left.\left\langle \left(1 - \prod_{\Lambda=1}^{m} (1 - (\mathcal{I}_{\Lambda_\beth})^{q})^{\Lambda_\varpi}\ \ (1 - (\mathcal{F}_{m+1_\beth})^{q})^{m+1_\varpi}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \left\langle \prod_{\Lambda=1}^{m+1} (\mathcal{T}\Lambda\beth)\,\Lambda\varpi, 1 - \prod_{\Lambda=1}^{m+1} (1 - (\mathcal{I}\Lambda\beth))\,\Lambda\varpi, \left(1 - \prod_{\Lambda=1}^{m+1} (1 - (F\Lambda\beth)^{q})\,\Lambda\varpi\right)^{\frac{1}{q}} \right\rangle,$$

$\Lambda = 1, 2, ..., m + 1$.
Therefore, we can conclude that when $\Lambda = m + 1$, Eq (10) remains valid. Based on the preceding Steps (1) and (2), it is evident that Eq (8) holds for all values of Λ. □

The $\mathfrak{q}$-RSNWGO possesses the following properties, which are listed here without proof since the proof is comparable to that of the $\mathfrak{q}$-RSNWAO.

Proposition 4.8. *Idempotency Property: Consider a collection of* $\mathfrak{q}$*-RSN numbers denoted by* ${}^{\Lambda}\beth = \langle \mathcal{T}_{^{\Lambda}\beth}, \mathcal{I}_{^{\Lambda}\beth}, \mathcal{F}_{^{\Lambda}\beth} : \Lambda = 1, ..., m\rangle$. *If* ${}^{\Lambda}\beth = \beth = \langle \mathcal{T}_{\beth}, \mathcal{I}_{\beth}, \mathcal{F}_{\beth}\rangle$, $\forall\Lambda = 1, ..., m$. *Then,* $\mathfrak{q}$*-RSNWG*$({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) = \beth = \langle \mathcal{T}_{\beth}, \mathcal{I}_{\beth}, \mathcal{F}_{\beth}\rangle$.

Proposition 4.9. *Boundedness Property: Consider a collection of* $\mathfrak{q}$*-RSN numbers denoted by* ${}^{\Lambda}\beth = \langle \mathcal{T}_{^{\Lambda}\beth}, \mathcal{I}_{^{\Lambda}\beth}, \mathcal{F}_{^{\Lambda}\beth} : \Lambda = 1, ..., m\rangle$. *If* ${}^{-}\beth = \langle \mathcal{T}_{^{-}\beth}, \mathcal{I}_{^{+}\beth}, \mathcal{F}_{^{+}\beth}\rangle$ *and* ${}^{+}\beth = \langle \mathcal{T}_{^{+}\beth}, \mathcal{I}_{^{-}\beth}, \mathcal{F}_{^{-}\beth}\rangle$, *where,* $\mathcal{T}_{^{-}\beth} = \min_{\Lambda}\{\mathcal{T}_{^{\Lambda}\beth}\}$, $\mathcal{T}_{^{+}\beth} = \max_{\Lambda}\{\mathcal{T}_{^{\Lambda}\beth}\}$, $\mathcal{I}_{^{-}\beth} = \min_{\Lambda}\{\mathcal{I}_{^{\Lambda}\beth}\}$, $\mathcal{I}_{^{+}\beth} = \max_{\Lambda}\{\mathcal{I}_{^{\Lambda}\beth}\}$, $\mathcal{F}_{^{-}\beth} = \min_{\Lambda}\{\mathcal{F}_{^{\Lambda}\beth}\}$ *and* $\mathcal{F}_{^{+}\beth} = \max_{\Lambda}\{\mathcal{F}_{^{\Lambda}\beth}\}$, *then* ${}^{-}\beth \leq \mathfrak{q} - RNWG({}^{1}\beth, {}^{2}\beth, ...,$
${}^{m}\beth) \leq^{+} \beth$.

Proposition 4.10. *Monotonicity Property: Consider two collections of* $\mathfrak{q}$*-RSN numbers, denoted by* ${}^{\Lambda}\beth = \langle \mathcal{T}_{^{\Lambda}\beth}, \mathcal{I}_{^{\Lambda}\beth}, \mathcal{F}_{^{\Lambda}\beth} : \Lambda = 1, ..., m\rangle$ *and* ${}^{\Lambda}\beth^{*} = \langle \mathcal{T}_{^{\Lambda}\beth^{*}}, \mathcal{I}_{^{\Lambda}\beth^{*}}, \mathcal{F}_{^{\Lambda}\beth^{*}} : \Lambda = 1, ..., m\rangle$. *If,* $\mathcal{T}_{^{\Lambda}\beth} \leq \mathcal{T}_{^{\Lambda}\beth^{*}}$, $\mathcal{I}_{^{\Lambda}\beth} \geq \mathcal{I}_{^{\Lambda}\beth^{*}}$ *and* $\mathcal{F}_{^{\Lambda}\beth} \geq \mathcal{F}_{^{\Lambda}\beth^{*}}$, $\forall\Lambda = 1, ..., m$, *then,* $\mathfrak{q} - RSNWG({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) \leq \mathfrak{q} - RSNWG({}^{1}\beth^{*}, {}^{2}\beth^{*}, ..., {}^{m}\beth^{*})$.

5. The MCDM method relies on the AOs derived from q-RSN sets

This section introduces a methodology for solving MCDM issues using AOs and SFs tailored for q-RSN sets within a q-RSN setting. To accomplish this, we formulate a problem in MCDM where the evaluation results are expressed using q-RSN numbers. To tackle this MCDM problem, we implement the q-RSNWAO and q-RSNWGO. In this context, we suppose that the alternatives $X_{k=1,2,\ldots,n}$ are derived from assessments conducted by decision makers concerning criteria $\zeta_{i=1,2,\ldots,m}$, along with corresponding weights $\varpi_{i=1,2,\ldots,m}$ that satisfy the condition $\sum_{i=1}^{m} \varpi_i = 1$. We extend an invitation to experts to evaluate the q-RSN data for each criterion in order to select the most favorable candidate. In this context, we introduce the following algorithm to facilitate the selection of the optimal candidate.

The criteria assessed for each alternative are provided as q-RSN numbers, which are organized in a matrix known as the decision matrix.

To ensure criterion consistency, the obtained decision matrix, containing two types of criteria, is normalized. This normalization process is achieved using the following equation:

$$\beth = \begin{cases} \langle \mathcal{T}_\beth, \mathcal{I}_\beth, \mathcal{F}_\beth \rangle \quad \textit{for benefit criteria} \\ \\ \langle \mathcal{F}_\beth, 1 - \mathcal{I}_\beth, \mathcal{T}_\beth \rangle \quad \textit{for cost criteria} \end{cases} \tag{11}$$

Through the utilization of AOs of q-RSN numbers, the diverse criterion values of each candidate are amalgamated into a singular representative value denoted as $\mathbb{Z}_{k=1,2,\ldots,n}$.

Calculate the score values of $\mathbb{Z}_{k=1,2,\ldots,n}$ for each candidate, using the SF defined in Definition 3.11. To select the optimal candidate, choose X_j if $\mathbb{Z}_j$ is the maximum value among all $\mathbb{Z}_k$.

In case of a tie in the score values between two candidates, the optimal candidate is determined using the AF from Definition 3.12.

Therefore, the following steps explain the algorithm that builds based on AOs and SFs tailored for q-RSN sets within a q-RSN setting to solve MCDM issues:

Step 1. Construct q-RSN data in matrix form.
Step 2. Standardize the initial matrix presented in Step 1.
Step 3. Combine criteria values using q-RSNWAO or q-RSNWGO.
Step 4. Calculate the score values for each candidate.
Step 5. Rank results based on the score values of each candidate.
Step 6. End algorithm.

Figure 1 showcases the detailed steps involved in the methodology introduced in the above algorithm of this research.

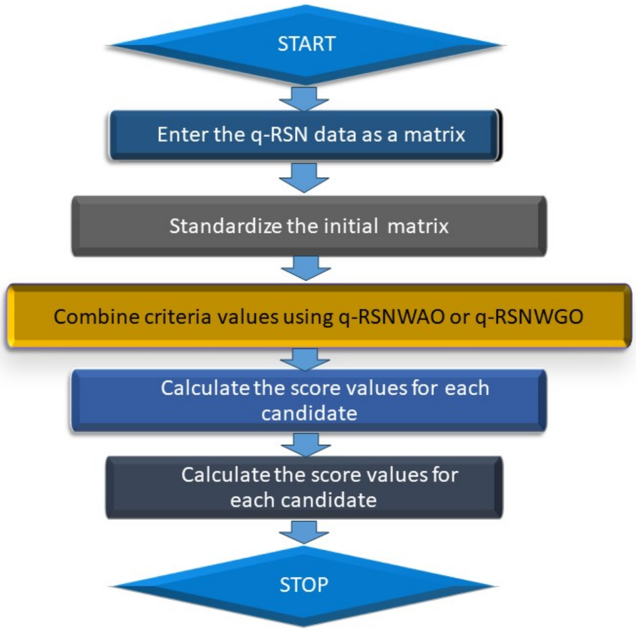

Figure 1. Visual representation of the methodology's sequential process.

6. Tangible implementation of q-RSN set

In this section, we harness the power of the aforementioned algorithm to conquer the formidable D-M dilemma at hand, where we will deal with one of the problems of MCDM, specifically choosing the best engineering company for constructing roads in metropolitan areas.

Example 6.1. Roads in metropolitan areas play an important part in many facets of daily life. They are critical for facilitating transportation, encouraging smooth movement, boosting economic activity, enabling public transit, assuring efficient emergency services, developing social connections, driving urban growth, and enhancing tourism. Road development and upkeep are the purview of government agencies. They work along with engineering firms to guarantee that road networks are developed, planned, and maintained appropriately. In order to make an informed judgment, government agencies use a variety of factors and criteria while evaluating engineering firms, such as ζ_1: Expertise in the field, ζ_2: Compliance with regulations, ζ_3: Control and guarantee of quality, ζ_4: Safety precautions and ζ_5: Proposed packages and pricing. Our goal in this suggested model is to select an engineering company from a set of four reputable engineering companies $X_{k=1,2,3,4}$ based on the given criteria. Suppose the weights for the criteria are 0.2 for expertise in the field, 0.1 for compliance with regulations, 0.1 for control and guarantee of quality, 0.3 for safety precautions, and 0.3 for proposed packages and pricing. With this decision-making scenario, q-RSN numbers, where $q = 4$, reflect the quality of the engineering companies with respect to the given criteria.

Then, as we will detail in the discussion that follows, we will utilize the recommended technique to choose the best engineering firm for the road building project.
Step 1: Using q-RSN data, government agencies assessed each engineering firm according to the specified standards. They created the decision matrix, which Table 2 displays.

Table 2. The initial decision matrix.

Criterion	X_1	X_2	X_3	X_4
ζ_1	$\langle 0.9, 0.8, 0.7\rangle$	$\langle 0.8, 0.1, 0.7\rangle$	$\langle 08, 0.7, 0.8\rangle$	$(\langle 0.5, 0.5, 0.9\rangle$
ζ_2	$\langle 0.5, 0.5, 0.3\rangle$	$\langle 0.9, 0.7, 0.2\rangle$	$\langle 0.7, 0.1, 0.9\rangle$	$\langle 0.3, 0.6, 0.9\rangle$
ζ_3	$\langle 0.8, 0.6, 0.8\rangle$	$\langle 0.7, 0.4, 0.2\rangle$	$\langle 0.6, 0.7, 0.8\rangle$	$\langle 0.9, 0.8, 0.8\rangle$
ζ_4	$(\langle 0.5, 0.4, 0.9\rangle$	$\langle 0.9, 0.5, 0.5\rangle$	$\langle 0.5, 0.6, 0.2\rangle$	$\langle 0.7, 0.9, 0.9\rangle$
ζ_5	$\langle 0.3, 0.9, 0.6\rangle$	$\langle 0.1, 0.8, 0.9\rangle$	$\langle 0.4, 0.1, 0.6\rangle$	$\langle 0.8, 0.1, 0.3\rangle$

Step 2: The next step is to normalize the initial matrix presented in Table 2 by taking the cost criterion's (ζ_5) complement. Table 3 shows the normalized decision matrix that was produced.

Table 3. The standardized decision matrix.

Criterion	X_1	X_2	X_3	X_4
ζ_1	$\langle 0.9, 0.8, 0.7\rangle$	$\langle 0.8, 0.1, 0.7\rangle$	$\langle 0.8, 0.7, 0.8\rangle$	$(\langle 0.5, 0.5, 0.9\rangle$
ζ_2	$\langle 0.5, 0.5, 0.3\rangle$	$\langle 0.9, 0.7, 0.2\rangle$	$\langle 0.7, 0.1, 0.9\rangle$	$\langle 0.3, 0.6, 0.9\rangle$
ζ_3	$\langle 0.8, 0.6, 0.8\rangle$	$\langle 0.7, 0.4, 0.2\rangle$	$\langle 0.6, 0.7, 0.8\rangle$	$\langle 0.9, 0.8, 0.8\rangle$
ζ_4	$(\langle 0.5, 0.4, 0.9\rangle$	$\langle 0.9, 0.5, 0.5\rangle$	$\langle 0.5, 0.6, 0.2\rangle$	$\langle 0.7, 0.9, 0.9\rangle$
ζ_5	$\langle 0.6, 0.1, 0.3\rangle$	$\langle 0.9, 0.2, 0.1\rangle$	$\langle 0.6, 0.9, 0.4\rangle$	$\langle 0.3, 0.9, 0.8\rangle$

Step 3: The criteria values for each alternative are combined using the $\mathfrak{q}$-RSNWAO. The aggregated values that were obtained are provided below.
$\mathbb{Z}_1 = \langle 0.7298, 0.3228, 0.5451\rangle$, $\mathbb{Z}_2 = \langle 0.8746, 0.2784, 0.2747\rangle$, $\mathbb{Z}_3 = \langle 0.6581, 0.5933, 0.4338\rangle$, and $\mathbb{Z}_4 = \langle 0.6563, 0.7594, 0.8586\rangle$.
Step 4: The score value of each alternative is calculated. We obtained $\Sigma(\mathbb{Z}_1) = 0.6242, \Sigma(\mathbb{Z}_2) = 0.7670, \Sigma(\mathbb{Z}_3) = 0.5196$ and $\Sigma(\mathbb{Z}_4) = 0.2943$.
Step 5. From Step 4, the ranking results are $X_2 \geq X_1 \geq X_3 \geq X_4$.

In Step 3, the $\mathfrak{q}$-RSNWGO can be applied. The outcomes are listed below.
$\mathbb{Z}_1 = \langle 0.6226, 0.4871, 0.7693\rangle$, $\mathbb{Z}_2 = \langle 0.8572, 0.3734, 0.5178\rangle$, $\mathbb{Z}_3 = \langle 0.6111, 0.7375, 0.6992\rangle$, and $\mathbb{Z}_4 = \langle 0.4782, 0.8301, 0.8701\rangle$.

Next, we determine each alternative's score value as follows: $\Sigma(\mathbb{Z}_1) = 0.4376, \Sigma(\mathbb{Z}_2) = 0.6982, \Sigma(\mathbb{Z}_3) = 0.3877$, and $\Sigma(\mathbb{Z}_4) = 0.2163$.

$X_2 \geq X_1 \geq X_3 \geq X_4$ is the order of the alternatives based on the aforementioned score values. These two suggested approaches obviously provide identical ranking outcomes.

7. Exploring the sensitivity of parameters

This subsection examines the effect of parameters on ranking outcomes in an effort to confirm the stability of the suggested MCDM approach.

Here, we examine the $\mathfrak{q}$-RSNWAO and $\mathfrak{q}$-RSNWGO, and analyze the resulting ranking outcomes. The alternative scores for different values of $\mathfrak{q}$ ranging from 4 to 50 are presented in Tables 4 and 5, and illustrated in Figure 2.

Table 4. Results for q-RSNWAO with varying q values.

q	SF	Order of alternatives
$q = 4$	$\Sigma(\mathbb{Z}_1) = 0.6242,\ \Sigma(\mathbb{Z}_2) = 0.7670,\ \Sigma(\mathbb{Z}_3) = 0.5196,\ \Sigma(\mathbb{Z}_4) = 0.2943$	$X_2 \geq X_1 \geq X_3 \geq X_4$
$q = 7$	$\Sigma(\mathbb{Z}_1) = 0.6051,\ \Sigma(\mathbb{Z}_2) = 0.7072,\ \Sigma(\mathbb{Z}_3) = 0.4903,\ \Sigma(\mathbb{Z}_4) = 0.3283$	$X_2 \geq X_1 \geq X_3 \geq X_4$
$q = 10$	$\Sigma(\mathbb{Z}_1) = 0.5898,\ \Sigma(\mathbb{Z}_2) = 0.6665,\ \Sigma(\mathbb{Z}_3) = 0.4781,\ \Sigma(\mathbb{Z}_4) = 0.3578$	$X_2 \geq X_1 \geq X_3 \geq X_4$
$q = 12$	$\Sigma(\mathbb{Z}_1) = 0.5827,\ \Sigma(\mathbb{Z}_2) = 0.6471,\ \Sigma(\mathbb{Z}_3) = 0.4744,\ \Sigma(\mathbb{Z}_4) = 0.3723$	$X_2 \geq X_1 \geq X_3 \geq X_4$
$q = 15$	$\Sigma(\mathbb{Z}_1) = 0.5752,\ \Sigma(\mathbb{Z}_2) = 0.6257,\ \Sigma(\mathbb{Z}_3) = 0.4715,\ \Sigma(\mathbb{Z}_4) = 0.3877$	$X_2 \geq X_1 \geq X_3 \geq X_4$
$q = 20$	$\Sigma(\mathbb{Z}_1) = 0.5620,\ \Sigma(\mathbb{Z}_2) = 0.6035,\ \Sigma(\mathbb{Z}_3) = 0.4697,\ \Sigma(\mathbb{Z}_4) = 0.4021$	$X_2 \geq X_1 \geq X_3 \geq X_4$
$q = 30$	$\Sigma(\mathbb{Z}_1) = 0.5680,\ \Sigma(\mathbb{Z}_2) = 0.5839,\ \Sigma(\mathbb{Z}_3) = 0.4690,\ \Sigma(\mathbb{Z}_4) = 0.4115$	$X_2 \geq X_1 \geq X_3 \geq X_4$
$q = 50$	$\Sigma(\mathbb{Z}_1) = 0.5594,\ \Sigma(\mathbb{Z}_2) = 0.5751,\ \Sigma(\mathbb{Z}_3) = 0.4690,\ \Sigma(\mathbb{Z}_4) = 0.4135$	$X_2 \geq X_1 \geq X_3 \geq X_4$

Table 5. Results for q-RSNWGO with varying q values.

q	SF	Order of alternatives
$q = 4$	$\Sigma(\mathbb{Z}_1) = 0.4376,\ \Sigma(\mathbb{Z}_2) = 0.6982,\ \Sigma(\mathbb{Z}_3) = 0.3877,\ \Sigma(\mathbb{Z}_4) = 0.2163$	$X_2 \geq X_1 \geq X_3 \geq X_4$
$q = 7$	$\Sigma(\mathbb{Z}_1) = 0.4463,\ \Sigma(\mathbb{Z}_2) = 0.6491,\ \Sigma(\mathbb{Z}_3) = 0.3890,\ \Sigma(\mathbb{Z}_4) = 0.2794$	$X_2 \geq X_1 \geq X_3 \geq X_4$
$q = 10$	$\Sigma(\mathbb{Z}_1) = 0.4620,\ \Sigma(\mathbb{Z}_2) = 0.6116,\ \Sigma(\mathbb{Z}_3) = 0.3986,\ \Sigma(\mathbb{Z}_4) = 0.3031$	$X_2 \geq X_1 \geq X_3 \geq X_4$
$q = 12$	$\Sigma(\mathbb{Z}_1) = 0.4709,\ \Sigma(\mathbb{Z}_2) = 0.5937,\ \Sigma(\mathbb{Z}_3) = 0.4041,\ \Sigma(\mathbb{Z}_4) = 0.3221$	$X_2 \geq X_1 \geq X_3 \geq X_4$
$q = 15$	$\Sigma(\mathbb{Z}_1) = 0.4809,\ \Sigma(\mathbb{Z}_2) = 0.5749,\ \Sigma(\mathbb{Z}_3) = 0.4100,\ \Sigma(\mathbb{Z}_4) = 0.3428$	$X_2 \geq X_1 \geq X_3 \geq X_4$
$q = 20$	$\Sigma(\mathbb{Z}_1) = 0.4912,\ \Sigma(\mathbb{Z}_2) = 0.5575,\ \Sigma(\mathbb{Z}_3) = 0.4154,\ \Sigma(\mathbb{Z}_4) = 0.3636$	$X_2 \geq X_1 \geq X_3 \geq X_4$
$q = 30$	$\Sigma(\mathbb{Z}_1) = 0.5000,\ \Sigma(\mathbb{Z}_2) = 0.5455,\ \Sigma(\mathbb{Z}_3) = 0.4193,\ \Sigma(\mathbb{Z}_4) = 0.3812$	$X_2 \geq X_1 \geq X_3 \geq X_4$
$q = 50$	$\Sigma(\mathbb{Z}_1) = 0.5038,\ \Sigma(\mathbb{Z}_2) = 0.5424,\ \Sigma(\mathbb{Z}_3) = 0.4207,\ \Sigma(\mathbb{Z}_4) = 0.3889$	$X_2 \geq X_1 \geq X_3 \geq X_4$

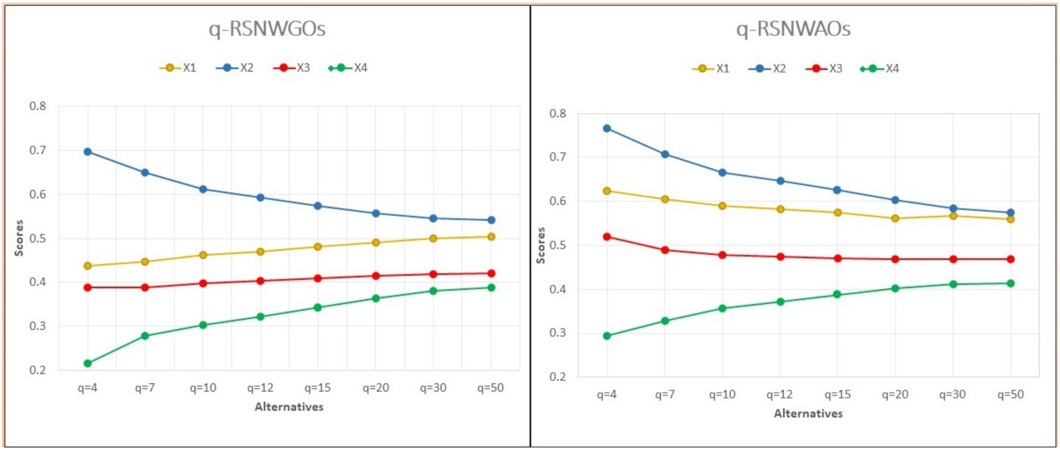

Figure 2. Accuracy of q-RSNWAO and q-RSNWGO with varying q values.

It is evident from examining Table 4 that the ranking results remain stable within the examined range. The dominant alternative is X_2, followed by X_1, X_3, and finally X_4. Interestingly, there is a noticeable inverse relationship between the values of q and the corresponding scores for alternatives X_1, X_2, and X_3. However, a direct relationship is observed for alternative X_4.

Upon reviewing Table 5, it is evident that the ranking results align precisely with those obtained in Table 3. The dominant alternative remains X_2, followed by X_1, X_3, and X_4, with this order remaining unchanged throughout the examined range. That is, Company X_2 represents the best option, followed by the rest of the companies in descending order.

It is worth mentioning that with the increasing values of q, there is a decrease in the score of X_2, while the scores of the remaining alternatives exhibit an upward trend. It is important to mention that in cases where two alternatives receive the same score, the accuracy value, as defined in Definition 3.12, is taken into consideration.

From the aforementioned discussion, we can observe a high level of uniformity and stability over the entire range. This exceptional stability provides solid proof of the suggested method's dependability and efficacy.

7.1. A comprehensive comparison of the proposed method with previous concepts

To evaluate the effectiveness of the suggested strategy, we will compare it with other frequently employed strategies in this section and go over their respective strengths and weaknesses.

In addition to the q-RSN model, numerous alternative models have been put up in the literature to handle MCDM issues. We will focus on the IN set [13], PyN set [16], q-ROF set [50], and simplified NS [4] since they are relevant in this comparison. In the following analysis, we will compare these variant models with the same data as in Example 6.1.

The first model we consider is the IN set, which comprises three components: $\mathcal{T}$, $\mathcal{I}$, and $\mathcal{F}$. However, it is important to note that this model imposes a constraint $\mathcal{T} + \mathcal{F} \leq 1$. Table 2 demonstrates that this model's requirement is violated in that it is unable to represent some data, such as $\langle 0.3, 0.9, 0.8\rangle$, as $0.3 + 0.8 \geq 1$. Thus, this kind of data is not effectively addressed by this paradigm.

We move on to the second model, the PyN set, which expands on the framework of the IN set, albeit with a limited scope, as it introduces the criterion $\mathcal{T}^2 + \mathcal{F}^2 \leq 1$. However, when considering the example of q-RSN value $\langle 0.9, 0.8, 0.7\rangle$ from Table 2, we find that this criterion is violated, as $0.9^2+0.7^2 = 1.3 > 1$. As a result, this approach cannot adequately solve the current MCDM challenge. On the other hand, the proposed q-RSN model exhibits its capability to handle this sort of data by considering $\mathfrak{q} = 4$. In this situation, we obtain $0.9^4 + 0.7^4 = 0.896 \leq 1$, which meets the criteria and allows for an effective representation. Significantly, the q-RSN set framework demonstrates a heightened level of comprehensiveness, as it encompasses IN set ($\mathfrak{q} = 1$)and PyN set ($\mathfrak{q} = 2$) as specific cases. The key attribute of the q-RSN set lies in its inherent flexibility to modify the parameter $\mathfrak{q}$. This dynamic adjustment enables the determination of the range of information expression, providing the ability to tailor it according to specific needs and circumstances.

Let us now explore the fourth model, the q-ROF set, which is defined by two membership degrees $\mathcal{T}$ and $\mathcal{F}$, subject to the constraint $\mathcal{T}^q + \mathcal{F}^q \leq 1$, where $q \geq 1$. However, it is essential to highlight that this model is not equipped to handle indeterminate situations. As a result, it is not applicable for solving the MCDM problem outlined in Section 6.

Lastly, let us examine the SN set, a powerful framework comprising three distinct components: $\mathcal{T}$, $\mathcal{I}$, and $\mathcal{F}$. Unlike the q-RSN model, the SN set treats these components as independent entities. This is in contrast to the q-RSN model, where both $\mathcal{T}$ and $\mathcal{F}$ exhibit a mutual dependency, while $\mathcal{I}$ retains its independence. The conditions governing the SN set dictate that $0 \leq \mathcal{T} + \mathcal{I} + \mathcal{F} \leq 3$, whereas for q-RSN set, the conditions are stipulated as $0 \leq (\mathcal{T})^{\mathfrak{q}} + \mathcal{I} + (\mathcal{F})^{\mathfrak{q}} \leq 2$.

It is important to highlight that the q-RSN set and the SN set share a similar structural foundation with the IFN set and the PyN set, albeit with distinct conditions. Nevertheless, both the SN set and the q-RSN set represent generalizations of the aforementioned methods. Importantly, the SN set is capable of representing the data in Table 3 and can be employed to solve the suggested MCDM problem. Notably, any values drawn from Table 3 conform to the conditions imposed by the SN set. Therefore, in the subsequent discussion, we will employ the AOs and SFs offered by the SN set to solve Example 6.1 and compare the results with those obtained using the suggested AOs and the SFs of the proposed method.

Peng et al. [4] defined the simplified neutrosophic aggregation operators (SNAOs) and SFs in the following way.

Definition 7.1. [4] Consider ${}^{\Lambda}\beth = \langle \mathcal{T}_{{}^{\Lambda}\beth}, \mathcal{I}_{{}^{\Lambda}\beth}, \mathcal{F}_{{}^{\Lambda}\beth} : \Lambda = 1, ..., m\rangle$, to be a family of SN numbers and ${}^{\Lambda}\varpi : \{\Lambda = 1, ..., m\}$, to be the weights of ${}^{\Lambda}\beth$.
(1) The simplified NWAO is defined as:

$$\mathrm{SNWA}({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) = \left\langle 1 - \prod_{\Lambda=1}^{m}\left(1 - \mathcal{T}_{{}^{\Lambda}\beth}\right)^{{}^{\Lambda}\varpi}, \prod_{\Lambda=1}^{m}\left(\mathcal{I}_{{}^{\Lambda}\beth}\right)^{{}^{\Lambda}\varpi}, \prod_{\Lambda=1}^{m}\left(\mathcal{F}_{{}^{\Lambda}\beth}\right)^{{}^{\Lambda}\varpi}\right\rangle. \tag{12}$$

(2) The simplified NWGO is defined as:

$$\mathrm{SNWG}({}^{1}\beth, {}^{2}\beth, ..., {}^{m}\beth) = \left\langle \prod_{\Lambda=1}^{m}\left(\mathcal{T}_{{}^{\Lambda}\beth}\right)^{{}^{\Lambda}\varpi}, 1 - \prod_{\Lambda=1}^{m}\left(1 - \mathcal{I}_{{}^{\Lambda}\beth}\right)^{{}^{\Lambda}\varpi}, 1 - \prod_{\Lambda=1}^{m}\left(1 - \mathcal{F}_{{}^{\Lambda}\beth}\right)^{{}^{\Lambda}\varpi}\right\rangle. \tag{13}$$

Definition 7.2. [4] Let $\beth = \langle \mathcal{T}_{\beth}, \mathcal{I}_{\beth}, \mathcal{F}_{\beth}\rangle$ be an SN number.

(1) A SF on $\beth$ is defined as $\mathcal{S}_{\beth} = \mathcal{S}(\beth) = \frac{1}{3}[\mathcal{T}_{\beth}+(1-\mathcal{I}_{\beth})+(1-\mathcal{F}_{\beth})]$. (14)

(2) The AF on $\beth$ is defined as $\mathcal{A}_{\beth} = \mathcal{A}(\beth) = \mathcal{T}_{\beth}-\mathcal{F}_{\beth}$. (15)

Let us now apply the AOs of SN numbers to solve Example 6.1. firstly, we will employ the simplified NWAO, as described in Eq (12), to combine the various criterion values for each candidate. This aggregation process will result in the single representative values: $\mathbb{Z}_1 = \langle 0.6908, 0.3228, 0.5451\rangle$, $\mathbb{Z}_2 = \langle 0.8718, 0.2784, 0.2747\rangle$, $\mathbb{Z}_3 = \langle 0.6382, 0.5933, 0.4338\rangle$, and $\mathbb{Z}_4 = \langle 0.5822, 0.7594, 0.8586\rangle$.

Next, we compute each alternative's score value as follows: $\Sigma(\mathbb{Z}_1) = 0.6076, \Sigma(\mathbb{Z}_2) = 0.7729, \Sigma(\mathbb{Z}_3) = 0.5371$, and $\Sigma(\mathbb{Z}_4) = 0.3214$. The ranking results are evident: $X_2 \geq X_1 \geq X_3 \geq X_4$, which align with the ranking results obtained using the q-RSNWAO.

Secondly, to compare the outcomes between the simplified NWGO and the q-RSNWGO, we examine the results obtained using the simplified NWGO as follows. Behold, the values of $\mathbb{Z}_{k=1,2,3,4}$ are revealed: $\mathbb{Z}_1 = \langle 0.6226, 0.4871, 0.7092\rangle$, $\mathbb{Z}_2 = \langle 0.8572, 0.3734, 0.4084\rangle$, $\mathbb{Z}_3 = \langle 0.6111, 0.6067, 0.6992\rangle$, and $\mathbb{Z}_4 = \langle 0.4782, 0.8301, 0.8680\rangle$, where the score values are: $\Sigma(\mathbb{Z}_1) = 0.4754, \Sigma(\mathbb{Z}_2) = 0.6918, \Sigma(\mathbb{Z}_3) = 0.4223$ and $\Sigma(\mathbb{Z}_4) = 0.26$. Therefore, the ranking results remain unchanged as $X_2 \geq X_1 \geq X_3 \geq X_4$, which perfectly align with the ranking results obtained using the q-RSNWGO. Based on the preceding discussion, we have observed that the suggested AOs in the new model, coupled with their associated SFs, generate the same outcomes as the SN set when applied to the same dataset. This observation confirms the efficacy of the proposed methodology. As previously stated, both the SN set and q-RSN set share a similar structure but with distinct conditions. This disparity in conditions serves as a driving force for introducing the q-RSN set, allowing it to be employed based on the specific circumstances of a problem, particularly when there exists a dependency between truth and falsity memberships. Table 6 presents a comparison of existing models using appropriate criteria, such as the presence of three membership degrees, dependency between the degrees, constraint presence, constraint flexibility, and ranking values.

Table 7 presents a summary of the generated results from both the proposed operators q-RSNWAO and q-RSNWGO as well as other established operators. This table shows that contributions as stated in the literature [45, 50, 59] cannot deal with the data shown in Example 6.1 because the value of q is limited to 1 and 2, i.e., $\mathcal{T}^1 + \mathcal{F}^1 > 1$ and $\mathcal{T}^2 + \mathcal{F}^2 > 1$ whereas the key factor in the design of each of q-RSNWAO, q-RSNWGO is the dynamics of the work of the $\mathrm{q} \geq 1$ as it is not defined by limits. Therefore, it is considered that the proposed AOs are more precise and flexible than current methods. As a practical application, these proposed AOs have proven highly efficient in dealing with the problem illustrated in Example 6.1, which was designed to select the best engineering company for road construction according to specific criteria explained in detail in Example 6.1. In addition, as for the mathematical structure of the proposed AOs in this work, each of them depends on one of the algebraic properties explained in Definition 3.8, specifically properties (1) and (2). Moreover, Figure 3 provides a visual representation of these results.

Table 6. Comparative examination of existing models using key criteria.

Method	Presence of three membership degrees	Dependency between the degrees	constraint presence	constraint flexibility	Ranking values
IN set [13]	✓	✓	✓	Low	Non-computable
PyN set [16]	✓	✓	✓	Moderate	Non-computable
q-ROF set [44]	x	✓	✓	Maximum	Non-computable
SN set [4]	✓	x	✓	Non-applicable	Algorithmic
The proposed method	✓	✓	✓	Maximum	Algorithmic

Table 7. Comparison of the suggested model with existing models based on different operators.

Methods	Score Values	Order of alternatives
INWAO [59]	Non-computable,Non-computable,Non-computable,Non-computable	Non-applicable
INWGO [59]	Non-computable,Non-computable,Non-computable,Non-computable	Non-applicable
PyNWAO set [45]	Non-computable,Non-computable,Non-computable,Non-computable	Non-applicable
PyNWGO set [45]	Non-computable,Non-computable,Non-computable,Non-computable	Non-applicable
q-ROFAO [50]	Non-computable,Non-computable,Non-computable,Non-computable	Non-applicable
q-ROFGO [50]	Non-computable,Non-computable,Non-computable,Non-computable	Non-applicable
Simplified NWAO [4]	$\Sigma(\mathbb{Z}_1) = 0.6076, \Sigma(\mathbb{Z}_2) = 0.7729, \Sigma(\mathbb{Z}_3) = 0.5371, \Sigma(\mathbb{Z}_4) = 0.3214$	$X_2 \geq X_1 \geq X_3 \geq X_4$
Simplified NWGO [4]	$\Sigma(\mathbb{Z}_1) = 0.4754, \Sigma(\mathbb{Z}_2) = 0.6918, \Sigma(\mathbb{Z}_3) = 0.4223$ and $\Sigma(\mathbb{Z}_4) = 0.26$	$X_2 \geq X_1 \geq X_3 \geq X_4$
The proposed q-RSNWAO	$\Sigma(\mathbb{Z}_1) = 0.6242, \Sigma(\mathbb{Z}_2) = 0.7670, \Sigma(\mathbb{Z}_3) = 0.5196, \Sigma(\mathbb{Z}_4) = 0.2943$	$X_2 \geq X_1 \geq X_3 \geq X_4$
The proposed q-RSNWGO	$\Sigma(\mathbb{Z}_1) = 0.4376, \Sigma(\mathbb{Z}_2) = 0.6982, \Sigma(\mathbb{Z}_3) = 0.3877, \Sigma(\mathbb{Z}_4) = 0.2163$	$X_2 \geq X_1 \geq X_3 \geq X_4$

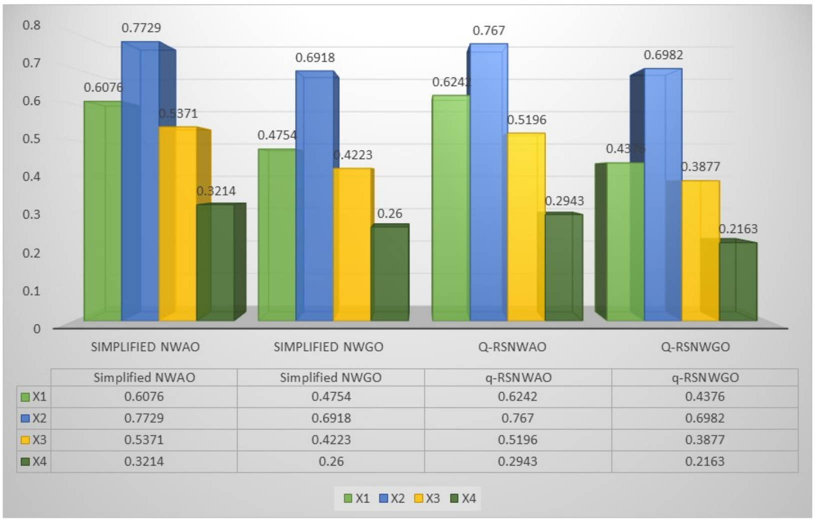

Figure 3. Visual representation of results from Table 7.

8. Conclusions

This article presents a comprehensive examination of the suggested theory of the q-RSN set, demonstrating its potential to include and expand existing approaches where the concept's novelty stands out because it enables it to cover the defects discovered in the IF, PyF, and q-ROF sets when these concepts fail to cope with complex data structures when dealing with D-M problems. The main reason behind this flexibility in the q-RSN set compared with previous theories is the introduction of the idea of raising $\mathcal{T}$ and $\mathcal{F}$ to the power of q, hence expanding the range of membership grades covered. The article defines the q-RSN set, develops operational concepts, and outlines several AOs in the q environment. These operators' intrinsic properties are rigorously verified. Further, a new MCDM technique is rigorously designed, based on the suggested operators and the effective use of SFs. A numerical evaluation of engineering businesses is undertaken using q-RSN numbers to determine their performance across multiple criteria. Next, q-RSNWAOs and q-RSNWGOs are used to aggregate criterion values, and SF is used to generate ranking results. The ideal solution's behavior is analyzed by varying the parameter q for the suggested operators and SFs. Notably, the ranking results were consistent over the measured range, and the ideal solution stayed constant despite these differences. This clearly indicates the suggested approaches' strong stability and robustness. Furthermore, this publication compared the suggested method in relation to established approaches, using a thorough and insightful discussion to explicate and interpret the results. Future research should explore broader generalizations of the q-RSN set, particularly in demonstrations where this structure is not apt. Interested researchers can focus on these demonstrations and work on improving our proposed notion: (a) The researchers who prepared this article did not address the basic set theory operations accompanying this concept, such as the processes of union and intersection operations. A future study could define these set theory operations, in addition to clarifying the common properties between them.

(b) Interested researchers can realize the bipolarity on the $\mathfrak{q}$-RSN set by investigating a new notion named the Bipolar - $\mathfrak{q}$-RSN set which pays attention to the positive and negative aspects of D-M. (c) This concept neglects to deal with attribute-valued sets and attribute-multiple-valued sets; therefore, this barrier can be covered by combining the $\mathfrak{q}$-RSN set with SS and hyper soft set (HSS). (d) The three membership degrees of the $\mathfrak{q}$-RSN set can be expanded to a complex space by following the mechanism explained in [60]. (e) Exploring some mathematical structures in algebra [61, 62], topology [63, 64], and graph theory [65] on the $\mathfrak{q}$-RSN set. (f) Some traditional D-M methods like TOPSIS, VIKOR, and combinative distance-based assessment (CODAS) based on the $\mathfrak{q}$-RSN set can be applied to handle real-life scenarios. (g) Study some measures such as distance, similarity, and entropy on $\mathfrak{q}$-RSN sets and test them in handling D-M problems. Finally, in a practical view, it is significant to note that after these future studies proposed in this section, which can be applied to our proposed concept in this work, our concept $\mathfrak{q}$-RSN set is of great significance in the fuzziness environment that deals with data uncertainty and inconsistency included in D-M problems.

Author contributions

Ashraf Al-Quran and Faisal Al-Sharqi: Conceptualization, methodology, writing original draft; Ashraf Al-Quran and Abdelhamid Mohammed Djaouti: Supervision, language editing, and funding acquisition; Faisal Al-Sharqi: Writing the review and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgements

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia. Project No. KFU251357.

Conflicts of interest

Authors declare no conflicts of interest.

References

1. F. Smarandache, *Neutrosophy: Neutrosophic probability, set, and logic: Analytic synthesis & synthetic analysis*, Rehoboth, NM: American Research Press, 1998.
2. H. B. Wang, F. Smarandache, Y. Q. Zhang, R. Sunderraman, *Single valued neutrosophic sets*, Hanko, Finland: North-European Scientific Publishers, **4** (2010), 410–413.
3. J. Ye, A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets, *J. Intell. Fuzzy Syst.*, **26** (2014), 2459–2466. https://doi.org/10.3233/IFS-130916

4. J. Peng, J. Q. Wang, J. Wang, H. Zhang, X. Chen, Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems, *Int. J. Syst. Sci.*, **47** (2016), 2342–2358. https://doi.org/10.1080/00207721.2014.994050

5. R. Mishra, P. Rani, R. S. Prajapati, Multi-criteria weighted aggregated sum product assessment method for sustainable biomass crop selection problem using single-valued neutrosophic sets, *Appl. Soft Comput.*, **113** (2021), 108038. https://doi.org/10.1016/j.asoc.2021.108038

6. M. Ali, F. Smarandache, Complex neutrosophic set, *Neural Comput. Appl.*, **28** (2017), 1817–1834. https://doi.org/10.1007/s00521-015-2154-y

7. A. Al-Quran, A. Ahmad, F. Al-Sharqi, A. Lutfi, Q-complex neutrosophic set, *Int. J. Neutrosophic Sci.*, **20** (2023), 8–19. https://doi.org/10.54216/IJNS.200201

8. J. Qiu, L. Jiang, H. Fan, P. Li, C. You, Dynamic nonlinear simplified neutrosophic sets for multiple-attribute group decision making, *Heliyon*, **10** (2024), e27493. https://doi.org/10.1016/j.heliyon.2024.e27493

9. J. Ye, Entropy measures of simplified neutrosophic sets and their decision-making approach with positive and negative arguments, *J. Manag. Anal.*, **8** (2021), 252–266. https://doi.org/10.1080/23270012.2021.1885513

10. J. Ye, S. Du, R. Yong, F. Zhang, Arccosine and arctangent similarity measures of refined simplified neutrosophic indeterminate sets and their multicriteria decision-making method, *J. Intell. Fuzzy Syst.*, **40** (2021), 9159–9171. https://doi.org/10.3233/JIFS-201571

11. P. K. Maji, Neutrosophic soft set, *Ann. Fuzzy Math. Inform.*, **5** (2013), 157–168.

12. S. Alkhazaleh, Neutrosophic vague set theory, *Crit. Rev.*, **10** (2015), 29–39.

13. M. Bhowmik, M. Pal, Intuitionistic neutrosophic set, *J. Inform. Comput. Sci.*, **4** (2009), 142–152.

14. V. Chinnadurai, A. Bobin, Simplified intuitionistic neutrosophic soft set and its application on diagnosing psychological disorder by using similarity measure, *Appl. Appl. Math.*, **16** (2021), 34.

15. R. Jansi, K. Mohana, F. Smarandache, Correlation measure for Pythagorean neutrosophic sets with T and F as dependent neutrosophic components, *Neutrosophic Sets Sy.,* **30** (2019), 202–212.

16. R. Radha, A. S. A. Mary, R. Prema, S. Broumi, Neutrosophic Pythagorean sets with dependent neutrosophic Pythagorean components and its improved correlation coefficients, *Neutrosophic Sets Sy.*, **46** (2021), 77–86.

17. R. Imran, K. Ullah, Z. Ali, M. Akram, A multi-criteria group decision-making approach for robot selection using interval-valued intuitionistic fuzzy information and aczel-alsina bonferroni means, *Spectr. Decis. Mak. Appl.*, **1** (2024), 1–32.

18. M. Riaz, H. M. A. Farid, S. Ashraf, H. Kamacı, Single-valued neutrosophic fairly aggregation operators with multi-criteria decision-making, *Comput. Appl. Math.*, **42** (2023), 104. https://doi.org/10.1007/s40314-023-02233-w

19. H. M. A. Farid, M. Riaz, Single-valued neutrosophic dynamic aggregation information with time sequence preference for IoT technology in supply chain management, *Eng. Appl. Artif. Intell.*, **126** (2023), 106940. https://doi.org/10.1016/j.engappai.2023.106940

20. H. Garg, Linguistic single-valued neutrosophic power aggregation operators and their applications to group decision-making problems, *IEEE-CAA J. Automatic.*, **7** (2020), 546–558. https://doi.org/10.1109/JAS.2019.1911522
21. C. Liu, Y. Luo, Power aggregation operators of simplified neutrosophic sets and their use in multi-attribute group decision making, *IEEE-CAA J. Automatic.*, **6** (2019), 575–583. https://doi.org/10.1109/JAS.2017.7510424
22. R. Yong, J. Ye, S. Du, A. Zhu, Y. Zhang, Aczel-Alsina weighted aggregation operators of simplified neutrosophic numbers and its application in multiple attribute decision making, *Comput. Model. Eng. Sci.*, **132** (2022), 569–584. https://doi.org/10.32604/cmes.2022.019509
23. R. Sahin, H. Zhang, Induced simplified neutrosophic correlated aggregation operators for multi-criteria group decision-making. *J. Exp. Theor. Artif. In.*, **30** (2018), 279–292. https://doi.org/10.1080/0952813X.2018.1430857
24. X. Lu, T. Zhang, Y. Fang, J. Ye, Einstein aggregation operators of simplified neutrosophic indeterminate elements and their decision-making method, *Neutrosophic Sets Syst.*, **47** (2021), 12–25.
25. D. Shigui, R. Yong, J. Ye, Subtraction operational aggregation operators of simplified neutrosophic numbers and their multi-attribute decision making approach, *Neutrosophic Sets Sy.*, **33** (2020), 157–168.
26. C. Jana, M. A. Pal, Robust single-valued neutrosophic soft aggregation operators in multi-criteria decision making, *Symmetry*, **11** (2019), 110. https://doi.org/10.3390/sym11010110
27. F. Al-Sharqi, M. U. Romdhini, A. Al-Quran, Group decision-making based on aggregation operator and score function of Q-neutrosophic soft matrix, *J. Intell. Fuzzy Sy.*, **45** (2023), 305–321. https://doi.org/10.3233/JIFS-224552
28. F. Al-Sharqi, A. Al-Quran, Z. M. Rodzi, Multi-attribute group decision-making based on aggregation operator and score function of bipolar neutrosophic hypersoft environment, *Neutrosophic Sets Sy.*, **61** (2023), 465–492.
29. H. Hashim, H. Garg, A. Al-Quran, N. A. Awang, L. Abdullah, Heronian mean operators considering Shapley fuzzy measure under interval neutrosophic vague environment for an investment decision, *Int. J. Fuzzy Syst.*, **24** (2022), 2068–2091. https://doi.org/10.1007/s40815-021-01247-y
30. H. Hashim, L. Abdullah, A. Al-Quran, A. Awang, Entropy measures for interval neutrosophic vague sets and their application in decision making, *Neutrosophic Sets Sy.*, **45** (2021), 74–95.
31. A. Al-Quran, H. Hashim, L. Abdullah, A hybrid approach of interval neutrosophic vague sets and DEMATEL with new linguistic variable, *Symmetry*, **12** (2020), 275. https://doi.org/10.3390/sym12020275
32. M. Unver, E. Türkarslan, N. Elik, M. Olgun, J. Ye, Intuitionistic fuzzy-valued neutrosophic multi-sets and numerical applications to classification, *Complex Intell. Syst.*, **8** (2022), 1703–1721. https://doi.org/10.1007/s40747-021-00621-5
33. S. Broumi, F. Smarandache, Intuitionistic neutrosophic soft set, *J. Inform. Comput. Sci.*, **8** (2013), 130–140.

34. M. Akram, A. Luqman, Intuitionistic single-valued neutrosophic hypergraphs, *Opsearch*, **54** (2017), 799–815. https://doi.org/10.1007/s12597-017-0306-9

35. M. Akram, M. Sitara, Certain concepts in intuitionistic neutrosophic graph structures, *Information*, **8** (2017), 154. https://doi.org/10.3390/info8040154

36. S. S. Hussain, I. Rosyida, H. Rashmanlou, F. Mofidnakhaei, Interval intuitionistic neutrosophic sets with its applications to interval intuitionistic neutrosophic graphs and climatic analysis, *Comput. Appl. Math.*, **40** (2021), 121. https://doi.org/10.1007/s40314-021-01504-8

37. P. Chellamani, D. Ajay, Pythagorean neutrosophic Dombi fuzzy graphs with an application to MCDM, *Neutrosophic Sets Sy.*, **47** (2021), 411–431.

38. J. N. Ismail, Z. Rodzi, F. Al-Sharqi, A. Al-Quran, H. Hashim, N. H. Sulaiman, Algebraic operations on Pythagorean neutrosophic sets: Extending applicability and decision-making capabilities, *Int. J. Neutrosophic Sci.*, **21** (2023), 127–134. https://doi.org/10.54216/IJNS.210412

39. J. N. Ismail, Z. Rodzi, H. Hashim, N. H. Sulaiman, F. Al-Sharqi, A. Al-Quran, et al., Enhancing decision accuracy in DEMATEL using Bonferroni mean aggregation under Pythagorean neutrosophic environment, *J. Fuzzy. Ext. Appl.*, **4** (2023), 281–298.

40. M. Palanikumar, K. Arulmozhi, C. Jana, Multiple attribute decision-making approach for Pythagorean neutrosophic normal interval-valued fuzzy aggregation operators, *Comput. Appl. Math.*, **41** (2022). https://doi.org/10.1007/s40314-022-01791-9

41. M. Asif, U. Ishtiaq, I. K. Argyros, Hamacher aggregation operators for pythagorean fuzzy set and its application in multi-attribute decision-making problem, *Spectr. Oper. Res.*, **2** (2025), 27–40. https://doi.org/10.31181/sor2120258

42. M. Palanikumar, K. Arulmozhi, T. Gunasekar, Possibility Pythagorean neutrosophic soft set and its application of decision making, *J. Int. Math. Virtual Inst.*, **12** (2022), 159–174.

43. S. Petchimuthu, M. F. Banu, C. Mahendiran, T. Premala, Power and energy transformation: Multi-criteria decision-making utilizing complex q-rung picture fuzzy generalized power prioritized Yager operators, *Spectr. Oper. Res.*, **2** (2025), 219–258. https://doi.org/10.31181/sor21202525

44. Z. Ali, Fairly aggregation operators based on complex p, q-rung orthopair fuzzy sets and their application in decision-making problems, *Spectr. Oper. Res.*, **2** (2025), 113–131. https://doi.org/10.31181/sor21202514

45. M. Palanikumar, K. Arulmozhi, MCGDM based on TOPSIS and VIKOR using Pythagorean neutrosophic soft with aggregation operators, *Neutrosophic Sets Sy.*, **51** (2022), 539–555.

46. S. A. Razak, Z. M. Rodzi, N. Ahmad, A. G. Ahmad, Exploring the boundaries of uncertainty: Interval valued Pythagorean neutrosophic set and their properties, *Malays. J. Fundam. Appl. Sci.*, **20** (2024), 813–824. https://doi.org/10.11113/mjfas.v20n4.3482

47. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Set. Syst.*, **20** (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3

48. R. R. Yager, *Pythagorean fuzzy subsets*, In: 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), Canada: IEEE, 2013, 57–61. https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375

49. R. R. Yager, Generalized orthopair fuzzy sets, *IEEE T. Fuzzy Syst.*, **25** (2016), 1222–1230. https://doi.org/10.1109/TFUZZ.2016.2604005
50. P. D. Liu, P. Wang, Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making, *Int. J. Intell. Syst.*, **33** (2017), 259–280. https://doi.org/10.1002/int.21927
51. M. R. Seikh, U. Mandal, q-Rung orthopair fuzzy Archimedean aggregation operators: Application in the site selection for software operating units, *Symmetry*, **15** (2023), 1680. https://doi.org/10.3390/sym15091680
52. Y. Du, X. Du, Y. Li, J. Cui, F. Hou, Complex q-rung orthopair fuzzy Frank aggregation operators and their application to multi-attribute decision making, *Soft Comput.*, **26** (2022), 11973–12008. https://doi.org/10.1007/s00500-022-07465-2
53. H. M. A. Farid, M. Riaz, V. Simic, X. Peng, q-Rung orthopair fuzzy dynamic aggregation operators with time sequence preference for dynamic decision-making, *PeerJ Comput. Sci.*, **10** (2024). https://doi.org/10.7717/peerj-cs.1742
54. H. M. A. Farid, M. Riaz, q-Rung orthopair fuzzy Aczel-Alsina aggregation operators with multi-criteria decision-making, *Eng. Appl. Artif. Intell.*, **122** (2023), 106105. https://doi.org/10.1016/j.engappai.2023.106105
55. A. P. Darko, D. Liang, Some q-rung orthopair fuzzy Hamacher aggregation operators and their application to multiple attribute group decision making with modified EDAS method, *Eng. Appl. Artif. Intell.*, **87** (2020), 103259. https://doi.org/10.1016/j.engappai.2019.103259
56. M. R. Seikh, U. Mandal, q-Rung orthopair fuzzy Frank aggregation operators and its application in multiple attribute decision-making with unknown attribute weights, *Granul. Comput.*, **7** (2022), 709–730. https://doi.org/10.1007/s41066-021-00290-2
57. A. F. Alrasheedi, J. Kim, R. Kausar, q-Rung orthopair fuzzy information aggregation and their application towards material selection, *AIMS Math.*, **8** (2023), 18780–18808. https://doi.org/10.3934/math.2023956
58. T. Senapati, G. Chen, I. Ullah, M. S. A. Khan, F. Hussain, A novel approach towards multiattribute decision making using q-rung orthopair fuzzy Dombi-Archimedean aggregation operators, *Heliyon*, **10** (2024), e27969. https://doi.org/10.1016/j.heliyon.2024.e27969
59. M. Unver, E. Turkarslan, N. Elik, M. Olgun, J. Ye, Intuitionistic fuzzy-valued neutrosophic multi-sets and numerical applications to classification, *Complex Intell. Syst.*, **8** (2022), 1703–1721. https://doi.org/10.1007/s40747-021-00621-5
60. Y. Al-Qudah, N. Hassan, Complex multi-fuzzy soft set: Its entropy and similarity measure, *IEEE Access*, **6** (2018), 65002–65017. https://doi.org/10.1109/ACCESS.2018.2877921
61. D. J. Martina, G. Deepa, Some algebraic properties on rough neutrosophic matrix and its application to multi-criteria decision-making, *AIMS Math.*, **11** (2023), 24132–24152. https://doi.org/10.3934/math.20231230
62. N. A. Khalid, M. Saeed, An approach for hybridizing N-subalgebra with quantified neutrosophic set using G-algebra, *Neutrosophic Syst. Appl.*, **25** (2025), 14–29. https://doi.org/10.61356/j.nswa.2025.25411

63. A. Bataihah, A. Hazaymeh, Quasi contractions and fixed point theorems in the context of neutrosophic fuzzy metric spaces, *Eur. J. Pure Appl. Math.*, **18** (2025), 5785. https://doi.org/10.29020/nybg.ejpam.v18i1.5785

64. Y. Saber, F. Alsharari, F. Smarandache, An introduction to single-valued neutrosophic soft topological structure, *Soft Comput.*, **26** (2022), 7107–7122. https://doi.org/10.1007/s00500-022-07150-4

65. M. U. Romdhini, F. Al-Sharqi, A. Nawawi, A. Al-Quran, H. Rashmanlou, Signless Laplacian energy of interval-valued fuzzy graph and its applications, *Sains Malays.*, **52** (2023), 2127–2137. https://doi.org/10.17576/jsm-2023-5207-18

AIMS Press

© 2025 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)