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Research article

Consimilarity of hybrid number matrices and hybrid number matrix equations $\mathbf{A\widetilde{X} - XB = C}$

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Abstract: This research aims to investigate the consimilarity of hybrid number matrices and to develop solutions for matrix equations associated with these numbers. Hybrid numbers are an innovative algebraic structure that unifies dual, complex, and hyperbolic (perplex) number systems. These numbers are isomorphic to split quaternions and hold significant importance in mathematical theory and physical applications, especially in the context of non-commutative algebraic structures. The paper demonstrates how linear matrix equations associated with hybrid numbers, such as $\mathbf{A\widetilde{X} - XB = C}$, can be solved by reducing them to Sylvester equations through real matrix representations. The concept of consimilarity, defined as a transformation preserving structural properties of matrices without requiring invertibility, is thoroughly examined. This concept is applied to analyze eigenvalues, diagonalization, and both linear and nonlinear matrix equations involving hybrid number matrices. By investigating the consimilarity of hybrid number matrices, the study introduces new algebraic methods and computational techniques, expanding classical results in matrix theory to hybrid numbers. This research not only advances theoretical insights into hybrid number systems but also opens avenues for practical applications in scientific and engineering fields.

Keywords: hybrid numbers; split quaternions; hybrid number matrices; consimilarity; matrix equations
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1. Introduction

The study of linear matrix equations represents a cornerstone of matrix theory, with broad applications spanning control and systems theory, stability analysis, optimal control, and neural networks [1–3]. Notably, matrix equations such as $AX - XB = C$ and $X - AXB = C$ hold critical importance in various fields, including control theory, stability analysis, and system dynamics. These equations are intimately connected to the classical Lyapunov and Stein equations in matrix theory [4,5].

Quaternion and split quaternion matrices play significant roles in both pure and applied mathematics, particularly in areas such as matrix theory, algebra, and physics [6–8]. The study of real quaternion matrices has been a subject of interest since the early 20th century, with several researchers investigating key properties such as eigenvalues and consimilarity [9–11]. While real quaternions are non-commutative, their structure presents unique challenges and opportunities in the analysis of matrix equations and linear systems. Consimilarity, a relationship between matrices, has been explored in the context of quaternion matrices, with results closely linked to their Jordan canonical forms and diagonalization properties [10–12]. Additionally, split quaternions, which extend the concept of real quaternions, have found applications in representing Lorentzian relations and solving geometric and physical problems [13–15].

Hybrid numbers, introduced by Özdemir in 2018, form an innovative algebraic structure that combines the dual, complex, and hyperbolic (perplex) number systems. This system is isomorphic to split quaternions and has gained significance in mathematical theory and physical applications, particularly in non-commutative algebraic structures [16,17]. Hybrid numbers provide a new approach to solving various problems in fields like theoretical physics, control theory, and stability analysis [16]. Öztürk and Özdemir investigated the concept of similarity for hybrid numbers by solving the linear equations $\mathbf{px} = \mathbf{xq}$ and $\mathbf{qx} - \mathbf{xp} = \mathbf{c}$ [18]. Subsequently, they discussed elliptic transformations using hybrid numbers and proved the Rodrigues and Cayley transformations [19]. Çakır and Özdemir studied hybrid number matrices using the properties of complex matrices [20].

Matrix equations play a significant role in control theory, system dynamics, and optimization, particularly those of the form $AX - XB = C$ and $X - AXB = C$, which are fundamental for analyzing linear systems. However, classical matrix theory primarily focuses on matrices with real or complex coefficients, leaving limited direct methods applicable to more complex algebraic structures, such as hybrid number matrices.

The motivation of this study lies in extending the classical concepts of similarity and consimilarity to hybrid number matrices, which exhibit non-commutative structures similar to quaternions. The primary goal is to investigate the eigenvalues, diagonalizability, and solutions to matrix equations involving hybrid numbers. Specifically, the matrix equation $\mathbf{A}\widetilde{\mathbf{X}} - \mathbf{XB} = \mathbf{C}$ is explored within the context of hybrid numbers, with the objective of relating it to Sylvester's equation. Given that hybrid numbers are isomorphic to split quaternions, their applications span areas such as geometric transformations, Lorentzian relations, and stability analysis in control systems. This research aims to contribute to the theoretical understanding of hybrid number matrices and provide new insights into the generalization of classical matrix theory to non-commutative algebraic structures.

In matrix theory, the concept of similarity plays a foundational role. Two square matrices A and B are said to be similar if there exists an invertible matrix P such that $B = P^{-1}AP$ [3, 21]. Similar matrices share many important properties, including eigenvalues, determinant, trace, and characteristic polynomial [3, 21]. However, when considering algebraic structures like quaternions and hybrid numbers, the notion of matrix similarity becomes more intricate due to the non-commutative nature of these structures.

Beyond similarity, the concept of consimilarity has emerged as an important structural relation between matrices [11]. Consimilarity refers to transformations that preserve the structural properties of matrices without requiring the existence of an invertible matrix. In contrast to similarity, which relies on invertibility, consimilarity focuses on preserving certain inherent characteristics of matrices, such as eigenvalues and block structure [11]. The concept of consimilarity is particularly relevant for matrices over non-commutative rings, such as those involving hybrid numbers or quaternions. Investigating the consimilarity of hybrid number matrices can provide deeper insights into their algebraic properties and applications.

The aim of this study is to explore the consimilarity of hybrid number matrices, a concept that concerns the structural similarity between matrices without necessarily requiring invertibility, and to investigate solutions to matrix equations involving these numbers. Consimilarity preserves the structural properties of matrices, making it a powerful tool for understanding the behavior of hybrid number systems. By examining the consimilarity of hybrid number matrices, we can gain a deeper understanding of their eigenvalues, matrix diagonalization, and solutions to both linear and nonlinear matrix equations. Moreover, this investigation can lead to the development of new algebraic methods and computational techniques for solving matrix equations involving hybrid numbers.

One of the key goals of this research is to extend classical results in matrix theory to hybrid number matrices, particularly in the context of consimilarity and matrix equations. The study of hybrid number matrices offers the opportunity to generalize existing theories and apply them to new and emerging mathematical structures. Furthermore, the geometric and physical interpretations of hybrid numbers, particularly in relation to Lorentz transformations and Minkowski space, offer significant opportunities for future research. By studying the consimilarity of hybrid number matrices, we can explore their potential applications in areas such as quantum mechanics, special relativity, and control systems. The insights gained from this research may not only advance mathematical theory but also contribute to the development of practical solutions in various scientific and engineering fields.

In this paper, we investigate the consimilarity of hybrid number matrices, exploring their algebraic properties and solutions to related matrix equations. By examining the connections between hybrid numbers, split quaternions, and matrix theory, we aim to extend classical results and propose new techniques for solving matrix equations involving hybrid numbers. Specifically, we solve the hybrid matrix equation $\mathbf{A}\widetilde{\mathbf{X}} - \mathbf{XB} = \mathbf{C}$ by reducing it to Sylvester's equation. This research is expected to contribute both to the theoretical understanding and the practical applications of hybrid numbers in various domains, including control theory and theoretical physics.

2. Algebraic properties of hybrid numbers

The set of hybrid numbers can be represented as

$$\mathbb{K} = \left\{ a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : \mathbf{i}^2 = -1,\ \varepsilon^2 = 0,\ \mathbf{h}^2 = 1,\ \mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon, a, b, c, d \in \mathbb{R} \right\}.$$

For every hybrid number $\mathbf{p} = p_1 + p_2\mathbf{i} + p_3\varepsilon + p_4\mathbf{h}$, we define $S_{\mathbf{p}} = p_1$ as the scalar part of $\mathbf{p}$, and $V_{\mathbf{p}} = p_2\mathbf{i} + p_3\varepsilon + p_4\mathbf{h}$ as its vector part. The hybrid number's conjugate is denoted by $\overline{\mathbf{p}}$, and it is expressed as $\overline{\mathbf{p}} = S_{\mathbf{p}} - V_{\mathbf{p}} = p_1 - p_2\mathbf{i} - p_3\varepsilon - p_4\mathbf{h}$.

Addition of two hybrid numbers $\mathbf{p} = p_1 + p_2\mathbf{i} + p_3\varepsilon + p_4\mathbf{h}$ and $\mathbf{q} = q_1 + q_2\mathbf{i} + q_3\varepsilon + q_4\mathbf{h}$ is defined by

$$\mathbf{p} + \mathbf{q} = (q_1 + q_1) + (p_2 + q_2)\,\mathbf{i} + (p_3 + q_3)\,\varepsilon + (p_4 + q_4)\,\mathbf{h}.$$

The multiplication table of the units $\mathbf{i}$, ε, and $\mathbf{h}$ is as follows, and the product of two hybrid numbers is done with the help of Table 1.

Table 1. The multiplication table of the units $\mathbf{i}$, ε, and $\mathbf{h}$.

$\bullet$	1	$\mathbf{i}$	ε	$\mathbf{h}$
1	1	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{1}-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	ε	$\mathbf{h}+\mathbf{1}$	0	$-\varepsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

Therefore, the multiplication of two hybrid numbers $\mathbf{p} = p_1 + p_2\mathbf{i} + p_3\varepsilon + p_4\mathbf{h}$ and $\mathbf{q} = q_1 + q_2\mathbf{i} + q_3\varepsilon + q_4\mathbf{h}$ can be found by

$$\begin{aligned}\mathbf{pq} &= p_1q_1 - p_2q_2 + p_4q_4 + p_2q_3 + p_3q_2 + \mathbf{i}\,(p_1q_2 + p_2q_1 + p_2q_4 - p_4q_2) \\ &\quad +\varepsilon\,(p_1q_3 + p_3q_1 + p_2q_4 - p_4q_2 - p_3q_4 + p_4q_3) + \mathbf{h}\,(p_4q_1 + p_1q_4 - p_2q_3 + p_3q_2)\,.\end{aligned}$$

The set of hybrid numbers constitutes a non-commutative ring; thus, in a general context, the non-equivalence of $\mathbf{pq} \neq \mathbf{qp}$ is anticipated for $\mathbf{p}, \mathbf{q} \in \mathbb{K}$.

The character of the hybrid number $\mathbf{p} = p_1 + p_2\mathbf{i} + p_3\varepsilon + p_4\mathbf{h}$ is defined as the real number $C(\mathbf{p}) = \mathbf{p}\overline{\mathbf{p}} = \overline{\mathbf{p}}\mathbf{p} = a^2 + (b-c)^2 - c^2 - d^2$. Moreover, a hybrid number's character is determined through the product $C(\mathbf{p}) = \mathbf{p}\overline{\mathbf{p}}$. If $C(\mathbf{p}) < 0$, $C(\mathbf{p}) > 0$, or $C(\mathbf{p}) = 0$, then $\mathbf{p}$ is categorized as spacelike, timelike, or lightlike. Additionally, one can demonstrate the equivalence $C(\mathbf{pq}) = C(\mathbf{p})\,C(\mathbf{q})$ using the product of hybrid numbers.

The norm of a hybrid number is defined as

$$\|\mathbf{p}\| = \sqrt{|C(\mathbf{p})|} = \sqrt{\left|p_1^2 + (p_2 - p_3)^2 - p_3^2 - p_4^2\right|}.$$

The inverse of the hybrid number $\mathbf{p}$ is

$$\mathbf{p}^{-1} = \frac{\overline{\mathbf{p}}}{C(\mathbf{p})}, \quad \|\mathbf{p}\| \neq 0.$$

Consequently, it can be said that lightlike hybrid numbers are not inverted. The article by Özdemir [16] provides comprehensive information about hybrid numbers.

Theorem 2.1. *[20] The following properties are satisfied for any* $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{K}$.

(1) $\mathbf{h}x_1 = \overline{x_1}\mathbf{h}$ *for* $x_1 \in \mathbb{C}$.
(2) $\mathbf{h}\,(x_1 + x_2\mathbf{h}) = \overline{x_2} + \overline{x_1}\mathbf{h}$ *for* $x_1, x_2 \in \mathbb{C}$.
(3) $\overline{\mathbf{ph}} = -\mathbf{h}\overline{\mathbf{p}}$.
(4) $\mathbf{p}^2 = S_{\mathbf{p}}^2 - \left\|V_{\mathbf{p}}\right\| + 2S_{\mathbf{p}}V_{\mathbf{p}}$.
(5) $\overline{\mathbf{pq}} = \overline{\mathbf{q}}\overline{\mathbf{p}}$.
(6) $(\mathbf{pq})\,\mathbf{r} = \mathbf{p}\,(\mathbf{qr})$.
(7) $\mathbf{pq} \neq \mathbf{qp}$ *in general.*
(8) $\mathbf{p} = \overline{\mathbf{p}} \Leftrightarrow \mathbf{p} \in \mathbb{R}$.
(9) If $p_1^2 + (p_2 - p_3)^2 \neq p_3^2 + p_4^2$, *then* $\mathbf{p}^{-1} = \dfrac{\overline{\mathbf{p}}}{\|\mathbf{p}\|^2}$.
(10) $\forall \mathbf{p} \in \mathbb{K}$ *there exists a unique representation of the form* $\mathbf{p} = x_1 + x_2\mathbf{h} \in \mathbb{K}$ *such that* $x_1, x_2 \in \mathbb{C}$.

The linear transformations designated by

$$\begin{array}{lcl} \varphi_{\mathbf{p}}:\mathbb{K}\to\mathbb{K} & & \tau_{\mathbf{p}}:\mathbb{K}\to\mathbb{K} \\ \mathbf{q}\to\varphi_{\mathbf{p}}(\mathbf{q})=\mathbf{pq} & \text{and} & \mathbf{q}\to\tau_{\mathbf{p}}(\mathbf{q})=\mathbf{qp} \end{array}$$

are called the left and right product matrices of the hybrid number algebra $\mathbb{K}$, respectively. Utilizing these transformations, the 4×4 real product matrices on the right and left sides of a hybrid number are respectively derived as follows:

$$\varphi_{\mathbf{p}}=\begin{bmatrix} p_1 & -p_2+p_3 & p_2 & p_4 \\ p_2 & p_1-p_4 & 0 & p_2 \\ p_3 & -p_4 & p_1+p_4 & p_2-p_3 \\ p_4 & p_3 & -p_2 & p_1 \end{bmatrix} \text{ and } \tau_{\mathbf{p}}=\begin{bmatrix} p_1 & p_3-p_2 & p_2 & p_4 \\ p_2 & p_1+p_4 & 0 & -p_2 \\ p_3 & p_4 & p_1-p_4 & p_3-p_2 \\ p_4 & -p_3 & p_2 & p_1 \end{bmatrix},$$

where $\mathbf{p}=p_1+p_2\mathbf{i}+p_3\varepsilon+p_4\mathbf{h}\in\mathbb{K}$. Moreover, we have $\det\varphi_{\mathbf{p}}=\det\tau_{\mathbf{p}}=\|\mathbf{p}\|^4$ [18].

Theorem 2.2. *[18] Let* $\mathbf{p},\mathbf{q}\in\mathbb{K}$ *and* $\lambda\in\mathbb{R}$. *Then*

(1) $\varphi_{\mathbf{p}}=\varphi_{\mathbf{q}}\Longleftrightarrow\mathbf{p}=\mathbf{q}\Longleftrightarrow\tau_{\mathbf{p}}=\tau_{\mathbf{q}}$,
(2) $\varphi_{\mathbf{p}+\mathbf{q}}=\varphi_{\mathbf{p}}+\varphi_{\mathbf{q}}$, $\tau_{\mathbf{p}+\mathbf{q}}=\tau_{\mathbf{p}}+\tau_{\mathbf{q}}$,
(3) $\varphi_{\mathbf{pq}}=\varphi_{\mathbf{p}}\varphi_{\mathbf{q}}$, $\tau_{\mathbf{pq}}=\tau_{\mathbf{p}}\tau_{\mathbf{q}}$,
(4) $\varphi_{\mathbf{p}}\tau_{\mathbf{q}}=\tau_{\mathbf{q}}\varphi_{\mathbf{p}}$,
(5) $\varphi_{\lambda\mathbf{p}}=\varphi_{\mathbf{p}\lambda}=\lambda\varphi_{\mathbf{p}}$, $\tau_{\lambda\mathbf{p}}=\tau_{\mathbf{p}\lambda}=\lambda\tau_{\mathbf{p}}$,
(6) $\varphi_{\mathbf{1}}=\tau_{\mathbf{1}}=\mathrm{I}_4$,
(7) $\varphi_{\mathbf{p}}+\varphi_{\overline{\mathbf{p}}}=2p_1\mathrm{I}_4$, $\tau_{\mathbf{p}}+\tau_{\overline{\mathbf{p}}}=2p_1\mathrm{I}_4$,
(8) $\varphi_{\mathbf{p}^{-1}}=\varphi_{\mathbf{p}}^{-1}$, $\quad\tau_{\mathbf{p}^{-1}}=\tau_{\mathbf{p}}^{-1}$ *where* $\|\mathbf{p}\|\neq0$,
(9) $\mu\tau_{\overline{\mathbf{p}}}=\varphi_{\mathbf{p}}\mu$, $\mu=\begin{bmatrix} 1 & 0 \\ 0 & -\mathbf{I}_3 \end{bmatrix}$.

In the work by Özdemir [16], an isomorphism between 2×2 real matrices and hybrid numbers is established, leading to the representation of hybrid numbers using 2×2 real matrices as denoted by

$$\aleph_{\mathbf{p}}=\begin{bmatrix} p_1+p_3 & p_2-p_3+p_4 \\ p_3-p_2+p_4 & p_1-p_3 \end{bmatrix},$$

for $\mathbf{p}=p_1+p_2\mathbf{i}+p_3\varepsilon+p_4\mathbf{h}\in\mathbb{K}$. Additionally, Çakır [20] elucidates certain attributes of this matrix $\aleph_{\mathbf{p}}$.

Theorem 2.3. *[20] Every hybrid number can be represented by a* 2×2 *complex matrix.*

3. Consimilarity of hybrid number matrices

The consimilarity of matrices using hybrid numbers as inputs will be the subject of this section. The set of $m\times n$ matrices with entries in the algebra of hybrid numbers, denoted by $\mathbb{M}_{m\times n}(\mathbb{K})$, forms a ring with unity under the standard operations of matrix addition and multiplication. If $m=n$, then the set of hybrid matrices is denoted $\mathbb{M}_n(\mathbb{K})$. The set of hybrid number matrices can be represented as

$$\mathbb{M}_{m\times n}(\mathbb{K})=\left\{\mathbf{A}=A_1+A_2\mathbf{i}+A_3\varepsilon+A_4\mathbf{h}:A_1,A_2,A_3,A_4\in\mathbb{M}_{m\times n}(\mathbb{R})\right\}.$$

$\mathbb{M}_{m\times n}(\mathbb{K})$ is a module over the ring $\mathbb{K}$ [20]. As with complex matrices, let $\overline{\mathbf{A}} = (\overline{\mathbf{a}_{st}}) \in \mathbb{M}_{m\times n}(\mathbb{K})$, $\mathbf{A}^T = (\mathbf{a}_{ts}) \in \mathbb{M}_{n\times m}(\mathbb{K})$ and $\mathbf{A}^* = \left(\overline{\mathbf{A}}\right)^T \in \mathbb{M}_{n\times m}(\mathbb{K})$ be the conjugate, transpose, and conjugate transpose of $\mathbf{A} = (\mathbf{a}_{st}) \in \mathbb{M}_{m\times n}(\mathbb{K})$, respectively.

Corollary 3.1. *[20] The following properties generally hold true for any* $\mathbf{A} \in \mathbb{M}_{m\times n}(\mathbb{K})$, $\mathbf{B} \in \mathbb{M}_{n\times k}(\mathbb{K})$, *and* $\mathbf{C} \in \mathbb{M}_n(\mathbb{K})$.

(1) $\left(\overline{\mathbf{C}}\right)^{-1} \neq \overline{(\mathbf{C}^{-1})}$,
(2) $\left(\mathbf{C}^T\right)^{-1} \neq \left(\mathbf{C}^{-1}\right)^T$,
(3) $\left(\overline{\mathbf{AB}}\right) \neq \overline{\mathbf{A}}\,\overline{\mathbf{B}}$,
(4) $(\mathbf{AB})^T \neq \mathbf{B}^T\mathbf{A}^T$.

Theorem 3.1. *[20] Let* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n(\mathbb{K})$ *and* $\mathbf{p} \in \mathbb{K}$. *Then the following are satisfied:*

(1) $\left(\overline{\mathbf{A}}\right)^T = \overline{(\mathbf{A}^T)}$,
(2) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ *if* $\mathbf{A}$ *and* $\mathbf{B}$ *are invertible.*
(3) $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$,
(4) $(\mathbf{A}^*)^{-1} = \left(\mathbf{A}^{-1}\right)^*$ *if* $\mathbf{A}$ *is invertible.*

Definition 3.1. *Let* $\mathbf{A}$, $\mathbf{B} \in \mathbb{M}_n(\mathbb{K})$. $\mathbf{A}$ *and* $\mathbf{B}$ *are called to be similar hybrid number matrices if there exists a hybrid number matrix* $\mathbf{P}, (\det \mathbf{P} \neq 0)$, *satisfying the equality* $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{B}$, *and it is denoted by* $\mathbf{A} \sim \mathbf{B}$. " $\sim$ " *is an equivalence relation on* $\mathbb{M}_n(\mathbb{K})$.

If $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n(\mathbb{K})$, it generally holds that $\overline{\mathbf{AB}} \neq \overline{\mathbf{A}}\,\overline{\mathbf{B}}$. Consequently, the mapping $\mathbf{A} \to \overline{\mathbf{P}}\mathbf{A}\mathbf{P}^{-1}$ does not define an equivalence relation on $\mathbb{M}_n(\mathbb{K})$. Therefore, it is necessary to introduce a new definition of consimilarity for matrices over hybrid numbers.

Definition 3.2. *Let* $\mathbf{A} \in \mathbb{M}_n(\mathbb{K})$*; then we define* $\widetilde{\mathbf{A}} = \mathbf{hAh} = A_1 - A_2\mathbf{i} - A_3\varepsilon + A_4\mathbf{h}$. *We say that* $\widetilde{\mathbf{A}}$ *is the* $\mathbf{h}$*-conjugate of* $\mathbf{A}$.

The following equalities are readily verifiable for any given matrices $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m\times n}(\mathbb{K})$ and $\mathbf{C} \in \mathbb{M}_{n\times s}(\mathbb{K})$:

(1) $\widetilde{\mathbf{A}} = \mathbf{A} \Leftrightarrow \mathbf{A} \in \mathbb{M}_n(\mathbb{P})$, where $\mathbb{M}_n(\mathbb{P})$ is the hyperbolic number matrices set,
(2) $\widetilde{\left(\widetilde{\mathbf{A}}\right)} = \mathbf{A}$,
(3) $(\widetilde{\mathbf{A} + \mathbf{B}}) = \widetilde{\mathbf{A}} + \widetilde{\mathbf{B}}$,
(4) $(\widetilde{\mathbf{AC}}) = \widetilde{\mathbf{A}}\widetilde{\mathbf{C}}$,
(5) $\overline{\left(\widetilde{\mathbf{A}}\right)} = \widetilde{\left(\overline{\mathbf{A}}\right)}$,
(6) $\left(\widetilde{\mathbf{A}}\right)^T = \widetilde{\mathbf{A}^T}$.

Definition 3.3. *Let* $\mathbf{A}$, $\mathbf{B} \in \mathbb{M}_n(\mathbb{K})$. $\mathbf{A}$ *and* $\mathbf{B}$ *are called to be consimilar hybrid number matrices if there exists a hybrid number matrix* $\mathbf{P}, (\det \mathbf{P} \neq 0)$, *satisfying the equality* $\widetilde{\mathbf{P}}\mathbf{A}\mathbf{P}^{-1} = \mathbf{B}$, *and it is denoted by* $\mathbf{A} \overset{c}{\sim} \mathbf{B}$. " $\overset{c}{\sim}$ " *is an equivalence relation on* $\mathbb{M}_n(\mathbb{K})$.

Theorem 3.2. *For three hybrid number matrices* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{M}_n(\mathbb{K})$ *the following statements hold:*
Reflexive relation: $\mathbf{A} \overset{c}{\sim} \mathbf{A}$,
Symmetric relation: if $\mathbf{A} \overset{c}{\sim} \mathbf{B}$, *then* $\mathbf{B} \overset{c}{\sim} \mathbf{A}$,
Transitive relation: if $\mathbf{A} \overset{c}{\sim} \mathbf{B}$, $\mathbf{B} \overset{c}{\sim} \mathbf{C}$, *then* $\mathbf{A} \overset{c}{\sim} \mathbf{C}$.

Proof. Reflexive: $\mathbf{IAI}^{-1} = \mathbf{A}$ trivially, for $\mathbf{A} \in \mathbb{M}_n(\mathbb{K})$ and unit matrix $\mathbf{I}$. Therefore, consimilarity is a reflexive relation.

Symmetric: Let $\widetilde{\mathbf{P}}\mathbf{AP^{-1}} = \mathbf{B}$. As $\mathbf{P}$ is invertible, we have

$$\left(\widetilde{\mathbf{P}}\right)^{-1}\mathbf{BP} = \left(\widetilde{\mathbf{P}}\right)^{-1}\widetilde{\mathbf{P}}\mathbf{AP^{-1}P} = \mathbf{A}.$$

Therefore, consimilarity is a symmetric relation.

Transitive: Let $\widetilde{\mathbf{P}}\mathbf{AP^{-1}} = \mathbf{B}$ and $\widetilde{\mathbf{Q}}\mathbf{BQ^{-1}} = \mathbf{C}$. Then

$$\mathbf{C} = \widetilde{\mathbf{Q}}\widetilde{\mathbf{P}}\mathbf{AP^{-1}Q^{-1}} = \widetilde{\mathbf{QP}}\mathbf{A}\left(\mathbf{QP}\right)^{-1}.$$

Therefore, consimilarity is a transitive relation. □

Thus, the relation $\overset{c}{\sim}$ defines an equivalence relation on hybrid matrices.

Theorem 3.3. *If* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n(\mathbb{K})$, *then*

$$\mathbf{A} \overset{c}{\sim} \mathbf{B} \Leftrightarrow \mathbf{hA} \sim \mathbf{hB} \Leftrightarrow \mathbf{Ah} \sim \mathbf{Bh} \Leftrightarrow \mathbf{hA} \sim \mathbf{Bh}.$$

Proof. Let $\mathbf{A} \overset{c}{\sim} \mathbf{B}$. Since $\mathbf{A} \overset{c}{\sim} \mathbf{B}$ $\Leftrightarrow$there exists a nonsingular matrix $\mathbf{P} \in \mathbb{M}_n(\mathbb{K})$ so that $\widetilde{\mathbf{P}}\mathbf{AP^{-1}} = \mathbf{hPhAP^{-1}} = \mathbf{B}$. Therefore, $\mathbf{A} \overset{c}{\sim} \mathbf{B} \Leftrightarrow \mathbf{PhAP^{-1}} = \mathbf{hB} \Leftrightarrow \mathbf{hA} \sim \mathbf{hB}$. Since $\mathbf{h}^{-1}\mathbf{hAh} = \mathbf{Ah}$, we get $\mathbf{hA} \sim \mathbf{Ah}$ and $\mathbf{hB} \sim \mathbf{Bh}$. Thus, $\mathbf{hA} \sim \mathbf{hB} \Leftrightarrow \mathbf{Ah} \sim \mathbf{Bh} \Leftrightarrow \mathbf{hA} \sim \mathbf{Bh}$. □

Definition 3.4. *Consider an arbitrary matrix* $\mathbf{A} \in \mathbb{M}_n(\mathbb{K})$ *and* $\lambda \in \mathbb{K}$. *In the event that* $\lambda \in \mathbb{K}$ *satisfies the equation* $\mathbf{A}\widetilde{\mathbf{x}} = \mathbf{x}\lambda$ (*or* $\mathbf{A}\widetilde{\mathbf{x}} = \lambda\mathbf{x}$) *for some nonzero hybrid number column vector* $\mathbf{x}$, *the term 'right (left) coneigenvalue' designates* λ *as the right (left) coneigenvalue of* $\mathbf{A}$, *while* $\mathbf{x}$ *assumes the role of the right (left) coneigenvector of* $\mathbf{A}$ *corresponding to the specific right (left) coneigenvalue. The set of the right coneigenvalues is defined by*

$$\widetilde{\sigma}_r(\mathbf{A}) = \{\lambda \in \mathbb{K} : \mathbf{A}\widetilde{\mathbf{x}} = \mathbf{x}\lambda, \textit{for some nonzero } \mathbf{x}\}.$$

The set of left coneigenvalues is similarly defined and is denoted by $\widetilde{\sigma}_l(\mathbf{A})$.

Theorem 3.4. *If* $\mathbf{A} \in \mathbb{M}_n(\mathbb{K})$ *is consimilar to* $\mathbf{B} \in \mathbb{M}_n(\mathbb{K})$, *the right coneigenvalues of* $\mathbf{A}$ *and* $\mathbf{B}$ *are the same.*

Proof. Let $\mathbf{A} \overset{c}{\sim} \mathbf{B}$. Then, there exists a nonsingular matrix $\mathbf{P} \in \mathbb{M}_n(\mathbb{K})$ such that $\mathbf{B} = \widetilde{\mathbf{P}}\mathbf{AP^{-1}}$. Suppose that $\lambda \in \mathbb{K}$ is a right coneigenvalue of the matrix $\mathbf{A}$. Then we can find a hybrid number column vector $\mathbf{x}$ such that $\mathbf{A}\widetilde{\mathbf{x}} = \mathbf{x}\lambda$, $\mathbf{x} \neq 0$. Let $\mathbf{y} = \mathbf{P}\widetilde{\mathbf{x}}$. Then

$$\mathbf{By} = \widetilde{\mathbf{P}}\mathbf{AP^{-1}y} = \widetilde{\mathbf{P}}\mathbf{AP^{-1}P}\widetilde{\mathbf{x}} = \widetilde{\mathbf{P}}\mathbf{A}\widetilde{\mathbf{x}} = \widetilde{\mathbf{P}}\mathbf{x}\lambda = \widetilde{\mathbf{y}}\lambda.$$

Therefore, right coneigenvalues of $\mathbf{A}$ and $\mathbf{B}$ are the same. □

Theorem 3.5. *If* $\mathbf{A} \in \mathbb{M}_n(\mathbb{K})$, *then* λ *is right coneigenvalue of* $\mathbf{A}$ *if and only if, for any* $\beta \in \mathbb{K}$ *with* $\|\beta\| \neq 0$, *the term* $\widetilde{\beta}\lambda\beta^{-1}$ *is a right eigenvalue of* $\mathbf{A}$.

Proof. From the definition of the coneigenvalue of $\mathbf{A}$ we find

$$\begin{aligned}\mathbf{A}\widetilde{\mathbf{x}} = \mathbf{x}\lambda &\Leftrightarrow \mathbf{A}\widetilde{\mathbf{x}}\beta^{-1} = \mathbf{x}\lambda\beta^{-1}\\ &\Leftrightarrow \mathbf{A}\widetilde{\mathbf{x}}\beta^{-1} = \mathbf{x}\left(\widetilde{\beta^{-1}\beta}\right)\lambda\beta^{-1}\\ &\Leftrightarrow \mathbf{A}\left(\widetilde{\mathbf{x}}\beta^{-1}\right) = \mathbf{x}\widetilde{\beta}^{-1}\left(\widetilde{\beta}\lambda\beta^{-1}\right)\\ &\Leftrightarrow \mathbf{A}\left(\widetilde{\mathbf{x}\widetilde{\beta}^{-1}}\right) = \mathbf{x}\widetilde{\beta}^{-1}\left(\widetilde{\beta}\lambda\beta^{-1}\right).\end{aligned}$$

Therefore, $\widetilde{\beta}\lambda\beta^{-1}$ is a right eigenvalue of $\mathbf{A}$. □

Definition 3.5. *[20] For* $\mathbf{A} = C_1 + C_2\mathbf{h} \in \mathbb{M}_n(\mathbb{K})$*, we shall call the* $2n \times 2n$ *complex matrix*

$$\begin{bmatrix} C_1 & C_2 \\ \overline{C_2} & \overline{C_1} \end{bmatrix}$$

uniquely determined by $\mathbf{A}$*, the complex adjoint matrix or adjoint of the hybrid number matrix* $\mathbf{A}$*, symbolized* $\chi_{\mathbf{A}}$.

Theorem 3.6. *[20] Let* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n(\mathbb{K})$*; then the following properties are satisfied:*

(1) $\chi_{I_n} = I_{2n}$,
(2) $\chi_{\mathbf{A}+\mathbf{B}} = \chi_{\mathbf{A}} + \chi_{\mathbf{B}}$,
(3) $\chi_{\mathbf{AB}} = \chi_{\mathbf{A}}\chi_{\mathbf{B}}$,
(4) $\chi_{\mathbf{A}^{-1}} = (\chi_{\mathbf{A}})^{-1}$ *if* $\mathbf{A}^{-1}$ *exist,*
(5) $\chi_{\mathbf{A}^*} \neq (\chi_{\mathbf{A}})^*$ *in general.*

Proof. Its proof can be shown easily. □

4. Real matrix representation of hybrid number matrices

Hybrid number matrices exhibit diverse real number depictions. This section inaugurates a novel real rendering for the hybrid number matrix, which shall be employed in the context of consimilarity, and deliberates upon the attributes inherent in the authentic portrayal.

Consider $\mathbf{A} = A_1 + A_2\mathbf{i} + A_3\varepsilon + A_4\mathbf{h} \in \mathbb{M}_{m\times n}(\mathbb{K})$ to be a hybrid number matrix. The linear transformation $\mathcal{R}_{\mathbf{A}}$ shall be delineated as

$$\begin{aligned}\mathcal{R}_{\mathbf{A}} &: \mathbb{M}_{m\times n}(\mathbb{K}) \to \mathbb{M}_{m\times n}(\mathbb{K})\\ \mathcal{R}_{\mathbf{A}}(\mathbf{B}) &= \mathbf{A}\widetilde{\mathbf{B}}.\end{aligned}$$

Using this operator and the basis $\{1, \mathbf{i}, \varepsilon, \mathbf{h}\}$ of the module $\mathbb{M}_n(\mathbb{K})$, we can write

$$\begin{aligned}\mathcal{R}_{\mathbf{A}}(1) &= \mathbf{A}1 = A_1 + A_2\mathbf{i} + A_3\varepsilon + A_4\mathbf{h},\\ \mathcal{R}_{\mathbf{A}}(\mathbf{i}) &= \mathbf{A}\widetilde{\mathbf{i}} = -A_1\mathbf{i} + A_2 - A_3(\mathbf{h}+1) - A_4(-\varepsilon - \mathbf{i})\\ &= A_2 - A_3 + (A_4 - A_1)\mathbf{i} + A_4\varepsilon - A_3\mathbf{h},\\ \mathcal{R}_{\mathbf{A}}(\varepsilon) &= \mathbf{A}\widetilde{\varepsilon} = -A_1\varepsilon - A_2(1-\mathbf{h}) - A_4\varepsilon\\ &= -A_2 - (A_1 + A_4)\varepsilon + A_2\mathbf{h},\end{aligned}$$

$$\begin{aligned}\mathcal{R}_{\mathbf{A}}(\mathbf{h}) &= \widetilde{\mathbf{A}\mathbf{h}} = A_1\mathbf{h} + A_2(\varepsilon+\mathbf{i}) - A_3\varepsilon + A_4 \\ &= A_4 + A_2\mathbf{i} + (A_2 - A_3)\varepsilon + A_1\mathbf{h}.\end{aligned}$$

Then, the following real matrix representation can be found as

$$\mathcal{R}_{\mathbf{A}} = \begin{bmatrix} A_1 & A_2 - A_3 & -A_2 & A_4 \\ A_2 & A_4 - A_1 & 0 & A_2 \\ A_3 & A_4 & -A_1 - A_4 & A_2 - A_3 \\ A_4 & -A_3 & A_2 & A_1 \end{bmatrix}_{4m\times 4n}.$$

Within this framework, $\mathcal{R}_{\mathbf{A}}$ is called the real matrix representation of $\mathbf{A}$ corresponding to the linear transformation $\mathcal{R}_{\mathbf{A}}(\mathbf{B}) = \widetilde{\mathbf{A}\mathbf{B}}$. It is feasible to establish a correspondence between a real matrix $A \in \mathbb{M}_{4m\times n}(\mathbb{R})$ and a hybrid number matrix $\mathbf{A} \in \mathbb{M}_{m\times n}(\mathbb{K})$. Via employment of the symbol $\cong$, we will denote

$$\mathbf{A} = A_1 + A_2\mathbf{i} + A_3\varepsilon + A_4\mathbf{h} \cong A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \in \mathbb{M}_{4m\times n}(\mathbb{R}).$$

Subsequently, the multiplication operation involving $\mathbf{A} \in \mathbb{M}_{m\times n}(\mathbb{K})$ and $\mathbf{B} \in \mathbb{M}_{n\times k}(\mathbb{K})$ can be equivalently expressed through the ordinary matrix multiplication $\widetilde{\mathbf{A}\mathbf{B}} \cong \mathcal{R}_{\mathbf{A}}B$.

Let

$$P_n = \begin{bmatrix} I_n & O_n & O_n & O_n \\ O_n & -I_n & O_n & O_n \\ O_n & O_n & -I_n & O_n \\ O_n & O_n & O_n & I_n \end{bmatrix}_{4n\times 4n}, \quad Q_n = \begin{bmatrix} O_n & -I_n & I_n & O_n \\ I_n & O_n & O_n & -I_n \\ O_n & O_n & O_n & -I_n \\ O_n & O_n & I_n & O_n \end{bmatrix}_{4n\times 4n},$$

$$R_n = \begin{bmatrix} O_n & O_n & O_n & I_n \\ O_n & I_n & O_n & O_n \\ O_n & I_n & -I_n & O_n \\ I_n & O_n & O_n & O_n \end{bmatrix}_{4n\times 4n}, \quad S_n = \begin{bmatrix} O_n & O_n & I_n & O_n \\ I_n & O_n & O_n & -I_n \\ I_n & O_n & O_n & O_n \\ O_n & -I_n & I_n & O_n \end{bmatrix}_{4n\times 4n},$$

where I_n and O_n are, respectively, an $n \times n$ identity matrix and a zero matrix.

By the operation of block matrices, we have

Proposition 4.1. *(1) If* $\mathbf{A} \in \mathbb{M}_{m\times n}(\mathbb{K})$, *then* $P_m^{-1}\mathcal{R}_{\mathbf{A}}P_n = \mathcal{R}_{\widetilde{\mathbf{A}}}$, $Q_n^{-1}\mathcal{R}_{\mathbf{A}}Q_n = -\mathcal{R}_{\mathbf{A}}$, $R_n^{-1}\mathcal{R}_{\mathbf{A}}R_n = \mathcal{R}_{\mathbf{A}}$, $S_n^{-1}\mathcal{R}_{\mathbf{A}}S_n = -\mathcal{R}_{\mathbf{A}}$.

(2) If $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m\times n}(\mathbb{K})$, *then* $\mathcal{R}_{\mathbf{A}} = \mathcal{R}_{\mathbf{B}} \Longleftrightarrow \mathbf{A} = \mathbf{B}$.

(3) If $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m\times n}(\mathbb{K})$, *then* $\mathcal{R}_{\mathbf{A}+\mathbf{B}} = \mathcal{R}_{\mathbf{A}} + \mathcal{R}_{\mathbf{B}}$.

(4) If $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m\times n}(\mathbb{K})$, *then* $\mathcal{R}_{\mathbf{A}\mathbf{B}} = \mathcal{R}_{\mathbf{A}}P_n\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{A}}\mathcal{R}_{\widetilde{\mathbf{B}}}P_n = P_n\mathcal{R}_{\widetilde{\mathbf{A}}}\mathcal{R}_{\mathbf{B}}$.

(5) If $\mathbf{A} \in \mathbb{M}_{m\times n}(\mathbb{K})$ *and* $\lambda \in \mathbb{R}$, *then* $\mathcal{R}_{\lambda\mathbf{A}} = \mathcal{R}_{\mathbf{A}\lambda} = \lambda\mathcal{R}_{\mathbf{A}}$.

(6) If $\mathbf{A} \in \mathbb{M}_m(\mathbb{K})$, *in this case* $\mathbf{A}$ *is nonsingular if and only if* $\mathcal{R}_{\mathbf{A}}$ *is nonsingular,* $\mathcal{R}_{\mathbf{A}}^{-1} = P_m\mathcal{R}_{\mathbf{A}^{-1}}P_m$.

(7) If $\mathbf{A} \in \mathbb{M}_{m\times n}(\mathbb{K})$, *then* $\mathcal{R}_{\mathbf{A}} = P_n\mathcal{R}_{\widetilde{\mathbf{A}}}P_n$.

(8) If $\mathbf{A} \in \mathbb{M}_n(\mathbb{K})$,

$$\mathbf{A} = A_1 + A_2\mathbf{i} + A_3\varepsilon + A_4\mathbf{h} = \frac{1}{3}\begin{bmatrix} I_n & \mathbf{i}I_n & \varepsilon I_n & \mathbf{h}I_n \end{bmatrix}\mathcal{R}_{\mathbf{A}}\begin{bmatrix} I_n \\ \mathbf{i}I_n \\ \varepsilon I_n \\ \mathbf{h}I_n \end{bmatrix}.$$

Proof. It can be easily seen by the definition of real matrix representation of hybrid number matrices. □

5. Hybrid number matrix equation $\mathbf{A}\widetilde{\mathbf{X}} - \mathbf{X}\mathbf{B} = \mathbf{C}$

In this section, we investigate the solutions to the hybrid number matrix equations

$$\mathbf{A}\widetilde{\mathbf{X}} - \mathbf{X}\mathbf{B} = \mathbf{C} \tag{5.1}$$

by means of real representation, where $\mathbf{A} \in \mathbb{M}_n(\mathbb{K}), \mathbf{B} \in \mathbb{M}_n(\mathbb{K})$, and $\mathbf{C} \in \mathbb{M}_n(\mathbb{K})$. We begin by defining the real representation matrix corresponding to Eq (5.1) by

$$\mathcal{R}_{\mathbf{A}}Y - Y\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{C}}. \tag{5.2}$$

By (4) in Proposition 4.1, the Eq (5.1) is equivalent to the equation

$$\mathcal{R}_{\mathbf{A}}(\mathcal{R}_{\mathbf{X}}P_n) - (\mathcal{R}_{\mathbf{X}}P_n)\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{C}}. \tag{5.3}$$

Proposition 5.1. *Hybrid number matrix equation $\mathbf{A}\widetilde{\mathbf{X}} - \mathbf{X}\mathbf{B} = \mathbf{C}$ has a solution $\mathbf{X} \in \mathbb{M}_{m\times n}(\mathbb{K})$ if and only if the real matrix equation $\mathcal{R}_{\mathbf{A}}Y - Y\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{C}}$ has a solution $Y = \mathcal{R}_{\mathbf{X}}P_n$.*

Proof. Let $\mathbf{X} \in \mathbb{M}_{m\times n}(\mathbb{K})$ be a solution to the hybrid number matrix equation $\mathbf{A}\widetilde{\mathbf{X}} - \mathbf{X}\mathbf{B} = \mathbf{C}$. Then, we get

$$\begin{aligned}
&\mathbf{A}\widetilde{\mathbf{X}} - \mathbf{X}\mathbf{B} = \mathbf{C}\\
&\Leftrightarrow \mathcal{R}_{\mathbf{A}\widetilde{\mathbf{X}}} - \mathcal{R}_{\mathbf{X}\mathbf{B}} = \mathcal{R}_{\mathbf{C}}\\
&\Leftrightarrow \mathcal{R}_{\mathbf{A}}P_n\mathcal{R}_{\widetilde{\mathbf{X}}} - \mathcal{R}_{\mathbf{X}}P_n\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{C}}\\
&\Leftrightarrow \mathcal{R}_{\mathbf{A}}\mathcal{R}_{\mathbf{X}}P_n - \mathcal{R}_{\mathbf{X}}P_n\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{C}}\\
&\Leftrightarrow \mathcal{R}_{\mathbf{A}}(\mathcal{R}_{\mathbf{X}}P_n) - (\mathcal{R}_{\mathbf{X}}P_n)\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{C}}.
\end{aligned}$$

Therefore, the solution of the hybrid number matrix equation $\mathbf{A}\widetilde{\mathbf{X}} - \mathbf{X}\mathbf{B} = \mathbf{C}$ is $\mathbf{X} \in \mathbb{M}_{m\times n}(\mathbb{K})$, if and only if the solution of the real matrix equation $\mathcal{R}_{\mathbf{A}}Y - Y\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{C}}$ is $Y = \mathcal{R}_{\mathbf{X}}P_n$. □

Theorem 5.1. *[3] (Sylvester) Let $A \in \mathbb{M}_n$ and $B \in \mathbb{M}_m$ be given. The equation $AX - XB = C$ has a unique solution $X \in \mathbb{M}_{n\times m}$ for each given $C \in \mathbb{M}_{n\times m}$ if and only if $\sigma(A) \cap \sigma(B) = \emptyset$, that is, if and only if A and B have no eigenvalue in common. In particular, if $\sigma(A) \cap \sigma(B) = \emptyset$, then the only X such that $AX - XB = 0$ is $X = 0$. If A and B are real, then $AX - XB = C$ has a unique solution $X \in \mathbb{M}_{n\times m}(\mathbb{R})$ for each given $C \in \mathbb{M}_{n\times m}(\mathbb{R})$.*

Lemma 5.1. *Let $\mathbf{A} = A_1 + A_2\mathbf{i} + A_3\varepsilon + A_4\mathbf{h} \in \mathbb{M}_n(\mathbb{K})$ and $\mathbf{B} = B_1 + B_2\mathbf{i} + B_3\varepsilon + B_4\mathbf{h} \in \mathbb{M}_m(\mathbb{K})$ be given. The equation $\mathbf{A}\widetilde{\mathbf{X}} - \mathbf{X}\mathbf{B} = \mathbf{C}$ has a unique solution $\mathbf{X} \in \mathbb{M}_{n\times m}(\mathbb{K})$ for each given $\mathbf{C} \in \mathbb{M}_{n\times m}(\mathbb{K})$ if and only if $\sigma(\mathcal{R}_{\mathbf{A}}) \cap \sigma(\mathcal{R}_{\mathbf{B}}) = \emptyset$.*

Proof. Let $\mathbf{A} = A_1 + A_2\mathbf{i} + A_3\varepsilon + A_4\mathbf{h} \in \mathbb{M}_n(\mathbb{K})$ and $\mathbf{B} = B_1 + B_2\mathbf{i} + B_3\varepsilon + B_4\mathbf{h} \in \mathbb{M}_m(\mathbb{K})$. Then, we have

$$\begin{aligned}
\mathbf{A}\widetilde{\mathbf{X}} - \mathbf{X}\mathbf{B} &= \mathbf{C} \Leftrightarrow \mathcal{R}_{\mathbf{A}\widetilde{\mathbf{X}}} - \mathcal{R}_{\mathbf{X}\mathbf{B}} = \mathcal{R}_{\mathbf{C}} \Leftrightarrow \mathcal{R}_{\mathbf{A}}P_n\mathcal{R}_{\widetilde{\mathbf{X}}} - \mathcal{R}_{\mathbf{X}}P_n\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{C}}\\
&\Leftrightarrow \mathcal{R}_{\mathbf{A}}\mathcal{R}_{\mathbf{X}}P_n - \mathcal{R}_{\mathbf{X}}P_n\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{C}} \Leftrightarrow \mathcal{R}_{\mathbf{A}}(\mathcal{R}_{\mathbf{X}}P_n) - (\mathcal{R}_{\mathbf{X}}P_n)\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{C}}.
\end{aligned}$$

In this case, according to Theorem 5.1, the necessary and sufficient condition for the solution of the equation $\mathcal{R}_{\mathbf{A}}(\mathcal{R}_{\mathbf{X}}P_n) - (\mathcal{R}_{\mathbf{X}}P_n)\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{C}}$ is

$$\sigma(\mathcal{R}_{\mathbf{A}}) \cap \sigma(\mathcal{R}_{\mathbf{B}}) = \varnothing.$$

Therefore, for the equation $\mathbf{A}\widetilde{\mathbf{X}} - \mathbf{XB} = \mathbf{C}$ to have a unique solution of $\mathbf{X} \in \mathbb{M}_{n\times m}(\mathbb{K})$ for each given $\mathbf{C} \in \mathbb{M}_{n\times m}(\mathbb{K})$, if and only if $\sigma(\mathcal{R}_{\mathbf{A}}) \cap \sigma(\mathcal{R}_{\mathbf{B}}) = \varnothing$. □

Theorem 5.2. *Let* $\mathbf{A} \in \mathbb{M}_m(\mathbb{K}), \mathbf{B} \in \mathbb{M}_n(\mathbb{K})$*, and* $\mathbf{C} \in \mathbb{M}_{m\times n}(\mathbb{K})$*. Then the hybrid number matrix Eq* (5.1) *has a solution* $\mathbf{X} \in \mathbb{M}_{m\times n}(\mathbb{K})$ *if and only if the real representation matrix Eq (5.2) has a solution* $Y \in \mathbb{M}_{4m\times 4n}(\mathbb{R})$*; in this case, if Y is a solution to* (5.2)*, then the following matrix:*

$$\mathbf{X} = \frac{1}{12}\begin{bmatrix} I_m & \mathbf{i}I_m & \varepsilon I_m & \mathbf{h}I_m \end{bmatrix}\left(Y + Q_m^{-1}YQ_n + R_m^{-1}YR_n + S_m^{-1}YS_n\right)\begin{bmatrix} I_n \\ -\mathbf{i}I_n \\ -\varepsilon I_n \\ \mathbf{h}I_n \end{bmatrix} \tag{5.4}$$

is a solution to the hybrid number matrix Eq (5.1).

Proof. We show that if the real matrix

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix},\ Y_{st} \in \mathbb{M}_{m\times n}(\mathbb{R}),\ s,t = 1,2,3,4 \tag{5.5}$$

is a solution to (5.2), then the matrix given in (5.4) is a solution to (5.1). Since $Q_m^{-1}\mathcal{R}_{\mathbf{X}}Q_n = -\mathcal{R}_{\mathbf{X}}$, $R_m^{-1}\mathcal{R}_{\mathbf{X}}R_n = \mathcal{R}_{\mathbf{X}}$, $S_m^{-1}\mathcal{R}_{\mathbf{X}}S_n = -\mathcal{R}_{\mathbf{X}}$, and $Y = \mathcal{R}_{\mathbf{X}}P_n$, we have

$$\begin{aligned} &\mathcal{R}_{\mathbf{A}}\left(-Q_m^{-1}YP_nQ_n\right)P_n - \left(-Q_m^{-1}YP_nQ_n\right)P_n\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{C}}, \\ &\mathcal{R}_{\mathbf{A}}\left(R_m^{-1}YP_nR_n\right)P_n - \left(R_m^{-1}YP_nR_n\right)P_n\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{C}}, \\ &\mathcal{R}_{\mathbf{A}}\left(-S_m^{-1}YP_nS_n\right)P_n - \left(-S_m^{-1}YP_nS_n\right)P_n\mathcal{R}_{\mathbf{B}} = \mathcal{R}_{\mathbf{C}}. \end{aligned} \tag{5.6}$$

The equations in (5.6) demonstrate that if Y is a solution of (5.2), then $\left(-Q_m^{-1}YP_nQ_n\right)P_n$, $\left(R_m^{-1}YP_nR_n\right)P_n$, and $\left(-S_m^{-1}YP_nS_n\right)P_n$ are also solutions to (5.2). Hence, the following real matrix:

$$\widehat{Y} = \frac{1}{4}\left(Y - \left(Q_m^{-1}YP_nQ_n - R_m^{-1}YP_nR_n + S_m^{-1}YP_nS_n\right)P_n\right) \tag{5.7}$$

is a solution to (5.2). By substituting (5.5) into (5.7) and simplifying the resulting expression, we readily obtain

$$\widehat{Y}P_n = \mathcal{R}_{\mathbf{X}} = \begin{bmatrix} X_1 & X_2 - X_3 & -X_2 & X_4 \\ X_2 & X_4 - X_1 & 0 & X_2 \\ X_3 & X_4 & -X_1 - X_4 & X_2 - X_3 \\ X_4 & -X_3 & X_2 & X_1 \end{bmatrix}, \tag{5.8}$$

where

$$X_1 = \frac{1}{4}(Y_{11} + Y_{22} + Y_{33} + Y_{44}), \qquad X_2 = \frac{1}{4}(Y_{21} + Y_{13} + Y_{24} - Y_{43}),$$
$$X_3 = \frac{1}{4}(Y_{12} + Y_{13} + Y_{31} + Y_{24} + Y_{42} - Y_{34}), \quad X_4 = \frac{1}{4}(Y_{14} - Y_{22} - Y_{23} + Y_{41} + Y_{33}).$$

Next, we construct a hybrid number matrix utilizing Proposition 4.1(8):

$$\mathbf{X} = X_1 + X_2\mathbf{i} + X_3\varepsilon + X_4\mathbf{h} = \frac{1}{3}\begin{bmatrix} I_m & \mathbf{i}I_m & \varepsilon I_m & \mathbf{h}I_m \end{bmatrix}\widehat{Y}\begin{bmatrix} I_n \\ -\mathbf{i}I_n \\ -\varepsilon I_n \\ \mathbf{h}I_n \end{bmatrix}$$
$$= \frac{1}{12}\begin{bmatrix} I_m & \mathbf{i}I_m & \varepsilon I_m & \mathbf{h}I_m \end{bmatrix}\left(Y + Q_m^{-1}YQ_n + R_m^{-1}YR_n + S_m^{-1}YS_n\right)\begin{bmatrix} I_n \\ -\mathbf{i}I_n \\ -\varepsilon I_n \\ \mathbf{h}I_n \end{bmatrix}.$$

Hence, $\mathbf{X}$ serves as a solution to the equation presented in (5.1). □

As a special case of Theorem 5.2 for $\mathbf{C} = 0$, we have the following result for consimilarity of hybrid number matrices.

Theorem 5.3. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n(\mathbb{K})$. *If* $\mathbf{C} = 0$ *in the equation* $\mathbf{A}\widetilde{\mathbf{X}} - \mathbf{X}\mathbf{B} = \mathbf{C}$, *and* $\mathbf{X} \in \mathbb{M}_{m\times n}(\mathbb{K})$ *is nonsingular, then the hybrid number matrix* $\mathbf{A}$ *is consimilar to* $\mathbf{B}$. *Furthermore, the associated real matrix* $\mathcal{R}_{\mathbf{A}}$ *is similar to* $\mathcal{R}_{\mathbf{B}}$.

Proof. If the matrix $\mathbf{C} = 0$ and $\mathbf{X}$ is nonsingular in the equation $\mathbf{A}\widetilde{\mathbf{X}} - \mathbf{X}\mathbf{B} = \mathbf{C}$, then we obtain

$$\mathbf{A}\widetilde{\mathbf{X}} = \mathbf{X}\mathbf{B} \Rightarrow \mathbf{X}^{-1}\mathbf{A}\widetilde{\mathbf{X}} = \mathbf{B},$$

and

$$\begin{aligned} &\mathbf{A}\widetilde{\mathbf{X}} = \mathbf{X}\mathbf{B} \Rightarrow \mathcal{R}_{\mathbf{A}\widetilde{\mathbf{X}}} = \mathcal{R}_{\mathbf{X}\mathbf{B}} \\ &\Rightarrow \mathcal{R}_{\mathbf{A}}P_n\mathcal{R}_{\widetilde{\mathbf{X}}} = \mathcal{R}_{\mathbf{X}}P_n\mathcal{R}_{\mathbf{B}} \\ &\Rightarrow (\mathcal{R}_{\mathbf{X}}P_n)^{-1}\mathcal{R}_{\mathbf{A}}P_n\mathcal{R}_{\widetilde{\mathbf{X}}} = \mathcal{R}_{\mathbf{B}} \\ &\Rightarrow (\mathcal{R}_{\mathbf{X}}P_n)^{-1}\mathcal{R}_{\mathbf{A}}(\mathcal{R}_{\mathbf{X}}P_n) = \mathcal{R}_{\mathbf{B}}. \end{aligned}$$

Therefore, the hybrid number matrix $\mathbf{A}$ is consimilar to $\mathbf{B}$, and the real matrix $\mathcal{R}_{\mathbf{A}}$ is similar to $\mathcal{R}_{\mathbf{B}}$. □

Example 5.1. *Solve hybrid number matrix equation*

$$\begin{bmatrix} 1-\varepsilon & \mathbf{i}+2\varepsilon-\mathbf{h} \\ 3-2\mathbf{h} & 2-\mathbf{i}+2\mathbf{h} \end{bmatrix}\widetilde{\mathbf{X}} - \mathbf{X}\begin{bmatrix} 2+\mathbf{i} & \varepsilon+\mathbf{h} \\ 1-\mathbf{h} & 3-2\mathbf{i}+\mathbf{h} \end{bmatrix} = \begin{bmatrix} -7+16\mathbf{i}+5\varepsilon-3\mathbf{h} & -4+\mathbf{i}-9\varepsilon-6\mathbf{h} \\ 4+15\mathbf{i}+5\varepsilon+6\mathbf{h} & -2-11\mathbf{i}-11\varepsilon-2\mathbf{h} \end{bmatrix}$$

by utilizing its real representation.

The real representation of given equation is

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 0 & -1 & 0 & -1 \\ 3 & 2 & 0 & -1 & 0 & 1 & -2 & 2 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & -5 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & 0 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & -2 & 2 & -1 & -4 & 0 & -1 \\ 0 & -1 & 1 & -2 & 0 & 1 & 1 & 0 \\ -2 & 2 & 0 & 0 & 0 & -1 & 3 & 2 \end{bmatrix} Y - Y \begin{bmatrix} 2 & 0 & 1 & -1 & -1 & 0 & 0 & 1 \\ 1 & 3 & 0 & -2 & 0 & 2 & -1 & 1 \\ 1 & 0 & -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & -2 & -2 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 & -2 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & -4 & 0 & -2 \\ 0 & 1 & 0 & -1 & 1 & 0 & 2 & 0 \\ -1 & 1 & 0 & 0 & 0 & -2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -4 & 11 & 10 & -16 & -1 & -3 & -6 \\ 4 & -2 & 10 & 0 & -15 & 11 & 6 & -2 \\ 16 & 1 & 4 & -2 & 0 & 0 & 16 & 1 \\ 15 & -11 & 2 & 0 & 0 & 0 & 15 & -11 \\ 5 & -9 & -3 & -6 & 10 & 10 & 11 & 10 \\ 5 & -11 & 6 & -2 & -10 & 4 & 10 & 0 \\ -3 & -6 & -5 & 9 & 16 & 1 & -7 & -4 \\ 6 & -2 & -5 & 11 & 15 & -11 & 4 & -2 \end{bmatrix}.$$

The solution to this equation yields

$$Y = \begin{bmatrix} 0 & 1 & 2 & 1 & -3 & 0 & 2 & 1 \\ 1 & -1 & 1 & -1 & -1 & 2 & 3 & 1 \\ -3 & 0 & -2 & 0 & 0 & 0 & -3 & 0 \\ -1 & 2 & -2 & -2 & 0 & 0 & -1 & 2 \\ -1 & 1 & -2 & -1 & 2 & 2 & -2 & -1 \\ 0 & 1 & -3 & -1 & 4 & 0 & -1 & 1 \\ 2 & 1 & -1 & 1 & 3 & 0 & 0 & 1 \\ 3 & 1 & 0 & 1 & 1 & -2 & 1 & -1 \end{bmatrix}.$$

Then,

$$\mathbf{X} = \frac{1}{12}\begin{bmatrix} I_2 & \mathbf{i}I_2 & \varepsilon I_2 & \mathbf{h}I_2 \end{bmatrix}\left(Y + Q_2^{-1}YQ_2 + R_2^{-1}YR_2 + S_2^{-1}YS_2\right)\begin{bmatrix} I_2 \\ -\mathbf{i}I_2 \\ -\varepsilon I_2 \\ \mathbf{h}I_2 \end{bmatrix} = \begin{bmatrix} -3\mathbf{i} - \varepsilon + 2\mathbf{h} & 1 + \varepsilon + \mathbf{h} \\ 1 - \mathbf{i} + 3\mathbf{h} & -1 + 2\mathbf{i} + \varepsilon + \mathbf{h} \end{bmatrix}.$$

6. Conclusions

In this study, we have explored the consimilarity of hybrid number matrices, an important structural relationship between matrices that preserves key matrix properties without requiring invertibility. By investigating the eigenvalues, diagonalizability, and solutions to matrix equations involving hybrid numbers, we have extended classical matrix theory concepts, specifically the notions of similarity and consimilarity, to non-commutative algebraic structures.

The primary focus of this research was to examine the matrix equation $\mathbf{A}\widetilde{\mathbf{X}} - \mathbf{XB} = \mathbf{C}$ in the context of hybrid numbers and relate it to Sylvester's equation. This approach provides new insights into solving matrix equations involving hybrid numbers, contributing to the development of new algebraic methods and computational techniques. Furthermore, the study of hybrid numbers, which are isomorphic to split quaternions, opens up opportunities for applications in geometric transformations, Lorentzian relations, and stability analysis in control systems.

The results obtained from this work not only enhance the theoretical understanding of hybrid number matrices but also have practical implications for areas such as quantum mechanics, special relativity, and control systems. Future research may explore further applications of these matrices, extending the current findings and potentially leading to new breakthroughs in both mathematics and physics.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflicts of interest.

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