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Research article

Some rigidity theorems for totally real submanifolds in complex space forms

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Abstract: A study of the relationship between pseudo-umbilical totally real submanifolds and minimal totally real submanifolds in complex space forms is presented in this paper. The paper studies totally real submanifolds in complex space forms. The moving-frame method and the DDVV inequality (a conjecture for the Wintgen inequality on Riemannian submanifolds in real space forms proven by P.J. De Smet, F. Dillen, L. Verstraelen, and L. Vrancken) are used to obtain some rigidity theorems and an integral inequality, improving the associated results.

Keywords: complex space form, totally real submanifolds; second fundamental form; pinching theorems
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1. Introduction

A key challenge in differential geometry is exploring the relationship between the geometry and topology of Riemannian manifolds. In submanifold theory, an exciting question is how the pinching conditions on intrinsic or extrinsic curvature invariants affect the geometry and topology of submanifolds in space forms. Simons first established a key result on minimal submanifolds of spheres with a sufficiently pinched second fundamental form in his seminal paper [18]. Later, Chern, et al. [6] proved a well-known rigidity theorem, which has since motivated numerous significant advances in the study of pinching conditions. The study of rigidity theorems is crucial in the theory of minimum submanifolds. Some pioneering work and substantial research on rigidity theorems for minimum submanifolds in spheres have been conducted by Lawson [8]. Let the square norm of the second fundamental form be represented by σ and a unit sphere be represented by $\mathbb{S}^{n+m}$ with the codimension m. If a compact minimal submanifold $\mathcal{N}^n$ in $\mathbb{S}^{n+m}$ with the following pinching condition:

$$0 \leq \sigma \leq \left(\frac{n}{2-\frac{1}{m}}\right)$$

then either

$$\sigma = 0, \quad \text{or} \quad \sigma = \left(\frac{n}{2-\frac{1}{m}}\right)$$

and $\mathcal{N}$ is the Clifford hypersurface or the Veronese surface in $\mathbb{S}^4$. Later, Li [9] and Chen [5] improved the pinching number $\frac{n}{(2-1/m)}$ to $\frac{2n}{3}$. They showed that if

$$0 \leq \sigma \leq \frac{2n}{3},$$

then either

$$\sigma = 0 \quad \text{or} \quad \sigma = \frac{2n}{3},$$

and $\mathcal{N}$ is the Veronese surface in $\mathbb{S}^4$.

After initial motivation by Simons [18] and preliminary developments (for example, [5, 8, 12, 16, 17]), this topic has received much attention. These underlying works reveal, in particular, several similarities between the free boundary minimal surfaces in a Euclidean unit ball and closed minimal surfaces in the sphere. In this respect, the classical results and tactics for obtaining rigidity results, in conclusion, may indicate the direction of interest in exploring similar progress in the free boundary case. This study was motivated by the rigidity theorems for minimal submanifolds and submanifolds with parallel mean curvature in space forms see [9, 11, 19, 20], etc.

On the other hand, the space forms are useful for understanding the geometric analysis. Several authors constructed the first eigenvalues for submanifolds in different space forms such as in C-totally real submanifolds in Sasakian space forms [1], Lagrangian submanifolds in complex space forms [2], slant submanifolds of Sasakian space form [13, 15], semi-slant submanifolds of Sasakian space forms [14] and totally real submanifolds in generalized complex space forms [3] that contain a p-laplacian operator. It should be noted that little work has been done on the rigidity theorems for totally real submanifolds in space form geometry. Therefore, motivated by some previous work, we constructed the rigidity for a totally real submanifold in complex space form and discuss their consequences in the present paper.

2. Basic formulas and definitions

Assume that $\widetilde{\mathcal{N}}^n$ is a complex space form of constant holomorphic sectional curvature 4κ, denoted $\widetilde{\mathcal{N}}^n(4\kappa)$. The curvature tensor $\widetilde{\mathrm{R}}$ of $\widetilde{\mathcal{N}}^n(4\kappa)$ can be expressed as:

$$\begin{aligned}\widetilde{\mathrm{R}}(\mathrm{V}_1, \mathrm{V}_2)\mathrm{V}_3 =\kappa\Big\{&g(\mathrm{V}_2, \mathrm{V}_3)\mathrm{V}_1 - g(\mathrm{V}_1, \mathrm{V}_3)\mathrm{V}_2 + g(\mathrm{V}_3, \mathrm{J}\mathrm{V}_2)\mathrm{J}\mathrm{V}_1 \\ &- g(\mathrm{V}_3, \mathrm{J}\mathrm{V}_1)\mathrm{J}\mathrm{V}_2 + 2g(\mathrm{V}_1, \mathrm{J}\mathrm{V}_2)\mathrm{J}\mathrm{V}_3\Big\}\end{aligned} \tag{2.1}$$

for all $\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3 \in \Gamma(T\widetilde{\mathcal{N}})$. Based on the cases, $\kappa < 0$, $\kappa = 0$, and $\kappa > 0$, $\widetilde{\mathbb{N}}^n(4\kappa)$ is the complex hyperbolic space $\mathbb{C}H^n$, the complex Euclidean space $\mathbb{C}^n$ and the complex projective space $\mathbb{C}P^n$. We call an m-dimensional Riemannian submanifold $\mathcal{N}^m$ of $\widetilde{\mathbb{N}}^n(4\kappa)$ as totally real if the standard complex structure J of $\widetilde{\mathcal{N}}^n(4\kappa)$ maps any tangent space of $\mathcal{N}^m$ into its corresponding normal space [4].

We considered an orthonormal frame

$$\{e_1 \cdots e_m, e_{m+1} \cdots e_{m+h}, e_1^* = \mathrm{J}e_1 \cdots e_m^* = \mathrm{J}e_m, e_{(m+1)^*} = \mathrm{J}e_{m+1} \cdots e_{(m+h)^*} = \mathrm{J}e_{m+h}\}$$

in $\widetilde{\mathcal{N}}^{m+h}(4\kappa)$ restricted to $\mathcal{N}^m$, $e_1 \cdots e_m$ is tangent to $\mathcal{N}^m$. We provided the indices as follows:

$$\mathcal{A}, \mathcal{B}, \mathcal{C} \cdots = 1, \cdots, m+h, 1^* \cdots, m+h^*$$
$$a, b, c \cdots = 1, \cdots, m; a^*, b^*, c^* = m+1, \cdots, m+h, 1^*, \cdots, m+h^*.$$

Let Π denote the squared length of the second fundamental form ζ of $\mathcal{N}^m$, which is defined by

$$\Pi = \sum_{abk} (\zeta_{ab}^k)^2. \tag{2.2}$$

Similarly, the mean curvature of $\mathcal{N}^m$ is calculated as:

$$\mathsf{H} = \frac{1}{m} \sum_{ak} \zeta_{aa}^k e_k. \tag{2.3}$$

If $\mathsf{H} = 0$ in (2.3), then $\mathcal{N}^m$ is minimal. From (2.1), we get the following equation for submanifold in complex space form:

$$\widetilde{\mathrm{K}}_{\mathcal{ABCD}} = \Big(\delta_{\mathcal{AC}}\delta_{\mathcal{BD}} - \delta_{\mathcal{AD}}\delta_{\mathcal{BC}}\Big)\kappa + \kappa\Big(\mathrm{J}_{\mathcal{AC}}\mathrm{J}_{\mathcal{BD}} - \mathrm{J}_{\mathcal{AD}}\mathrm{J}_{\mathcal{BC}} + 2\mathrm{J}_{\mathcal{AB}}\mathrm{J}_{\mathcal{CD}}\Big) \tag{2.4}$$

where $\widetilde{\mathrm{K}}$ is the sectional curvature of $\widetilde{\mathbb{N}}^n(4\kappa)$. The curvature tensor of indices for the submanifold is

$$\mathrm{R}_{abcl} = \widetilde{\mathrm{K}}_{abcl} + \sum_{\alpha} \Big(\zeta_{ac}^{\alpha}\zeta_{bl}^{\alpha} - \zeta_{al}^{\alpha}\zeta_{bc}^{\alpha}\Big). \tag{2.5}$$

We define the Ricci curvature for a totally real submanifold:

$$\mathrm{R}_{ab} = (m-1)\kappa\delta_{ab} + \sum_{a} \Big(\zeta_{ab}^k \sum_{c} \zeta_{cc}^k - \sum_{c} \zeta_{ac}^k \zeta_{cb}^k\Big). \tag{2.6}$$

From the above, we can establish some notation

$$\Pi = \|\zeta\|^2, \quad \mathsf{H} = |\xi|, \quad \mathsf{H}_{\alpha} = (\zeta_{ab}^{\alpha})_{m\times m}. \tag{2.7}$$

Let us assume that e_{m+1} is parallel to H in which case we have

$$tr\mathsf{H}_{m+1} = m\mathsf{H}, \quad \mathsf{H}_{\alpha} = 0, \quad \alpha \neq m+1 \tag{2.8}$$

where tr stands for the trace of the matrix $\mathsf{H}_{\alpha} = (\zeta_{ab}^{\alpha})$. Taking account of (2.5) and (2.8), we have the scalar curvature as

$$\mathrm{R} = m(m-1)\kappa + m^2\mathsf{H}^2 - \Pi \tag{2.9}$$

where H stands for the mean curvature vector of $\mathcal{N}^m$. Since H is constant, it can be concluded that the scalar curvature R is constant if and only if Π is constant by (2.9). Let ζ^k_{abc} denote the second covariant derivative of ζ^k_{ab} in which case we have

$$\sum_c \zeta^\alpha_{abc}\omega_c = \mathrm{d}\zeta^\alpha_{ab} - \sum_c \zeta^\alpha_{cb}\omega_{ca} - \sum_c \zeta^\alpha_{ac}\omega_{cb} + \sum_t \zeta^\alpha_{ab}\omega_{tk} \tag{2.10}$$

where $\{\omega_a\}$ is the dual frame of $\mathcal{N}^m$. Taking the exterior derivative of the equation (2.10), we obtain

$$\sum_l \zeta^\alpha_{abcl}\omega_l = \mathrm{d}\zeta^\alpha_{abc} - \sum_l \zeta^\alpha_{abc}\omega_{la} + \sum_t \zeta^t_{abc}\omega_{tk} - \sum_l \zeta^\alpha_{alc}\omega_{ab} - \sum_l \zeta^\alpha_{abl}\omega_{lc}. \tag{2.11}$$

Moreover, the Laplacian of ζ^α_{ab} is

$$\Delta\zeta^\alpha_{ab} = \sum_c \zeta^\alpha_{abcc} = \sum_c \zeta^\alpha_{ccab} + \sum_{cd}\left(\zeta^\alpha_{cd}\mathcal{R}_{dabc} + \zeta^\alpha_{da}\mathcal{R}_{dcbc}\right) - \sum_{\beta c}\zeta^\beta_{ca}\mathcal{R}_{\alpha\beta bc}. \tag{2.12}$$

Lemma 2.1. *[10] Let* $\mathrm{T}_1, \cdots, \mathrm{T}_n$ *be symmetric* $(m\times m)$*-matrices, in which case*

$$\sum_{r,s=1}^n \|[\mathrm{T}_r, \mathrm{T}_s]\|^2 \le \left(\sum_{r=1}^n \|\mathrm{T}_r\|^2\right)^2$$

such that equality holds if and only if the following matrices are satisfied:

$$\mathrm{T}_r = P\begin{pmatrix} 0 & \mu & 0\cdots0 \\ \mu & 0 & 0\cdots0 \\ 0 & 0 & 0\cdots0 \\ \vdots & \vdots & \vdots\ddots\vdots \\ 0 & 0 & 0\cdots0 \end{pmatrix}P^t, \quad \mathrm{T}_s == P\begin{pmatrix} \mu & 0 & 0\cdots0 \\ 0 & -\mu & 0\cdots0 \\ 0 & 0 & 0\cdots0 \\ \vdots & \vdots & \vdots\ddots\vdots \\ 0 & 0 & 0\cdots0 \end{pmatrix}P^t$$

where P is an orthogonal $(m\times m)$*-matrix and* $[\mathrm{T}_r, \mathrm{T}_s] = \mathrm{T}_r\mathrm{T}_s - \mathrm{T}_s\mathrm{T}_r$ *is the commutator of the matrices* $\mathrm{T}_r, \mathrm{T}_s$.

Lemma 2.2. *Let us* $\mathcal{T}_1, \mathcal{T}_2, \cdots \mathcal{T}_m (n \ge 2)$ *be symmetric* $(m\times m)$*-matrices, in this case*

$$-2\sum_{\alpha,\beta=1}^m \left(tr(\mathcal{T}^2_\alpha\mathcal{T}^2_\beta - tr(\mathcal{T}_\alpha\mathcal{T}_\beta)^2\right) \ge \sum_{\alpha,\beta=1}^m [tr(\mathcal{T}_\alpha\mathcal{T}_\beta)]^2 - \frac{3}{2}\Big(\sum_{\alpha=1}^m tr(\mathcal{T}^2_\alpha)\Big)^2. \tag{2.13}$$

We can now estimate our first main result, which is as follows.

Theorem 2.1. *If the mean curvature vector of an m-dimensional compact totally real submanifold* $\mathcal{N}^m$ *in complex space form* $\widetilde{\mathcal{N}}^{m+h}(4\kappa)$ *is parallel and satisfies the following inequality*

$$\mathrm{R}_{\mathcal{N}} \ge \left(\frac{m+2h-1}{2(m+2h)}\right)\left(\kappa + \mathsf{H}^2\right), \tag{2.14}$$

then $\mathcal{N}^m$ *is a totally umbilical sphere* $\mathbb{S}^m\Big(\frac{1}{\sqrt{\kappa+\mathsf{H}^2}}\Big)$, *where* H *denotes the mean curvature of* $\mathcal{N}^m$.

Proof. Assume that $\mathcal{N}^m$ is a totally real submanifold of complex space form $\widetilde{\mathcal{N}}^{m+h}(4\kappa)$ with the parallel mean curvature vector H. Consider an e_{m+1} that it is parallel to H and

$$tr\mathsf{H}_{m+1} = m\mathsf{H}, \quad tr\mathsf{H}_\alpha = 0, \quad \alpha = m+1. \tag{2.15}$$

We assume that the mean curvature vector H is parallel, so we have

$$\mathrm{D}^\perp \mathsf{H} = \mathrm{d}\mathsf{H}e_{m+1} + \mathsf{H}\mathrm{D}^\perp e_{m+1} = \mathrm{d}\mathsf{H}e_{m+1} + \mathsf{H}\sum_\beta \omega_{m+1\beta}e_\beta = 0 \tag{2.16}$$

where D is a Levi-Civita connection. From the structure equation and (2.16), we derive

$$\begin{aligned} \mathrm{d}\omega_{m+1\beta} &= -\sum_\gamma \omega_{m+1\gamma}\wedge\omega_{\gamma\beta} + \frac{1}{2}\sum_{cl}\mathrm{R}_{m+1\beta cl}\omega_c\wedge\omega_l \\ &= \frac{1}{2}\sum_{cl}\mathrm{R}_{m+1\beta cl}\omega_c\wedge\omega_l = 0. \end{aligned} \tag{2.17}$$

If we consider, from (2.12), that $\mathcal{N}^m$ has as parallel mean curvature vector and $\sum_c \mathsf{H}^\alpha_{ccab} = 0$, we can derive

$$\begin{aligned} \frac{1}{2}\Delta\Pi_H &= \sum_{abc}(\zeta^{m+1}_{abc})^2 + \sum_{ij}\zeta^{m+1}_{ab}\Delta\zeta^{m+1}_{ab} \\ &= \sum_{abc}(\zeta^{m+1}_{abc})^2 + \sum_{abcl}\zeta^{m+1}_{ab}\Big(\zeta^{m+1}_{cl}R_{labc} + \zeta^{m+1}_{la}\mathrm{R}_{lcbc}\Big). \end{aligned} \tag{2.18}$$

Let $\mathrm{R}_\mathcal{N}(p,\pi)$ the represent the sectional curvature of $\mathcal{N}^m$ for the 2-plane $\pi \subset T_p\mathcal{N}$ at the point $p \in \mathcal{N}^m$. Then set

$$\mathrm{R}_{\min}(p) = \min_{\pi\subset T_p\mathcal{N}} \mathrm{R}_\mathcal{N}(p,\pi).$$

Therefore, the orthonormal fields are $\{e_i\}$ such that $\zeta^{m+1}_{ab} = \lambda_i\delta_{ab}$, where λ_i represents the eigenvalues; hence, we get

$$\begin{aligned} \sum_{abcl}\zeta^{m+1}_{ab}\Big(\zeta^{m+1}_{cl}\mathrm{R}_{labc} + \zeta^{m+1}_{la}\mathrm{R}_{mlcbc}\Big) &= \frac{1}{2}\sum_{ab}(\lambda_a-\lambda_b)^2\mathrm{R}_{abab} \\ &\geq \frac{1}{2}\sum_{ab}\Big(\lambda_a-\lambda_b\Big)^2\mathrm{R}_{\min}. \end{aligned} \tag{2.19}$$

Taking (2.18) and (2.19), we have

$$\frac{1}{2}\Delta\Pi_\mathsf{H} \geq \sum_{abc}(\zeta^{n+1}_{abc})^2 + \frac{1}{2}\sum_{ab}(\lambda_a-\lambda_b)^2\mathrm{R}_{\min}. \tag{2.20}$$

It follows from $\mathrm{R}_\mathcal{N} \geq \frac{m+2h-1}{2(m+2h)}(\kappa+\mathsf{H}^2)$ and the lemma of Hopf that Π_H is a constant [21], and we derive

$$\frac{1}{2}\sum_{ab}(\lambda_a-\lambda_b)^2\mathrm{R}_{\min} = 0. \tag{2.21}$$

It is implied that $\lambda_a = \lambda_b$. In this case, $\mathcal{N}^m$ is pseudo-umbilical. Again, from (2.12), $\sum_c \mathsf{H}^{\alpha}_{ccab} = 0$ and mean curvature of $\mathcal{N}^m$ is parallel; we can constructs

$$\frac{1}{2}\Delta\tau = \sum_{\alpha\neq m+1}\sum_{abc}(\zeta^{\alpha}_{abc})^2 + \sum_{\alpha\neq m+1}\sum_{abcl}\zeta^{\alpha}_{ab}\Big(\zeta^{\alpha}_{cl}\mathrm{R}_{labc} + \zeta^{\alpha}_{la}\mathrm{R}_{lcbc}\Big) - \sum_{\alpha\neq m+1}\sum_{\beta abc}\zeta^{\alpha}_{ab}\zeta^{\beta}_{ca}\mathrm{R}_{\alpha\beta bc} \tag{2.22}$$

where τ is the scalar curvature of $\mathcal{N}^m$. From (2.5) and (2.15), we get

$$\sum_{\alpha\neq m+1}\sum_{abcl}\zeta^{\alpha}_{ab}\Big(\zeta^{\alpha}_{cl}\mathrm{R}_{labc} + \zeta^{\alpha}_{la}\mathrm{R}_{lcbc}\Big) = m(\kappa + \mathsf{H}^2)\tau + \sum_{\alpha\beta\neq m+1}\Big(\mathrm{tr}(\mathsf{H}_\alpha\mathsf{H}_\beta)^2 - \mathrm{tr}(\mathsf{H}^2_\alpha\mathsf{H}^2_\beta)\Big) - \sum_{\alpha\beta\neq m+1}\Big(\mathrm{tr}(\mathsf{H}_\alpha\mathsf{H}_\beta)\Big)^2. \tag{2.23}$$

Again (2.5), we derive

$$\sum_{\alpha\neq m+1}\sum_{\beta abc}\zeta^{\alpha}_{ab}\zeta^{\beta}_{ca}\mathrm{R}_{\alpha\beta bc} = \sum_i tr\mathsf{H}^2_{i^*} - \sum_{\alpha,\beta\neq m+1}\Big(tr(\mathsf{H}_\alpha\mathsf{H}_\beta)^2 - tr(\mathsf{H}^2_\alpha\mathsf{H}^2_\beta)\Big). \tag{2.24}$$

Inserting (2.24) and (2.23) into (2.22), we obtain

$$\begin{aligned}\frac{1}{2}\Delta\tau = &\sum_{\alpha\neq m+1}\sum_{abc}(\zeta^{\alpha}_{abc})^2 + \sum_i tr\mathsf{H}^2_{i^*} - a'm(1 + \mathsf{H}^2)\tau\\ &+ (1 + a')\sum_{\alpha\neq m+1}\sum_{abcl}\zeta^{\alpha}_{ab}(\zeta^{\alpha}_{cl}\mathrm{R}_{labc} + \zeta^{\alpha}_{la}\mathrm{R}_{lcbc}) + a'\sum_{\alpha\beta\neq m+1}\Big(\mathrm{tr}(\mathsf{H}_\alpha\mathsf{H}_\beta)\Big)^2\\ &+ (1 - a')\sum_{\alpha\beta\neq m+1}\Big(\mathrm{tr}(\mathsf{H}_\alpha\mathsf{H}_\beta)^2 - \mathrm{tr}(\mathsf{H}^2_\alpha\mathsf{H}^2_\beta)\Big).\end{aligned} \tag{2.25}$$

For a fixed α, we choose the orthonormal frame field $\{e_a\}$ such that $\zeta^{\alpha}_{ab} = \lambda^{\alpha}_a\delta_{ab}$. From (2.15), we get

$$\begin{aligned}\sum_{abcl}\zeta^{\alpha}_{ab}\Big(\zeta^{\alpha}_{cl}\mathrm{R}_{labc} + \zeta^{\alpha}_{la}\mathrm{R}_{lcbc}\Big) =&\frac{1}{2}\sum_{ab}(\lambda^{\alpha}_a - \lambda^{\alpha}_b)^2\mathrm{R}_{abab}\\ \geq&\ \frac{1}{2}\sum_{ab}(\lambda^{\alpha}_i - \lambda^{\alpha}_j)^2\mathrm{R}_{\min}\\ =&m\mathrm{tr}\mathsf{H}^2_\alpha\mathrm{R}_{\min}\end{aligned}$$

which implies that

$$\sum_{\alpha\neq m+1}\sum_{abcl}\zeta^{\alpha}_{ab}\Big(\zeta^{\alpha}_{cl}\mathrm{R}_{labc} + \zeta^{\alpha}_{la}\mathrm{R}_{lcbc}\Big) \geq m\tau\mathrm{R}_{min}. \tag{2.26}$$

In the implementation of DDVV (a conjecture for the Wintgen inequality on Riemannian submanifolds in real space forms proven by P.J. De Smet, F. Dillen, L. Verstraelen, and L. Vrancken), demonstrated by the article [7], inequality of Lemma 2.2, we construct the following:

$$\sum_{\alpha\beta\neq m+1}\Big\{\mathrm{tr}(\mathsf{H}^2_\alpha\mathsf{H}^2_\beta) - \mathrm{tr}(\mathsf{H}_\alpha\mathsf{H}_\beta)^2\Big\} = \frac{1}{2}\sum_{\alpha\beta\neq m+1}\mathrm{tr}(\mathsf{H}_\alpha\mathsf{H}_\beta - \mathsf{H}_\beta\mathsf{H}_\alpha)^2$$

$$\leq \frac{1}{2}(\sum_{\alpha\neq m+1} \mathrm{tr}\mathsf{H}_\alpha^2)^2$$
$$= \frac{1}{2}\tau^2. \tag{2.27}$$

On the other hand, we have

$$\sum_{\alpha\beta\neq m+1} \Big(\mathrm{tr}(\mathsf{H}_\alpha \mathsf{H}_\beta)\Big)^2 \geq \frac{\tau^2}{m+2h-1}. \tag{2.28}$$

Setting $a' = \frac{m+2h-1}{m+2h+1}$ in (2.25), and combining (2.26)–(2.28), we derive

$$\frac{1}{2}\Delta\tau \geq \left\{ -\Big(\frac{m+2h-1}{m+2h+1}\Big)\Big(\kappa + \mathsf{H}^2\Big) + \Big(\frac{2m+4h}{m+2h+1}\Big)\mathrm{R}_{min}\right\} m\tau. \tag{2.29}$$

If our assumption (2.14) is satisfied, then we have

$$\frac{1}{2}\Delta\tau \geq 0$$

by Hopf's lemma. This concludes that $\Delta\tau = 0$. Hence, we get the following:

$$\tau = 0 \quad or \quad \mathrm{R}_{\mathcal{N}} = \Big(\frac{m+2h-1}{2(m+2h)}\Big)\Big(\kappa + \mathsf{H}^2\Big).$$

For the first case, $\tau = 0$, and thus $\mathcal{N}^m$ is totally umbilical. For the second case, on the basis of (2.5), we derive

$$\mathrm{R}_{abab} = \kappa + \mathsf{H}^2$$

and conclude that $\mathcal{N}^m$ is a totally umbilical sphere

$$\mathbb{S}^m\Big(\frac{1}{\sqrt{\kappa + \mathsf{H}^2}}\Big).$$

Moreover, all inequalities (2.26)–(2.29) changed to equalities if

$$\mathrm{R}_{\mathcal{N}} = \Big(\frac{m+2h-1}{2(m+2h)}\Big)\Big(\kappa + \mathsf{H}^2\Big).$$

Now, we will show that the second case can not occur. For this, we consider the equality (2.27) implies that either all $\mathsf{H}'_\alpha s$ are zero or two of the $\mathsf{H}'_\alpha s$ are nonzero $\alpha \neq m+1$. We estimate the following if the inequality in (2.28) and (2.29) converts into equalities:

$$tr\mathsf{H}_\alpha^2 = tr\mathsf{H}_\beta^2 \quad (\alpha, \beta \neq m+1), \quad \text{and} \quad \sum_t tr\mathsf{H}_{t^*}^2 = 0.$$

Thus $\mathcal{N}^m$ is totally umbilical with $\mathrm{R}_{abab} = \kappa + \mathsf{H}^2$, as the $\mathsf{H}_\alpha^{'s}$ are zero ($\alpha \neq m+1$). This leads to a contradiction. This completes the proof of the theorem. □

In the results below, we have the following:

Theorem 2.2. *Let* $J\xi$ *be normal to an* $(m \geq 2)$*-dimensional totally real submanifold* $\mathcal{N}^m$ *in the complex space form* $\widetilde{\mathcal{N}}^{m+h}(4\kappa)$*. Then either* $\mathcal{N}^m$ *is totally umbilical or it satisfies the inequality*

$$\inf \rho \leq m(\kappa + \mathsf{H}^2)(m - \frac{5}{3}) \tag{2.30}$$

where the scalar curvature, the mean curvature, and the mean curvature vector are represented by ρ, H*, and* ξ*, respectively.*

Proof. Let $J\xi$ be normal to $\mathcal{N}^m$. We can consider e_{m+1} is parallel to ξ, and we have

$$tr\mathsf{H}_{m+1} = m\mathsf{H}, \quad tr\mathsf{H}_\alpha = 0, \quad \alpha \neq m+1. \tag{2.31}$$

From (2.12), we have

$$\frac{1}{2}\Delta\Pi = \sum_{\alpha ijk} (\zeta^\alpha_{abc})^2 + \sum_{\alpha abc} \zeta^\alpha_{ab}\zeta^\alpha_{ccab} + \sum_{\alpha abcl} \zeta^\alpha_{ab}\Big(\zeta^\alpha_{cl}\mathrm{R}_{labc} + \zeta^\alpha_{la}\mathrm{R}_{lcbc}\Big) - \sum_{\alpha\beta abc} \zeta^\alpha_{ab}\zeta^\beta_{ca}\mathrm{R}_{\alpha\beta bc}. \tag{2.32}$$

From (2.5), (2.31), and if $\mathcal{N}^m$ is totally umbilical, we obtain

$$\begin{aligned}\sum_{\alpha abcl} \zeta^\alpha_{ab}\Big(\zeta^\alpha_{cl}\mathrm{R}_{labc} + \zeta^\alpha_{la}\mathrm{R}_{lcbc}\Big) =& m(\kappa + \mathsf{H}^2)\Pi - m^2\mathsf{H}^2 + \sum_{\alpha\beta}\Big\{\mathrm{tr}\Big(\mathsf{H}_\alpha\mathsf{H}_\beta\Big)^2 - \mathrm{tr}\Big(\mathsf{H}^2_\alpha\mathsf{H}^2_\beta\Big)\Big\} \\ &- \sum_{\alpha\beta}\Big(\mathrm{tr}\Big(\mathsf{H}_\alpha\mathsf{H}_\beta\Big)\Big)^2, \end{aligned} \tag{2.33}$$

$$\sum_{\alpha\beta abc} \zeta^\alpha_{ab}\zeta^\beta_{ca}\mathrm{R}_{\alpha\beta bc} = -\sum_{\alpha\beta}\Big\{\mathrm{tr}(\mathsf{H}_\alpha\mathsf{H}_\beta)^2 - \mathrm{tr}(\mathsf{H}^2_\alpha\mathsf{H}^2_\beta)\Big\} - \sum_a \mathrm{tr}\mathsf{H}^2_{a^*}. \tag{2.34}$$

In view of (2.31), and the pseudo-umbilical condition such that $\zeta^{m+1}_{ab} = \mathsf{H}\delta_{ab}$, we derive

$$\sum_{\alpha abc} \zeta^\alpha_{ab}\zeta^\alpha_{ccab} = m\mathsf{H}\Delta\mathsf{H}, \tag{2.35}$$

$$\sum_{\alpha abc}(\zeta^\alpha_{abc})^2 \geq \sum_{ac}(\zeta^{m+1}_{aac})^2 = m\sum_a(\nabla_a\mathsf{H})^2, \tag{2.36}$$

$$\frac{1}{2}\Delta\mathsf{H}^2 = \mathsf{H}\Delta\mathsf{H} + \sum_a(\nabla_a\mathsf{H})^2. \tag{2.37}$$

By Lemma 2.2 and pseudo-umbilical condition $\zeta^{m+1}_{ab} = \mathsf{H}\delta_{ab}$, we have

$$\begin{aligned} 2\sum_{\alpha\beta}\Big\{\mathrm{tr}\Big(\mathsf{H}_\alpha\mathsf{H}_\beta\Big)^2 - \mathrm{tr}\Big(\mathsf{H}^2_\alpha\mathsf{H}^2_\beta\Big)\Big\} - \sum_{\alpha\beta}\mathrm{tr}\Big(\mathsf{H}_\alpha\mathsf{H}_\beta\Big)^2 \\ = 2\sum_{\alpha\beta\neq m+1}\Big\{\mathrm{tr}\Big(\mathsf{H}_\alpha\mathsf{H}_\beta\Big)^2 - \mathrm{tr}\Big(\mathsf{H}^2_\alpha\mathsf{H}^2_\beta\Big)\Big\} \\ - \sum_{\alpha\beta\neq m+1} tr\Big(\mathsf{H}_\alpha\mathsf{H}_\beta\Big)^2 - \Big(\mathrm{tr}H^2_{m+1}\Big)^2 \end{aligned}$$

$$\geq -\frac{3}{2}\tau^2 - m^2\mathsf{H}^4$$
$$= -\frac{3}{2}(\Pi - m\mathsf{H}^2)^2 - m^2\mathsf{H}^4. \tag{2.38}$$

Substituting (2.33)–(2.38) into (2.32), we have

$$\begin{aligned}\frac{1}{2}\Delta\Pi \geq& \frac{1}{2}m\Delta\mathsf{H}^2 + m\Big(\kappa + \mathsf{H}^2\Big)\Pi - \frac{3}{2}\Big(\Pi - m\mathsf{H}^2\Big)^2 - m^2\mathsf{H}^4 - m^2\mathsf{H}^2\\ =& \frac{1}{2}m\Delta\mathsf{H}^2 + \Big(\Pi - m\mathsf{H}^2\Big)\Big\{m\Big(\kappa + \mathsf{H}^2\Big) - \frac{3}{2}\Big(\Pi - m\mathsf{H}^2\Big)\Big\}\\ =& \frac{1}{2}m\Delta\mathsf{H}^2 + \tau\Big\{m\Big(\kappa + \mathsf{H}^2\Big) - \frac{3}{2}\tau\Big\}.\end{aligned} \tag{2.39}$$

By the same argument as in [16], we conclude that either M^n is totally umbilical or

$$\inf\rho \leq m\Big(\kappa + \mathsf{H}^2\Big)\Big(m - \frac{5}{3}\Big).$$

This completes the proof of the theorem. □

Theorem 2.3. *Let* Jξ *be normal to an* ($m \geq 2$)*-dimensional compact totally real submanifold* $\mathcal{N}^m$ *in the complex space form* $\widetilde{\mathcal{N}}^{m+h}(4\kappa)$*. Then we have the inequality*

$$\int\Big\{2(\kappa + \mathsf{H}^2)\Pi - 3\Pi^2 - 5m^2\mathsf{H}^4 - 4m^2\mathsf{H}^2 + 2m\mathsf{H}^2\Big\}\mathrm{dV} \leq 0 \tag{2.40}$$

where H *and* Π *denote the mean curvature of* $\mathcal{N}^m$ *and the squared norm of the length of the second fundamental form of* $\mathcal{N}^m$*, respectively.*

Proof. Without loss of generality, we consider e_{1^*} such that it is parallel to ξ and $tr\mathsf{H}_{1^*} = m\mathsf{H}$. In this case $tr\mathsf{H}_\alpha = 0$, for $\alpha \neq 1^*$, and Jξ is normal to $\mathcal{N}^m$. Taking this together with (2.5), we have

$$\begin{aligned}\sum_{\alpha\beta abc}\zeta^\alpha_{ab}\zeta^\beta_{ca}\mathrm{R}_{\alpha\beta bc} =& m^2\mathsf{H}^2 - \sum_a \mathrm{tr}\mathsf{H}^2_{a^*} - \sum_{\alpha\beta}\Big\{\mathrm{tr}(\mathsf{H}_\alpha\mathsf{H}_\beta)^2 - \mathrm{tr}(\mathsf{H}^2_\alpha\mathsf{H}^2_\beta)\Big\}\\ \leq& m^2\mathsf{H}^2 - \mathrm{tr}\mathsf{H}^2_{1^*} - \sum_{\alpha\beta}\Big\{\mathrm{tr}(\mathsf{H}_\alpha\mathsf{H}_\beta)^2 - \mathrm{tr}(\mathsf{H}^2_\alpha\mathsf{H}^2_\beta)\Big\}\\ =& m^2\mathsf{H}^2 - m\mathsf{H}^2 - \sum_{\alpha\beta}\Big\{\mathrm{tr}(\mathsf{H}_\alpha\mathsf{H}_\beta)^2 - \mathrm{tr}(\mathsf{H}^2_\alpha\mathsf{H}^2_\beta)\Big\}.\end{aligned} \tag{2.41}$$

By the same argument as in Theorem 2.2, we conclude that

$$\frac{1}{2}\Delta\Pi \geq \frac{1}{2}m\Delta\mathsf{H}^2 + m(\kappa + \mathsf{H}^2)\Pi - \frac{3}{2}(\Pi - m\mathsf{H}^2)^2 - m^2\mathsf{H}^4 - 2m^2\mathsf{H}^2 + m\mathsf{H}^2.$$

The boundary of $\mathcal{N}^m$ is compact, by Stokes's theorem, we obtain

$$\int\Big\{m(\kappa + \mathsf{H}^2)\Pi - \frac{3}{2}(\Pi - m\mathsf{H}^2)^2 - m^2\mathsf{H}^4 - 2m^2\mathsf{H}^2 + m\mathsf{H}^2\Big\} \leq 0$$

which implies (2.40). This completes the proof of the theorem. □

Remark 2.1. *Note that if* $\kappa = 1$ *in* (2.36), *then the complex space form* $\widetilde{\mathcal{N}}^{m+h}(4\kappa)$ *turns to a complex projective space with constant curvature of* 1.

From the hypothesis, Theorem 2.1, we rewrite that

Theorem 2.4. *Let* $\mathcal{N}^m$ *be an m-dimensional compact totally real submanifold in complex projective space* $\mathbb{C}P^{m+h}$ *with parallel mean curvature. In this case, we have*

$$\mathrm{R}_{\mathcal{N}} \geq \left(\frac{m+2h-1}{2(m+2h)}\right)\left((1+\mathsf{H}^2\right),$$

and thus $\mathcal{N}^m$ *is a totally umbilical sphere* $\mathbb{S}^m\left(\frac{1}{\sqrt{1+\mathsf{H}^2}}\right)$, *where* H *and* Π *denote the mean curvature of* $\mathcal{N}^m$ *and the squared norm of the length of the second fundamental form of* $\mathcal{N}^m$, *respectively. Moreover,* $\mathbb{C}P^{m+h}$ *has the constant sectional curvature* 1.

Theorem 2.5. *Let* $\mathcal{N}^m$ *be an* $(m \geq 2)$*-dimensional totally real submanifold in complex projective spaces* $\mathbb{C}P^{m+h}$. *If* $\mathrm{J}\xi$ *is normal to* $\mathcal{N}^m$, *then either* $\mathcal{N}^m$ *is totally umbilical or satisfies the following inequality:*

$$\inf \rho \leq m\left\{1+\mathsf{H}^2\right\}\left(m-\frac{5}{3}\right).$$

Similarly, Theorem 2.3 can be written as if the mean curvature is minimal from Theorem 2.4

Corollary 2.1. *Let* $\mathcal{N}^m$ *be an* $(m \geq 2)$*-dimensional compact totally real submanifold in complex projective spaces* $\mathbb{C}P^{m+h}$. *If* $\mathrm{J}\xi$ *is normal to* $\mathcal{N}^m$, *then we have the following inequality:*

$$\int \left\{2\left(1+\mathsf{H}^2\right)\Pi - 3\Pi^2 - 5m^2\mathsf{H}^4 - 4m^2\mathsf{H}^2 + 2m\mathsf{H}^2\right\}d\mathrm{V} \leq 0.$$

Remark 2.2. *Using Remark 2.1 in Theorems 2.4 and 2.5, then Theorems 2.4 and 2.5 coincided with Theorems 1 and 2 in [21].*

3. Conclusions

The study of totally real submanifolds in complex space forms is a rich area in differential geometry, with deep connections to complex geometry, curvature theory, and minimal submanifold theory. Their extrinsic curvature properties (like second fundamental form, mean curvature) are deeply influenced by the complex structure of the ambient space. This leads to classification results that help understand the geometric landscape of complex manifolds. In theoretical physics (e.g., string theory), totally real submanifolds relate to real slices of complexified spaces. In all these, totally real submanifolds offer tools to probe the nature of curvature, symmetry, and submanifold geometry.

Author contributions

Fatimah Alghamdi: Conceptualization, Investigation; Fatemah Mofarreh: Conceptualization, Methodology, Writing-review and editing Writing-original draft preparation, Investigation, Funding acquisition; Akram Ali: Conceptualization, Methodology, Investigation, Writing-review and editing; Mohamed Lemine Bouleryah: Methodology, Writing-original draft preparation, Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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