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Research article

Analytical insight into fractional Fornberg-Whitham equations using novel transform methods

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Abstract: This work addressed one of the most essential evolutionary equations that was widely used in describing various nonlinear wave propagation and dispersive phenomena in various scientific and engineering applications, which was called the nonlinear fractional Fornberg-Whitham (FFW). Due to the importance of this equation, we examined it by employing two highly effective techniques: the residual power series method (RPSM) and the new iterative approach (NIM), both distinguished by their efficacy in solving more complicated nonlinear fractional evolutionary equations. Moreover, we integrated the Elzaki transform with both approaches to create Elzaki RPSM (ERPSM) and Elzaki NIM (ENIM) to ease the calculations. The ERPSM effectively combined the power series approach with residual error analysis to generate highly accurate series solutions, while ENIM provided alternative frameworks for handling nonlinearities and achieving rapid convergence. Comparative studies of the obtained solutions highlighted these methods' efficiency, accuracy, and reliability in solving fractional-order differential equations. The results underscored the potential of these analytical techniques for modeling and solving complex fractional wave equations, contributing to the advancement of mathematical physics and computational fluid dynamics.

Keywords: Elzaki transform; residual power series transform method; new iterative transform method; fractional Fornberg-Whitham equations
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1. Introduction

Fractional calculus (FC) explores generalized versions of classical derivatives and integrals through non-integer orders since it has captured scientific attention in the last fifty years. Fractional derivatives occur in technical and physical systems such as viscoelasticity and groundwater flow, wave propagation, finance, fluid mechanics, and plasma physics [1–4]. FC presents a more flexible method than classical calculus since it works with differentiation and integration of non-integer orders, accurately representing complex systems displaying memory dependencies and hereditary characteristics. Due to the significance of FC and its significant role in providing a more accurate vision to explain many mysterious phenomena or to bridge the gap between the theoretical results of classical FC and practical results, FC has attracted significant efforts of many researchers, either to provide more accurate methods for analyzing various types of fractional differential equations (FDEs) or by applying reliable techniques in addressing various engineering, physical, and scientific challenges.

Several researchers have examined theoretical outcomes derived from the presence of FDEs solutions and their uniqueness in different formats; some examples of these investigations can be found in Refs. [6–10]. Many fractional differential problems clearly either do not have closed-form solutions or are too complex for an analytical solution to be useful. Because of this, numerous authors have developed various numerical approximation techniques. These include the Fourier spectral methods, finite difference schemes, the homotopy analysis approach, He's variational iteration method, Adomian's decomposition, and many more that are categorized in Ref. [2]. We direct our readers to the classic books and papers [1, 2] for comprehensive research on the solution of FDEs.

Whitham first introduced the Fornberg–Whitham (FW) equation in 1967 for discussing the wave-breaking's qualitative behavior [11]:

$$\frac{\partial\varphi}{\partial\tau} - \frac{\partial^3\varphi}{\partial\vartheta^2\partial\tau} + \frac{\partial\varphi}{\partial\vartheta} = \varphi\frac{\partial^3\varphi}{\partial\vartheta^3} - \varphi\frac{\partial\varphi}{\partial\vartheta} + 3\frac{\partial\varphi}{\partial\vartheta}\frac{\partial^2\varphi}{\partial\vartheta^2}, \tag{1.1}$$

where $\varphi \equiv \varphi(\vartheta, \tau)$ represents a fluid velocity function. This equation supports the following exact solution [12]

$$\varphi(\vartheta, \tau) = Ae^{\left(\frac{\vartheta}{2} - \frac{2\tau}{3}\right)}, \tag{1.2}$$

where A is an arbitrary nonzero constant.

Equation (1.1) is one of the most significant differential equations extensively utilized to analyze many nonlinear phenomena in different mediums. Thus, numerous research studies were conducted to analyze Eq (1.1) using various analytical and numerical methods and derive traveling wave solutions, as evidenced in Refs. [13–17].

In our current study, we will examine the nonlinear fractional FW (FFW) equation

$$D_\tau^p\varphi - \frac{\partial^3\varphi}{\partial\vartheta^2\partial\tau} + \frac{\partial\varphi}{\partial\vartheta} = \varphi\frac{\partial^3\varphi}{\partial\vartheta^3} - \varphi\frac{\partial\varphi}{\partial\vartheta} + 3\frac{\partial\varphi}{\partial\vartheta}\frac{\partial^2\varphi}{\partial\vartheta^2}, \tag{1.3}$$

where $\varphi \equiv \varphi(\vartheta, \tau)$, $D_\tau^p \equiv \partial_{\tau^p}^p \equiv \partial^\alpha/\partial\tau^\alpha$ indicates the fractional derivative, and $0 < p \leq 1$ represents the fractional parameter (fractionality).

Numerous researchers examined Eq (1.3) and employed diverse approximate approaches to formulate approximations that model numerous nonlinear phenomena characterized by this equation.

For instance, the homotopy analysis method (HAM) was utilized to derive the analytical approximation to the FFW Eq (1.3) [18]. The HAM-derived approximations were also compared with those from homotopy perturbation method (HPM) and Adomian decomposition method (ADM) [18]. The Laplace-ADM (LADM) was employed to determine a closed form of the series solution to the time-FFW-type equations [19]. The q-Homotopy analysis Shehu transform method was utilized to solve and analyze the time-FFW equation in the Caputo sense [20].

An analytical technique called the residual power series method (RPSM) was put out as illustrated in Ref. [21] to ascertain the coefficients of a class of differential equations (DEs) power series solutions. The basic idea lies in solving several linear and nonlinear equations using power series methods without linearity or perturbation. In addition, the technique requires calculating a derivative of the remaining function at each step of the coefficient-finding process. The method in question was first introduced by Eriqat et al. [22], and its benefits stem from the fact that it requires less computing resources to extract the form of solutions in a power series of values defined by a series of algebraic procedures. The proposed method leverages the idea of limits at infinity to accomplish its primary objective, even though the RPSM technique does not require the concept of the derivative to be used to determine the parameters of the series's solution. Actually, accurate results have been produced for several types of nonlinear and linear DEs using the RPSM. The suggested method is unique because it can handle nonlinear equations, which is impossible with typical Elzaki transform-based methods. The RPSM approach has recently been modified to address a variety of fractional DEs, such as hyperbolic systems of Caputo-time-fractional partial differential equations (PDEs) with changing coefficients [23], nonlinear time-fractional dispersive PDEs [24], time-fractional Navier-Stokes equations [25], fuzzy Quadratic Riccati DEs [26], Lane-Emden equations [27], the logistic system and Fisher's models [28], as well as nonlinear fractional reaction-diffusion for bacteria growth models [29].

Daftardar-Gejji and Jafari [30] introduced the new iterative method (NIM) in 2006, and it has since become an efficient mathematical tool for solving both linear and nonlinear FDEs. NIM's effectiveness is demonstrated by the numerous nonlinear problems it has been used to solve, including algebraic equations, integral equations, and ordinary DEs (ODEs) or PDEs of both fractional and integer order. The unique features of NIM include its ease of use and comprehension, which make it available to a broad range of scholars and practitioners. Numerous scholars extensively employed this method to study and analyze various evolutionary equations and to compare its outcomes with other methods. For example, within the Caputo framework, time-FDEs employing the Atangana–Baleanu fractional differential operator were analyzed via the Aboodh transform iterative method [31]. The iterative Laplace transform approach, integrating the Laplace transform with the iterative technique, was employed to provide exact and approximate analytical solutions for fractional PDEs (FPDEs) system [32]. The q-homotopy analysis method (q-HAM) and the natural transform iterative method (NTIM) were utilized to analyze and solve more complicated and strong nonlinear FPDEs, including the fractional Sawada–Kotera and KDV-Burger equations [33]. The NIM was applied to investigate the fractional-order inhomogeneous PDEs and fractional-order Roseau-Hyman. The obtained findings were contrasted with those derived by the Laplace variational iteration method (LVIM), HPM, and LADM, [34]. The authors found that the obtained results by NIM demonstrated more accuracy than those using HPM, LADM, and LVIM [34]. Recently, the Laplace novel iterative method (LNIM) was utilized to analyze the fractional Korteweg-de Vries (FKdV) and fractional modified KdV (FmKdV) equations and to model positron-acoustic cnoidal waves (PACWs) in multicomponent plasmas comprising inertialess non-Maxwellian electrons and hot positrons following the Kaniadakis distribution, alongside inertial fluid cold positrons and stationary ions [35]. The authors employed derived approximations to examine the influence of the fractional parameter on the behavior of FKdV-PACWs and FmKdV-PACWs [35]. Additionally, the Aboodh RPSM (ARPSM) and Aboodh NIM (ANIM) were employed to analyze fractional Burgers-type equations, including the fractional Burgers and KdV-Burgers equations, which are widely used to describe shock waves in plasma physics and marine environments [36]. Furthermore, the authors employed the "Tantawy Technique" to address the proposed problems and obtain highly precise approximations, thereafter comparing these approximations with the results generated from ARPSM and ANIM [36]. The fractional third- and fifth-order KdV-type equations were analyzed using the ARPSM and ANIM to derive high-accuracy approximations in order to model nonlinear phenomena in a plasma and fluid mediums [37]. Also, the multidimensional time-fractional Navier–Stokes equations were solved utilizing ARPSM and ANIM, in the framework of the Caputo operator [38]. Moreover, the fractional Hirota–Satsuma coupled KdV problem was addressed via ARPSM ANIM, and high-precision approximations were formulated and examined [39]. Additionally, the fractional damped Burger's equation was solved using ARPSM and ANIM, and several highly accurate analytical approximations were obtained to characterize one-dimensional nonlinear shock waves in fluid dynamics, plasma physics, and other disciplines [40]. Additionally, the Laplace RPSM (LRPSM) was utilized to analyze the fractional generalized Burger–Fisher equation, resulting in an analytical approximation for modeling shock waves and examining the influence of fractionality on the profile of these waves [41]. Compared to established techniques, such as the ADM [42], the HPM [43], and the VIM [44], NIM has shown improved performance and better efficiency. That's why people dealing with complex nonlinear situations find it popular [45]. In general, there are many studies that used both RPSM and NIM with various transforms to examine various evolutionary wave equations (EWEs), which succeeded in providing results that highlighted the ambiguity of some theoretical results that integer differential equations failed to reveal. This work aims to employ the RPSM and NIM with the Elzaki transform to analyze more complicated and strong nonlinearity FDEs, particularly the FFW Eq (1.3), and derive some highly accurate analytical approximations for this model.

2. Basic definitions

This section provides a detailed introduction to the Elzaki transform (ET) and the Caputo fractional concept.

Definition 2.1. *The Riemann-Liouville (RL) fractional integral operator for order $\delta \geq 0$, is defined as follows [46, 47]*

$$G^{\delta} f(\tau) = \begin{cases} \frac{1}{\Gamma(\delta)} \int_0^{\tau} \frac{f(s)}{(\tau - s)^{1-\delta}} ds = \frac{1}{\Gamma(\delta)} \tau^{\delta - 1} \times f(\tau) & \delta > 0, \tau > 0, \\ f(\tau) & \delta = 0. \end{cases} \tag{2.1}$$

The following properties are fulfilled for the fractional integral of RL:

$$\begin{aligned} &(1)\; G^{\delta} \tau^{\chi} = \frac{\Gamma(\chi + 1)}{\Gamma(\chi + \delta + 1)} \tau^{\delta + \chi}, \chi > -1, \\ &(2)\; G^{\delta}(\lambda f(\tau) + \mu g(\tau)) = \lambda G^{\delta} f(\tau) + \mu G^{\delta} g(\tau), \end{aligned} \tag{2.2}$$

where (λ, μ) are real constants.

Definition 2.2. *Let the function be $f(\tau) : [0, +\infty) \to R$, and m represents the upper positive integer of $\delta(\delta > 0)$, then the definition of the Caputo fractional derivative (CFD) is given as [48, 49]*

$$D^{\delta} f(\tau) = \frac{1}{\Gamma(m-\delta)} \int_0^{\tau} \frac{f^{(m)}(s)}{(\tau - s)^{(\delta+1-m)}} ds, \tag{2.3}$$

where $(m-1) < \delta \leq m, \ \ m \in N$.

CFD's operator achieves the following characteristics:

$$\begin{aligned} &(1)\ D^{\delta} G^{\delta} f(\tau) = f(\tau), \\ &(2)\ G^{\delta} D^{\delta} f(\tau) = f(\tau) - \textstyle\sum_{i=0}^{m-1} y^{(i)}(0) \frac{\tau^i}{i!}, \\ &(3)\ D^{\delta} \tau^{\chi} = \begin{cases} \frac{\Gamma(\chi+1)}{\Gamma(\chi+1-\delta)} \tau^{\chi-\delta} & \chi \geq \delta, \\ 0 & \chi < \delta, \end{cases} \\ &(4)\ D^{\delta} c = 0, \\ &(5)\ D^{\delta}(\lambda f(\tau) + \mu g(\tau)) = \lambda D^{\alpha} f(\tau) + \mu D^{\alpha} g(\tau), \end{aligned} \tag{2.4}$$

where c, δ, and μ represent real constants.

Definition 2.3. *The form of the power series about $\tau = \tau_0$, is given as [50]*

$$\sum_{k=0}^{\infty} f_k(\vartheta)(\tau - \tau_0)^{k\delta} = f_0(\vartheta) + f_1(\vartheta)(\tau - \tau_0)^{\delta} + f_2(\vartheta)(\tau - \tau_0)^{2\delta} + \cdots, \tag{2.5}$$

where $0 < (m-1) < \delta \leq m, \ \ \tau \geq \tau_0$, and this form is referred to as the multiple fractional power series (MFPS).

Theorem 2.1. *Suppose that the function $\varphi(\vartheta, \tau)$ possesses the following MFPS representation at $\tau = \tau_0$ [47],*

$$\begin{gathered} \varphi(\vartheta, \tau) = \sum_{n=0}^{\infty} \varphi_n(\vartheta, \tau) = \sum_{n=0}^{\infty} f_n(\vartheta)(\tau - \tau_0)^{n\delta}, \\ 0 < (m-1) < \delta \leq m, \ \ \vartheta \in I, \ \ \tau_o \leq \tau < \tau_0 + R. \end{gathered} \tag{2.6}$$

If $D_{\tau}^{n\delta} \varphi(\vartheta, \tau)$ are continuous on $I \times (\tau_0, \tau_0 + R), m = 0, 1, 2, \cdots$, then coefficients $f_n(\vartheta)$ of Eq (2.6) read

$$f_n(\vartheta) = \frac{D_{\tau}^{n\delta} \varphi(\vartheta, \tau_o)}{\Gamma(n\delta + 1)}, \ \ n = 0, 1, 2, \cdots, \tag{2.7}$$

where $D_{\tau}^{n\delta} = \partial_{\tau^{n\delta}}^{n\delta} = \partial_{\tau^{\delta}}^{\delta} \times \partial_{\tau^{\delta}}^{\delta} \times ... \times \partial_{\tau^{\delta}}^{\delta}$ (n-times), and $R = min_{c \in I} R_c$, where the fractional power series' radius of convergence is indicated R_c by $\sum_{n=0}^{\infty} f_n(c)(\tau - \tau_0)^{n\delta}$.

Definition 2.4. *We examine functions in the set B defined by using a new transform termed the ET, designed for functions of exponential order [51, 52]:*

$$B = f(\tau) : \exists M, k_1, k_2 > 0, \ \ | f(\tau) | < M \exp^{\frac{|\tau|}{kj}}, \ \ if\ \tau \in (-1)^j \times [0, \infty). \tag{2.8}$$

Here, the constant M must be a finite number for each function in the set, where k_1, k_2 may be infinite or finite. The integral form to ET reads

$$E[f(\tau)] = T(v) = v\int_0^\infty f(\tau)\exp^{\frac{-\tau}{v}}\, d\tau, \quad \tau\geq 0, \quad k_1 \leq v \leq k_2. \tag{2.9}$$

Theorem 2.2. *Let us defined ET to the RL derivative; if $T(v)$ is ET to "τ", then the following definition is considered [53]:*

$$E\left[D^{\delta} f(\tau)\right] = v^{-\delta}\left[T(v) - \sum_{k=1}^{m} v^{\delta-k+2}[D^{\delta-k} f(0)]\right], \tag{2.10}$$

where $-1 < m-1 \leq \delta < m$.

Definition 2.5. *According to Theorem 2, ET to the CFD $D_\tau^{\delta} f(\tau)$ can be expressed as follows [53]*

$$E\left[D_\tau^{\delta} f(\tau)\right] = v^{-\delta} E[f(\tau)] - \sum_{k=0}^{n-1} v^{2-\delta+k} f^{(k)}(0), \tag{2.11}$$

where $n-1 < \delta < n$.

3. Methodology

To demonstrate the fundamental concept of the suggested methods (RPSM with ET (ERPSM)), we examine a general form of a nonlinear nonhomogeneous PDE as follows:

$$D_\tau^{\delta}\varphi(\vartheta,\tau) = \mathcal{L}(\varphi(\vartheta,\tau)) + \mathcal{N}(\varphi(\vartheta,\tau)) + h\left(\varphi\left(\vartheta,\tau\right)\right), \tag{3.1}$$

with the initial conditions (ICs)

$$\varphi_i(\vartheta,0) = g_k, \ \forall\ k = 0,\cdots m-1. \tag{3.2}$$

Here, $\varphi(\vartheta,\tau)$ indicates the wave function, whereas $\mathcal{L}$ and $\mathcal{N}$ are, respectively, the general linear and nonlinear fractional differential operators; and $h\left(\varphi\left(\vartheta,\tau\right)\right)$ represents the inhomogeneous part.

3.1. Residual power series method (RPSM) with ET (ERPSM)

For the ERPSM, the following procedures are recommended,

Step 1. Applying ET to both sides of Eq (3.1) yields

$$E[D_\tau^{\delta}\varphi(\vartheta,\tau)] = E[\mathcal{L}(\varphi(\vartheta,\tau)) + \mathcal{N}(\varphi(\vartheta,\tau)) + h\left(\varphi\left(\vartheta,\tau\right)\right)]. \tag{3.3}$$

Step 2. Using ICs (3.2) with the following definition of the ET for CFD

$$E\left[D_\tau^{\delta}\varphi(\vartheta,\tau)\right] = v^{-\delta} E[\varphi(\vartheta,\tau)] - \sum_{k=0}^{n-1} v^{2-\delta+k} f^{(k)}(0), \tag{3.4}$$

in Eq (3.1), we get

$$E[\varphi(\vartheta,\tau)] = \sum_{k=0}^{n-1} v^{2+k} f^{(k)}(0) + v^{\delta} E\left[\mathcal{L}(\varphi(\vartheta,\tau)) + \mathcal{N}\varphi(\vartheta,\tau) + h\left(\varphi\left(\vartheta,\tau\right)\right)\right]. \tag{3.5}$$

Step 3. Utilizing the inverse of ET on both sides of Eq (3.5) implies

$$\varphi(\vartheta,\tau) = g(\vartheta) + v^{\delta} E\left[\begin{array}{c} E^{-1}\left[\mathcal{L}\left(\varphi(\vartheta,\tau)\right)\right] + E^{-1}\left[\mathcal{N}\left(\varphi(\vartheta,\tau)\right)\right] \\ +E^{-1}\left[h\left(\varphi\left(\vartheta,\tau\right)\right)\right] \end{array}\right]. \tag{3.6}$$

where $g(\vartheta)$ denotes the IC in this case.

Step 4. According to the classic RPSM, the following series solution is considered

$$\varphi(\vartheta,\tau) = \sum_{m=0}^{\infty} f_m(\vartheta)\frac{\tau^{m\delta}}{\Gamma(1+m\delta)}, \tag{3.7}$$

and for certain terms, the approximate value of Eq (3.7), can be expressed as follows

$$S_i = \sum_{m=0}^{i} \varphi_m(\vartheta,\tau) = \sum_{m=0}^{i} f_m(\vartheta)\frac{\tau^{m\delta}}{\Gamma(1+m\delta)}. \tag{3.8}$$

Step 5. From Eq (3.6), we can get the following Elzaki residual function (ERF)

$$Res_i(\vartheta,\tau) = \varphi_i(\vartheta,\tau) - \left\{\begin{array}{c} g(\vartheta) + E^{-1}[v^{\delta}E[\mathcal{L}(\varphi_{i-1}(\vartheta,\tau)) \\ +\mathcal{N}\varphi_{i-1}(\vartheta,\tau)) + h\left(\varphi_{i-1}\left(\vartheta,\tau\right)\right)]] \end{array}\right\} \tag{3.9}$$

Step 6. Inserting Eq (3.8) into Eq (3.9), we can get the values of the values of $f_m(\vartheta)$, $\forall\ m = 0,1,2,\cdots$, by solving the following equation for $m = 0,1,2,\cdots$,

$$\tau^{-m\delta} Res_m(\vartheta,\tau)\ |_{\tau=0} = 0,\ \ \forall\ \tau \in N^{*}, \tag{3.10}$$

Step 7. According to the ERPSM, the following approximation up to the i^{th}-order is considered

$$S_i = \varphi_0 + \varphi_1 + \varphi_3 + \cdots + \varphi_i, \tag{3.11}$$

with

$$\begin{aligned} \varphi_0 &= f_0(\vartheta), \\ \varphi_1 &= f_1(\vartheta)\frac{\tau^{\delta}}{\Gamma(1+\delta)}, \\ \varphi_2 &= f_2(\vartheta)\frac{\tau^{2\delta}}{\Gamma(1+2\delta)}, \\ &\vdots \\ \varphi_i &= f_i(\vartheta)\frac{\tau^{i\delta}}{\Gamma(1+i\delta)}. \end{aligned} \tag{3.12}$$

For all $\delta \in (0;1]$, $|Res(\vartheta,\tau)|_{exact}$ generally equals to zero and this relation can be used to get the residual error for the derived approximation as follows:

$$|\ Res_i(\vartheta,\tau)\ | = |\ D_{\tau}^{\delta}\varphi(\vartheta,\tau) - \mathcal{L}(\varphi_i(\vartheta,\tau)) - \mathcal{N}(\varphi_i(\vartheta,\tau)) - h\left(\varphi_i\left(\vartheta,\tau\right)\right)\ |\ . \tag{3.13}$$

3.2. Elzaki new iterative method (ENIM)

We consider the following differential equation to demonstrate the fundamental concept of the ENIM [30, 54]:

$$D_\tau^\delta \varphi(\vartheta,\tau) + \mathcal{L}(\varphi(\vartheta,\tau)) + \mathcal{N}(\varphi(\vartheta,\tau)) = 0, \ \forall m \in \mathbf{N}, \tag{3.14}$$

with ICs

$$\partial_{\tau^k}^k \varphi(\vartheta,0) = \varphi_k(\vartheta), \ \ \forall\, k = 0,1,2,\cdots,m-1. \tag{3.15}$$

Here, $\mathcal{L}$ and $\mathcal{N}$ indicate the general linear and nonlinear operators, and $D_\tau^\delta = \partial_{\tau^\delta}^\delta$ represents CFD, where $\delta \in R^+$, $m-1 < \delta \le m, m = 0,1,...,n$.

Step 1: Rearrange Eq (3.14) in the following manner

$$D_\tau^\delta \varphi(\vartheta,\tau) = -\mathcal{L}(\varphi(\vartheta,\tau)) - \mathcal{N}(\varphi(\vartheta,\tau)). \tag{3.16}$$

Step 2: Applying ET to both sides of Eq (3.16) implies

$$E\left[D_\tau^\delta \varphi(\vartheta,\tau)\right] = E\left[-\mathcal{L}(\varphi(\vartheta,\tau)) - \mathcal{N}(\varphi(\vartheta,\tau))\right]. \tag{3.17}$$

Step 3: By using the definition of $E\left[D_\tau^\delta \varphi(\vartheta,\tau)\right]$ as given in Eq (2.11), we get

$$v^{-\delta} E[\varphi(\vartheta,\tau)] - \sum_{k=0}^{n-1} v^{2-\delta+k} \partial_{\tau^k}^k \varphi(\vartheta,0) = E\left[-\mathcal{L}(\varphi(\vartheta,\tau)) - \mathcal{N}(\varphi(\vartheta,\tau))\right], \tag{3.18}$$

which leads to

$$E[\varphi(\vartheta,\tau)] = \sum_{k=0}^{n-1} v^{2+k} \partial_{\tau^k}^k \varphi(\vartheta,0) + v^\delta E\left[-\mathcal{L}(\varphi(\vartheta,\tau)) - \mathcal{N}(\varphi(\vartheta,\tau))\right]. \tag{3.19}$$

Step 4: Applying the inverse of ET on Eq (3.19) and using IC (3.15), we get

$$\begin{aligned} \varphi(\vartheta,\tau) &= E^{-1}\left[\sum_{k=0}^{n-1} v^{2+k} \partial_{\tau^k}^k \varphi(\vartheta,0) + v^\delta E\left[-\mathcal{L}(\varphi(\vartheta,\tau)) - \mathcal{N}(\varphi(\vartheta,\tau))\right]\right] \\ &= E^{-1}\left[\sum_{k=0}^{n-1} v^{2+k} \partial_{\tau^k}^k \varphi(\vartheta,0)\right] - E^{-1}\left[v^\delta E\left[\mathcal{L}(\varphi(\vartheta,\tau))\right]\right] \\ &\quad - E^{-1}\left[v^\delta E\left[\mathcal{N}(\varphi(\vartheta,\tau))\right]\right]. \end{aligned} \tag{3.20}$$

Step 5: According to the NIM, the following approximation is utilized

$$\varphi(\vartheta,\tau) = \sum_{i=0}^{\infty} \varphi_i(\vartheta,\tau). \tag{3.21}$$

Step 6: According to Eq (3.21), the nonlinear operator $\mathcal{N}(\varphi(\vartheta,\tau))$ can be decomposed as follows

$$\mathcal{N}(\varphi(\vartheta,\tau) = \mathcal{N}(\varphi_0(\vartheta,\tau)) + \sum_{n=1}^{\infty}\left[\mathcal{N}\left(\sum_{i=0}^{n} \varphi_i(\vartheta,\tau)\right) - \mathcal{N}\left(\sum_{i=0}^{n-1} \varphi_i(\vartheta,\tau)\right)\right]. \tag{3.22}$$

Step 7: By inserting Eqs (3.21) and (3.22) into Eq (3.20), we get

$$\begin{aligned}\sum_{i=0}^{\infty}\varphi_i(\vartheta,\tau) = {}& E^{-1}\left[\sum_{k=0}^{n-1} v^{2+k}\partial^k_{\tau^k}\varphi(\vartheta,0)\right]\\ &- E^{-1}\left[v^{\delta}E\left[\mathcal{N}\left(\varphi_0(\vartheta,\tau)\right)\right]\right] - E^{-1}\left[v^{\delta}E\left[\mathcal{L}(\varphi_0(\vartheta,\tau))\right]\right]\\ &- E^{-1}\left[v^{\delta}E\left[\sum_{n=1}^{\infty}\left[\mathcal{N}\left(\sum_{i=0}^{n}\varphi_i(\vartheta,\tau)\right) - \mathcal{N}\left(\sum_{i=0}^{n-1}\varphi_i(\vartheta,\tau)\right)\right]\right]\right]\\ &- E^{-1}\left[v^{\delta}E\left[\mathcal{L}\left(\sum_{i=1}^{n}\varphi_i(\vartheta,\tau)\right)\right]\right]. \end{aligned} \tag{3.23}$$

Step 8: From Eq (3.23), we have

$$\begin{cases} \varphi_0(\vartheta,\tau) = E^{-1}\left[\sum_{k=0}^{n-1} v^{2+k}\partial^k_{\tau^k}\varphi(\vartheta,0)\right],\\ \varphi_1(\vartheta,\tau) = -E^{-1}\left[v^{\delta}E\left[\mathcal{N}\left(\varphi_0(\vartheta,\tau)\right)\right]\right] - E^{-1}\left[v^{\delta}E\left[\mathcal{L}(\varphi_0(\vartheta,\tau))\right]\right],\\ \varphi_i(\vartheta,\tau) = -E^{-1}\left[v^{\delta}E\left[\sum_{n=1}^{\infty}\left[\mathcal{N}\left(\sum_{i=0}^{n}\varphi_i(\vartheta,\tau)\right) - \mathcal{N}\left(\sum_{i=0}^{n-1}\varphi_i(\vartheta,\tau)\right)\right]\right]\right]\\ \qquad - E^{-1}\left[v^{\delta}E\left[\mathcal{L}\left(\sum_{i=1}^{n}\varphi_i(\vartheta,\tau)\right)\right]\right]. \end{cases} \tag{3.24}$$

Step 9: Now, by collecting the obtained values of $\varphi_0(\vartheta,\tau)$, $\varphi_1(\vartheta,\tau)$, $\cdots$, $\varphi_i(\vartheta,\tau)$, we finally get the analytical approximation in the following form

$$\varphi = \sum_{n=0}^{\infty}\varphi_i. \tag{3.25}$$

4. Numerical examples

This section will offer two numerical examples for fractional nonlinear FW equations and analyze them utilizing ERPSM and ENIM.

4.1. Analyzing Example I using ERPSM

Let us consider the following fractional nonlinear FW equation

$$D^p_\tau\varphi - \frac{\partial^3\varphi}{\partial\vartheta^2\partial\tau} + \frac{\partial\varphi}{\partial\vartheta} - \varphi\frac{\partial^3\varphi}{\partial\vartheta^3} - 3\frac{\partial\varphi}{\partial\vartheta}\frac{\partial^2\varphi}{\partial\vartheta^2} + \varphi\frac{\partial\varphi}{\partial\vartheta} = 0,\quad 0 < p \le 1, \tag{4.1}$$

with the IC

$$\varphi(\vartheta,0) \equiv \varphi_0 = e^{\vartheta/2}. \tag{4.2}$$

The exact solution for Eq (4.1) at $p = 1$, reads

$$\varphi = e^{\frac{\vartheta}{2}-\frac{2\tau}{3}}. \tag{4.3}$$

The brief steps for analyzing problem (4.1) using ERPSM are introduced in the following manner:

Step 1: Applying ET to Eq (4.1) yields

$$E\left[D_\tau^p \varphi\right] = E\left[\frac{\partial^3 \varphi}{\partial \vartheta^2 \partial \tau} - \frac{\partial \varphi}{\partial \vartheta} + \varphi \frac{\partial^3 \varphi}{\partial \vartheta^3} + 3\frac{\partial \varphi}{\partial \vartheta}\frac{\partial^2 \varphi}{\partial \vartheta^2} - \varphi \frac{\partial \varphi}{\partial \vartheta}\right]. \tag{4.4}$$

Step 2: Inserting the definition of ET for the CFD as given in Eq (2.11) into Eq (4.4) implies

$$v^{-p} E[\varphi] - v^{2-p}\varphi_0 = E\left[\frac{\partial^3 \varphi}{\partial \vartheta^2 \partial \tau} - \frac{\partial \varphi}{\partial \vartheta} + \varphi \frac{\partial^3 \varphi}{\partial \vartheta^3} + 3\frac{\partial \varphi}{\partial \vartheta}\frac{\partial^2 \varphi}{\partial \vartheta^2} - \varphi \frac{\partial \varphi}{\partial \vartheta}\right], \tag{4.5}$$

which leads to

$$E[\varphi] = v^2 \varphi_0 + v^p E\left[\frac{\partial^3 \varphi}{\partial \vartheta^2 \partial \tau} - \frac{\partial \varphi}{\partial \vartheta} + \varphi \frac{\partial^3 \varphi}{\partial \vartheta^3} + 3\frac{\partial \varphi}{\partial \vartheta}\frac{\partial^2 \varphi}{\partial \vartheta^2} - \varphi \frac{\partial \varphi}{\partial \vartheta}\right]. \tag{4.6}$$

Note that $E[\varphi] \equiv E[\varphi(\vartheta, \tau)] = \varphi(\vartheta, v)$.

Step 3: Applying the inverse of ET "E^{-1}" to Eq (4.6) yields

$$\begin{aligned}
\varphi = \varphi_0 &+ v^p E\left[E^{-1}\left[\frac{\partial^3 \varphi}{\partial \vartheta^2 \partial \tau}\right]\right] + v^p E\left[E^{-1}[\varphi] \times E^{-1}\left[\frac{\partial^3 \varphi}{\partial \vartheta^3}\right]\right] \\
&+ 3v^p E\left[E^{-1}\left[\frac{\partial \varphi}{\partial \vartheta}\right] \times E^{-1}\left[\frac{\partial^2 \varphi}{\partial \vartheta^2}\right]\right] && (4.7) \\
&- v^p E\left[E^{-1}[\varphi] \times E^{-1}\left[\frac{\partial \varphi}{\partial \vartheta}\right]\right] - v^p\left[\frac{\partial \varphi}{\partial \vartheta}\right]. && (4.8)
\end{aligned}$$

Step 4: According to RPSM, the series solution is given up to the k^{th}-truncated terms as follows

$$S_i = \sum_{m=0}^{i} \varphi_m(\vartheta, \tau) = \sum_{m=0}^{i} f_m(\vartheta) \frac{\tau^{mp}}{\Gamma(1 + mp)}. \tag{4.9}$$

Step 5: ERFs for Eq (4.7) reads

$$\begin{aligned}
ERes(\vartheta, \tau) = (\varphi(\vartheta, v) - \varphi_0) &- v^p E\left[E^{-1}[\varphi]\, E^{-1}\left[\frac{\partial^3 \varphi}{\partial \vartheta^2 \partial \tau}\right]\right] \\
&- v^p E\left[E^{-1}[\varphi]\, E^{-1}\left[\frac{\partial^3 \varphi}{\partial \vartheta^3}\right]\right] \\
&- 3v^p E\left[E^{-1}\left[\frac{\partial \varphi}{\partial \vartheta}\right] E^{-1}\left[\frac{\partial^2 \varphi}{\partial \vartheta^2}\right]\right] && (4.10) \\
&+ v^p E\left[E^{-1}[\varphi]\, E^{-1}\left[\frac{\partial \varphi}{\partial \vartheta}\right]\right] + v^p\left[\frac{\partial \varphi}{\partial \vartheta}\right], && (4.11)
\end{aligned}$$

along with the following k^{th}-truncated ERFs:

$$
\begin{aligned}
ERes_k(\vartheta,\tau) = (\varphi_k(\vartheta,\tau)-\varphi_0) - v^p E\left[E_\tau^{-1}\left[\varphi_{k-1}\right]E^{-1}\left[\frac{\partial^3\varphi_{k-1}}{\partial\vartheta^2\partial\tau}\right]\right]\\
- v^p E\left[E^{-1}\left[\varphi_{k-1}\right]E^{-1}\left[\frac{\partial^3\varphi_{k-1}}{\partial\vartheta^3}\right]\right]\\
- 3v^p E\left[E^{-1}\left[\frac{\partial\varphi_{k-1}}{\partial\vartheta}\right]E^{-1}\left[\frac{\partial^2\varphi_{k-1}}{\partial\vartheta^2}\right]\right] \qquad (4.12)\\
+ v^p E\left[E^{-1}\left[\varphi_{k-1}\right]E^{-1}\left[\frac{\partial\varphi_{k-1}}{\partial\vartheta}\right]\right] + v^p\left[\frac{\partial\varphi_{k-1}}{\partial\vartheta}\right]. \qquad (4.13)
\end{aligned}
$$

Step 6: Now, to find the values of the functions $f_k(\vartheta)\ \forall\ k = 1, 2, 3, \cdots$, we should solve the following equation for $k = 0, 1, 2, \cdots$:

$$\tau^{-kp} Res_k(\vartheta,\tau)\mid_{\tau=0} = 0, \qquad (4.14)$$

we finally get the following values

$$
\begin{cases} f_1(\vartheta) = -\frac{e^{\vartheta/2}}{2},\ f_2(\vartheta) = \frac{e^{\vartheta/2}}{4},\\ f_3(\vartheta) = -\frac{e^{\vartheta/2}}{8},\ f_4(\vartheta) = \frac{e^{\vartheta/2}}{16}, \end{cases} \qquad (4.15)
$$

and so on.

Step 7: By substituting the obtained values of $f_k(\vartheta)$, $\forall\ k = 1, 2, 3, \cdots$, into Eq (4.9), we get

$$\varphi = \frac{1}{16}e^{x/2}\left(16 - \frac{8\tau^p}{\Gamma(p+1)} + \frac{4\tau^{2p}}{\Gamma(2p+1)} - \frac{2\tau^{3p}}{\Gamma(3p+1)} + \frac{\tau^{4p}}{\Gamma(4p+1)}\right) + \cdots. \qquad (4.16)$$

4.2. Analyzing Example I using ENIM

To apply ENIM for analyzing Eq (4.1), we follow the same steps introduced above in order to get the values of $\varphi_0 \equiv \varphi_0(\vartheta,\tau)$, $\varphi_1 \equiv \varphi_1(\vartheta,\tau)$, $\cdots$, $\varphi_i \equiv \varphi_i(\vartheta,\tau)$. By inserting the value of the IC given is Eq (4.2) into Eq (3.24), we get

$$
\begin{aligned}
\varphi_0 &= E^{-1}\left[\sum_{k=0}^{n-1} v^{2+k}\partial_{\tau^k}^k\varphi(\vartheta,0)\right]\\
&= E^{-1}\left[v^2\varphi(\vartheta,0)\right] = E^{-1}\left[v^2 e^{\vartheta/2}\right] = e^{\vartheta/2},\\
\varphi_1 &= -E^{-1}\left[v^p E\left[\mathcal{N}(\varphi_0)\right]\right] - E^{-1}\left[v^p E\left[\mathcal{L}(\varphi_0)\right]\right] = -\frac{\tau^p}{2\Gamma(p+1)}e^{\vartheta/2},\\
\varphi_2 &= -E^{-1}\left[v^p E\left[\mathcal{N}(\varphi_0+\varphi_1) - \mathcal{N}(\varphi_0)\right]\right] - E^{-1}\left[v^p E\left[\mathcal{L}(\varphi_1)\right]\right]\\
&= -\frac{(p-\tau)\,\tau^{2p-1}}{4\Gamma(2p+1)}e^{\vartheta/2},
\end{aligned}
$$

and

$$\varphi_3 = -E^{-1}\left[v^p E\left[\mathcal{N}(\varphi_0+\varphi_1+\varphi_2) - \mathcal{N}(\varphi_0+\varphi_1)\right]\right] - E^{-1}\left[v^p E\left[\mathcal{L}(\varphi_2)\right]\right]$$
$$= -\frac{\left(4\tau^2+9p^2-3p(4\tau+1)\right)\tau^{3p-2}}{32\Gamma(3p+1)}e^{\vartheta/2}.$$

By collecting the obtained values of φ_0, φ_1, φ_2,. . ., φ_i, we finally get the analytical approximation to Eq (4.1) upto the third approximations as follows

$$\varphi = \varphi_0+\varphi_1+\varphi_2+\varphi_3+\cdots$$
$$= e^{\vartheta/2}\left[1-\frac{\tau^p}{2\Gamma(p+1)}+\frac{(\tau-p)\tau^{2p-1}}{4\Gamma(2p+1)}+\frac{\left(-9p^2-4\tau^2+3p(1+4\tau)\right)\tau^{3p-2}}{32\Gamma(3p+1)}+\cdots\right]. \tag{4.17}$$

4.3. Analyzing Example II using RPSTM

Here, we consider the same problem (4.1) with different IC and exact solutions as follows:

$$\varphi(\vartheta,0) = \cosh^2\left(\frac{\vartheta}{4}\right), \tag{4.18}$$

and

$$\varphi(\vartheta,\tau) = \cosh^2\left(\frac{\vartheta}{4}-\frac{11\tau}{24}\right). \tag{4.19}$$

The brief procedures for analyzing problem (4.1) with IC (4.18) utilizing ERPSM are presented in the following manner:

Step 1: Applying ET "E" to Eq (4.1) yields

$$E\left[D_\tau^p\varphi\right] = E\left[\frac{\partial^3\varphi}{\partial\vartheta^2\partial\tau}-\frac{\partial\varphi}{\partial\vartheta}+\varphi\frac{\partial^3\varphi}{\partial\vartheta^3}+3\frac{\partial\varphi}{\partial\vartheta}\frac{\partial^2\varphi}{\partial\vartheta^2}-\varphi\frac{\partial\varphi}{\partial\vartheta}\right]. \tag{4.20}$$

Step 2: Inserting the definition of ET for the CFD as given in Eq (2.11) into Eq (4.20) implies

$$v^{-p}E[\varphi]-v^{2-p}\varphi_0 = E\left[\frac{\partial^3\varphi}{\partial\vartheta^2\partial\tau}-\frac{\partial\varphi}{\partial\vartheta}+\varphi\frac{\partial^3\varphi}{\partial\vartheta^3}+3\frac{\partial\varphi}{\partial\vartheta}\frac{\partial^2\varphi}{\partial\vartheta^2}-\varphi\frac{\partial\varphi}{\partial\vartheta}\right], \tag{4.21}$$

which leads to

$$E[\varphi] = v^2\varphi_0+v^pE\left[\frac{\partial^3\varphi}{\partial\vartheta^2\partial\tau}-\frac{\partial\varphi}{\partial\vartheta}+\varphi\frac{\partial^3\varphi}{\partial\vartheta^3}+3\frac{\partial\varphi}{\partial\vartheta}\frac{\partial^2\varphi}{\partial\vartheta^2}-\varphi\frac{\partial\varphi}{\partial\vartheta}\right]. \tag{4.22}$$

Note that $E[\varphi] \equiv E[\varphi(\vartheta,\tau)] = \varphi(\vartheta,v)$.

Step 3: Applying the inverse of ET "E^{-1}" to Eq (4.22) yields

$$\begin{aligned}\varphi = \varphi_0 &+ v^p E\left[E^{-1}\left[\frac{\partial^3\varphi}{\partial\vartheta^2\partial\tau}\right]\right] + v^p E\left[E^{-1}[\varphi]\times E^{-1}\left[\frac{\partial^3\varphi}{\partial\vartheta^3}\right]\right]\\ &+ 3v^p E\left[E^{-1}\left[\frac{\partial\varphi}{\partial\vartheta}\right]\times E^{-1}\left[\frac{\partial^2\varphi}{\partial\vartheta^2}\right]\right]\\ &- v^p E\left[E^{-1}[\varphi]\times E^{-1}\left[\frac{\partial\varphi}{\partial\vartheta}\right]\right] - v^p\left[\frac{\partial\varphi}{\partial\vartheta}\right].\end{aligned} \tag{4.23}$$

Step 4: According to RPSM, the series solution is given up to the k^{th}-truncated terms as follows

$$S_i = \sum_{m=0}^{i}\varphi_m(\vartheta,\tau) = \sum_{m=0}^{i} f_m(\vartheta)\frac{\tau^{mp}}{\Gamma(1+mp)}. \tag{4.24}$$

Step 5: ERFs for Eq (4.23) reads

$$\begin{aligned}ERes(\vartheta,\tau) = (\varphi(\vartheta,v)-\varphi_0) &- v^p E\left[E^{-1}[\varphi]\,E^{-1}\left[\frac{\partial^3\varphi}{\partial\vartheta^2\partial\tau}\right]\right]\\ &- v^p E\left[E^{-1}[\varphi]\,E^{-1}\left[\frac{\partial^3\varphi}{\partial\vartheta^3}\right]\right]\\ &- 3v^p E\left[E^{-1}\left[\frac{\partial\varphi}{\partial\vartheta}\right]E^{-1}\left[\frac{\partial^2\varphi}{\partial\vartheta^2}\right]\right]\\ &+ v^p E\left[E^{-1}[\varphi]\,E^{-1}\left[\frac{\partial\varphi}{\partial\vartheta}\right]\right] + v^p\left[\frac{\partial\varphi}{\partial\vartheta}\right],\end{aligned} \tag{4.25}$$

along with the following k^{th}-truncated ERFs:

$$\begin{aligned}ERes_k(\vartheta,\tau) = (\varphi_k(\vartheta,\tau)-\varphi_0) &- v^p E\left[E_\tau^{-1}[\varphi_{k-1}]\,E^{-1}\left[\frac{\partial^3\varphi_{k-1}}{\partial\vartheta^2\partial\tau}\right]\right]\\ &- v^p E\left[E^{-1}[\varphi_{k-1}]\,E^{-1}\left[\frac{\partial^3\varphi_{k-1}}{\partial\vartheta^3}\right]\right]\\ &- 3v^p E\left[E^{-1}\left[\frac{\partial\varphi_{k-1}}{\partial\vartheta}\right]E^{-1}\left[\frac{\partial^2\varphi_{k-1}}{\partial\vartheta^2}\right]\right]\\ &+ v^p E\left[E^{-1}[\varphi_{k-1}]\,E^{-1}\left[\frac{\partial\varphi_{k-1}}{\partial\vartheta}\right]\right] + v^p\left[\frac{\partial\varphi_{k-1}}{\partial\vartheta}\right].\end{aligned} \tag{4.26}$$

Step 6: Now, to find the values of the functions $f_k(\vartheta)$ $\forall$ $k = 1,2,3,\cdots$, we should solve the following equation for $k = 0,1,2,\cdots$:

$$\tau^{-kp}Res_k(\vartheta,\tau)\,|_{\tau=0}= 0, \tag{4.27}$$

and we finally get the following values:

$$\begin{cases} f_1(\vartheta) = -\frac{11}{32}\sinh\left(\frac{\vartheta}{2}\right),\\ f_2(\vartheta) = \frac{121}{512}\cosh\left(\frac{\vartheta}{2}\right),\\ f_3(\vartheta) = -\frac{1331}{8192}\sinh\left(\frac{\vartheta}{2}\right),\end{cases}$$

and so on.

Step 7: By substituting the obtained values of $f_k(\vartheta)$, $\forall\, k = 1, 2, 3, \cdots$, into Eq (4.24), we have

$$\varphi(\vartheta,\tau) = \cosh^2\left(\frac{\vartheta}{4}\right) - \frac{11\sinh\left(\frac{\tau}{2}\right)}{32}\frac{\tau^p}{\Gamma(p+1)} + \frac{121\cosh\left(\frac{\vartheta}{2}\right)}{512}\frac{\tau^{2p}}{\Gamma(2p+1)} + \cdots. \tag{4.28}$$

4.4. Analyzing Example II using ENIM

Here, the ENIM is employed to find an approximation to Eq (4.1) with the IC (4.18). For this purpose, we directly can use relation (3.24), to find the values of $\varphi_0 \equiv \varphi_0(\vartheta,\tau)$, $\varphi_1 \equiv \varphi_1(\vartheta,\tau)$, $\cdots$, $\varphi_i \equiv \varphi_i(\vartheta,\tau)$:

$$\begin{aligned}
\varphi_0 &= E^{-1}\left[\sum_{k=0}^{n-1} v^{2+k}\partial^k_{\tau^k}\varphi(\vartheta,0)\right] = E^{-1}\left[v^2\varphi(\vartheta,0)\right]\\
&= E^{-1}\left[v^2\cosh^2\left(\frac{\vartheta}{4}\right)\right] = \cosh^2\left(\frac{\vartheta}{4}\right), && (4.29)\\
\varphi_1 &= -E^{-1}\left[v^p E\left[\mathcal{N}(\varphi_0)\right]\right] - E^{-1}\left[v^p E\left[\mathcal{L}(\varphi_0)\right]\right]\\
&= -\frac{11\tau^p\sinh\left(\frac{\vartheta}{2}\right)}{32\Gamma(p+1)}, && (4.30)\\
\varphi_2 &= -E^{-1}\left[v^p E\left[\mathcal{N}(\varphi_0+\varphi_1) - \mathcal{N}(\varphi_0)\right]\right] - E^{-1}\left[v^p E\left[\mathcal{L}(\varphi_1)\right]\right]\\
&= \frac{11\tau^{2p-1}\left(11\tau\cosh\left(\frac{\vartheta}{2}\right) - 8p\sinh\left(\frac{\vartheta}{2}\right)\right)}{512\Gamma(2p+1)}, && (4.31)
\end{aligned}$$

and

$$\begin{aligned}
\varphi_3 &= -E^{-1}\left[v^p E\left[\mathcal{N}(\varphi_0+\varphi_1+\varphi_2) - \mathcal{N}(\varphi_0+\varphi_1)\right]\right] - E^{-1}\left[v^p E\left[\mathcal{L}(\varphi_2)\right]\right]\\
&= -\frac{11\tau^{3p-2}\left[-264p\tau\cosh\left(\frac{\vartheta}{2}\right) + \left(48p(3p-1) + 121\tau^2\right)\sinh\left(\frac{\vartheta}{2}\right)\right]}{8192\Gamma(3p+1)}. && (4.32)
\end{aligned}$$

By collecting the obtained values of φ_0, φ_1, φ_2,..., φ_i, we finally get the analytical approximation to Eq (4.1) up to the third approximations as follows:

$$\begin{aligned}
\varphi &= \varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 + \cdots\\
&= \cosh^2\left(\frac{\vartheta}{4}\right) - \frac{11\tau^p\sinh\left(\frac{\vartheta}{2}\right)}{32\Gamma(p+1)} + \frac{11\tau^{2p-1}\left(11\tau\cosh\left(\frac{\vartheta}{2}\right) - 8p\sinh\left(\frac{\vartheta}{2}\right)\right)}{512\Gamma(2p+1)}\\
&\quad - \frac{11\tau^{3p-2}\left[-264p\tau\cosh\left(\frac{\vartheta}{2}\right) + \left(48p(3p-1) + 121\tau^2\right)\sinh\left(\frac{\vartheta}{2}\right)\right]}{8192\Gamma(3p+1)} + \cdots. && (4.33)
\end{aligned}$$

5. Numerical discussions

The numerical results obtained through the ERPSM and ENIM are presented in Tables 1-2 and Figures 1–8, which provide insights into the behavior of the derived approximations at different fractionality. Figure 1 presents a comparative visualization between the derived approximation (4.16) using ERPSM and the exact solution (4.3) for the integer case, i.e., $p = 1$ at $\tau = 0.1$. As is clear, the great harmony between the two solutions enhances the accuracy of the derived approximations. Furthermore, we analyzed the approximation (4.16) against the fractionality p to understand the impact of p on the dynamical behavior of this approximation as illustrated in Figure 2. This result illustrates the smooth transition from the derived approximation to the exact solution when p approaches one, demonstrating the obtained solutions' high accuracy. Likewise, we compared the derived approximation (4.17) using ENIM with the exact solution (4.3) for $p = 1$, as illustrated in Figure 3. Furthermore, we examined the approximation (4.17) against the fractionality p, as illustrated in Figure 4. The derived approximation (4.17) exhibits concordance with the exact solution (4.3), which enhances augmenting the precision of the derived approximation (4.17) through ENIM. Additionally, we computed the absolute errors for both the first and second approximations, as presented in Table 1. The obtained results demonstrate that ENIM is superior to ERPSM in this instance; however, this is not universally applicable, as documented in many published studies.

Table 1. A comparison between the absolute errors of the approximation (4.16) using ERPSM and the approximation (4.17) using ENIM for $p = 1$ at $\tau = 0.1$.

ϑ	$Exact$	$RPSTM_{p=1}$	$NITM_{p=1}$	R_{ERPSM}	R_{ENIM}
-4.	0.126607	0.128735	0.126705	0.00212777	0.0000977367
-3.5	0.162567	0.165299	0.162692	0.00273211	0.000125496
-3.	0.20874	0.212248	0.208901	0.00350809	0.000161141
-2.5	0.268027	0.272532	0.268234	0.00450448	0.000206909
-2.	0.344154	0.349938	0.344419	0.00578387	0.000265676
-1.5	0.441902	0.449329	0.442243	0.00742663	0.000341134
-1.	0.567414	0.57695	0.567852	0.00953599	0.000438025
-0.5	0.728574	0.740818	0.729136	0.0122444	0.000562436
0.	0.935507	0.951229	0.936229	0.0157222	0.000722182
0.5	1.20121	1.2214	1.20214	0.0201877	0.0009273
1.	1.54239	1.56831	1.54358	0.0259215	0.00119068
1.5	1.98047	2.01375	1.982	0.0332839	0.00152886
2.	2.54297	2.58571	2.54493	0.0427373	0.00196309
2.5	3.26524	3.32012	3.26776	0.0548758	0.00252066
3.	4.19265	4.26311	4.19589	0.0704619	0.00323659
3.5	5.38347	5.47395	5.38763	0.0904749	0.00415587
4.	6.91251	7.02869	6.91785	0.116172	0.00533624

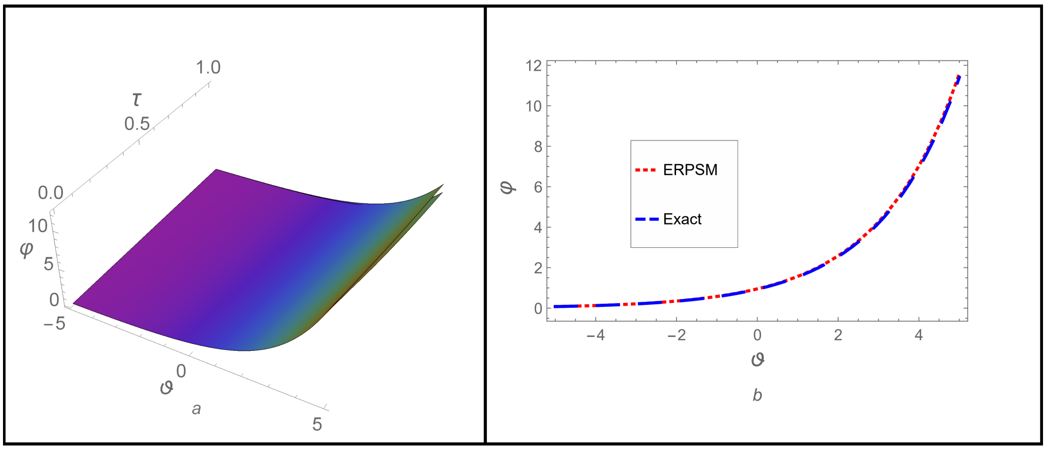

Figure 1. This figure compares the exact solution (4.3) and the approximate solution (4.16) obtained by ERPSM for $p = 1$ at $\tau = 0.1$.

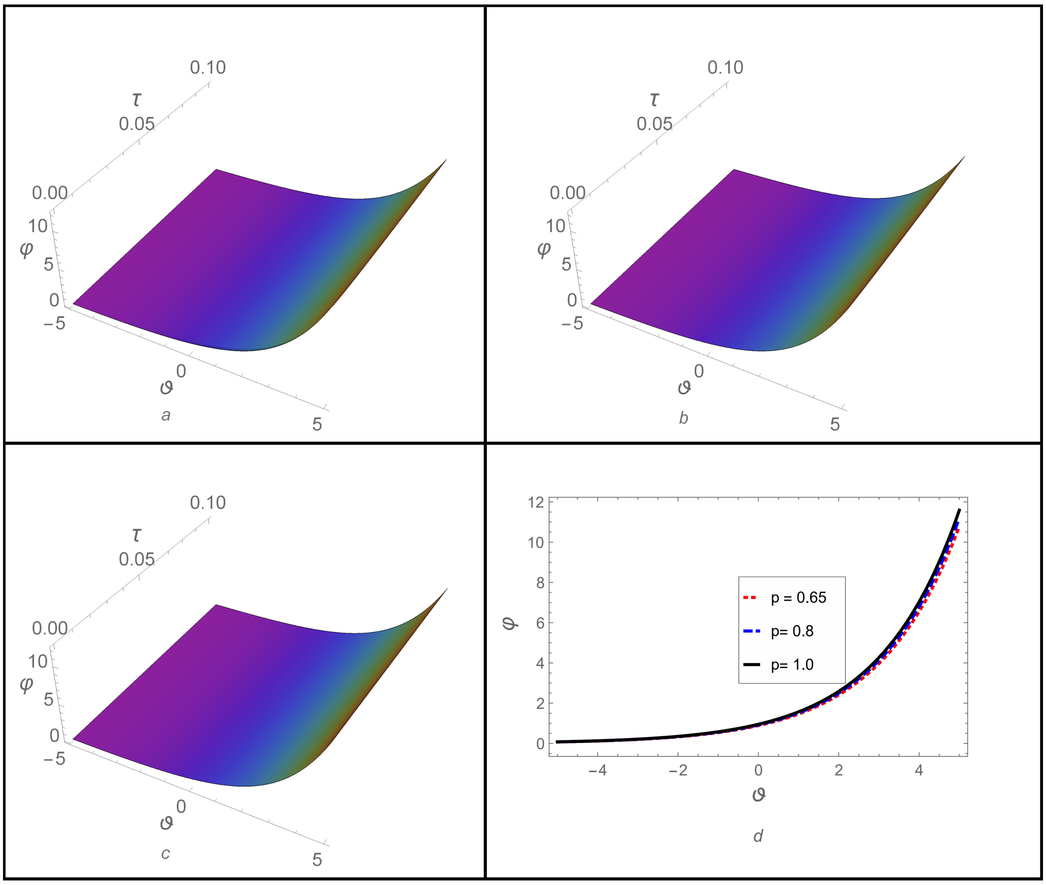

Figure 2. The approximate solution (4.16) obtained by ERPSM is investigated against the fractionality p: (a) 3D-graph in the plane-(ϑ, τ) at $p = 0.65$, (b) 3D-graph in the plane-(ϑ, τ) at $p = 0.8$, (c) 3D-graph in the plane-(ϑ, τ) at $p = 1$, and (d) 2D-graph at $t = 0.1$.

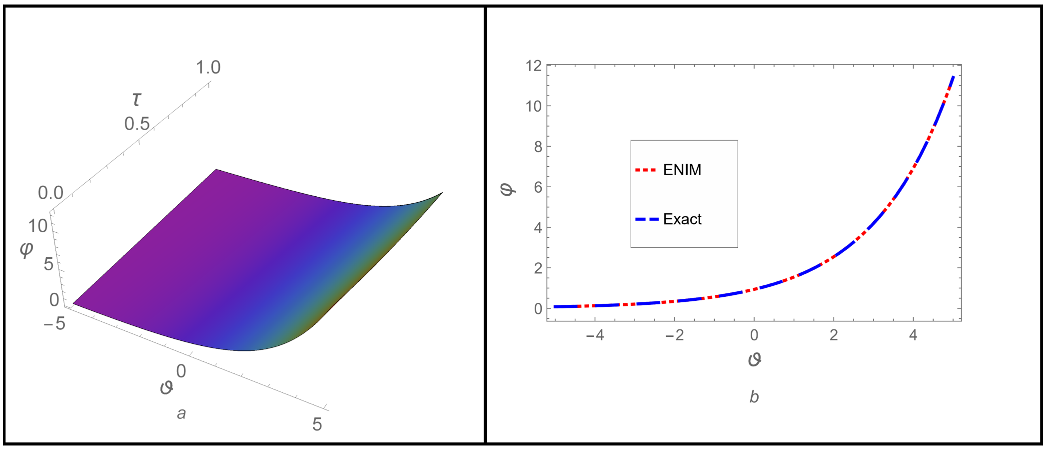

Figure 3. This figure compares the exact solution (4.3) and the approximate solution (4.17) obtained by ENIM for $p = 1$ at $\tau = 0.1$.

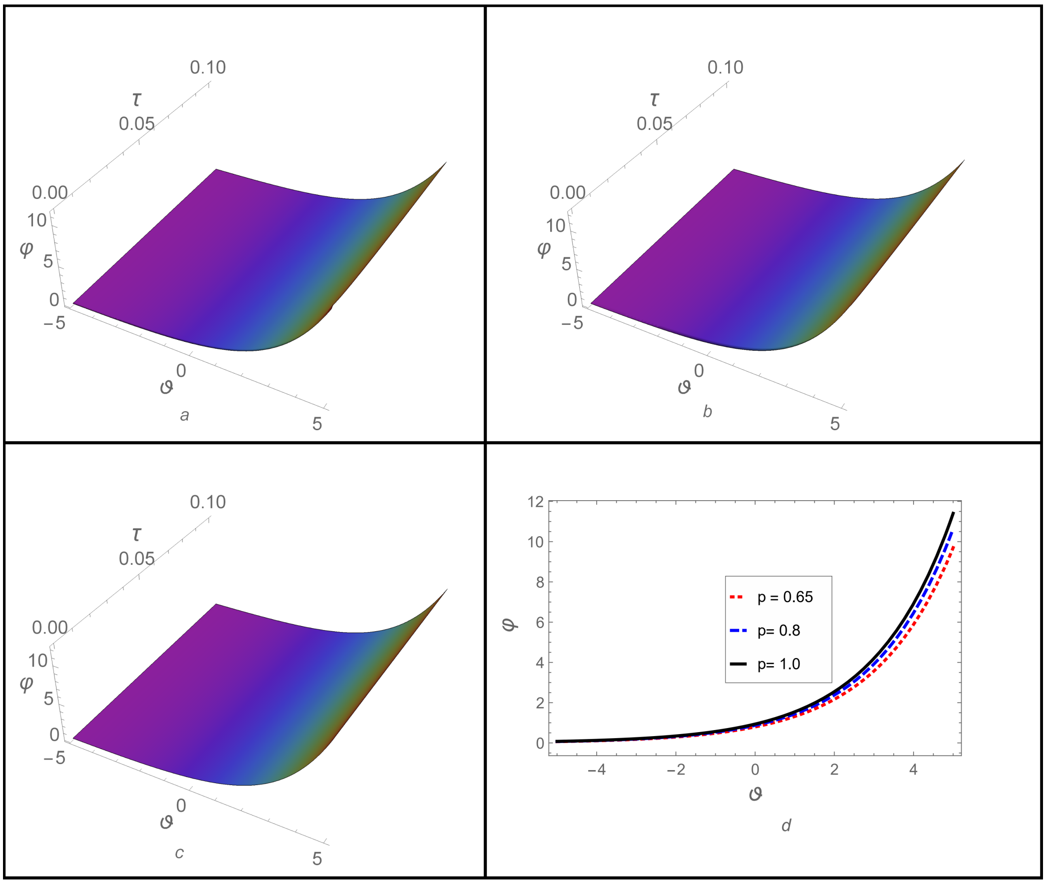

Figure 4. The approximate solution (4.17) obtained by ENIM is investigated against the fractionality p: (a) 3D-graph in the plane-(ϑ, τ) at $p = 0.65$, (b) 3D-graph in the plane-(ϑ, τ) at $p = 0.8$, (c) 3D-graph in the plane-(ϑ, τ) at $p = 1$, and (d) 2D-graph at $t = 0.1$.

Furthermore, in Example II, we examined the derived approximations (4.28) and (4.33) using ERPSM and ENIM, respectively, against the fractionality p to investigate the influence of p on the behavior of these approximations, as illustrated in Figures 5 and 6. Moreover, we compared the generated approximations (4.28) and (4.33) against the exact solution (4.19) for the integer case to verify and validate the high precision of these approximations, as illustrated in Figures 7 and 8, respectively. Additionally, the absolute errors of approximations (4.28) and (4.33) are computed numerically, as presented in Table 2. The analysis results indicate the incredible precision of these approximations, which enhances the efficacy of the proposed methods in examining more intricate problems. Nonetheless, ENIM demonstrated superiority over ERPSM in this instance.

Overall, the graphical results and tables confirm the effectiveness of ERPSM and ENIM in solving the time-fractional FW equations. Both methods exhibit excellent accuracy and convergence behavior, with solutions aligning closely with exact values as the fractional order approaches unity. The 3D and 2D analyses further validate the applicability of these methods for FDEs, providing deep insights into the role of fractional parameters in governing solution dynamics.

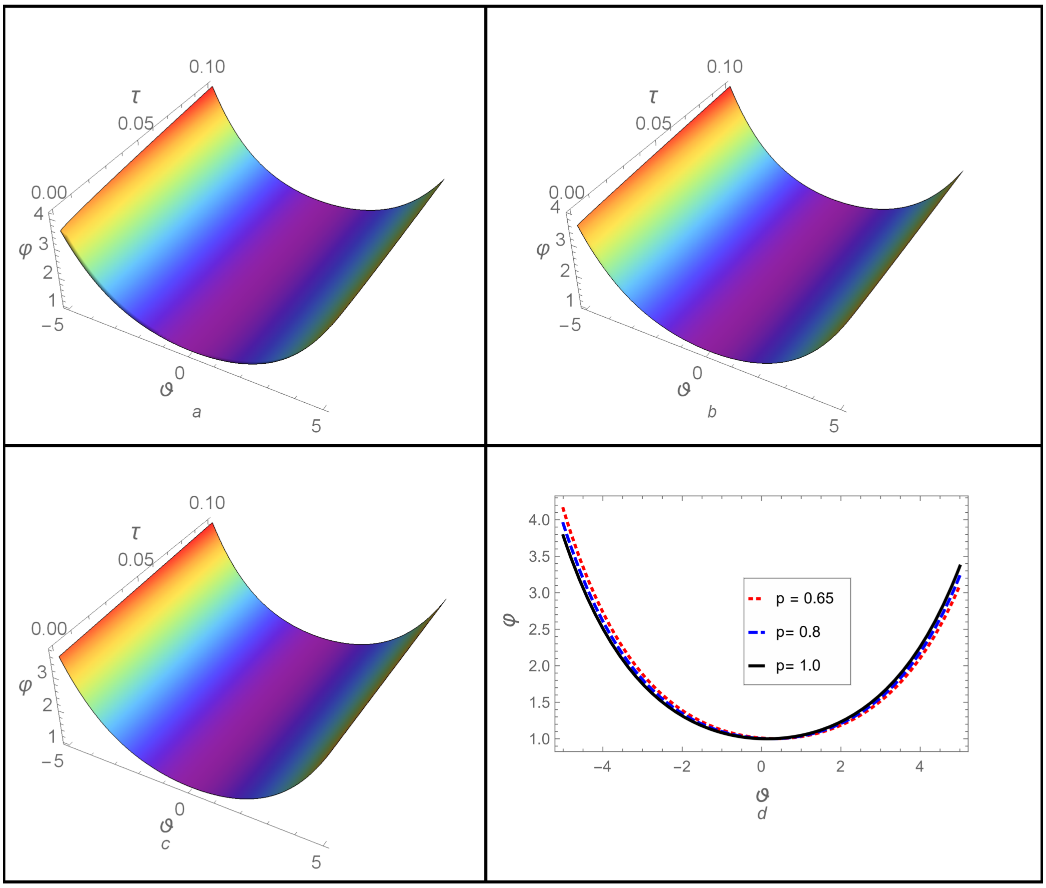

Figure 5. The approximate solution (4.28) obtained by ERPSM is investigated against the fractionality p: (a) 3D-graph in the plane-(ϑ, τ) at $p = 0.65$, (b) 3D-graph in the plane-(ϑ, τ) at $p = 0.8$, (c) 3D-graph in the plane-(ϑ, τ) at $p = 1$, and (d) 2D-graph at $t = 0.1$.

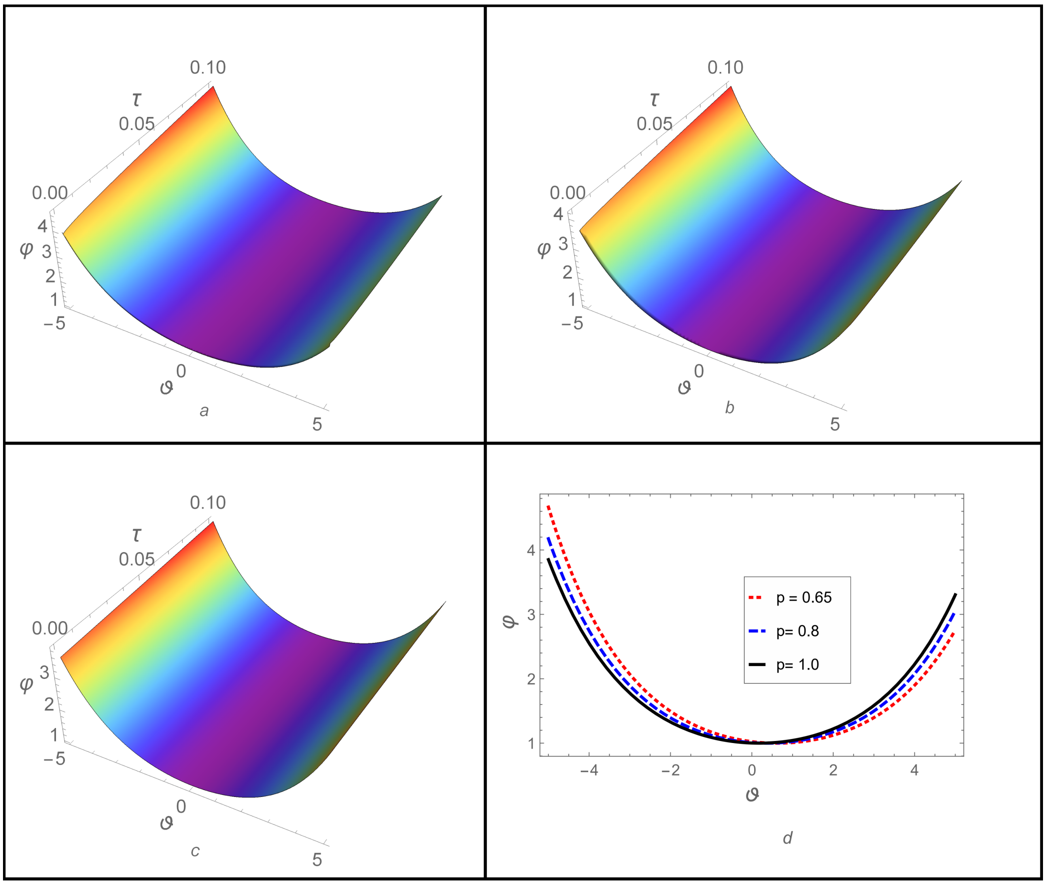

Figure 6. The approximate solution (4.33) obtained by ENIM is investigated against the fractionality p: (a) 3D-graph in the plane-(ϑ, τ) at $p = 0.65$, (b) 3D-graph in the plane-(ϑ, τ) at $p = 0.8$, (c) 3D-graph in the plane-(ϑ, τ) at $p = 1$, and (d) 2D-graph at $t = 0.1$.

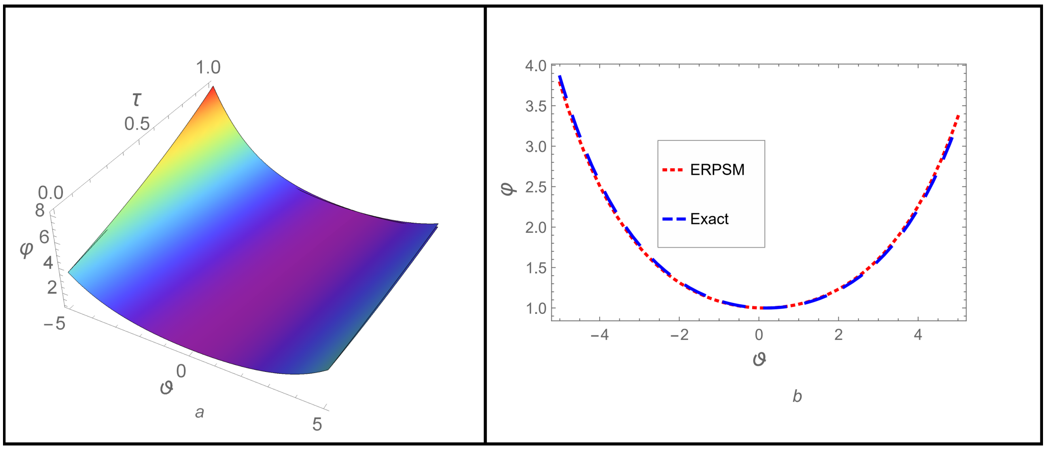

Figure 7. This figure compares the exact solution (4.19) and the approximate solution (4.28) obtained by ERPSM for $p = 1$ at $\tau = 0.1$.

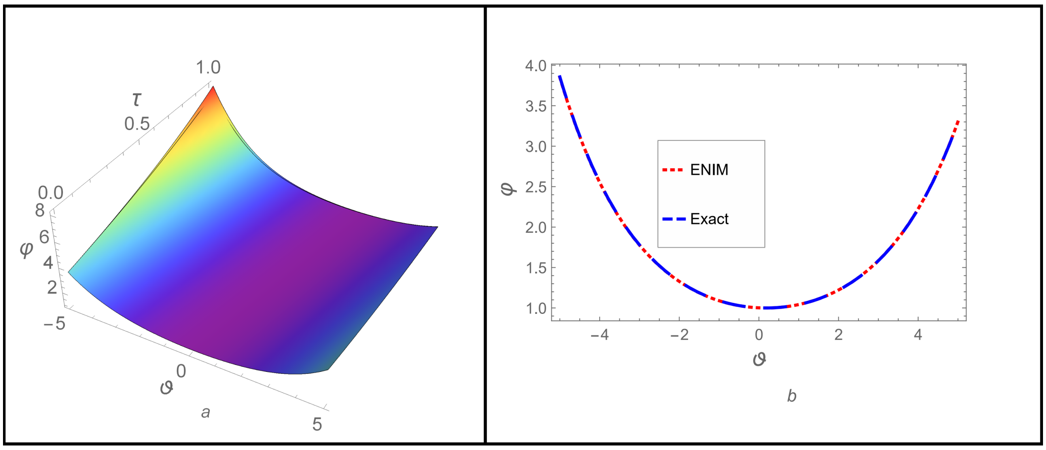

Figure 8. This figure compares the exact solution (4.19) and the approximate solution (4.33) obtained by ENIM for $p = 1$ at $\tau = 0.1$.

Table 2. A comparison between the absolute errors of the approximation (4.28) using ERPSM and the approximation (4.33) using ENIM for $p = 1$ at $\tau = 0.1$.

ϑ	$Exact$	$RPSTM_{p=1}$ =	$NITM_{p=1}$	R_{ERPSM}	R_{ENIM}
-4.	2.55547	2.51031	2.5515	0.0451557	0.00397246
-3.5	2.1164	2.08159	2.11332	0.0348057	0.00307927
-3.	1.77888	1.75224	1.7765	0.0266425	0.00237955
-2.5	1.52171	1.50155	1.51988	0.0201532	0.00182932
-2.	1.32872	1.31379	1.32733	0.0149299	0.00139402
-1.5	1.18781	1.17716	1.18676	0.0106447	0.0010463
-1.	1.0901	1.08307	1.08934	0.00702824	0.000764315
-0.5	1.02947	1.02562	1.02894	0.00385334	0.00053035
0.	1.0021	1.00118	1.00177	0.000920525	0.000329705
0.5	1.00628	1.00823	1.00613	0.00195446	0.000149774
1.	1.04227	1.04722	1.04229	0.00495223	0.0000207476
1.5	1.11232	1.12058	1.11251	0.00826113	0.000192573
2.	1.22085	1.23293	1.22122	0.012089	0.000376496
2.5	1.37466	1.39133	1.37524	0.0166765	0.000584073
3.	1.58342	1.60573	1.58425	0.0223116	0.000828346
3.5	1.86025	1.8896	1.86138	0.0293485	0.00112466
4.	2.22254	2.26077	2.22403	0.0382293	0.00149163

6. Conclusions

In this study, we have derived some analytical approximations to the nonlinear fractional FW equations using the residual power series method and the NIM in the framework of the ET. Combining the RPSM and NIM with ET resulted in the ERPSM and ENIM. We successfully captured the nonlocal and memory-dependent effects inherent in the fractional model under consideration by employing the CFD. Some analytical approximations have been derived using the suggested approaches. This analysis established the applicability and efficacy of the proposed methods to achieve highly accurate approximations for a wide range of strong and more complicated fractional nonlinear partial differential equations. All derived approximations have been graphically compared with the exact solutions for integer cases to evaluate the dependability of the used approaches. Furthermore, the absolute errors of the derived approximations have been assessed, revealing that ENIM produces approximations with superior accuracy compared to ERPSM. One of the most significant findings is that for low values of the fractional parameter, ERPSM yields more accurate and reliable results compared to ENIM. Furthermore, whether employing the Elzaki or Laplace transform or an alternative, the outcomes remain consistent, as any transform simplifies the computational process without altering the conclusions derived from one transform to another.

Comparative analyses confirmed the reliability and precision of these methodologies, showcasing their applicability to complex FDEs arising in mathematical physics and engineering. The findings contribute to understanding wave dynamics in dispersive media and nonlinear systems.

Future work: Future research may extend these techniques to more generalized nonlinear models, incorporating multidimensional and variable-order fractional derivatives for further advancements in theoretical and applied sciences, especially in fluid mechanics and plasma physics. Also, the modified FFW equation [55] poses a significant challenge and remains the subject of substantial research. In future endeavors, we will analyze this equation and some related fractional equations utilizing the methods applied in this study alongside contemporary techniques, such as the Tantawy technique [56–58], which has effectively analyzed numerous complex cases and produced encouraging outcomes.

Author contributions

Safyan Mukhtar: Conceptualization, Writing-review & editing, Project administration, Funding; Wedad Albalawi: Visualization, Writing-review & editing, Data curation; Faisal Haroon: Data curation, Project administration; Samir A. El-Tantawy: Resources, Validation, Software; Investigation, Resources. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used AI tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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