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Research article

Specific types of Lindelöfness and compactness based on novel supra soft operator

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Abstract: The aim of this study was to originate six types of generalized compactness and Lindelöfness in the frame of supra soft topological spaces (or SSTSs) based on the approaches of supra soft somewhere dense sets (or SS-sd-sets) and the SS-sd-closure operator, named SS-sd-almost compact (Lindelöf) spaces, SS-sd-approximately compact (Lindelöf) spaces and SS-sd-mildly compact (Lindelöf) spaces. The essential properties of each type of the aforementioned notions were studied. Specifically, we studied the invariance of these notions under specific types of soft mappings. Moreover, the relationships among these notions and between corresponding notions were discussed. Furthermore, the equivalence among them was proved under the SS-sd-partition condition. Finally, we provided a diagram to summarize these relationships with the support of concrete counterexamples.

Keywords: supra soft sd-closure operator; SS-sd-almost compactness; SS-sd-hyperconnectedness; SS-sd-approximately Lindelöfness; SS-sd-mildly compactness; SS-sd-partition
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1. Introduction

Soft set theory [1] is an essential alternative tool of crisp, fuzzy, and rough set theories which have several difficulties for dealing with problems involving uncertainties which were improved by Maji et al. [2]. The researchers had success in presenting concrete applications in rough set models [3, 4], decision making problems [5, 6], and medical sciences [7].

The approaches of soft continuity were defined by Ahmad and Kharal [8] in 2011. More advanced studies related to these notion were introduced in [9–11]. In the same year, the notion of soft topological spaces (or STSs) [12, 13] were defined. After that, several investigations related to weaker types of soft sets were studied. Examples include the following: Soft pre- (respectively, β-, α-) open sets [14, 15], soft semi-open sets [16–18], soft b-open sets [19, 20], nearly soft β-open sets [21], soft somewhat open sets [22], and soft sd-sets [23–25]. More investigation into the soft sd-continuity was introduced in [26, 27]. Azzam et al. [28] used soft set operators to generate new soft topologies. Compactness and connectedness based on soft somewhat open sets were introduced in [29].

In 2014, the notion of soft ideals [30, 31] was presented and employed to introduce the category of soft ideal topological spaces. By using the approach of soft semi-open sets, these notions have been generalized in [32, 33]. An application on soft ideal rough topological spaces in Diabetes mellitus [34] was introduced by Abd El-latif in 2018. Based on the soft ideals notions, several types of categories of soft open sets have been generalized in [35–38]. Also, generalized versions of soft separation axioms [39, 40] in STSs have been presented.

In 2013, Aygünoglu and Aygüun [41] presented the notion of soft compactness. In 2014, generalized notions of soft compactness were investigated in [42]. Kandil et al. defined the notions of soft-I-compact spaces [43] and soft-I-connected spaces [44]. Recently, Al-Shami et al. [45] used the concept of soft sd-sets to present six types of compactness in STSs in 2021.

The notion of SSTSs [46] was presented in 2014. Also, they defined many types of weaker SS-open sets and continuity. In 2015, continuous (respectively, open, closed) maps [47] were defined by Abd El-latif and Karataş. In 2024, Abd El-latif [48] defined several operators and maps based on SS-δ_i-open sets. In the same year, He originate many novel types of operators based on SS-sd-sets [49]. Moreover, he and his co-author used these notions to present many types of weaker soft maps [50]. In 2025, Abd El-latif et al. [51] introduced the concept of SS-somewhat open sets.

The concepts of closed spaces, generalized compactness, and compactness (Lindelöf) in SSTSs were initially introduced in [52] in 2018. In 2024, Abd El-latif et al. [53] used the SS-sd-sets to generalized these notions. Specifically, they presented the approach of SS-sd-compact (Lindelöf) space.

This study generally seeks to present six types of generalized Lindelöf and compact spaces in the frame of SSTSs. We organize it as follows: In the Preliminaries, we provide the fundamental notions and terminologies in STSs and SSTSs which will be needed in the subsequent sections. In Section 3, we formulate the concept of SS-sd-almost compact (Lindelöf) spaces and discuss their essential characterizations. Moreover, the pre-image (image) of each kind of approach under specific types of soft maps is studied. Furthermore, the navigation between SS-almost compact (Lindelöf) SSTSs and their parametric supra topological spaces is studied.

In Section 4, we present SS-sd-approximately compact (Lindelöf) spaces which are weaker than SS-almost compact (Lindelöf) spaces. Moreover, based on a special type of the soft finite (countable) intersection property (or SFIP), namely the conditions ω_1 and ω_2, we investigate more interesting properties of these notions.

In Section 5, the notion of SS-sd-mildly compact (Lindelöf) spaces is defined and their main properties are discussed. Also, we study the sufficient condition for the equivalence among the six types of compact (Lindelöf) SSTSs. Finally, we provide a diagram to summarize the relationships among the aforementioned notions, supported by several concrete examples in Figure 1.

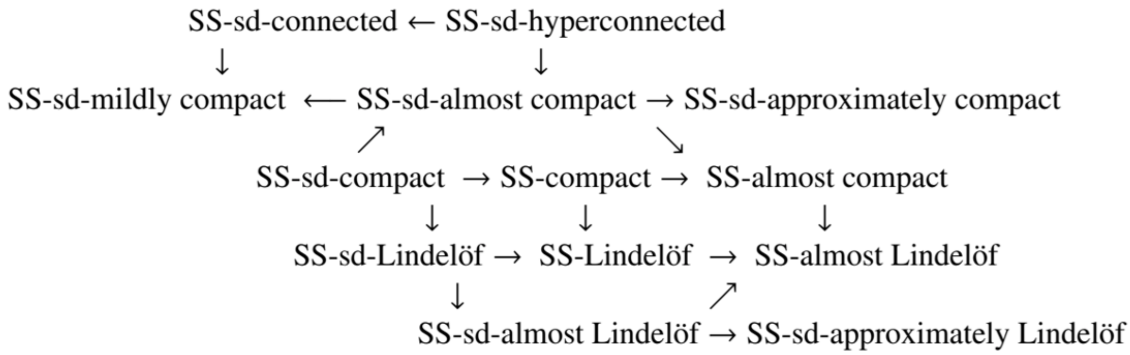

Figure 1. The relationships between diverse types of connectedness and compactness (Lindelöfness) based on SS-sd-sets.

2. Preliminaries

In this section, we introduce the fundamental terminologies and concepts in STSs and SSTSs which will be needed in this manuscript, for more details see [12, 46, 49, 50].

Definition 2.1. *[1] A pair (K, ϑ), denoted by K_ϑ, over initial universe C and the set of parameters ϑ is called a soft set, which is a parameterized family of subsets of the universe C, i.e., $K_\vartheta = \{K(\upsilon) : \upsilon \in \vartheta,\ K : \vartheta \to P(C)\}$. If $K(\upsilon) = \varphi$ (respectively, $K(\upsilon) = U$) for all $\upsilon \in \vartheta$, then (K, ϑ) is called a null (respectively, an absolute) soft set and will denoted by $\tilde{\varphi}$ (respectively, $\tilde{C}$). Henceforth, we be denote the class of all soft sets by $S(C)_\vartheta$.*

Definition 2.2. *[12] The collection $\tau \subseteq S(C)_\vartheta$ is called a soft topology on C if τ contains $\tilde{C}, \tilde{\varphi}$ and is closed under arbitrary soft union and finite soft intersection. The triplet (C, τ, ϑ) is called a soft topological space (or STS) over C.*

Definition 2.3. *[12] Let (C, τ, ϑ) be an STS and $(K, \vartheta) \in S(C)_\vartheta$. The soft closure of (K, ϑ), is denoted by $cl(K, \vartheta)$, is the intersection of all soft closed supersets of (K, ϑ). Also, the soft interior of (G, ϑ), is denoted by $int(G, \vartheta)$, is the union of all soft open subsets of (G, ϑ).*

Definition 2.4. *[12, 16] The soft set $(G, \vartheta) \in S(C)_\vartheta$ is called a soft point in $\tilde{C}$; is denoted by s_υ, if there exist $s \in U$ and $\upsilon \in \vartheta$ such that $G(\upsilon) = \{s\}$ and $G(\upsilon') = \varphi$ for each $\upsilon' \in \vartheta - \{\upsilon\}$. Also, $s_\upsilon \tilde{\in} (F, \vartheta)$, if for the element $\upsilon \in \vartheta$, $G(\upsilon) \subseteq F(\upsilon)$.*

Theorem 2.5. *[8] For the soft map $\psi_{sd} : (C, \tau, \vartheta_1) \to (D, \sigma, \vartheta_2)$, the following statements hold.*

(1) $\psi_{sd}^{-1}((N^{\tilde{c}}, \vartheta_2)) = (\psi_{sd}^{-1}(N, \vartheta_2))^{\tilde{c}}\ \forall\ (N, \vartheta_2) \in S(D)_{\vartheta_2}$.

(2) $\psi_{sd}(\psi_{sd}^{-1}((N, \vartheta_2))) \tilde{\sqsubseteq} (N, \vartheta_2)\ \forall\ (N, \vartheta_2) \in S(D)_{\vartheta_2}$. The equality holds if ψ_{sd} is surjective.

(3) $(N, \vartheta_1) \tilde{\sqsubseteq} \psi_{sd}^{-1}(\psi_{sd}((N, \vartheta_1)))\ \forall\ (N, \vartheta_1) \in S(C)_{\vartheta_1}$. The equality holds if ψ_{sd} is injective.

(4) $\psi_{sd}(\tilde{C}) \tilde{\sqsubseteq} \tilde{V}$. The equality holds if ψ_{sd} is surjective.

Definition 2.6. *[46] The collection $\nu \subseteq S(C)_\vartheta$ is called an SSTS on C if ν contains $\tilde{C}, \tilde{\varphi}$ and is closed under arbitrary soft union. The SS-interior of a soft subset (G, ϑ), is denoted by $int^s(G, \vartheta)$, is the soft union of all SS-open subsets of (G, ϑ). Also, the SS-closure of (K, ϑ), is denoted by $cl^s(K, \vartheta)$, is the soft intersection of all SS-closed supersets of (K, ϑ).*

Definition 2.7. *[46] Let* $((C, \nu, \vartheta)$ *be an SSTS, and then* $(G, \vartheta) \in S(C)_\vartheta$ *is called an SS-semi open set if* $(G, \vartheta) \tilde{\sqsubseteq} cl^s(int^s(G, \vartheta))$*. The soft complement of an SS-semi-open set is called SS-semi-closed.*

Definition 2.8. *[46] Let* (C, τ, ϑ) *be an STS and* (C, ν, ϑ) *be an SSTS. We say that* ν *is an SSTS associated with* τ *if* $\tau \subset \nu$*.*

Definition 2.9. *[46] A soft map* $\psi_{sd} : (C, \tau, \vartheta) \rightarrow (V, \sigma, \vartheta)$ *with* ν *as an associated SSTS with* τ *is said to be SS-continuous if* $\psi_{sd}^{-1}(G, \vartheta) \in \nu \ \forall \ (G, \vartheta) \in \sigma$*.*

Definition 2.10. *[49] Let* (C, ν, ϑ) *be an SSTS and* $(K, \vartheta) \in S(C)_\vartheta$*. Then* (K, ϑ) *is called an SS-sd-set if there exists* $\tilde{\varphi} \neq (O, \vartheta) \in \nu$ *such that*

$$(O, \vartheta) \tilde{\sqsubseteq} cl^s[(O, \vartheta) \tilde{\sqcap} (K, \vartheta)].$$

The soft complement of an SS-sd-set is said to be an SS-sc-set. The family of all SS-sd-sets (respectively, SS-sc-sets) is denoted by $SD(C)_\vartheta$ (respectively, $SC(C)_\vartheta$). Also, if $cl^s_{sd}(K, \vartheta) = \tilde{C}$, then (K, ϑ) is called SS-sd-dense.

Theorem 2.11. *[49] Let* (C, ν, ϑ) *be an SSTS and* $(G, \vartheta) \in S(C)_\vartheta$*. Then* $(K, \vartheta) \in SD(C)_\vartheta$ *if and only if* $int^s(cl^s(K, \vartheta)) \neq \tilde{\varphi}$*.*

Theorem 2.12. *[49] Let* (C, ν, ϑ) *be an SSTS and* $(G, \vartheta) \in S(C)_\vartheta$*. Then* $(K, \vartheta) \in SC(C)_\vartheta$ *if and only if* $\exists$ *a proper SS-closed subset* (H, ϑ) *of* $\tilde{C}$ *such that* $int^s(K, \vartheta) \tilde{\sqsubseteq} (H, \vartheta)$*.*

Corollary 2.13. *[49] Every soft subset (superset) of an SS-sc-set (SS-sd-set) is an SS-sc-set (SS-sd-set).*

Proposition 2.14. *[49] A soft subset* (L, ϑ) *of an SSTS* (C, ν, ϑ) *is either an SS-sd-set or SS-sc-set.*

Definition 2.15. *[49] The SS-sd-interior of a soft subset* (G, ϑ) *of an SSTS* (C, ν, ϑ) *denoted by* $int^s_{sd}(G, \vartheta)$*, is the largest SS-sd-subset of* (G, ϑ)*. Also, the SS-sd-closure of a soft subset* (H, ϑ) *of an SSTS* (C, ν, ϑ) *denoted by* $cl^s_{sd}(H, \vartheta)$*, is the smallest SS-sc-superset of* (H, ϑ)*.*

Theorem 2.16. *[49] For a soft subset* (T, ϑ) *of an SSTS* (C, ν, ϑ)*, we have*

(1) $cl^s_{sd}(T^{\tilde{c}}, \vartheta) = [int^s_{sd}(T, \vartheta)]^{\tilde{c}}$ and $int^s_{sd}(T^{\tilde{c}}, \vartheta) = [cl^s_{sd}(T, \vartheta)]^{\tilde{c}}$.

(2) $cl^s_{sd}(T, \vartheta) \tilde{\sqsubseteq} cl^s(T, \vartheta)$.

(3) $int^s(T, \vartheta) \tilde{\sqsubseteq} int^s_{sd}(T, \vartheta)$.

Theorem 2.17. *[50] A soft map* $\psi_{sd} : (C, \tau, \vartheta) \rightarrow (V, \sigma, \vartheta)$ *with* ν *as an associated SSTS with* τ *is SS-sd-cts iff either* $\psi_{sd}^{-1}(G, \vartheta) = \tilde{\varphi}$ *or* $\psi_{sd}^{-1}(G, \vartheta) \in SD(C)_\vartheta$ *for each* $(G, \vartheta) \in \sigma$*.*

Definition 2.18. *[52] A family* $\Gamma = \{(\Omega_\iota, \vartheta) : \iota \in \mathrm{I}\}$ *of soft sets is said to be an SS-open cover, if each member of* I *is an SS-open set.*

Definition 2.19. *[53] A class* $\Gamma = \{(\Omega_\iota, \vartheta) : \iota \in \mathrm{I}\}$ *of SS-sd-subsets of an SSTS* (C, ν, ϑ) *is said to be an SS-sd-cover of the soft subset* (G, ϑ) *of* $\tilde{C}$*, if* $(G, \vartheta) \tilde{\sqsubseteq} \Gamma$*.*

Definition 2.20. *[53] A soft subset* (G, ϑ) *of an SSTS* (C, ν, ϑ) *is said to be SS-sd-compact (Lindelöf), if every SS-sd-cover* $\{(\Omega_\iota, \vartheta) : \iota \in \mathrm{I}\}$ *of* (G, ϑ) *has a finite (countable) subclass* I_o *of* I *such that*

$$(G, \vartheta) \tilde{\sqsubseteq} \tilde{\sqcup}_{\iota \in \mathrm{I}_o} (\Omega_\iota, \vartheta).$$

The space (C, ν, ϑ) is said to be SS-sd-compact (Lindelöf) if $\tilde{C}$ is SS-sd-compact (Lindelöf) as a soft subset.

3. Almost compactness and Lindelöfness based on the supra soft sd-closure operator

In this section, we apply the SS-sd-closure operator to define new types of compactness in the frame of SSTSs, named SS-sd-almost compact (Lindelöf) spaces. The relationships with corresponding notions like SS-compact (Lindelöf) and SS-sd-hyperconnected are discussed, with the help of concrete examples and counterexamples. Also, based on a special types of the SFIP, namely the condition ω_1, we investigate more interesting properties of these notions. Moreover, the pre-image (image) of each SS-sd-almost Lindelöf (compact) set under specific types of soft maps is studied. Furthermore, the navigation between SS-almost compact (Lindelöf) SSTSs and their parametric supra topological spaces is studied.

Definition 3.1. *A soft subset* (G, ϑ) *of an SSTS* (C, ν, ϑ) *is said to be SS-sd-almost compact (Lindelöf), if every SS-sd-cover* $\{(\Omega_\iota, \vartheta) : \iota \in \mathrm{I}\}$ *of* (G, ϑ) *has a finite (countable) subclass* I_o *of* I *such that*

$$(G, \vartheta)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in \mathrm{I}_o} cl^s_{sd}(\Omega_\iota, \vartheta).$$

The space (C, ν, ϑ) is said to be SS-sd-almost compact (Lindelöf) if $\tilde{C}$ is SS-sd-almost compact (Lindelöf) as a soft subset.

Theorem 3.2. *Every SS-sd-almost compact (Lindelöf) space is SS-almost compact (Lindelöf).*

Proof: Suppose that $\Gamma = \{(\Omega_\iota, \vartheta) : \iota \in I\}$ is an SS-open cover for an SS-sd-almost compact SSTS (C, ν, ϑ). Then Γ is an SS-sd-cover for $\tilde{C}$. Since $\tilde{C}$ is SS-sd-almost compact, there is a finite subclass I_o of I such that

$$\tilde{C} = \tilde{\sqcup}_{\iota\in I_o} cl^s_{sd}(\Omega_\iota, \vartheta)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I_o} cl^s(\Omega_\iota, \vartheta), \text{ (according to Theorem 2.16).}$$

Hence, $\tilde{C}$ is SS-almost compact. The other case is handled in a similar way.

Remark 3.3. The following example shall confirm that the converse of Theorem 3.2 is not satisfied in general.

Example 3.4. *Consider the natural numbers N the universal set and consider* $n_1, n_2 \in N$ *any two distinct natural numbers. Let* $\vartheta = \{\upsilon_1, \upsilon_2\}$ *and* $\nu = \{\tilde{N}, \tilde{\varphi}, (A_j, \vartheta), j = 1, 2, 3\}$ *be an SSTS on U, where:*

$$A_1(\upsilon_1) = \{n_1\}, \quad A_1(\upsilon_2) = \{n_2\};$$
$$A_2(\upsilon_1) = N - \{n_1\}, \quad A_2(\upsilon_2) = N - \{n_2\};$$
$$A_3(\upsilon_1) = N - \{n_1\}, \quad A_3(\upsilon_2) = N.$$

It is clear that $\tilde{N}$ is SS-almost compact. On the other hand, the class

$$\Gamma = \{n_\upsilon : n_\upsilon \text{ } is \text{ } soft \text{ } point \text{ } in \text{ } \tilde{N}\} \cup \{(G, \vartheta) : (G, \vartheta) \text{ } is \text{ } an \text{ } SS - sd - neighborhood \text{ } for \text{ } n_{1_{\upsilon_1}}\}$$

is an SS-sd-cover for $\tilde{N}$, since every soft subset of $\tilde{N}$ is both an SS-sd-set and SS-sc-set [except $\{\tilde{N}, \tilde{\varphi}, (A_3, \vartheta), (A_3^{\tilde{c}}, \vartheta)\}$]. However, there is no subclass of Γ in which the soft union of the SS-sd-closure of its elements covers $\tilde{N}$. Thus, $\tilde{N}$ is not SS-sd-almost compact.

Theorem 3.5. **(1)** *Every SS-sd-compact space is SS-sd-almost compact.*

(2) *Every SS-sd-almost compact space is SS-sd-almost Lindelöf.*

Proof:

(1) Assume that $\{(\Omega_\iota, \vartheta) : \iota \in I\}$ is SS-sd-cover for $\tilde{C}$. Since $\tilde{C}$ is SS-sd-compact, there is a finite subclass I_o of I such that

$$\tilde{C} = \tilde{\sqcup}_{\iota \in I_o}(\Omega_\iota, \vartheta) \tilde{\sqsubseteq} \tilde{\sqcup}_{\iota \in I_o} cl^s_{sd}(\Omega_\iota, \vartheta).$$

Hence, $\tilde{C}$ is SS-sd-almost compact.

(2) Clear.

Examples 3.6.

(1) Consider $1 \in R$. Let $\vartheta = \{\upsilon_1, \upsilon_2, \upsilon_3, ...\}$ and $\nu = \{\tilde{R}, \tilde{\varphi}, (Y, \vartheta), (Z, \vartheta), (W, \vartheta)\}$ be an SSTS on R, where:

$$(Y, \vartheta) = \{(\upsilon_1, \{1\}), (\upsilon_2, \{1\}), (\upsilon_3, \{1\}), (\upsilon_4, \{1\}), ...\};$$

$$(Z, \vartheta) = \{(\upsilon_1, \{1\}), (\upsilon_2, \varphi), (\upsilon_3, \{1\}), (\upsilon_4, \{1\}), ...\};$$

$$(W, \vartheta) = \{(\upsilon_1, \varphi), (\upsilon_2, \{1\}), (\upsilon_3, \{1\}), (\upsilon_4, \{1\}), ...\}.$$

Since every SS-sd-subset of $\tilde{R}$ is supra soft-sd-dense, $\tilde{R}$ is SS-sd-almost compact (Lindelöf). On the other side, the class

$$\Gamma = \{(H_\iota, \vartheta) : H(\upsilon_i) = \{1, k\}, \ for\ \upsilon_i \in \vartheta, \ \iota \in I, \ and\ k \in R\}$$

forms an SS-sd-cover for $\tilde{R}$. However, there is no finite (countable) subclass of Γ that covers $\tilde{R}$. Thus, $\tilde{R}$ is not SS-sd-compact (Lindelöf).

(2) Consider the SSTS in Example 3.4, and we have that $\tilde{N}$ is SS-sd-almost Lindelöf, however it is not SS-sd-almost compact.

Definition 3.7. *[55] If every pair of SS-sd-subsets (G, ϑ) and (H, ϑ) of an SSTS (C, ν, ϑ) satisfies $(G, \vartheta) \tilde{\sqcap} (H, \vartheta) \neq \tilde{\varphi}$, then $\tilde{C}$ is said to be SS-sd-hyperconnected.*

Theorem 3.8. *[55] An SSTS (C, ν, ϑ) is SS-sd-hyperconnected if, and only if, every SS-sd-set (G, ϑ) is SS-sd-dense.*

Theorem 3.9. Every SS-sd-hyperconnected SSTS is SS-sd-almost compact.

Proof: Assume that $\{(\Omega_\iota, \vartheta) : \iota \in I\}$ is an SS-sd-cover for an SS-sd-hyperconnected SSTS (C, ν, ϑ), and then $cl^s_{sd}(\Omega_\iota, \vartheta) = \tilde{C}$ for each $\iota \in I$, from Theorem 3.8. Therefore, there is a finite subclass I_o of I such that

$$\tilde{C} = \tilde{\sqcup}_{\iota \in I_o} cl^s_{sd}(\Omega_\iota, \vartheta).$$

Thus, $\tilde{C}$ is SS-sd-almost compact.

Remark 3.10. *The following example shall confirm that the converse of Theorem 3.9 is not satisfied in general.*

Example 3.11. *Consider $r_1, r_2 \in R$ any two distinct real numbers. Let $\vartheta = \{\upsilon_1, \upsilon_2\}$ and $\nu = \{\tilde{R}, \tilde{\varphi}, (B_j, \vartheta), j = 1, .., 4\}$ be an SSTS on R, where:*

$$B_1(\upsilon_1) = \{r_1\}, \quad B_1(\upsilon_2) = \varphi;$$
$$B_2(\upsilon_1) = \{r_1\}, \quad B_2(\upsilon_2) = \{r_2\};$$
$$B_3(\upsilon_1) = \{r_2\}, \quad B_3(\upsilon_2) = \{r_2\};$$
$$B_4(\upsilon_1) = \{r_1, r_2\}, \quad B_4(\upsilon_2) = \{r_2\}.$$

It is clear to see that $\tilde{R}$ is an SS-sd-almost compact space. On the other hand, for the soft sets (B_1, ϑ) and $(B_1^{\tilde{c}}, \vartheta)$, we have $(B_1, \vartheta), (B_1^{\tilde{c}}, \vartheta) \in SD(R)_\vartheta$, whereas $(B_1, \vartheta)\tilde{\sqcap}(B_1^{\tilde{c}}, \vartheta) = \tilde{\varphi}$. Thus, $\tilde{R}$ is not SS-sd-hyperconnected.

Proposition 3.12. (1) *Every SSTS (C, ν, ϑ) defined on a finite (countable) universal set C is SS-sd-almost compact (Lindelöf).*

(2) *A countable (finite) soft union of SS-sd-almost Lindelöf (compact) subsets of an SSTS (C, ν, ϑ) is SS-sd-almost Lindelöf (compact).*

Proof: It follows from Definition 3.1.

Theorem 3.13. *Every SS-sc-subset of an SS-sd-almost Lindelöf (compact) SSTS (C, ν, ϑ) is SS-sd-almost Lindelöf (compact).*

Proof: Let $\{(\Omega_\iota, \vartheta) : \iota \in I\}$ be an SS-sd-cover for an SS-sc-subset (X, ϑ) of an SS-sd-almost Lindelöf SSTS (C, ν, ϑ), and then $(X, \vartheta)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I_o} cl^s_{sd}(\Omega_\iota, \vartheta)$. So, it follows that

$$\tilde{C} = (X, \vartheta)\tilde{\sqcup}(X^{\tilde{c}}, \vartheta)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I_o} cl^s_{sd}(\Omega_\iota, \vartheta)\tilde{\sqcup}(X^{\tilde{c}}, \vartheta).$$

Now, we have that

$$\{(\Omega_\iota, \vartheta) : \iota \in I\}\tilde{\sqcup}(X^{\tilde{c}}, \vartheta) \text{ is also an SS-sd-cover for } \tilde{C}.$$

Since $\tilde{C}$ is an SS-sd-almost Lindelöf, there is a countable subclass I_o of I such that

$$(X, \vartheta)\tilde{\sqsubseteq}\tilde{C} = \tilde{\sqcup}_{\iota\in I_o}(\Omega_\iota, \vartheta)\tilde{\sqcup}(X^{\tilde{c}}, \vartheta).$$

Hence,

$$(X, \vartheta)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I_o}(\Omega_\iota, \vartheta).$$

Thus, (X, ϑ) is SS-sd-almost Lindelöf. The case of SS-sd-compactness is similar.

Theorem 3.14. *If (L, ϑ) and (N, ϑ) are SS-sd-almost Lindelöf (compact) and SS-sc-subsets of an SSTS (U, ν, ϑ), respectively, then $(L, \vartheta)\tilde{\sqcap}(N, \vartheta)$ is SS-sd-almost Lindelöf (compact).*

Proof: Suppose $\{(\Omega_\iota, \vartheta) : \iota \in I\}$ is an SS-sd-cover for $(L, \vartheta)\tilde{\sqcap}(N, \vartheta)$, and then

$$(L, \vartheta)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I}(\Omega_\iota, \vartheta)\tilde{\sqcup}(N^{\tilde{c}}, \vartheta).$$

Since (L, ϑ) is an SS-sd-almost Lindelöf, there is a countable subclass I_o of I such that

$$(L, \vartheta)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I_o}(\Omega_\iota, \vartheta)\tilde{\sqcup}(N^{\tilde{c}}, \vartheta),$$

and it follows that

$$(L, \vartheta)\tilde{\sqcap}(N, \vartheta)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I_o}(\Omega_\iota, \vartheta).$$

Therefore, $(L, \vartheta)\tilde{\sqcap}(N, \vartheta)$ is SS-sd-almost Lindelöf. The case of SS-sd-almost compactness can be obtained similarly.

Remark 3.15. *The following example shall show that the converse of Proposition 3.14 is not necessarily satisfied in general.*

Example 3.16. *Assume that $C = \{5, 6\}$. Let $\vartheta = \{\upsilon_1, \upsilon_2\}$ be the set of parameters. Let (M_j, ϑ), $j = 1, .., 4$, be soft sets over the universe U, where*

$$\begin{aligned} M_1(\upsilon_1) &= C, & M_1(\upsilon_2) &= \{6\}; \\ M_2(\upsilon_1) &= \varphi, & M_2(\upsilon_2) &= \{6\}; \\ M_3(\upsilon_1) &= \{5\}, & M_3(\upsilon_2) &= \{6\}; \\ M_4(\upsilon_1) &= C, & M_4(\upsilon_2) &= \varphi. \end{aligned}$$

Then, $\nu = \{\tilde{C}, \tilde{\varphi}, (M_j, \vartheta), j = 1, .., 4\}$ defines an SSTS on U. According to Proposition 3.12, it is clear that $\tilde{C}$ is SS-sd-almost compact (Lindelöf). Also the soft set (W, ϑ), where

$$W(\upsilon_1) = \{5\}, \quad W(\upsilon_2) = \{6\},$$

is SS-sd-almost compact (Lindelöf). However, $\tilde{C}\tilde{\sqcap}(W, \vartheta)$ is not an SS-sc-set.

Corollary 3.17. *If (L, ϑ) and (N, ϑ) are SS-sd-almost Lindelöf (compact) and SS-sd-subsets of an SSTS (C, ν, ϑ), respectively, then $(L, \vartheta)\backslash(N, \vartheta)$ is SS-sd-almost Lindelöf (compact).*

Proof: It follows from Theorem 3.14.

Definition 3.18. *[54] A collection Γ of soft sets has the SFIP, if the soft intersection of the finite (countable) subfamily of Γ is non-empty.*

Definition 3.19. *A class $\Gamma = \{(\Omega_\iota, \vartheta) : \iota \in \mathrm{I}\}$ of soft subsets of an SSTS (C, ν, ϑ), in which for any finite (countable) subclass I_o of I we have that*

$$\tilde{\sqcap}_{\iota\in I_o} int^s_{sd}(\Omega_\iota, \vartheta) \neq \tilde{\varphi},$$

is said to satisfy the condition ω_1. It is obvious that any class satisfying the condition ω_1 also satisfies the SFIP.

Theorem 3.20. *An SSTS (C, ν, ϑ) is SS-sd-almost compact (Lindelöf) if, and only if, every family of SS-sc-subsets of $\tilde{C}$ satisfying the condition ω_1 has a non-null soft intersection.*

Proof: **Necessity:** Assume that $\{(\Omega_\iota, \vartheta) : \iota \in I\}$ is a class of SS-sc-sets, which satisfying the condition ω_1, and assume conversely that $\tilde{\sqcap}_{\iota\in I}(\Omega_\iota, \vartheta) = \tilde{\varphi}$. It follows that

$$\tilde{C} = \tilde{\sqcup}_{\iota\in I}(\Omega^{\tilde{c}}_\iota, \vartheta)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I} cl^s_{sd}(\Omega^{\tilde{c}}_\iota, \vartheta).$$

Hence,

$$\tilde{\varphi} = [\tilde{\sqcup}_{\iota\in I} cl^s_{sd}(\Omega^{\tilde{c}}_\iota, \vartheta)]^{\tilde{c}} = \tilde{\sqcap}_{\iota\in I} int^s_{sd}(\Omega_\iota, \vartheta),$$

which opposes the condition ω_1.

Sufficiency: Assume that $\Gamma = \{(\Omega_\iota, \vartheta) : \iota \in I\}$ is an SS-sd-cover for $\tilde{C}$ and assume to the contrary that $\tilde{C}$ is not SS-sd-almost Lindelöf. It follows that, for every countable subclass I_o of I, we have that

$$\tilde{\sqcup}_{\iota\in I_o} cl^s_{sd}(\Omega_\iota, \vartheta) \neq \tilde{C}; \text{ hence, } \tilde{\sqcap}_{\iota\in I_o} int^s_{sd}(\Omega^{\tilde{c}}_\iota, \vartheta) \neq \tilde{\varphi}.$$

Hence, $\{(\Omega^{\tilde{c}}_\iota, \vartheta) : \iota \in I\}$ is a class of SS-sc-subsets of $\tilde{C}$ with the condition ω_1. According to our assumption, $\tilde{\sqcap}_{\iota\in I}(\Omega^{\tilde{c}}_\iota, \vartheta) \neq \tilde{\varphi}$; so, $\tilde{\sqcup}_{\iota\in I}(\Omega_\iota, \vartheta) \neq \tilde{C}$, which opposes that Γ is an SS-sd-cover for $\tilde{C}$. Thus, $\tilde{C}$ is SS-sd-almost Lindelöf.

The case of SS-sd-almost compactness can be obtained in a similar way.

Theorem 3.21. *An SSTS (C, ν, ϑ) is SS-sd-almost Lindelöf (compact) if, and only if, for every class $\Gamma = \{(\Omega_\iota, \vartheta) : \iota \in \mathrm{I}\}$ of soft subsets of $\tilde{C}$ satisfying the condition ω_1, we have that $\tilde{\sqcap}_{\iota\in\mathrm{I}}\{cl^s_{sd}(\Omega_\iota, \vartheta) : (\Omega_\iota, \vartheta) \in \Gamma\} \neq \tilde{\varphi}$.*

Proof: We prove the case of SS-sd-almost Lindelöfness, where the case between parentheses can be obtained in the same way.

Necessity: Assume that $\Gamma = \{(\Omega_\iota, \vartheta) : \iota \in I\}$ is a class of soft sets which satisfy the condition ω_1 and suppose conversely that $\tilde{\sqcap}_{\iota\in I}\{cl^s_{sd}(\Omega_\iota, \vartheta) : (\Omega_\iota, \vartheta) \in \Gamma\} = \tilde{\varphi}$; then,

$$\tilde{\sqcup}_{\iota\in I}\{(cl^s_{sd}(\Omega_\iota, \vartheta))^{\tilde{c}} : (\Omega_\iota, \vartheta) \in \Gamma\} = \tilde{C},$$

which means

$$\{(cl^s_{sd}(\Omega_\iota, \vartheta))^{\tilde{c}} : (\Omega_\iota, \vartheta) \in \Gamma\}$$

is an SS-sd-cover for $\tilde{C}$.

Since $\tilde{C}$ is an SS-sd-almost Lindelöf, there is a constable subclass I_o of I such that

$$\tilde{C} = \tilde{\sqcup}_{\iota\in I_o} cl^s_{sd}[(cl^s_{sd}(\Omega_\iota, \vartheta))^{\tilde{c}}] : (\Omega_\iota, \vartheta) \in \Gamma\} = \tilde{\sqcup}_{\iota\in I_o}\{cl^s_{sd}[int^s_{sd}(\Omega^{\tilde{c}}_\iota, \vartheta)] : (\Omega_\iota, \vartheta) \in \Gamma\} \tilde{\sqsubseteq} \tilde{\sqcup}_{\iota\in I_o}\{cl^s_{sd}(\Omega^{\tilde{c}}_\iota, \vartheta) : (\Omega_\iota, \vartheta) \in \Gamma\}.$$

Hence,

$$\tilde{\varphi} = \tilde{\sqcap}_{\iota\in I_o}\{[cl^s_{sd}(\Omega^{\tilde{c}}_\iota, \vartheta)]^{\tilde{c}} : (\Omega_\iota, \vartheta) \in \Gamma\} = \tilde{\sqcap}_{\iota\in I_o}\{int^s_{sd}(\Omega_\iota, \vartheta) : (\Omega_\iota, \vartheta) \in \Gamma\} \text{ (given Theorem 2.16)},$$

which contradicts the condition ω_1. Thus, $\tilde{\sqcup}_{\iota\in I}\{cl^s_{sd}(\Omega_\iota, \vartheta) : (\Omega_\iota, \vartheta) \in \Gamma\} \neq \tilde{\varphi}$.

Sufficiency: Suppose that $\Gamma = \{(\Omega_\iota, \vartheta) : \iota \in I\}$ is an SS-sd-cover for $\tilde{C}$ and assume to the contrary that $\tilde{C}$ is not SS-sd-almost Lindelöf. Hence, for each countable subclass I_o of I we have that

$$\tilde{\sqcup}_{\iota\in I_o} cl^s_{sd}(\Omega_\iota, \vartheta) \neq \tilde{C} \text{ and it follows that } \tilde{\sqcap}_{\iota\in I_o} int^s_{sd}(\Omega^{\tilde{c}}_\iota, \vartheta) \neq \tilde{\varphi}.$$

Therefore, $\{(\Omega^{\tilde{c}}_\iota, \vartheta) : (\Omega_\iota, \vartheta) \in \Gamma\}$ is a class of SS-sc-sets satisfying the condition ω_1. According to the hypothesis,

$$\tilde{\sqcap}_{\iota\in I}\{cl^s_{sd}(\Omega^{\tilde{c}}_\iota, \vartheta) = (\Omega^{\tilde{c}}_\iota, \vartheta) : (\Omega_\iota, \vartheta) \in \Gamma\} \neq \tilde{\varphi}.$$

Hence,

$$\tilde{\sqcup}_{\iota\in I}\{(\Omega_\iota,\vartheta):(\Omega_\iota,\vartheta)\tilde{\in}\Gamma\}\neq\tilde{C},$$

which opposes that Γ is an SS-sd-cover for $\tilde{C}$.

Thus, $\tilde{C}$ is SS-sd-almost Lindelöf.

Theorem 3.22. *[50] The following statements are equivalent for a soft function $\psi_{sd}:(C,\tau,\vartheta_1)\to(D,\sigma,\vartheta_2)$ with ν as an associated SSTS with τ:*

(1) *ψ_{sd} is SS-sd-cts.*

(2) *For each $(J,\vartheta_2)\in\tau_2^c$, either $\psi_{sd}^{-1}(J,\vartheta_2)\in SC(C)_{\vartheta_1}$ or $\psi_{sd}^{-1}(J,\vartheta_2)=\tilde{C}$.*

(3) *$cl_{sd}^s(\psi_{sd}^{-1}(J,\vartheta_2))\tilde{\sqsubseteq}\psi_{sd}^{-1}(cl(J,\vartheta_2))\ \forall\ (J,\vartheta_2)\tilde{\sqsubseteq}\tilde{D}$.*

(4) *$\psi_{sd}(cl_{sd}^s(J,\vartheta_1))\tilde{\sqsubseteq}cl(\psi_{sd}(J,\vartheta_1))\ \forall\ (J,\vartheta_1)\tilde{\sqsubseteq}\tilde{C}$.*

(5) *$\psi_{sd}^{-1}(int(J,\vartheta_2))\tilde{\sqsubseteq}int_{sd}^s(\psi_{sd}^{-1}(J,\vartheta_2))\ \forall\ (J,\vartheta_2)\tilde{\sqsubseteq}\tilde{D}$.*

Theorem 3.23. *The image of each SS-sd-almost compact (Lindelöf) set is SS-almost compact (Lindelöf) under a surjective and an SS-sd-continuous map.*

Proof: Let $\psi_{sd}:(C,\tau,\vartheta_1)\to(D,\sigma,\vartheta_2)$ be an SS-sd-continuous map with ν,ν^* as associated SSTSs with τ,σ, respectively, (H,ϑ_1) is an SS-sd-almost compact subset of $\tilde{C}$ and $\{(\Omega_\iota,\vartheta_2):\iota\in I\}$ is an SS-open cover for $\psi_{sd}(H,\vartheta_1)$. Given ψ_{sd} is surjective,

$$(H,\vartheta_1)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I}\psi_{sd}^{-1}(\Omega_\iota,\vartheta_2)].$$

Given ψ_{sd} is SS-sd-continuous, $\psi_{sd}^{-1}(\Omega_\iota,\vartheta_2)\in SD(C)_{\vartheta_1}$ for each $\iota\in I$. Since (H,ϑ_1) is SS-sd-almost compact, there is a finite subclasses I_o of I such that

$$(H,\vartheta_1)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I_o}cl_{sd}^s[\psi_{sd}^{-1}(\Omega_\iota,\vartheta_2)],$$

and it follows that

$$\psi_{sd}(H,\vartheta_1)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I_o}\psi_{sd}(cl_{sd}^s[\psi_{sd}^{-1}(\Omega_\iota,\vartheta_2)])\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I_o}cl^s(\psi_{sd}[\psi_{sd}^{-1}(\Omega_\iota,\vartheta_2)])=\tilde{\sqcup}_{\iota\in I_o}cl^s(\Omega_\iota,\vartheta_2)\text{ (according to Theorem 3.22)}.$$

Therefore, $\psi_{sd}(H,\vartheta_1)$ is SS-almost compact. Similarly, one can get the proof of SS-almost Lindelöfness.

Definition 3.24. *[50] A soft mapping $\psi_{sd}:(C,\tau,\vartheta_1)\to(D,\sigma,\vartheta_2)$ with ν,ν^* as associated SSTSs with τ,σ, respectively, is said to be SS-sd-open if $\psi_{sd}(G,\vartheta_1)\in SD(D)_{\vartheta_2}$ for $\tilde{\varphi}\neq(G,\vartheta_1)\in\vartheta_1$.*

Proposition 3.25. *[55] A bijective soft map $\psi_{sd}:(C,\tau,\vartheta_1)\to(D,\sigma,\vartheta_2)$ with ν as an associated SSTS with σ is SS-sd-open if and only if it is SS-sd-closed if and only if $cl_{sd}^s[\psi_{sd}(G,\vartheta_1)]\tilde{\sqsubseteq}\psi_{sd}(cl(G,\vartheta_1))$ for each $(G,\vartheta_1)\tilde{\sqsubseteq}\tilde{C}$.*

Theorem 3.26. *The pre-image of each SS-sd-almost Lindelöf (compact) set is SS-almost Lindelöf (compact) under an injective and an SS-sd-open map.*

Proof: Let $\psi_{sd}:(C,\tau,\vartheta_1)\to(D,\sigma,\vartheta_2)$ be an SS-sd-open map with ν,ν^* as associated SSTSs with τ,σ, respectively, and (V,ϑ_2) is SS-sd-Lindelöf. Suppose that $\{(\Omega_\iota,\vartheta_1):\iota\in I\}$ is an SS-open cover for $\psi_{sd}^{-1}(V,\vartheta_2)$. It follows that $\psi_{sd}(\Omega_\iota,\vartheta_1)\in SD(D)_{\vartheta_2}$ for each $\iota\in I$ with

$$(V,\vartheta_2)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I}[\psi_{sd}(\Omega_\iota,\vartheta_1)].$$

Since (V,ϑ_2) is SS-sd-almost Lindelöf, there is a countable subclasses I_o of I such that

$$(V,\vartheta_2)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I_o}cl^s_{sd}[\psi_{sd}(\Omega_\iota,\vartheta_1)],$$

and it follows that

$$\psi^{-1}_{sd}(V,\vartheta_2)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I_o}\psi^{-1}_{sd}(cl^s_{sd}[\psi_{sd}(\Omega_\iota,\vartheta_1)])\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I_o}\psi^{-1}_{sd}(\psi_{sd}[cl^s(\Omega_\iota,\vartheta_1)]) = \tilde{\sqcup}_{\iota\in I_o}cl^s(\Omega_\iota,\vartheta_1) \text{ (according to Proposition 3.25).}$$

Therefore, $\psi^{-1}_{sd}(V,\vartheta_2)$ is SS-almost Lindelöf. Similarly the proof of SS-sd-almost compactness can be obtained.

Theorem 3.27. *Let (C,ν,ϑ) be an SSTS defined on a finite parameter set ϑ and universal set C. If for each $\upsilon\in\vartheta$ we have that the υ-parameter supra topological space is supra-sd-almost compact, then $\tilde{C}$ is SS-sd-almost compact.*

Proof: Assume that $\{(\Omega_\iota,\vartheta):\iota\in I\}$ is an SS-sd-cover for $\tilde{C}$. Then, for each $\upsilon\in\vartheta$ $C=\bigcup_{\iota\in I}\Omega_\iota(\upsilon)$, since (C,ν_υ) is supra-sd-almost compact for each $\upsilon\in\vartheta$,

$$C=\bigcup_{\iota=1}^{n_1}cl^s_{sd}(\Omega_\iota(\upsilon_1))=\bigcup_{\iota=n_1+1}^{n_2}cl^s_{sd}(\Omega_\iota(\upsilon_2))=.......=\bigcup_{\iota=n_{k-1}+1}^{n_k}cl^s_{sd}(\Omega_\iota(\upsilon_k)).$$

Hence, $\tilde{C}=\tilde{\sqcup}_{\iota=1}^{n_k}cl^s_{sd}(\Omega_\iota,\vartheta)$. Thus, $\tilde{C}$ is SS-sd-almost compact.

Corollary 3.28. *Let (C,ν,ϑ) be an SSTS defined on a finite parameter set ϑ and universal set C. If for each $\upsilon\in\vartheta$ we have that each υ-parameter supra topological space is supra-sd-almost Lindelöf, then $\tilde{C}$ is SS-sd-almost Lindelöf.*

Proof: It is similar to the proof of Theorem 3.27.

4. Supra soft-sd-approximately compactness and Lindelöfness

This section is devoted to originating the notion of SS-sd-approximately compact (Lindelöf) spaces and to study their essential characterizations and effects. Moreover, we study the behavior of SS-sd-approximately compact (Lindelöf) spaces under SS-sd-open and SS-sd-continuous maps.

Definition 4.1. *A soft subset (G,ϑ) of an SSTS (C,ν,ϑ) is said to be an SS-sd-approximately compact (Lindelöf) space, if every SS-sd-cover $\{(\Omega_\iota,\vartheta):\iota\in \mathrm{I}\}$ of (G,ϑ) has a finite (countable) subclass I_o of I such that*

$$(G,\vartheta)\tilde{\sqsubseteq}cl^s_{sd}[\tilde{\sqcup}_{\iota\in I_o}(\Omega_\iota,\vartheta)].$$

The space (C,ν,ϑ) is said to be SS-sd-approximately compact (Lindelöf) if $\tilde{C}$ is SS-sd-approximately compact (Lindelöf) as a soft subset.

Theorem 4.2. *Let (C,ν,ϑ) be an SSTS defined on a finite (countable) set of parameters ϑ. If there is an SS-sd-dense subset (A,ϑ) of $\tilde{C}$, then $\tilde{C}$ is SS-sd-approximately Lindelöf (compact).*

Proof: Assume that (A,ϑ) is an SS-sd-dense subset of $\tilde{C}$ and the class $\Gamma=\{(\Omega_\iota,\vartheta):\iota\in I\}$ is an SS-sd-cover for $\tilde{C}$. It follows that, for any soft point $s_\upsilon\tilde{\in}(A,\vartheta)$, there is an SS-sd-neighborhood $(G_{s_\upsilon},\vartheta)\in\Gamma$ for s_υ such that $cl^s_{sd}[\tilde{\sqcup}(G_{s_\upsilon},\vartheta)]=\tilde{C}$ and the class $\{(G_{s_\upsilon},\vartheta)\}$ is finite. Therefore, $\tilde{C}$ is SS-sd-approximately Lindelöf (compact).

Proposition 4.3. *If an SSTS (C, ν, ϑ) is SS-sd-approximately compact, then it is SS-sd-approximately Lindelöf.*

Proof: It is clear from Definition 4.1.

Note 4.4. *The converse of Proposition 4.3 is not satisfied in general as declared in Examples 3.6.*

Proposition 4.5. *A countable (finite) soft union of SS-sd-approximately Lindelöf (compact) subsets of an SSTS (C, ν, ϑ) is SS-sd-approximately Lindelöf (compact).*

Proof: Assume that $\Gamma = \{(\Omega_\iota, \vartheta) : \iota \in I\}$ is an SS-sd-cover for $(X, \Lambda)\tilde{\sqcup}(Y, \vartheta)$ where (X, ϑ) and (Y, ϑ) are SS-sd-approximately Lindelöf. It follows that Γ is an SS-sd-cover for (X, ϑ) and (Y, ϑ). Hence, there are countable subclasses I_1 and I_2 of I such that

$$(X, \vartheta)\tilde{\sqsubseteq}cl^s_{sd}[\tilde{\sqcup}_{n\in I_1}(\Omega_n, \vartheta)] \text{ and } (Y, \vartheta)\tilde{\sqsubseteq}cl^s_{sd}[\tilde{\sqcup}_{n\in I_2}(\Omega_n, \vartheta)].$$

Therefore,

$$(X, \vartheta)\tilde{\sqcup}(Y, \vartheta)\tilde{\sqsubseteq}(cl^s_{sd}[\tilde{\sqcup}_{n\in I_1}(\Omega_n, \vartheta)])\tilde{\sqcup}(cl^s_{sd}[\tilde{\sqcup}_{n\in I_2}(\Omega_n, \vartheta)])\tilde{\sqsubseteq}cl^s_{sd}[(\tilde{\sqcup}_{n\in I_1\cup I_2}(\Omega_n, \vartheta))],$$

where $I_1 \cup I_2$ is countable.

Thus, $(X, \vartheta)\tilde{\sqcup}(Y, \vartheta)$ is SS-sd-approximately Lindelöf. The case of SS-sd-approximate compactness can be obtained in a similar way.

Theorem 4.6. *[49] Let (C, ν, ϑ) be an SSTS and $(C, \vartheta), (D, \vartheta) \in S(U)_\vartheta$. Then,*

$$cl^s_{sd}(C, \vartheta)\tilde{\cup}cl^s_{sd}(D, \vartheta)\tilde{\subseteq}cl^s_{sd}[(C, \vartheta)\tilde{\cup}(D, \vartheta)].$$

Lemma 4.7. *According to Theorem 4.6, we have that every SS-sd-almost compact (Lindelöf) subset of an SSTS (C, ν, ϑ) is SS-sd-approximately compact (Lindelöf). The following example will show that the converse is not true in general.*

Example 4.8. *Consider $r_1, r_2 \in R$ two arbitrary real numbers. Let $\vartheta = \{\upsilon_1, \upsilon_2\}$ and $\nu = \{\tilde{R}, \tilde{\varphi}, (X, \vartheta), (Y, \vartheta), (Z, \vartheta), (W, \vartheta)\}$ be an SSTS on R, where*

$$(X, \vartheta) = \{(\upsilon_1, \{r_1, r_2\}), (\upsilon_2, \{r_1, r_2\});$$

$$(Y, \vartheta) = \{(\upsilon_1, \{r_1, r_2\}), (\upsilon_2, \{r_1\});$$

$$(Z, \vartheta) = \{(\upsilon_1, \{r_1\}), (\upsilon_2, \{r_1\});$$

$$(W, \vartheta) = \{(\upsilon_1, \{r_2\}), (\upsilon_2, \{r_2\}).$$

Consider the SS-sd-cover Γ for $\tilde{R}$ as follows:

$$\Gamma = \{(A, \vartheta) : (A, \vartheta) \text{ which is coutable in that}$$

there is only one parameter in ϑ say υ_1 such that $r_1 \in A(\upsilon_1)$ or $r_2 \in A(\upsilon_1)\}$.

Now, any SS-sd-cover for $\tilde{R}$ will contains (X, ϑ) where (X, ϑ) is supra soft-sd-dense. Hence, $\tilde{R}$ is SS-sd-approximately Lindelöf according to Theorem 4.2.

On the other side, $\tilde{R}$ does not have any countable subcover of Γ in which its SS-sd-closure of its elements covers it. Thus, $\tilde{R}$ is not SS-sd-almost Lindelöf.

Theorem 4.9. *Every SS-sd- hyperconnected SSTS is SS-sd-approximately compact.*

Proof: It is similar to the proof of Theorem 3.9.

Note 4.10. *The converse of Theorem 4.9 is not satisfied in general as illustrated in Example 3.11.*

Definition 4.11. *A class* $\Gamma = \{(\Omega_\iota, \vartheta) : \iota \in \mathrm{I}\}$ *of soft subsets of an SSTS* (C, ν, ϑ) *in which for any finite (countable) subclass* I_o *of* I*, we have that*

$$int^s_{sd}[\tilde{\sqcap}_{\iota \in I_o}(\Omega_\iota, \vartheta)] \neq \tilde{\varphi},$$

is said to satisfy the condition ω_2. It is clear that any class that satisfies the condition ω_2 also satisfies ω_1.

Theorem 4.12. *An SSTS* (C, ν, ϑ) *is SS-sd-approximately Lindelöf (compact) if, and only if, every family of SS-sc-subsets of* $\tilde{C}$ *satisfying the condition* ω_2 *has a non-null soft intersection.*

Proof: We prove the case of SS-sd-approximately Lindelöfness. The other case can be achieved in a similar way.
Necessity: Assume conversely that $\tilde{\sqcap}_{\iota \in I}(\Omega_\iota, \vartheta) = \tilde{\varphi}$ where $\{(\Omega_\iota, \vartheta) : \iota \in I\}$ is a class of SS-sc-sets satisfying the condition ω_2. Then,

$$\tilde{C} = \tilde{\bigcup}_{\iota \in I}(\Omega^{\tilde{c}}_\iota, \vartheta).$$

Given $\tilde{C}$ is SS-sd-approximately Lindelöf, there is a countable subclass I_o of I such that

$$\tilde{C} = cl^s_{sd}[\tilde{\sqcup}_{\iota \in I_o}(\Omega^{\tilde{c}}_\iota, \vartheta)].$$

Hence,

$$\tilde{\varphi} = [cl^s_{sd}(\tilde{\sqcup}_{\iota \in I_o}(\Omega^{\tilde{c}}_\iota, \vartheta))]^{\tilde{c}} = int^s_{sd}[\tilde{\sqcap}_{\iota \in I_o}(\Omega_\iota, \vartheta)] \text{ (given Theorem 2.16)},$$

which opposes the condition ω_2.
Sufficiency: Assume to the contrary that $\tilde{C}$ is not SS-sd-approximately Lindelöf and $\Gamma = \{(\Omega_\iota, \vartheta) : \iota \in I\}$ is an SS-sd-cover for $\tilde{C}$. Then, for each countable subclass I_o of I, we have that

$$cl^s_{sd}[\tilde{\sqcup}_{\iota \in I_o}(\Omega_\iota, \vartheta)] \neq \tilde{C}; \text{ then, } int^s_{sd}\tilde{\sqcap}_{\iota \in I_o}(\Omega^{\tilde{c}}_\iota, \vartheta) \neq \tilde{\varphi}.$$

Therefore, the class $\{(\Omega^{\tilde{c}}_\iota, \vartheta) : \iota \in I\}$ contains SS-sc-subsets of $\tilde{C}$ that satisfy the condition ω_2. According to our hypothesis, $\tilde{\sqcap}_{\iota \in I}(\Omega^{\tilde{c}}_\iota, \vartheta) \neq \tilde{\varphi}$; hence, $\tilde{\sqcup}_{\iota \in I}(\Omega_\iota, \vartheta) \neq \tilde{C}$, which contradicts that Γ is an SS-sd-cover for $\tilde{C}$. Thus, $\tilde{C}$ is SS-sd-approximately Lindelöf.

The proof of the following two theorems are easy to obtained by a similar technique to the proof of Theorems 3.23 and 3.26; so they will be omitted.

Theorem 4.13. *The image of each SS-sd-approximately compact (Lindelöf) set is SS-sd-approximately compact (Lindelöf) under a surjective and an SS-sd-continuous map.*

Theorem 4.14. *The pre-image of each SS-sd-approximately Lindelöf (compact) set is SS-approximately Lindelöf (compact) under an injective and an SS-sd-open map.*

5. More types of compactness and Lindelöfness based on supra soft scd-sets and relationships

Herein, we concentrate on the approaches of SS-sd-mildly compact (Lindelöf) spaces. The relationships with SS-sd-hyperconnected SSTSs are introduced. Moreover, we show that the six types of generalized compact (Lindelöf) spaces given in this paper are equivalent to the case of an SS-sd-hyperconnected SSTS. Furthermore, we originate the concept of SS-sd-partition SSTSs and use it to prove the equivalence of three types of them. Finally, we provide a diagram to summarize the relationships among the aforementioned notions, supported by several concrete examples in Figure 2.

Definition 5.1. *A soft subset (H, ϑ) of an SSTS (C, ν, ϑ) is said to be an SS-scd-set if it is both an SS-sd-set and SS-sc-set.*

Definition 5.2. *A class $\Gamma = \{(\Omega_\iota, \vartheta) : \iota \in \mathrm{I}\}$ of SS-scd-subsets of an SSTS (C, ν, ϑ) is said to be an SS-scd-cover of soft subset (G, ϑ) of $\tilde{C}$, if $(G, \vartheta)\tilde{\sqsubseteq}\Gamma$.*

A soft subset (G, ϑ) is said to be SS-sd-mildly compact (Lindelöf), if every SS-scd-cover $\{(\Omega_\iota, \vartheta) : \iota \in I\}$ of (G, ϑ) has a finite (countable) subclass I_o of I such that

$$(G, \vartheta)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota\in I_o}(\Omega_\iota, \vartheta).$$

The space (C, ν, ϑ) is said to be SS-sd-mildly compact (Lindelöf) if $\tilde{C}$ is SS-sd-mildly compact (Lindelöf) as a soft subset.

Proposition 5.3. *Every SS-sd-mildly compact space is SS-mildly Lindelöf.*

Proof: It is immediate from Definition 5.2.

Note 5.4. *Referring to Examples 3.6, it can be confirmed that the converse of Proposition 5.3 is not true in general.*

Theorem 5.5. *[55]*

(1) An SSTS (C, ν, ϑ) is SS-sd-connected if and only if there is no proper SS-scd-subset of $\tilde{C}$.

(2) Every SS-sd-hyperconnected SSTS is SS-sd-connected.

Proposition 5.6. *Every SS-sd-connected space is SS-sd-mildly compact.*

Proof: It follows from Theorem 5.5.

Remark 5.7. *The following example shall show that the converse of Proposition 5.6 is not necessarily satisfied in general.*

Example 5.8. *Assume that $C = \{7, 8, 9\}$. Let $\vartheta = \{\upsilon_1, \upsilon_2\}$ be the set of parameters. Let (T_j, ϑ), $j = 1, 2, 3$ be soft sets over the universe C, where*

$$T_1(\upsilon_1) = \{7\}, \quad T_1(\upsilon_2) = \{8\};$$
$$T_2(\upsilon_1) = \{8, 9\}, \quad T_2(\upsilon_2) = \{7, 9\};$$
$$T_3(\upsilon_1) = \{8, 9\}, \quad T_3(\upsilon_2) = C.$$

Then, $\nu = \{\tilde{C}, \tilde{\varphi}, (T_j, \vartheta), j = 1, 2, 3\}$ defines an SSTS on U. It is clear that $\tilde{C}$ is SS-sd-mildly compact. On the other side, we have that $(T_1, \vartheta), (T_2, \vartheta) \in SD(C)_\vartheta$ in which $\tilde{C} = (T_1, \vartheta)\tilde{\cup}(T_2, \vartheta)$ and $(T_1, \vartheta), (T_2, \vartheta)$ are disjoint. Thus, $\tilde{C}$ is SS-sd-disconnected.

Proposition 5.9. *A countable (finite) soft union of SS-sd-mildly Lindelöf (compact) subsets of an SSTS* (C, ν, ϑ) *is SS-sd-mildly Lindelöf (compact).*

Proof: It is similar to the proof of Proposition 4.5.

Proposition 5.10. *[49] For a soft subset* (T, ϑ) *of an SSTS* (C, ν, ϑ)*, we have that* $cl^s_{sd}(T, \vartheta) = (T, \vartheta) \Leftrightarrow$ (T, ϑ) *is a proper SS-sc-set.*

Proposition 5.11. *Every SS-sd-almost compact (Lindelöf) subset of an SSTS* (C, ν, ϑ) *is SS-sd-mildly compact (Lindelöf).*

Proof: It follows from Proposition 5.10.

Corollary 5.12. *If* (C, ν, ϑ) *is SS-sd-hyperconnected, then there are no proper SS-scd-sets.*

Proof: It is obvious from Theorem 3.8.

Theorem 5.13. *For an SS-sd-hyperconnected SSTS* (C, ν, ϑ)*, the following are equivalent:*

(1) $\tilde{C}$ *is SS-sd-mildly compact.*

(2) $\tilde{C}$ *is SS-sd-mildly Lindelöf.*

(3) $\tilde{C}$ *is SS-sd-almost compact.*

(4) $\tilde{C}$ *is SS-sd-almost Lindelöf.*

(5) $\tilde{C}$ *is SS-sd-approximately compact.*

(6) $\tilde{C}$ *is SS-sd-approximately Lindelöf.*

Proof: It follows from Theorem 3.8 and Corollary 5.12.

Theorem 5.14. *An SSTS* (C, ν, ϑ) *is SS-sd-mildly compact (Lindelöf) if, and only if, every family of SS-scd-subsets of* $\tilde{C}$ *with the SFIP has a non-empty intersection.*

Proof: **Necessity:** Suppose that $\{(\Omega_\iota, \vartheta) : \iota \in I\}$ is a collection of SS-scd-sets and assume to the contrary that $\tilde{\sqcap}_{\iota \in I}(\Omega_\iota, \vartheta) = \tilde{\varphi}$. It follows that

$$\tilde{C} = \tilde{\sqcup}_{\iota \in I}(\Omega^{\tilde{c}}_\iota, \vartheta).$$

Since $\tilde{C}$ is SS-sd-mildly compact, there is a finite subclass I_o of I such that

$$\tilde{C} = \tilde{\sqcup}_{\iota \in I_o}(\Omega^{\tilde{c}}_\iota, \vartheta) \text{ which contradicts the SFIP.}$$

Sufficiency: Suppose that $\{\Gamma = (\Omega_\iota, \vartheta) : \iota \in I\}$ is an SS-scd-cover for $\tilde{C}$ and assume conversely that $\tilde{C}$ is not SS-sd-mildly compact. It follows that, for every finite subclass I_o of I, we have

$$\tilde{\sqcup}_{\iota \in I_o}(\Omega_\iota, \vartheta) \neq \tilde{C} \text{ and so } \tilde{\sqcap}_{\iota \in I_o}(\Omega^{\tilde{c}}_\iota, \vartheta) \neq \tilde{\varphi}.$$

Hence, $\{(\Omega^{\tilde{c}}_\iota, \vartheta) : \iota \in I\}$ is a class of SS-scd-subsets of $\tilde{C}$ that has the SFIP. By assumption, $\tilde{\sqcap}_{\iota \in I}(\Omega^{\tilde{c}}_\iota, \vartheta) \neq \tilde{\varphi}$ and so $\tilde{\sqcup}_{\iota \in I}(\Omega_\iota, \vartheta) \neq \tilde{C}$, which contradicts that Γ is an SS-scd-cover for $\tilde{C}$. Thus, $\tilde{C}$ is SS-sd-mildly compact.

The case of SS-sd-mildly Lindelöfness can be obtained in a similar way.

The proof of the following propositions can be obtained by a similar technique to the one used in Section 3, so it is omitted.

Proposition 5.15. *Every SS-scd-subset of an SS-sd-mildly Lindelöf (compact) SSTS (C, ν, ϑ) is SS-sd-mildly Lindelöf (compact).*

Proposition 5.16. *If (L, ϑ) and (N, ϑ) are SS-sd-mildly Lindelöf (compact) and SS-scd-subsets of an SSTS (C, ν, ϑ), respectively, then $(L, \vartheta)\tilde{\sqcap}(N, \vartheta)$ is SS-sd-mildly Lindelöf (compact).*

Proposition 5.17. *The image of each SS-sd-mildly compact (Lindelöf) set is SS-mildly compact (Lindelöf) under a surjective and an SS-sd-continuous map.*

Proposition 5.18. *The pre-image of each SS-sd-mildly Lindelöf (compact) set is SS-mildly Lindelöf (compact) under an injective and an SS-sd-open map.*

Definition 5.19. *If a soft subset (T, ϑ) of an SSTS (C, ν, ϑ) is an SS-sd-set if and only if it is an SS-sc-set, then $\tilde{C}$ is called an SS-sd-partition space.*

Theorem 5.20. *For an SS-sd-partition SSTS (C, ν, ϑ), the following are equivalent:*

(1) $\tilde{C}$ is SS-sd-mildly compact (Lindelöf).

(2) $\tilde{C}$ is SS-sd-almost compact (Lindelöf).

(3) $\tilde{C}$ is SS-sd-approximately compact (Lindelöf).

Proof:

(1) ⇒ (2) Assume that $\Gamma = (\Omega_\iota, \vartheta) : \iota \in I\}$ is an SS-sd-cover for $\tilde{C}$. Since $\tilde{C}$ is an SS-sd-partition space, Γ is an SS-scd-cover for $\tilde{C}$. Given $\tilde{C}$ is SS-sd-mildly compact, there is a finite subclass I_o of I such that

$$\tilde{C} = \tilde{\sqcup}_{\iota \in I_o}(\Omega_\iota, \vartheta)\tilde{\sqsubseteq}\tilde{\sqcup}_{\iota \in I_o} cl^s_{sd}(\Omega_\iota, \vartheta).$$

Thus, $\tilde{C}$ is SS-sd-almost compact.

(2) ⇒ (3) It is obvious from Lemma 4.7.

(3) ⇒ (1) Assume that $\Gamma = (\Omega_\iota, \vartheta) : \iota \in I\}$ is an scd-cover for $\tilde{C}$. Then, Γ is an SS-sd-cover for $\tilde{C}$. Given $\tilde{C}$ is SS-sd-approximately compact, there is a finite subclass I_o of I such that

$$\tilde{C} = cl^s_{sd}[\tilde{\sqcup}_{\iota \in I_o}(\Omega_\iota, \vartheta)] = \tilde{\sqcup}_{\iota \in I_o}(\Omega_\iota, \vartheta).$$

Thus, $\tilde{C}$ is SS-sd-mildly compact.

Corollary 5.21. Let (C, ν, ϑ) be an SSTS, and then the following implications hold from Theorems 3.2, 3.5, 3.9, 5.5, and Lemma 4.7, and Proposition 5.6, which are not reversible.

SS-sd-connected ← SS-sd-hyperconnected
↓ ↓
SS-sd-mildly compact ⟵ SS-sd-almost compact → SS-sd-approximately compact
↗ ↘
SS-sd-compact → SS-compact → SS-almost compact
↓ ↓ ↓
SS-sd-Lindelöf → SS-Lindelöf → SS-almost Lindelöf
↓ ↗
SS-sd-almost Lindelöf → SS-sd-approximately Lindelöf

Figure 2. The relationships between diverse types of connectedness and compactness (Lindelöfness) based on SS-sd-sets.

6. Conclusions and upcoming work

In this manuscript, we originate six types of generalized compactness and Lindelofness in the frame of SSTSs, named SS-sd-almost compact (Lindelöf) spaces, SS-sd-approximately compact (Lindelöf) spaces, and SS-sd-mildly compact (Lindelöf) spaces. We discuss the essential properties of each type of these generalized compactness and Lindelofness spaces and study the relationships among them with the support of concrete counterexamples. Also, based on a special type of the SFIP, named the condition ω_1 and the condition ω_2, we investigate more interesting properties of these notions. Moreover, the pre-image (image) of each kind of approach under specific types of soft maps is studied. Furthermore, the equivalence among them is proved under the SS-sd-partition condition. Finally, with the confirmations of concrete counterexamples a diagram summarizing the relationships among the aforementioned approaches is provided.

Several kinds of generalized compactness and Lindelofness spaces which are provided herein will help us to classify specific soft structures into new categories, which are enough to model some real-life problems like those given in [4, 56], the application to soft ideal rough topological spaces in Diabetes mellitus [34], control of linear multi-agent systems under intermittent communication [57], control of multi-agent systems with switching networks and incomplete leader measurement [58], improving accuracy measures of rough sets [59], and applications in decision-making [5]. So, our future work will be in this direction. Also, the generalizations of the aforementioned notions to fuzzy supra soft topological spaces [60] will be considered. These extensions would enhance the relevance and applicability of our proposed approaches, bridging the gap between theoretical exploration and practical implementation.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares he has no conflict of interest regarding the publication of this paper.

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