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Research article

Evaluations of some Euler-type series via powers of the arcsin function

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Abstract: In this paper, by using special integrals and integer powers of the arcsin function, we obtained the recurrences or explicit expressions of some parametric Euler-type series involving multiple harmonic sums and multiple t-harmonic sums. According to our results, these Euler-type series are all expressible in terms of π, $\ln(2)$, and zeta values. In particular, by specifying the parameters, we presented as examples the evaluations of some special series, including some known ones in the literature and some new ones.

Keywords: harmonic numbers; infinite series identities; Euler-type series; inverse sine expansions
Mathematics Subject Classification: 05A19, 40A25

1. Introduction

Generalized harmonic numbers and *odd harmonic numbers* are defined by

$$H_n^{(s)} := \sum_{k=1}^{n} \frac{1}{k^s} \quad \text{and} \quad O_n^{(s)} := \sum_{k=1}^{n} \frac{1}{(2k-1)^s},$$

for $n \in \mathbb{N}_0$ and $s \in \mathbb{N}$. *Multiple harmonic sums* (MHSs) $\zeta_n(\boldsymbol{s})$ and *multiple t-harmonic sums* (MtSs) $t_n(\boldsymbol{s})$ are defined by

$$\zeta_n(\boldsymbol{s}) := \sum_{n \geq n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} \quad \text{and} \quad t_n(\boldsymbol{s}) := \sum_{n \geq n_1 > \cdots > n_k \geq 1} \prod_{j=1}^{k} \frac{1}{(2n_j - 1)^{s_j}}, \tag{1.1}$$

for $n \in \mathbb{N}_0$ and $\boldsymbol{s} = (s_1, \ldots, s_k) \in \mathbb{N}^k$. By convention, $\zeta_n(\emptyset) = t_n(\emptyset) := 1$ for $n \geq 0$, and $\zeta_n(\boldsymbol{s}) = t_n(\boldsymbol{s}) := 0$ for $n < k$. When $k = 1$, the MHSs and MtSs reduce respectively, to harmonic numbers and odd harmonic numbers, i.e., $\zeta_n(s) = H_n^{(s)}$ and $t_n(s) = O_n^{(s)}$. Moreover, for $s_1 > 1$, define the *multiple zeta values* (MZVs) and *multiple t-values* (MtVs) by

$$\zeta(\boldsymbol{s}) := \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} \quad \text{and} \quad t(\boldsymbol{s}) := \sum_{n_1 > \cdots > n_k \geq 1} \prod_{j=1}^{k} \frac{1}{(2n_j - 1)^{s_j}}, \tag{1.2}$$

which can be viewed as the limits of MHSs and MtSs, respectively. The concept of MZVs was introduced by Hoffman [1] and Zagier [2] in the early 1990s, and the concept of MtVs was formally introduced by Hoffman [3] in 2019. Similarly to MZVs and MtVs, let $\boldsymbol{s} = (s_1, \dots, s_k) \in \mathbb{N}^k$ and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_k) \in \{\pm 1\}^k$ with $(s_1, \sigma_1) \neq (1, 1)$, and we define the *alternating multiple zeta values* (AMZVs) by

$$\zeta(\boldsymbol{s}; \boldsymbol{\sigma}) := \sum_{n_1 > \cdots > n_k \geq 1} \frac{\sigma_1^{n_1} \cdots \sigma_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}.$$

Summation formulas on the (alternating) multiple zeta values have attracted extensive attention, and the readers are referred to the papers due to, e.g., Alegri et al. [4], Hoffman [1,5], and Shen and Cai [6].

The linear Euler sums $S_{p,q} := \sum_{n=1}^{\infty} H_n^{(p)}/n^q$ were first considered by Euler in correspondence with Goldbach in 1742 (see [7, p. 253]). Euler discovered that the linear sums $S_{p,q}$ can be evaluated in terms of zeta values when $p = 1$, $p = q$, $p + q$ is odd, and $(p, q) \in \{(2, 4), (4, 2)\}$. The nonlinear Euler sums were introduced by Flajolet and Salvy [8] in 1998. These sums are infinite series of the form

$$S_{p_1 p_2 \dots p_k, q} := \sum_{n=1}^{\infty} \frac{H_n^{(p_1)} H_n^{(p_2)} \cdots H_n^{(p_k)}}{n^q},$$

where $p_1, p_2, \dots, p_k, q \in \mathbb{N}$, with $p_1 \leq p_2 \leq \cdots \leq p_k$ and $q \geq 2$. Flajolet and Salvy used the contour integral representations and residue computation to establish explicit formulas of several classes of Euler sums. In 2020, by permutations and compositions, Xu and Wang [9] established explicit formulas of Euler sums $S_{p_1 p_2 \dots p_k, q}$ and various alternating Euler sums, and showed that these infinite series are expressible in terms of MZVs or AMZVs, respectively. More classical results on Euler sums can be found in, e.g., the works due to Bailey et al. [10], Borwein et al. [11], Chu [12,13], Wang and Lyu [14], Xu [15], Xu et al. [16], and Zheng [17].

Recently, various Euler-type series have been investigated, including Euler sums involving negative powers of two [18], Euler-type series on odd harmonic numbers [19], and Euler-type series on hyperharmonic numbers [20,21]. Additionally, in 2017, Hoffman [22] established some general series identities by symmetric functions and MZVs. For example, he showed that

$$\sum_{n=1}^{\infty} \frac{\mathcal{P}_k(H_n, H_n^{(2)}, \dots, H_n^{(k)})}{n^2} = \frac{k+3}{2} \zeta(k+2) - \frac{1}{2} \sum_{j=2}^{k} \zeta(j) \zeta(k+2-j),$$

$$\sum_{n=0}^{\infty} \frac{\mathcal{P}_k(H_n, H_n^{(2)}, \dots, H_n^{(k)})}{(n+1)(n+2)\cdots(n+q)} = \frac{1}{(q-1)!(q-1)^{k+1}},$$

for integers $k \geq 0$ and $q \geq 2$. According to the definition of the multivariate polynomials $\mathcal{P}_k$, it can be verified that $\zeta_n(\{1\}_k) = \mathcal{P}_k(H_n, H_n^{(2)}, \dots, H_n^{(k)})$, where $\{\cdots\}_k$ denotes k repetitions of a substring, with $\{\cdots\}_0 = \emptyset$. Therefore, Hoffman's results can also be viewed as Euler-type series identities. Kuba and Panholzer [23], and Ma and Wang [24], further generalized Hoffman's results, and established more infinite series identities on Euler-type series involving $\zeta_n(\{1\}_k)$ and $\zeta_n^\star(\{1\}_k)$, where the star version $\zeta_n^\star(\boldsymbol{s})$ was obtained from the definition of $\zeta_n(\boldsymbol{s})$ by replacing $n > n_1 > \cdots > n_k \geq 1$ by $n \geq n_1 \geq \cdots \geq n_k \geq 1$. More recent works on various Euler-type series can be found in the papers of Campbell and Chen [25], Chen and Wang [26], Liu and Wang [27], and Liu et al. [28]. Note that infinite series involving (odd) harmonic numbers and central binomial coefficients have also attracted extensive attention recently.

For example, Campbell et al. [29] used the theory of colored multiple zeta values and various special integrals to evaluate this kind of infinite series, and Nimbran et al. [30] used the series expansions of integer powers of the arcsin function to determine the values of such series. More recent results of this type can be found in, e.g., the papers of Wang and Xu [31], Wang and Yuan [32], and Wei and Xu [33].

In this paper, we will use the series expansions of $\arcsin^n(x)$ as well as some integrals of trigonometric functions and inverse trigonometric functions to obtain the explicit evaluations of more Euler-type series. For $n = 1, 2, 3$, the series expansions of $\arcsin^n(x)$ were given in Edwards' textbook [34, pp. 77–82], and the expansion of $\arcsin^4(x)$ was recorded in Ramanujan's notebooks [7, Proposition 15]. In the classical book [35, Eq (46)] of Schwatt, a general formula for the series expansion of $\arcsin^n(x)$, for $n \in \mathbb{N}$, was presented. However, two elegant formulas of $\arcsin^n(x)$ were presented by Borwein and Chamberland [36, Eqs (1.1) and (1.3)], which can be reformulated as

$$\frac{\arcsin^{2p+1}(x)}{(2p+1)!} = \sum_{n=p}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{t_n(\{2\}_p)}{(2n+1)} x^{2n+1}, \quad \text{for } p \geq 0, \tag{1.3}$$

$$\frac{\arcsin^{2p}(x)}{(2p)!} = \sum_{n=p}^{\infty} \frac{4^{n-p}}{\binom{2n}{n}} \frac{\zeta_{n-1}(\{2\}_{p-1})}{n^2} x^{2n}, \quad \text{for } p \geq 1. \tag{1.4}$$

By means of the expansions (1.3) and (1.4), the following parametric series involving MHSs $\zeta_n(\{2\}_p)$ and MtSs $t_n(\{2\}_p)$ will be studied in this paper:

$$\sum_{n=p}^{\infty} \frac{\zeta_n(\{2\}_p)}{(n+1)(n+m+1)}, \quad \sum_{n=p}^{\infty} \frac{t_n(\{2\}_p)}{(2n+1)(2n+2m+1)}, \tag{1.5}$$

$$\sum_{n=p}^{\infty} \frac{\zeta_n(\{2\}_p)}{(2n+k)(2n+k+1)}, \quad \sum_{n=p}^{\infty} \frac{t_n(\{2\}_p)}{(2n+k-1)(2n+k)}, \tag{1.6}$$

for $m \in \mathbb{N}$, $p \in \mathbb{N}_0$, and $k = 0, 1, 2$. Note that according to our previous results [37, Lemma 2.1], $\zeta_n(\{r\}_k)$ and $t_n(\{r\}_k)$ can be expressed in terms of harmonic numbers and odd harmonic numbers by the complete Bell polynomials:

$$\zeta_n(\{r\}_k) = \frac{1}{k!} Y_k(H_n^{(r)}, -1!H_n^{(2r)}, \ldots, (-1)^{k-1}(k-1)!H_n^{(kr)}),$$
$$t_n(\{r\}_k) = \frac{1}{k!} Y_k(O_n^{(r)}, -1!O_n^{(2r)}, \ldots, (-1)^{k-1}(k-1)!O_n^{(kr)}).$$

For example,

$$\zeta_n(r) = H_n^{(r)}, \quad \zeta_n(r,r) = \frac{1}{2}\{(H_n^{(r)})^2 - H_n^{(2r)}\},$$
$$t_n(r) = O_n^{(r)}, \quad t_n(r,r) = \frac{1}{2}\{(O_n^{(r)})^2 - O_n^{(2r)}\}.$$

Therefore, the parametric series in (1.5) and (1.6) are indeed Euler-type. We establish in Section 2 the recurrences of the series in (1.5) by the integral $\int_0^{\frac{\pi}{2}} \theta^n \sin^{2m}(\theta) \mathrm{d}\theta$, and derive in Sections 3–5 the explicit expressions of the series in (1.6) by the integrals $\int_0^{\theta} \arcsin^n(x)/x^3 \mathrm{d}x$, $\int_0^{\frac{\pi}{2}} \theta^n / \sin^2(\theta) \mathrm{d}\theta$, and $\int_0^{\theta} t^n \cos(t) \mathrm{d}t$.

The results show that the above general Euler-type series are all expressible in terms of powers of π, ln(2), and zeta values. By specifying the parameters, we present as examples the evaluations of some special Euler-type series, including some known ones appearing in recent papers due to Hoffman [22], Campbell and Chen [25], and Chen and Wang [26].

2. Series related to the integral of $\theta^n \sin^{2m}(\theta)$

In this section, we consider two kinds of infinite series $\mathcal{Z}_m^{(p)}$ and $\mathcal{T}_m^{(p)}$, defined by

$$\mathcal{Z}_m^{(p)} := \sum_{n=p}^{\infty} \frac{\zeta_n(\{2\}_p)}{(n+1)(n+m+1)} \quad \text{and} \quad \mathcal{T}_m^{(p)} := \sum_{n=p}^{\infty} \frac{t_n(\{2\}_p)}{(2n+1)(2n+2m+1)},$$

for integers $m \geq 1$ and $p \geq 0$. These series are closely related to the integral

$$\mathcal{I}(n,m) := \frac{1}{n!} \int_0^{\frac{\pi}{2}} \theta^n \sin^{2m}(\theta) \mathrm{d}\theta, \quad \text{for } n,m \geq 0.$$

According to Orr's result [38, Eq (14)], the following expression for $\mathcal{I}(n,m)$ holds:

$$\begin{aligned}\mathcal{I}(n,m) = &\frac{\pi^{n+1}\binom{2m}{m}}{2^{2m+n+1}(n+1)!} - \sum_{k=1}^{m} \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^j \binom{2m}{m+k} \pi^{n-2j+1}}{2^{2m+n} k^{2j} (n+1-2j)!} \\ &+ \delta_{\lfloor \frac{n+1}{2} \rfloor, \frac{n+1}{2}} \sum_{k=1}^{m} \frac{(-1)^{k+\lfloor \frac{n+1}{2} \rfloor} \binom{2m}{m+k}}{2^{2m+n} k^{n+1}}.\end{aligned}$$

In the next two theorems, we will show that by means of the integral $\mathcal{I}(n,m)$, the recurrences of the series $\mathcal{Z}_m^{(p)}$ and $\mathcal{T}_m^{(p)}$ can be established.

Theorem 2.1. *For integers $m \geq 1$ and $p \geq 0$, the series $\mathcal{Z}_m^{(p)}$ satisfy the following recurrence:*

$$\sum_{k=1}^{m} \frac{a(m,k)}{k} \mathcal{Z}_k^{(p)} = \frac{\binom{2m}{m} \pi^{2p+2}}{2^{2m-1}(2p+3)!} - \frac{4^{p+2}}{\pi} \mathcal{I}(2p+2,m), \tag{2.1}$$

where

$$a(m,k) = -\frac{2k}{4^m} \binom{2m-2k}{m-k} \binom{2k}{k}.$$

Therefore, the series $\mathcal{Z}_m^{(p)}$ is expressible in terms of powers of π.

Proof. Applying a change of variable $x \to \sin(\theta)$ to Eq (1.4) gives

$$\frac{\theta^{2p}}{(2p)!} = \sum_{n=p}^{\infty} \frac{4^{n-p}}{\binom{2n}{n}} \frac{\zeta_{n-1}(\{2\}_{p-1})}{n^2} \sin^{2n}(\theta). \tag{2.2}$$

Multiply (2.2) with $\sin^{2m}(\theta)$, integrate from 0 to $\frac{\pi}{2}$, and use Wallis' formula. Then we obtain

$$\mathcal{I}(2p,m) = \sum_{n=p}^{\infty} \frac{4^{n-p}}{\binom{2n}{n}} \frac{\zeta_{n-1}(\{2\}_{p-1})}{n^2} \int_0^{\frac{\pi}{2}} \sin^{2n+2m}(\theta) \mathrm{d}\theta = \frac{\pi}{2} \sum_{n=p}^{\infty} \frac{\binom{2n+2m}{n+m}}{\binom{2n}{n}} \frac{\zeta_{n-1}(\{2\}_{p-1})}{4^{p+m} n^2}$$

$$= \frac{\pi}{2}\sum_{n=p}^{\infty}\frac{\zeta_{n-1}(\{2\}_{p-1})}{4^p n^2}\prod_{k=1}^{m}\left(1-\frac{1}{2n+2k}\right).$$

Define $f_{2n}(m) = \prod_{k=1}^{m}(1-\frac{1}{2n+2k})$. Based on one of our previous results [37, Lemma 4.1], $f_l(m)$ can be expanded as $f_l(m) = 1 + \sum_{k=1}^{m}\frac{a(m,k)}{l+2k}$, with $a(m,k) = -\frac{2k}{4^m}\binom{2m-2k}{m-k}\binom{2k}{k}$. Substituting this expansion back, and using the definition of MZVs as well as the method of partial fraction decomposition, we have

$$\begin{aligned}\mathcal{I}(2p,m) &= \frac{\pi}{2}\sum_{n=p}^{\infty}\frac{\zeta_{n-1}(\{2\}_{p-1})}{4^p n^2}\left(1+\sum_{k=1}^{m}\frac{a(m,k)}{2n+2k}\right)\\ &= \frac{\pi}{2\cdot 4^p}\sum_{n=p}^{\infty}\frac{\zeta_{n-1}(\{2\}_{p-1})}{n^2}+\frac{\pi}{4^{p+1}}\sum_{n=p}^{\infty}\sum_{k=1}^{m}\frac{a(m,k)}{k}\zeta_{n-1}(\{2\}_{p-1})\left(\frac{1}{n^2}-\frac{1}{n(n+k)}\right)\\ &= \frac{\pi\zeta(\{2\}_p)}{2\cdot 4^p}+\frac{\pi\zeta(\{2\}_p)}{4^{p+1}}\sum_{k=1}^{m}\frac{a(m,k)}{k}-\frac{\pi}{4^{p+1}}\sum_{k=1}^{m}\frac{a(m,k)}{k}\mathcal{Z}_k^{(p-1)}.\end{aligned}$$

The summation in the second term can be computed by Gould [39, Eq (3.90)]:

$$\sum_{k=1}^{m}\frac{a(m,k)}{k} = -\frac{2}{4^m}\sum_{k=1}^{m}\binom{2m-2k}{m-k}\binom{2k}{k} = \frac{2\binom{2m}{m}}{4^m}-2, \tag{2.3}$$

which, together with the fact that $\zeta(\{2\}_p) = \frac{\pi^{2p}}{(2p+1)!}$ (see [40, Proposition 2.3]), gives the desired recurrence. □

Example 2.1. When $m = 1$, Eq (2.1) gives the following explicit expression:

$$\sum_{n=p}^{\infty}\frac{\zeta_n(\{2\}_p)}{(n+1)(n+2)} = \sum_{j=1}^{p+1}(-1)^{j+1}\frac{\pi^{2p+2-2j}}{(2p+3-2j)!}. \tag{2.4}$$

Setting $p = 1, 2$ in the above expression yields

$$\sum_{n=0}^{\infty}\frac{H_n^{(2)}}{(n+1)(n+2)} = \frac{\pi^2}{6}-1, \tag{2.5}$$
$$\sum_{n=0}^{\infty}\frac{\zeta_n(2,2)}{(n+1)(n+2)} = \frac{\pi^4}{120}-\frac{\pi^2}{6}+1.$$

Note that Eq (2.5) was presented by Hoffman [22, Corollary 1]. Similarly, when $m = 2, 3$, setting $p = 1, 2$ gives

$$\sum_{n=0}^{\infty}\frac{H_n^{(2)}}{(n+1)(n+3)} = \frac{\pi^2}{8}-\frac{11}{16}, \tag{2.6}$$
$$\sum_{n=0}^{\infty}\frac{\zeta_n(2,2)}{(n+1)(n+3)} = \frac{\pi^4}{160}-\frac{11\pi^2}{96}+\frac{43}{64},$$

$$\sum_{n=0}^{\infty}\frac{H_n^{(2)}}{(n+1)(n+4)}=\frac{11\pi^2}{108}-\frac{341}{648},\tag{2.7}$$

$$\sum_{n=0}^{\infty}\frac{\zeta_n(2,2)}{(n+1)(n+4)}=\frac{11\pi^4}{2160}-\frac{341\pi^2}{3888}+\frac{11813}{23328},$$

where Eqs (2.6) and (2.7) can be found in Sofo [41, Lemma 1.1].

In the next theorem, we derive the recurrence of the series $\mathcal{T}_m^{(p)}$.

Theorem 2.2. *For integers $m \geq 1$ and $p \geq 0$, the series $\mathcal{T}_m^{(p)}$ satisfies the following recurrence:*

$$\sum_{k=1}^{m}\frac{a(m,k)}{k}\mathcal{T}_k^{(p)}=\frac{\binom{2m}{m}(\frac{\pi}{2})^{2p+2}}{2^{2m-1}(2p+2)!}-2\mathcal{I}(2p+1,m),\tag{2.8}$$

where $a(m,k)=-\frac{2k}{4^m}\binom{2m-2k}{m-k}\binom{2k}{k}$. Therefore, the series $\mathcal{T}_m^{(p)}$ is expressible in terms of powers of π.

Proof. Applying a change of variable $x \to \sin(\theta)$ to Eq (1.3), the series expansion turns into

$$\frac{\theta^{2p+1}}{(2p+1)!}=\sum_{n=p}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{t_n(\{2\}_p)}{(2n+1)}\sin^{2n+1}(\theta).\tag{2.9}$$

Multiply (2.9) with $\sin^{2m}(\theta)$, integrate from 0 to $\frac{\pi}{2}$, and use Wallis' formula as well as the expansion of $f_l(m)$. Then we obtain

$$\begin{aligned}\mathcal{I}(2p+1,m)&=\sum_{n=p}^{\infty}\frac{\binom{2n}{n}t_n(\{2\}_p)}{4^n(2n+1)}\int_0^{\frac{\pi}{2}}\sin^{2n+2m+1}(\theta)\mathrm{d}\theta\\&=4^m\sum_{n=p}^{\infty}\frac{\binom{2n}{n}t_n(\{2\}_p)}{\binom{2n+2m}{n+m}(2n+1)(2n+2m+1)}=\sum_{n=p}^{\infty}\frac{t_n(\{2\}_p)}{(2n+1)^2}f_{2n+1}(m)\\&=\sum_{n=p}^{\infty}\frac{t_n(\{2\}_p)}{(2n+1)^2}+\sum_{k=1}^{m}a(m,k)\sum_{n=p}^{\infty}\frac{t_n(\{2\}_p)}{(2n+1)^2(2n+2k+1)}.\end{aligned}\tag{2.10}$$

By partial fraction decomposition and the definition of MtVs, we have

$$\sum_{n=p}^{\infty}\frac{t_n(\{2\})_p}{(2n+1)^2}=t(\{2\}_{p+1}),$$

$$\sum_{n=p}^{\infty}\frac{t_n(\{2\}_p)}{(2n+1)^2(2n+2k+1)}=\frac{1}{2k}(t(\{2\}_{p+1})-\mathcal{T}_k^{(p)}).$$

Setting $m=0$ in (2.10) gives $t(\{2\}_p)=\frac{1}{(2p)!}(\frac{\pi}{2})^{2p}$ (see also [42, Lemma 1]). Combining the above results with (2.3) yields Eq (2.8). □

Example 2.2. When $m = 1$, Eq (2.8) can be reformulated as

$$\sum_{n=0}^{\infty}\frac{t_n(\{2\}_p)}{(2n+1)(2n+3)}=\sum_{j=1}^{p}(-1)^j\frac{\pi^{2p+2-2j}}{4^{p+1}(2p+2-2j)!}+\frac{(-1)^p}{4^{p+1}}.$$

In particular, setting $p = 1, 2$ yields

$$\sum_{n=0}^{\infty}\frac{O_n^{(2)}}{(2n+1)(2n+3)}=\frac{\pi^2}{32}-\frac{1}{8}, \tag{2.11}$$

$$\sum_{n=0}^{\infty}\frac{t_n(2,2)}{(2n+1)(2n+3)}=\frac{\pi^4}{1536}-\frac{\pi^2}{128}+\frac{1}{32}.$$

Equation (2.11) was presented by Chen and Wang [26, Example 4.6]. Similarly, setting $m = 2$, $p = 1, 2$ and $m = 3$, $p = 1, 2$ in Eq (2.8) gives

$$\sum_{n=0}^{\infty}\frac{O_n^{(2)}}{(2n+1)(2n+5)}=\frac{3\pi^2}{128}-\frac{1}{12},$$

$$\sum_{n=0}^{\infty}\frac{t_n(2,2)}{(2n+1)(2n+5)}=\frac{\pi^4}{2048}-\frac{11\pi^2}{2048}+\frac{1}{48},$$

$$\sum_{n=0}^{\infty}\frac{O_n^{(2)}}{(2n+1)(2n+7)}=\frac{11\pi^2}{576}-\frac{203}{3240},$$

$$\sum_{n=0}^{\infty}\frac{t_n(2,2)}{(2n+1)(2n+7)}=\frac{11\pi^4}{27648}-\frac{341\pi^2}{82944}+\frac{1823}{116640}.$$

3. Series related to the integral of $\arcsin^n(x)/x^3$

Based on Eqs (1.3) and (1.4) as well as the integral $\int_0^{\sin(\theta)}\frac{\arcsin^n(x)}{x^3}\mathrm{d}x$, we can establish the expressions of another two kinds of infinite series.

Theorem 3.1. *For an integer $p \geq 0$, the following series are expressible in terms of powers of π, $\ln(2)$, and zeta values:*

$$\sum_{n=p}^{\infty}\frac{\zeta_n(\{2\}_p)}{2n(2n+1)}=\sum_{k=1}^{p}(-1)^k\frac{(2k+1)(4^{-k}-1)\pi^{2p}\zeta(2k+1)}{(2p-2k)!}+(-1)^p\zeta(2p+1)-\frac{\pi\ln(2)}{(2p+1)!} \tag{3.1}$$

and

$$\sum_{n=p}^{\infty}\frac{t_n(\{2\}_p)}{2n(2n-1)}=\sum_{k=1}^{p}(-1)^k\frac{(2k+1)(1-4^k)(\frac{\pi}{2})^{2p-2k}\zeta(2k+1)}{(2p-2k)!16^k}+(-1)^{p-1}\frac{(2p+1)\zeta(2p+1)}{4^p}-\frac{(\frac{\pi}{2})^{2p}\ln(2)}{(2p)!}. \tag{3.2}$$

Proof. Consider the integral $\frac{1}{(2p)!}\int_0^{\sin(\theta)}\frac{\arcsin^{2p}(x)}{x^3}\mathrm{d}x$. On the one hand, by integration by parts twice, we have

$$\begin{aligned}&\frac{1}{(2p)!}\int_0^{\sin(\theta)}\frac{\arcsin^{2p}(x)}{x^3}\mathrm{d}x\\&=-\frac{\theta^{2p}}{2(2p)!\sin^2(\theta)}-\frac{\theta^{2p-1}\cot(\theta)}{2(2p-1)!}+\frac{1}{2(2p-2)!}\int_0^{\sin(\theta)}\frac{\arcsin^{2p-2}(x)}{x}\mathrm{d}x.\end{aligned}$$

On the other hand, expanding $\arcsin^{2p}(x)$ by (1.4), using partial fraction decomposition, and applying Eq (2.2), the integral on the left can be rewritten as

$$\begin{aligned}\frac{1}{(2p)!}\int_0^{\sin(\theta)}\frac{\arcsin^{2p}(x)}{x^3}\mathrm{d}x&=\sum_{n=p}^{\infty}\frac{4^{n-p}}{\binom{2n}{n}}\frac{\zeta_{n-1}(\{2\}_{p-1})\sin^{2n-2}(\theta)}{n^2(2n-2)}\\&=\sum_{n=p}^{\infty}\frac{4^{n-p}}{\binom{2n}{n}}\left(\frac{\zeta_{n-1}(\{2\}_{p-1})\sin^{2n-2}(\theta)}{n(2n-2)}-\frac{\zeta_{n-1}(\{2\}_{p-1})\sin^{2n-2}(\theta)}{2n^2}\right)\\&=\sum_{n=p}^{\infty}\frac{4^{n-p}}{\binom{2n}{n}}\frac{\zeta_{n-1}(\{2\}_{p-1})\sin^{2n-2}(\theta)}{n(2n-2)}-\frac{\theta^{2p}}{2(2p)!\sin^2(\theta)}.\end{aligned}$$

Combining the above two expressions gives

$$\begin{aligned}&\sum_{n=p}^{\infty}\frac{4^{n-p}}{\binom{2n}{n}}\frac{\zeta_{n-1}(\{2\}_{p-1})\sin^{2n-2}(\theta)}{n(2n-2)}\\&=\frac{1}{2(2p-2)!}\int_0^{\sin(\theta)}\frac{\arcsin^{2p-2}(x)}{x}\mathrm{d}x-\frac{\theta^{2p-1}\cot(\theta)}{2(2p-1)!}.\end{aligned}\tag{3.3}$$

Now, integrate both sides of Eq (3.3) from 0 to $\frac{\pi}{2}$. Then by Wallis' formula, the left-hand side of (3.3) turns into

$$\sum_{n=p}^{\infty}\frac{4^{n-p}}{\binom{2n}{n}}\frac{\zeta_{n-1}(\{2\}_{p-1})}{n(2n-2)}\int_0^{\frac{\pi}{2}}\sin^{2n-2}(\theta)\mathrm{d}\theta=\frac{\pi}{4^p}\sum_{n=p}^{\infty}\frac{\zeta_{n-1}(\{2\}_{p-1})}{(2n-2)(2n-1)}.$$

The integral of the first term on the right-hand side of (3.3) can be computed by the following formula (see [37, Eq (5.2)]):

$$\begin{aligned}\zeta(3,\{2\}_p)&=\frac{4^{p+2}}{(2p+2)!\pi}\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{\sin(\theta)}\frac{\arcsin^{2p+2}(x)}{x}\mathrm{d}x\\&=\sum_{n=1}^{p+1}(-1)^n\frac{\pi^{2p+2-2n}(1-2^{2n})}{(2p+3-2n)!2^{2n-2}}n\zeta(2n+1)+(-1)^{p+1}2\zeta(2p+3),\end{aligned}$$

and the integral of the second term can be evaluated by Orr's result [43, Eq (2.2)]:

$$\int_0^{\frac{\pi}{2}}\theta^n\cot(\theta)\mathrm{d}\theta=\left(\frac{\pi}{2}\right)^n\ln(2)+\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\frac{n!(1-4^{-k})\pi^{n-2k}\zeta(2k+1)}{(n-2k)!2^n}+\delta_{\lfloor\frac{n}{2}\rfloor,\frac{n}{2}}(-1)^{\frac{n}{2}}\frac{n!\zeta(n+1)}{2^n}.\tag{3.4}$$

Substituting the above formulas into (3.3) gives (3.1).

Similarly, expanding the integral $\frac{1}{(2p+1)!}\int_0^{\sin(\theta)}\frac{\arcsin^{2p+1}(x)}{x^3}\mathrm{d}x$ in two different ways yields

$$\sum_{n=p}^{\infty}\frac{\binom{2n}{n}}{4^n}\frac{t_n(\{2\}_p)\sin^{2n-1}(\theta)}{2n-1}=\frac{1}{(2p-1)!}\int_0^{\sin(\theta)}\frac{\arcsin(x)^{2p-1}}{x}\mathrm{d}x-\frac{\theta^{2p}\cot(\theta)}{(2p)!} \tag{3.5}$$

(see also [37, Theorem 3.7]). Integrate both sides of (3.5) from 0 to $\frac{\pi}{2}$. Using Wallis' formula for the left, and the integral expression of $t(3,\{2\}_p)$ [37, Eq (5.1)] as well as the integral (3.4) for the right, we obtain Eq (3.2). □

Example 3.1. Setting $p=1,2,3$ in Eq (3.1) gives

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{2n(2n+1)}=-\frac{\pi^2\ln(2)}{6}+\frac{5\zeta(3)}{4},$$

$$\sum_{n=1}^{\infty}\frac{\zeta_n(2,2)}{2n(2n+1)}=-\frac{\pi^4\ln(2)}{120}+\frac{3\pi^2\zeta(3)}{8}-\frac{59\zeta(5)}{16},$$

$$\sum_{n=1}^{\infty}\frac{\zeta_n(2,2,2)}{2n(2n+1)}=-\frac{\pi^6\ln(2)}{5040}+\frac{3\pi^4\zeta(3)}{160}-\frac{25\pi^2\zeta(5)}{32}+\frac{377\zeta(7)}{64}.$$

Setting $p=1,2,3$ in Eq (3.2) yields

$$\sum_{n=1}^{\infty}\frac{O_n^{(2)}}{2n(2n-1)}=-\frac{\pi^2\ln(2)}{8}+\frac{21\zeta(3)}{16},$$

$$\sum_{n=1}^{\infty}\frac{t_n(2,2)}{2n(2n-1)}=-\frac{\pi^4\ln(2)}{384}+\frac{9\pi^2\zeta(3)}{128}-\frac{155\zeta(5)}{256},$$

$$\sum_{n=1}^{\infty}\frac{t_n(2,2,2)}{2n(2n-1)}=-\frac{\pi^6\ln(2)}{46080}+\frac{3\pi^4\zeta(3)}{2048}-\frac{75\pi^2\zeta(5)}{2048}+\frac{889\zeta(7)}{4096}.$$

4. Series related to the integral of $\theta^n/\sin^2(\theta)$

In this section, we establish the expressions of two Euler-type series by the integral of $\theta^n/\sin^2(\theta)$.

Theorem 4.1. *For an integer $p\geq 0$, the following series are expressible in terms of powers of π, $\ln(2)$, and zeta values:*

$$\sum_{n=p}^{\infty}\frac{\zeta_n(\{2\}_p)}{(2n+1)(2n+2)}=\frac{\pi^{2p}\ln(2)}{(2p+1)!}+\sum_{k=1}^{p}(-1)^k\frac{(1-4^{-k})\pi^{2p-2k}\zeta(2k+1)}{(2p+1-2k)!} \tag{4.1}$$

and

$$\sum_{n=p}^{\infty}\frac{t_n(\{2\}_p)}{2n(2n+1)}=\sum_{k=1}^{p}(-1)^k\frac{(4^k-1)\left(\frac{\pi}{2}\right)^{2p-2k}\zeta(2k+1)}{(2p-2k)!16^k}+\frac{\left(\frac{\pi}{2}\right)^{2p}\ln(2)}{(2p)!}+(-1)^p\frac{\zeta(2p+1)}{4^p}. \tag{4.2}$$

Proof. Consider the integral $\int_0^{\frac{\pi}{2}} \frac{\theta^n}{\sin^2(\theta)} \mathrm{d}\theta$. On the one hand, by integration by parts, we have

$$\int_0^{\frac{\pi}{2}} \frac{\theta^n}{\sin^2(\theta)} \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \theta^n \mathrm{d}(-\cot(\theta)) = n \int_0^{\frac{\pi}{2}} \theta^{n-1} \cot(\theta) \mathrm{d}\theta, \tag{4.3}$$

which can be computed by Eq (3.4). On the other hand, expanding θ^{2p} by Eq (2.2) and using Wallis' formula, we obtain

$$\int_0^{\frac{\pi}{2}} \frac{\theta^{2p}}{\sin^2(\theta)} \mathrm{d}\theta = \frac{\pi}{2} \cdot \frac{(2p)!}{4^{p-1}} \sum_{n=p}^{\infty} \frac{\zeta_{n-1}(\{2\}_{p-1})}{2n(2n-1)},$$

which, together with (4.3), gives Eq (4.1). Similarly, expanding θ^{2p+1} by Eq (2.9), using Wallis' formula, and applying Eq (4.3), we obtain Eq (4.2). □

Example 4.1. Setting $p = 1, 2, 3$ in Eq (4.1) gives

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{H_n^{(2)}}{(2n+1)(2n+2)} &= \frac{\pi^2 \ln(2)}{6} - \frac{3\zeta(3)}{4}, \\
\sum_{n=0}^{\infty} \frac{\zeta_n(2,2)}{(2n+1)(2n+2)} &= \frac{\pi^4 \ln(2)}{120} - \frac{\pi^2\zeta(3)}{8} + \frac{15\zeta(5)}{16}, \\
\sum_{n=0}^{\infty} \frac{\zeta_n(2,2,2)}{(2n+1)(2n+2)} &= \frac{\pi^6 \ln(2)}{5040} - \frac{\pi^4\zeta(3)}{160} + \frac{5\pi^2\zeta(5)}{32} - \frac{63\zeta(7)}{64}.
\end{aligned} \tag{4.4}$$

Here, Eq (4.4) can be found in Campbell and Chen [25, p. 15], and Chen and Wang [26, Example 4.1]. Setting $p = 1, 2, 3$ in Eq (4.2) yields

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{O_n^{(2)}}{2n(2n+1)} &= \frac{\pi^2 \ln(2)}{8} - \frac{7\zeta(3)}{16}, \\
\sum_{n=1}^{\infty} \frac{t_n(2,2)}{2n(2n+1)} &= \frac{\pi^4 \ln(2)}{384} - \frac{3\pi^2\zeta(3)}{128} + \frac{31\zeta(5)}{256}, \\
\sum_{n=1}^{\infty} \frac{t_n(2,2,2)}{2n(2n+1)} &= \frac{\pi^6 \ln(2)}{46090} - \frac{\pi^4\zeta(3)}{2048} + \frac{15\pi^2\zeta(5)}{2048} - \frac{127\zeta(7)}{4097}.
\end{aligned}$$

5. Series related to the integral of $t^n \cos(t)$

In this section, we establish two series identities based on Eqs (2.2) and (2.9), which give the evaluations of the following two Euler-type series.

Theorem 5.1. *For an integer $p \geq 0$, the following series are expressible in terms of powers of π, $\ln(2)$, and zeta values:*

$$\sum_{n=p}^{\infty}\frac{\zeta_n(\{2\}_p)}{(2n+2)(2n+3)}=\sum_{k=0}^{p}\sum_{j=1}^{p-k}(-1)^{k+j}\frac{(4^{k-j}-4^k)\pi^{2p-2k-2j}\zeta(2j+1)}{(2p+1-2k-2j)!}+(1-\ln(2))\sum_{k=0}^{p}(-4)^k\frac{\pi^{2p-2k}}{(2p+1-2k)!} \tag{5.1}$$

and

$$\sum_{n=p}^{\infty}\frac{t_n(\{2\}_p)}{(2n+1)(2n+2)}=(-1)^p\ln(2)+(1-\ln(2))\sum_{k=0}^{p-1}(-1)^k\frac{(\frac{\pi}{2})^{2p-2k}}{(2p-2k)!}-(-1)^p\sum_{k=1}^{p}\frac{\zeta(2k+1)}{4^k}+\sum_{k=0}^{p-1}\sum_{j=1}^{p-k}(-1)^{k+j}\frac{(1-4^j)(\frac{\pi}{2})^{2p-2k-2j}\zeta(2j+1)}{(2p-2k-2j)!16^j}. \tag{5.2}$$

Proof. On the one hand, by integration by parts, we have

$$\frac{1}{(2p)!}\int_0^{\theta}t^{2p}\cos(t)\mathrm{d}t=\sum_{k=0}^{p}(-1)^k\frac{\theta^{2p-2k}\sin(\theta)}{(2p-2k)!}+\sum_{k=0}^{p-1}(-1)^k\frac{\theta^{2p-2j-1}\cos(\theta)}{(2p-2k-1)!}.$$

On the other hand, replace θ by t in (2.2), multiply both sides by $\cos(t)$, and integrate from 0 to θ. Then we have

$$\begin{aligned}\frac{1}{(2p)!}\int_0^{\theta}t^{2p}\cos(t)\mathrm{d}t&=\sum_{n=p}^{\infty}\frac{4^{n-p}}{\binom{2n}{n}}\frac{\zeta_{n-1}(\{2\}_{p-1})}{n^2(2n+1)}\sin^{2n+1}(\theta)\\&=\sum_{n=p}^{\infty}\frac{4^{n-p}}{\binom{2n}{n}}\frac{\zeta_{n-1}(\{2\}_{p-1})}{n^2}\sin^{2n+1}(\theta)-2\sum_{n=p}^{\infty}\frac{4^{n-p}}{\binom{2n}{n}}\frac{\zeta_{n-1}(\{2\}_{p-1})}{n(2n+1)}\sin^{2n+1}(\theta)\\&=\frac{\theta^{2p}\sin(\theta)}{(2p)!}-2\sum_{n=p}^{\infty}\frac{4^{n-p}}{\binom{2n}{n}}\frac{\zeta_{n-1}(\{2\}_{p-1})}{n(2n+1)}\sin^{2n+1}(\theta),\end{aligned}$$

where the last two steps can be obtained by partial fraction decomposition and Eq (2.9). Combining the above two results gives us an identity. Divide both sides of this identity by $\sin(\theta)$, and integrate from 0 to $\frac{\pi}{2}$. Then we have

$$\begin{aligned}&2\sum_{n=p}^{\infty}\frac{4^{n-p}}{\binom{2n}{n}}\frac{\zeta_{n-1}(\{2\}_{p-1})}{n(2n+1)}\int_0^{\frac{\pi}{2}}\sin^{2n}(\theta)\mathrm{d}\theta\\&=\sum_{k=1}^{p}\frac{(-1)^{k+1}}{(2p-2k)!}\int_0^{\frac{\pi}{2}}\theta^{2p-2k}\mathrm{d}\theta+\sum_{k=0}^{p-1}\frac{(-1)^{k+1}}{(2p-2k-1)!}\int_0^{\frac{\pi}{2}}\theta^{2p-2k-1}\cot(\theta)\mathrm{d}\theta.\end{aligned}$$

Using Wallis' formula for the left, and applying the integral (3.4) for the second term on the right, we obtain Eq (5.1). Similarly, by means of the integral $\int_0^{\frac{\pi}{2}}t^{2p+1}\cos(t)\mathrm{d}t$, we have

$$\sum_{n=p}^{\infty} \frac{\binom{2n}{n}}{4^n} \frac{t_n(\{2\}_p) \sin^{2n+2}(\theta)}{2n+2}$$
$$= \sum_{k=1}^{p} (-1)^{k+1} \frac{\theta^{2p-2k+1} \sin(\theta)}{(2p-2k+1)!} + \sum_{k=0}^{p} (-1)^{k+1} \frac{\theta^{2p-2k} \cos(\theta)}{(2p-2k)!} + (-1)^p \tag{5.3}$$

(see also [37, Theorem 3.8]). Divide (5.3) by $\sin(\theta)$ and integrate from 0 to $\frac{\pi}{2}$. The integral of the left-hand side is easy to calculate by Wallis' formula. The right-hand side turns into

$$\sum_{k=1}^{p} \frac{(-1)^{k+1}}{(2p-2k+1)!} \int_0^{\frac{\pi}{2}} \theta^{2p-2k+1} \mathrm{d}\theta$$
$$+ \sum_{k=0}^{p-1} \frac{(-1)^{k+1}}{(2p-2k)!} \int_0^{\frac{\pi}{2}} \theta^{2p-2k} \cot(\theta) \mathrm{d}\theta + (-1)^{p+1} \int_0^{\frac{\pi}{2}} \frac{\cos(\theta)-1}{\sin(\theta)} \mathrm{d}\theta.$$

The first and third terms can be computed directly. For the second term, use the integral (3.4). Finally, we obtain Eq (5.2). □

Example 5.1. Setting $p = 1, 2$ in Eq (5.1) gives

$$\sum_{n=0}^{\infty} \frac{H_n^{(2)}}{(2n+2)(2n+3)} = -\frac{\pi^2 \ln(2)}{6} + \frac{\pi^2}{6} + \frac{3\zeta(3)}{4} + 4\ln(2) - 4, \tag{5.4}$$
$$\sum_{n=0}^{\infty} \frac{\zeta_n(2,2)}{(2n+2)(2n+3)} = -\frac{\pi^4 \ln(2)}{120} + \frac{\pi^4}{120} - \frac{15\zeta(5)}{16} + \frac{\pi^2\zeta(3)}{8} - \frac{2\pi^2}{3} - 3\zeta(3)$$
$$+ \frac{2\pi^2 \ln(2)}{3} - 16\ln(2) + 16,$$

where Eq (5.4) can be found in Chen and Wang [26, Example 4.2]. Setting $p = 1, 2$ in Eq (5.2) yields

$$\sum_{n=0}^{\infty} \frac{O_n^{(2)}}{(2n+1)(2n+2)} = -\frac{\pi^2 \ln(2)}{8} + \frac{\pi^2}{8} + \frac{7\zeta(3)}{16} - \ln(2),$$
$$\sum_{n=0}^{\infty} \frac{t_n(2,2)}{(2n+1)(2n+2)} = -\frac{\pi^4 \ln(2)}{384} + \frac{\pi^4}{384} - \frac{7\zeta(3)}{16} + \frac{3\pi^2\zeta(3)}{128} + \frac{\pi^2 \ln(2)}{8} - \frac{\pi^2}{8}$$
$$- \frac{31\zeta(5)}{256} + \ln(2).$$

6. Conclusions

In this paper, by using integer powers of the arcsin function and special integrals, we obtained the explicit expressions or recurrences of some parametric Euler-type series:

$$\sum_{n=p}^{\infty} \frac{\zeta_n(\{2\}_p)}{(n+1)(n+m+1)}, \quad \sum_{n=p}^{\infty} \frac{t_n(\{2\}_p)}{(2n+1)(2n+2m+1)},$$

$$\sum_{n=p}^{\infty} \frac{\zeta_n(\{2\}_p)}{(2n+k)(2n+k+1)}, \quad \sum_{n=p}^{\infty} \frac{t_n(\{2\}_p)}{(2n+k-1)(2n+k)},$$

for $m \in \mathbb{N}$, $p \in \mathbb{N}_0$, and $k = 0, 1, 2$. These Euler-type series are all expressible in terms of π, ln(2), and zeta values.

Use of Generative-AI tools declaration

The author declares he (she) have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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