Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(4): 8066–8094.
DOI: 10.3934/math.2025370
Received: 17 January 2025
Revised: 19 March 2025
Accepted: 24 March 2025
Published: 07 April 2025

Research article

Pointwise potential estimates for solutions to a class of nonlinear elliptic equations with measure data

Zhaoyue Sui and Feng Zhou*

School of Mathematics and Statistics, Shandong Normal University, Jinan, Shandong, 250358, China

* **Correspondence:** Email: zhoufeng@u.nus.edu.

Abstract: In this article, we investigate the regularities of solutions to a class of nonlinear elliptic equations with measure data. These equations involve the N-functions, and the solutions belong to the Sobolev-Orlicz spaces. Through the application of comparison arguments, Caccioppoli-type inequality, and maximal estimate, we derive pointwise Riesz potential estimates for both the gradient of the solutions and the solutions themselves. Furthermore, we establish Hölder continuity estimates for the solutions.

Keywords: Riesz potential; nonlinear elliptic equations; regularity; Sobolev-Orlicz spaces
Mathematics Subject Classification: 35J60, 35R05, 35R06

1. Introduction

This article deals with the nonlinear elliptic equation with measure data of the type

$$-\operatorname{div}\mathcal{A}(x,\nabla u)=\mu \quad \text{in } \Omega. \tag{1.1}$$

Here $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ is a bounded domain, the unknown $u\in W^{1,P}(\Omega)$ with an N-function $P(\cdot)$ which will be introduced in Section 2, and $\mathcal{A}(x,\nabla u)\in L^1_{\text{loc}}(\Omega,\mathbb{R}^n)$. In (1.1), μ is a Radon measure defined on Ω with finite total mass $|\mu|(\Omega)<\infty$. Moreover, if the measure μ is actually an L^1-function, then

$$|\mu|(Q):=\int_Q|\mu(x)|\,\mathrm{d}x \tag{1.2}$$

for a measurable subset $Q\subset\Omega$. We assume that measure μ satisfies the following condition:
(M) There exists some $\theta_0\in(0,n-1)$ such that

$$|\mu|(B(x_0,2R))\leq(2R)^{\theta_0} \tag{1.3}$$

holds for all $x_0\in\Omega$ and every $R>0$.

In this article, we assume that the Carathéodory vector field $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1-regular in the gradient variable ξ, and satisfies $\mathcal{A}(x, 0) = 0$, and the growth, ellipticity, and continuity conditions, i.e., there are constants $0 \le s \le 1$, $0 < \nu \le L$, and $K \ge 1$ such that

$$
\text{(A1)} \qquad \begin{cases} |\mathcal{A}(x,\xi)| + \left|\mathcal{A}_\xi(x,\xi)\right| \left(|\xi|^2 + s^2\right)^{\frac{1}{2}} \le L \dfrac{P\left(\left(|\xi|^2+s^2\right)^{\frac{1}{2}}\right)}{(|\xi|^2+s^2)^{\frac{1}{2}}}, \\ \nu^{-1} P\left(|\xi_2 - \xi_1|\right) \le \langle \mathcal{A}(x,\xi_2) - \mathcal{A}(x,\xi_1),\, \xi_2 - \xi_1 \rangle, \\ |\mathcal{A}(x,\xi) - \mathcal{A}(y,\xi)| \le K\omega\left(|x-y|\right) \dfrac{P\left(\left(|\xi|^2+s^2\right)^{\frac{1}{2}}\right)}{(|\xi|^2+s^2)^{\frac{1}{2}}} \end{cases}
$$

for any $x, y \in \Omega$ and $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$. In (A1), the function $\omega : [0,\infty) \to [0,\infty)$ has the following hypothesis.

(A2) ω is a non-decreasing concave function such that $\omega(0) = \lim_{\rho \searrow 0} \omega(\rho) = 0$ and $\omega(\cdot) \le 1$.

Moreover, ω is assumed to satisfy the Dini-continuous condition:

$$
\text{(A3)} \qquad d(R) := \int_0^R \omega(\rho) \frac{d\rho}{\rho} < \infty
$$

for every $0 < R \le 1$.

A significant example of (1.1) is the p-Laplacian type equation, for which $s = 0$, $p \in (1,\infty)$, $P(x) = x^p$, and $\mathcal{A}(\nabla u) = |\nabla u|^{p-2}\nabla u$. Then Eq (1.1) can be expressed as

$$
-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \mu.
$$

The relevant research on the regularities of solutions to elliptic equations starts with Kilpeläinen and Malý [1, 2], and extends with a different technique by Trudinger and Wang [3]. Later, Duzaar and Mingione make a further study in [4]. Those results show a standard fact that solutions to non-homogeneous p-Laplacian-type equations with measure data can be pointwise estimated in a natural way by involving the classical nonlinear Wolff potential $\mathbf{W}^{\mu}_{\beta,p}(x,R)$ [5], that is,

$$
\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B(x,\rho))}{\rho^{n-\beta p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}, \qquad \beta \in \left(0, \frac{n}{p}\right].
$$

Based on the relationship between Wolff potential and Riesz potential, Mingione et al. [6, 7] find out pointwise gradient estimates hold for general quasilinear degenerate equations by applying the Riesz potential

$$
\mathbf{I}^{\mu}_{\beta}(x,R) := \mathbf{W}^{\mu}_{\frac{\beta}{2},2}(x,R) = \int_0^R \frac{|\mu|(B(x,\rho))}{\rho^{n-\beta}} \frac{d\rho}{\rho}, \quad \beta > 0. \tag{1.4}
$$

Baroni proves pointwise gradient bounds for solutions in terms of linear Riesz potentials in [8]. In addition, the caloric Riesz potential serves as a means for pointwise estimation of the spatial gradient of solutions to nonlinear degenerate parabolic equations [9].

Further, pointwise gradient estimates via the nonlinear Wolff potentials for weak solutions to various quasilinear elliptic equations with measure data are obtained by Mingione [10] and Yao [11]. More generally, pointwise potential estimates for elliptic equations and systems with Orlicz growth are studied in [12–14], respectively.

In recent years, a great deal of effort has gone into investigating nonlinear elliptic equations and systems involving measure data. Chilebicka et al. [15] study estimates including precise continuity and Hölder continuity criteria by the means of potential of a Wolff type; they also provide regularity estimates of the solutions and their gradients in the generalized Marcinkiewicz scale [16]. The existence of solutions in the framework of renormalized solutions is introduced in [17]. There are many interesting results in [18–20].

In this article, a weak solution to (1.1) is a function $u \in W^{1,P}(\Omega)$ such that

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla\varphi \rangle \, \mathrm{d}\zeta = \int_\Omega \varphi \, \mathrm{d}\mu, \tag{1.5}$$

whenever $\varphi \in C_0^\infty(\Omega)$. Inspired by Mingione et al. [4, 10, 21], the main objective is to present pointwise potential estimates and interior Hölder continuity of weak solutions to (1.1) by using (1.4) in the Sobolev-Orlicz spaces.

We state our pointwise estimates in Theorems 1.1 and 1.2. It is important to note that θ_0 in the following theorems will be introduced in (1.3). The first main result is the gradient pointwise estimates of u as follows.

Theorem 1.1. *Let* $u \in C^1(\Omega) \cap W^{1,P}(\Omega)$ *be a weak solution to* (1.1) *under the assumptions* (A1), (A2), (A3)*, Radon measure* μ *satisfy* $\mu \in L^1(\Omega)$ *and* (M)*, and* P *be an N-function with* $\Delta_2(P, \widetilde{P}) < \infty$ *satisfying Assumption 2.4 and* (2.12)*. There exists a constant* $C \equiv C(n, \nu, L, K, s, \theta_0, C_{\Delta_2}, C_1) > 0$ *and a positive radius* $\widetilde{R} < 1$ *such that the pointwise estimate*

$$P(|\nabla u(x_0)|) \le C \fint_{B(x_0,R)} [P(|\nabla u|) + P(s)] \, \mathrm{d}x + C \, \mathbf{I}^{\mu}_{n-\theta_0}(x_0, 4R) \tag{1.6}$$

holds whenever $B(x_0, R) \subset \Omega$ *and* $0 < R \le \widetilde{R}$*. In* (1.6)*,* $\fint$ *denotes integral average,* Δ_2*-condition,* C_{Δ_2}*, and* C_1 *will be described in Definition 2.3 and Lemma 2.5, respectively.*

On the basis of Theorem 1.1, we demonstrate the pointwise estimate of u.

Theorem 1.2. *Let* $u \in C^0(\Omega) \cap W^{1,P}(\Omega)$ *be a weak solution to* (1.1) *under the assumptions* $(A1)_1$, $(A1)_2$, *Radon measure* μ *satisfy* $\mu \in L^1(\Omega)$ *and* (M)*, and* P *be an N-function with* $\Delta_2(P, \widetilde{P}) < \infty$ *satisfying Assumption 2.4, and* (2.12)*. There exists a constant* $C \equiv C(n, \nu, L, \theta_0, C_{\Delta_2}, C_1, \mathrm{diam}(\Omega)) > 0$ *and a positive radius* $R < 1$ *such that for every* $r \le R$ *the pointwise estimate*

$$P(|u(x_0)|) \le C \, r^{1+\varepsilon_0} \fint_{B(x_0,r)} P\left(\frac{|u|}{r}\right) \mathrm{d}x + C \, r^{\alpha} \, \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x_0, 2r) \tag{1.7}$$

holds, where ε_0 *and* α *will be introduced in Lemmas 2.5 and 3.3, respectively.*

Inspired by Mingione's result [10], the following theorem expounds the Hölder continuity of the solution u.

Theorem 1.3 (Interior Hölder type estimate)**.** *Let $u \in C^0(\Omega) \cap W^{1,P}(\Omega)$ be a weak solution to* (1.1) *under the assumptions* (A1), (A2)*, let μ be a Radon measure satisfying $\mu \in L^1(\Omega)$ and* (M)*, and let P be an N-function satisfying $\Delta_2(P,\widetilde{P}) < \infty$, Assumption 2.4, and* (2.12)*. Then there exists constants $\alpha \in [0,1)$ and $0 < R < 1$ such that for every x, $y \in B(x_0, 2R) \subset \Omega$, there holds*

$$\begin{aligned} & P\left(|u(x)-u(y)|\right) \\ \leq\ & C\left[\mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x,2R) + \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(y,2R) + \fint_{B(x_0,2R)} P\left(\frac{|u|}{R}\right) \mathrm{d}\zeta + P(s)\right] \cdot |x-y|^{\alpha}, \end{aligned}$$

where the constant C depends on n, ν, L, θ_0, $\omega(\cdot)$, C_{Δ_2}, C_1, and diam(Ω).

Remark 1.4. *Drawing upon the Riesz potential, our results present pointwise estimates and Hölder continuity within the more generalized framework of Sobolev-Orlicz spaces. Notably,* (1.6) *provides an estimate of ∇u, whereas* (1.7) *estimates u itself. We shall leverage Lemma 2.11 to relate u to ∇u.*

In this article, we adopt several technical tools and methods in Sobolev-Orlicz spaces, and explore the properties of solutions of homogeneous equations to those of inhomogeneous equations with measure data. We first establish Proposition 3.2, which reveals the density of the Riesz potential and serves as a crucial conclusion in facilitating the proof of subsequent comparison lemmas. The proof of the comparison estimate is divided into two intricate steps; the first step requires Sobolev-type embedding, while the second primarily employs scaling changes. Our primary objective is to prove pointwise potential estimates and Hölder continuity, which is shown essentially by oscillation estimates of solutions. By utilizing summation methods and involving the Riesz potential, we proceed from deriving the estimates of the gradient of u to those of u itself. Ultimately, we employ the sharp maximal functions, and achieve the interior Hölder estimates of solutions.

This article is organized as follows: In Section 2, we state fundamental tools and definitions such as N-functions and maximal functions. Section 3 is devoted to the proof of Lemmas 3.4 and 3.9, and Section 4 presents supporting results, to gather Caccioppoli-type inequality and maximal estimate towards the proof of the main theorems. In the last section, we present the proofs of Theorems 1.1–1.3, respectively.

2. Preliminaries

In this section, we give the definitions and tools of N-functions, function spaces; classical inequalities, and maximal functions.

2.1. N-functions

The following definitions and results are standard in the context of N-function; see [22].

Definition 2.1. *A function $P : [0,\infty) \to [0,\infty)$ is said to be an N-function, if P is convex, differentiable, and its derivative P' is a right continuous, non-decreasing function satisfying that $P'(0) = 0$ and $P'(t) > 0$ for $t > 0$.*

Definition 2.2. *The complementary function $\widetilde{P} : [0,\infty) \to [0,\infty)$ is defined by*

$$\widetilde{P}(x) = \sup_{t\geq 0}\left[\,x\,t - P(t)\,\right].$$

Definition 2.3. *We say that P satisfied the Δ_2-condition, if there exists $C_{\Delta_2} > 0$ such that*

$$P(2t) \le C_{\Delta_2} P(t)$$

for all $t \ge 0$. By $\Delta_2(P)$ we denote the smallest constant C_{Δ_2}. The function P is said to satisfy the ∇_2-condition, if $\widetilde{P} \in \Delta_2$. Then we define

$$\Delta_2\left(P, \widetilde{P}\right) := \max\left\{\Delta_2(P),\ \Delta_2(\widetilde{P})\right\}. \tag{2.1}$$

By the Δ_2-condition of Definition 2.3, we can easily obtain

$$P(x+y) \le C_{\Delta_2}\left[P(x) + P(y)\right] \tag{2.2}$$

for every $x,\ y \ge 0$.

Assumption 2.4. *Let P be a convex function that satisfies $\Delta_2(P, \widetilde{P}) < \infty$ as* (2.1)*, and P is C^2 on* $(0, \infty)$*. Moreover, let $P'(0) = 0$, $\lim_{t\to\infty} P'(t) = \infty$ and uniformly in $t \ge 0$*

$$P'(t) \sim t\, P''(t).$$

Assumption 2.4 assures that P is an N-function.

We let $P^{-1} : [0, \infty) \to [0, \infty)$ be the right-continuous inverse function of P, and $(P')^{-1} : [0, \infty) \to [0, \infty)$ the inverse function of P'. Then $\widetilde{\widetilde{P}(t)} = P(t)$ and $(\widetilde{P})'(t) = (P')^{-1}(t)$ hold. By [23], one has

$$P(t) \sim tP'(t), \ \text{ and } \ \widetilde{P}\left(P'(t)\right) \sim P(t) \tag{2.3}$$

hold uniformly in $t \ge 0$. By (2.2) and (2.3), we have

$$P'(x+y) \le C_{\Delta_2}\left[P'(x) + P'(y)\right]. \tag{2.4}$$

Lemma 2.5. [23] *Let P be an N-function with $\Delta_2(P, \widetilde{P}) < \infty$. Then there exist $\varepsilon_0 > 0$, $C_1 > 0$ which only depend on $\Delta_2(P, \widetilde{P})$ such that for all $t \ge 0$ and all $\lambda \in [0, 1]$, one has*

$$P(\lambda t) \le C_1 \lambda^{1+\varepsilon_0} P(t). \tag{2.5}$$

From Definition 2.2 of N-function $\widetilde{P}(\cdot)$, it is easy to obtain that (2.5) holds for $\widetilde{P}$, i.e.,

$$\widetilde{P}(\lambda t) \le C_1 \lambda^{1+\varepsilon_0} \widetilde{P}(t). \tag{2.6}$$

Using (2.3) and (2.5), it is not difficult to obtain

$$P'(\lambda t) \le C_1 \lambda^{\varepsilon_0} P'(t). \tag{2.7}$$

In this article, we denote $a \sim b$ by $C\, b \le a \le C'\, b$ for two constants C and C'.

Lemma 2.6. [23] *Let P be an N-function under Assumption 2.4, and $\mathcal{A}$ satisfies the continuity and growth condition of* $(\mathrm{A1})_1$, $(\mathrm{A1})_2$*. Then*

$$(\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)) \cdot (\xi_1 - \xi_2) \ \sim \ P(|\xi_1 - \xi_2|) \ \sim \ |\xi_1 - \xi_2|\, P'(|\xi_1| + |\xi_2|).$$

Moreover,

$$\mathcal{A}(x, \xi) \cdot \xi \sim P(|\xi|)$$

uniformly in $\xi \in \mathbb{R}^n$ and $x \in \Omega$.

Example 2.7. [24] *Assume that P is a Young function such that*

$$P(t) \approx t^{p_1}(\log t)^{p_2} \quad , t \gg 1,$$

where $p_1 > 1$ *and* $p_2 \in \mathbb{R}$. *The derivative* P' *of* P *is as follow:*

$$P'(t) \approx t^{p_1-1} (\log(1+t))^{p_2} \quad \text{near infinity.}$$

The complementary function $\widetilde{P}$ *satisfies*

$$\widetilde{P}(t) \approx t^{\frac{p_1}{p_1-1}} (\log(t))^{-\frac{p_2}{p_1-1}} \quad \text{near infinity.}$$

It is not difficult to verify that $P(\cdot)$ *is an N-function satisfying Definitions 2.1–2.3, and Assumption 2.4.*

2.2. Function spaces

In this article, we need the following definitions of function spaces. The classical Orlicz spaces $L^P(\mathbb{R}^n)$ with its norm are given via [22]

$$L^P(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) \,\Big|\, \int_{\mathbb{R}^n} P(|f(x)|)\,\mathrm{d}x < \infty \right\},$$

and

$$\|f(x)\|_{L^P(\mathbb{R}^n)} = \inf \left\{ k > 0 \,\Big|\, \int_{\mathbb{R}^n} P\left(\frac{f(x)}{k}\right) \mathrm{d}x \le 1 \right\}. \tag{2.8}$$

If $\|f(x)\|_{L^P(\mathbb{R}^n)}$ is finite, then $f(x) \in L^P(\mathbb{R}^n)$. The Sobolev-Orlicz spaces $W^{1,P}(\mathbb{R}^n)$ and its norm are given by [25]

$$W^{1,P}(\mathbb{R}^n) = \left\{ f \in L^P(\mathbb{R}^n) \,\middle|\, \nabla f \in L^P(\mathbb{R}^n) \right\},$$

and

$$\|f(x)\|_{W^{1,P}(\mathbb{R}^n)} = \|f\|_{L^P(\mathbb{R}^n)} + \|\nabla f\|_{L^P(\mathbb{R}^n)}.$$

If $\|f(x)\|_{W^{1,P}(\mathbb{R}^n)}$ is finite, then $f(x) \in W^{1,P}(\mathbb{R}^n)$. Both $L^P(\mathbb{R}^n)$ and $W^{1,P}(\mathbb{R}^n)$ are Banach spaces.

2.3. Classical inequalities

In this subsection, we recall several classical inequalities.

Lemma 2.8. (Young's inequality [26]) *For all* $\varepsilon > 0$, *there exist* C_ε, $\widetilde{C}_\varepsilon$ *depending on* $\Delta_2(P, \widetilde{P})$, *such that for all* $\zeta_1, \zeta_2 \ge 0$, *there holds*

$$\zeta_1 \cdot \zeta_2 \le \varepsilon P(\zeta_1) + C_\varepsilon \widetilde{P}(\zeta_2), \tag{2.9}$$

and

$$\zeta_1 \cdot \zeta_2 \le \varepsilon \widetilde{P}(\zeta_1) + \widetilde{C}_\varepsilon P(\zeta_2). \tag{2.10}$$

Lemma 2.9. *Let P be an N-function with the* Δ_2*-condition. Then for all* $\varepsilon > 0$, *there exists a constant C such that*

$$|P(x) - P(y)| \le \varepsilon\, C_{\Delta_2} P(y) + CP(|x-y|)$$

for $x, y > 0$, *where the constant C depends on* C_{Δ_2} *and* C_1.

Proof. By the mean value theorem, for $x > y > 0$, there exists $\lambda_0 \in (0, 1)$ such that

$$P(x) - P(y) = P'\left[\lambda_0 x + (1-\lambda_0)y\right](x-y) = P'\left[y + \lambda_0(x-y)\right](x-y).$$

We use (2.4) and (2.7) to get

$$P(x) - P(y) \le C_{\Delta_2}\left[P'(y)(x-y) + C_1\lambda_0^{\varepsilon_0}P'(x-y)(x-y)\right].$$

Then applying (2.3) and (2.10), we obtain that

$$\begin{aligned}|P(x) - P(y)| &\le C_{\Delta_2}\left[P'(y)\,|x-y| + C_1\lambda_0^{\varepsilon_0}P\left(|x-y|\right)\right]\\ &\le \varepsilon\, C_{\Delta_2}\widetilde{P}\left(P'(y)\right) + \widetilde{C}_\varepsilon P(|x-y|) + CP(|x-y|)\\ &\le \varepsilon\, C_{\Delta_2}P(y) + CP(|x-y|).\end{aligned}$$

We complete the proof of Lemma 2.9. □

Let B be a measurable set with positive measure, and $f : B \to \mathbb{R}^n$ a measurable function. We denote the integral average of f by

$$(f)_B = ⨍_B f(x)\,\mathrm{d}x = \frac{1}{|B|}\int_B f(x)\,\mathrm{d}x.$$

Lemma 2.10. (Jesen's inequality [27]) *Let P be an N-function with $\Delta_2(P, \widetilde{P}) < \infty$. If $f \in W^{1,P}(B(x,R))$, then there exists $C \equiv C(n)$ for $B(x,R) \subset \Omega$ such that*

$$P\left(\left|⨍_{B(x,R)} f\,\mathrm{d}\zeta\right|\right) \le C ⨍_{B(x,R)} P(|f|)\,\mathrm{d}\zeta.$$

Lemma 2.11. (Sobolev-Poincaré's inequality [27]) *Let P be an N-function with $\Delta_2(P, \widetilde{P}) < \infty$ and satisfy Assumption 2.4. If $f \in W^{1,P}(B(x,R))$, then there exist $0 < \theta_1 < 1$ and $C > 0$ such that*

$$⨍_{B(x,R)} P\left(\frac{|f - (f)_{B(x,R)}|}{R}\right)\mathrm{d}\zeta \le C\left(⨍_{B(x,R)} P^{\theta_1}(|\nabla f|)\,\mathrm{d}\zeta\right)^{\frac{1}{\theta_1}} \tag{2.11}$$

holds whenever $B(x,R) \subset \Omega$.

The following lemma describes an embedding into a space of continuous functions; see Theorem 8.39 in [28] and (2.22) in [24].

Lemma 2.12. *If an N-function P satisfies that*

$$\int_1^\infty \frac{P^{-1}(x)}{x^{\frac{n+1}{n}}}\,\mathrm{d}x < \infty, \tag{2.12}$$

then $W_0^{1,P}(\Omega) \hookrightarrow C^0(\Omega) \cap L^\infty(\Omega)$, that is, there exists a constant $C \equiv C(n)$ such that

$$\|f\|_{L^\infty(\Omega)} \le C\|f\|_{W_0^{1,P}(\Omega)} \sim C\|\nabla f\|_{L^P(\Omega)}$$

for all $f \in W^{1,P}(\Omega)$.

The following iteration lemma plays an essential role in proving the Caccioppoli-type inequality (4.1).

Lemma 2.13. (Iteration lemma [29]) *Let $f : [\gamma R, R] \to [0, \infty)$ be a bounded function such that the inequality*

$$f(\varrho) \leq \frac{1}{2} f(r) + \frac{C_2}{(r-\varrho)^{\kappa}}$$

holds for fixed constants C_2, $\kappa \geq 0$, and $\gamma R \leq \varrho \leq r \leq R$ with $0 < \gamma < 1$. Then we have

$$f(\gamma R) \leq \frac{C}{[(1-\gamma)R]^{\kappa}}$$

for a constant C depending only on κ.

2.4. Maximal functions and sharp maximal functions

Throughout this subsection, we provide powerful tools for Hölder estimate of Theorem 1.3. Analogously to the definition of the classical maximal operator in [30], we define the generalized maximal operator and sharp maximal function in Sobolev-Orlicz spaces.

Definition 2.14. [31] *Let $-1 < \tau < n$, $R < \mathrm{dist}(x, \partial\Omega)$, $x \in \Omega$, where $\partial\Omega$ is the boundary of Ω. Let f be a function in Orlicz space $L^P(\Omega)$ or a measure with finite mass, and P be an N-function with $\Delta_2(P, \widetilde{P}) < \infty$. The function defined by*

$$M^{P}_{\tau,R}(f)(x) := \sup_{0<r\leq R} r^{\tau} \fint_{B(x,r)} P(|f|)\, \mathrm{d}\zeta \tag{2.13}$$

is called the restricted fractional τ generalized maximal function of f.

Definition 2.15. [31] *Let $\beta \in (0,1)$, $x \in \Omega$, and $R < \mathrm{dist}(x, \partial\Omega)$, let $f \in L^P(\Omega)$ and P be an N-function with $\Delta_2(P, \widetilde{P}) < \infty$. The function defined by*

$$M^{\#,P}_{\beta,R}(f)(x) := \sup_{0<r\leq R} r^{-\beta} \fint_{B(x,r)} P\left(|f - (f)_{B(x,r)}|\right) \mathrm{d}\zeta$$

is called the restricted fractional β generalized sharp maximal function of f. For $\theta > 0$, we also denote

$$\widetilde{M}^{\#,P}_{\theta,R}(f)(x) := \sup_{0<r\leq R} r^{\theta} \fint_{B(x,r)} P\left(\frac{|f - (f)_{B(x,r)}|}{r}\right) \mathrm{d}\zeta. \tag{2.14}$$

The following note gives us a connection between maximal functions and sharp maximal functions.

Remark 2.16. *Combining the generalized sharp maximal functions and Lemma 2.11, we see that if Assumption 2.4 holds, then it follows that*

$$\widetilde{M}^{\#,P}_{\theta,R}(f)(x) \leq C \sup_{0<r\leq R} r^{\theta} \left(\fint_{B(x,r)} P^{\theta_1}\left(|\nabla f|\right) \mathrm{d}\zeta\right)^{\frac{1}{\theta_1}}. \tag{2.15}$$

3. A comparison argument

In this section, we collect the relevant difference estimates and decay estimates, and consider the density of the Riesz potential $\mathbf{I}_\beta^\mu$ explicitly. We define $v \in u + W_0^{1,P}(B(x_0, 2R))$ as the unique solution to the homogeneous Dirichlet problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla v) &= 0 \quad \text{in } B(x_0, 2R), \\ v &= u \quad \text{on } \partial B(x_0, 2R). \end{cases} \tag{3.1}$$

The existence of v is guaranteed by a standard monotonicity argument; see [13]. One obtains the following control estimate.

Lemma 3.1. *Let $u \in W^{1,P}(\Omega)$ be as in Theorem 1.1 satisfying the continuity and growth condition of* (A1)*, $v \in u + W_0^{1,P}(B(x_0, 2R))$ be a solution to* $(3.13)_1$*, let P be an N-function with $\Delta_2(P, \widetilde{P}) < \infty$ and satisfy Assumption 2.4. Then the following estimate*

$$\fint_{B(x_0,R)} P(|\nabla v|)\,\mathrm{d}x \le C \fint_{B(x_0,2R)} P(|\nabla u| + s)\,\mathrm{d}x \tag{3.2}$$

holds, where C depends on n, ν, s, L.

We consider the estimate of the difference of a solution to (1.1), and that of the corresponding solution to the Dirichlet problem (3.1).

Proposition 3.2. *Let $u \in W^{1,P}(\Omega)$ be as in Theorem 1.1 satisfying* $(\mathrm{A1})_2$*, $v \in u + W_0^{1,P}(B(x_0, 2R))$ be a solution to* (3.1)*, let P be an N-function with $\Delta_2(P, \widetilde{P}) < \infty$, and satisfy Assumption 2.4,* (2.12)*. Radon measure μ satisfies* (M)*. There exists a constant $C \equiv C(n, \nu, C_1)$ such that*

$$\fint_{B(x_0,2R)} P(|\nabla u - \nabla v|)\,\mathrm{d}x \le C\, \frac{|\mu|(B(x_0, 2R))}{(2R)^{\theta_0}}. \tag{3.3}$$

Proof. Without loss of generality, we first assume that $B(x_0, 2R) \equiv B(0, 1)$ and $|\mu|(B(0, 1)) = 1$. Then we shall remove these two conditions for general situations.

Step 1. We assume that $B(x_0, 2R) \equiv B(0, 1)$ with $|\mu|(B(0, 1)) = 1$. By choosing $\varphi \equiv u - v$ as a test function in (1.5), and using (1.1) and $(3.1)_1$, we obtain

$$\int_{B(0,1)} \langle \mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla v), \nabla u - \nabla v\rangle\,\mathrm{d}x = \int_{B(0,1)} (u - v)\,\mathrm{d}\mu. \tag{3.4}$$

By using the ellipticity assumption $(\mathrm{A1})_2$, we deduce that

$$\nu^{-1} \int_{B(0,1)} P\left(|\nabla u - \nabla v|\right)\mathrm{d}x \le \int_{B(0,1)} \langle \mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla v), \nabla u - \nabla v\rangle\,\mathrm{d}x.$$

By (3.4) and Lemma 2.12 , it follows that

$$\nu^{-1} \int_{B(0,1)} P\left(|\nabla u - \nabla v|\right)\mathrm{d}x \;\le\; \int_{B(0,1)} (u - v)\,\mathrm{d}\mu \;\le\; \sup_{B(0,1)} |u - v| \cdot |\mu|(B(0, 1)) \;\le\; C\,\|\nabla u - \nabla v\|_{L^P(B(0,1))}.$$

That is, there exists a constant C_3 such that

$$\int_{B(0,1)} P(|\nabla u - \nabla v|)\,\mathrm{d}x \le C_3\,\|\nabla u - \nabla v\|_{L^P(B(0,1))}. \tag{3.5}$$

We claim that there exists a positive constant $C \equiv C(n,\ \nu)$ such that

$$\int_{B(0,1)} P\,(|\nabla u - \nabla v|)\,\mathrm{d}x \le C \tag{3.6}$$

with $2R = 1$ and $|\mu|(B(0,1)) = 1$. According to (2.8), we write $\|\nabla u - \nabla v\|_{L^P(B(0,1))} = k$. We shall prove (3.6) in the following two scenarios. On the one hand, if $0 < k \le 1$, then it follows from (3.5) that

$$\int_{B(0,1)} P(|\nabla u - \nabla v|)\,\mathrm{d}x \le C_3\,k \le C.$$

On the other hand, if $k > 1$, then we set $k = (C_1C_3)^{-(2+\varepsilon_0)}$, where C_1 and C_3 are given in (2.5) and (3.5), respectively. It is obvious that $\frac{1}{C_1C_3} > 1$, and so

$$\int_{B(0,1)} (C_1C_3)^{1+\varepsilon_0}\,P(|\nabla u - \nabla v|)\,\mathrm{d}x \le (C_1C_3)^{1+\varepsilon_0}\ C_3\,\frac{1}{(C_1C_3)^{2+\varepsilon_0}} = \frac{1}{C_1}.$$

By involving $\lambda = C_1C_3 < 1$ in (2.5), we obtain that

$$\int_{B(0,1)} P\,(C_1C_3|\nabla u - \nabla v|)\,\mathrm{d}x \le C_1 \int_{B(0,1)} (C_1C_3)^{1+\varepsilon_0} P(|\nabla u - \nabla v|)\,\mathrm{d}x \le 1. \tag{3.7}$$

By (3.7) and (2.8), one finds that the norm $k \le \dfrac{1}{C_1C_3}$, that is, $\dfrac{1}{C_1C_3} \le 1$, which is contradictory to the condition $\dfrac{1}{C_1C_3} > 1$. Therefore, the inequality (3.6) holds.

Step 2. Scaling procedures.

We assume that $|\mu|(B(x_0,2R)) = 1$, and we shall reduce to the case $B(x_0,2R) \equiv B(0,1)$ by a standard scaling argument. By letting

$$\tilde{u}(y) := \frac{u(x_0 + 2Ry)}{2R}, \qquad \tilde{v}(y) := \frac{v(x_0 + 2Ry)}{2R},$$
$$\tilde{\mathcal{A}}(y,\xi) := (2R)^{n-\theta_0-1}\mathcal{A}(x_0 + 2Ry,\xi), \quad \tilde{\mu}(y) := (2R)^{n-\theta_0}\mu(x_0 + 2Ry)$$

for $y \in B(0,1)$, one has the following equations:

$$\begin{aligned} -\operatorname{div}\tilde{\mathcal{A}}(y,\nabla\tilde{u}) &= \tilde{\mu} \ \text{ on } B(0,1),\\ \operatorname{div}\tilde{\mathcal{A}}(y,\nabla\tilde{v}) &= 0 \ \text{ on } B(0,1). \end{aligned}$$

With the definition of Radon measure (1.2), we find the following relation between $|\tilde{\mu}|(B(0,1))$ and $|\mu|(B(x_0,2R))$, namely,

$$\begin{aligned} |\tilde{\mu}|(B(0,1)) &= \int_{B(0,1)} \tilde{\mu}(y)\,\mathrm{d}y = \int_{B(0,1)} (2R)^{n-\theta_0}\mu(x_0 + 2Ry)\,\mathrm{d}y\\ &= \frac{1}{(2R)^n}\int_{B(x_0,2R)} (2R)^{n-\theta_0}\mu(x)\,\mathrm{d}x = \frac{|\mu|(B(x_0,2R))}{(2R)^{\theta_0}}. \end{aligned}$$

Next, we shall reduce the general case to the special case $|\mu|(B(0,1)) = 1$. We define

$$M = \left[\, |\tilde{\mu}|(B(0,1))\,\right]^{\frac{1}{1+\varepsilon_0}} = \left[\frac{|\mu|(B(x_0,2R))}{(2R)^{\theta_0}} \right]^{\frac{1}{1+\varepsilon_0}}. \tag{3.8}$$

By (1.3), one has $M \le 1$. Hence the new solution, coefficient, and datum become

$$\bar{u} := \frac{\tilde{u}}{M}, \qquad \bar{v} := \frac{\tilde{v}}{M},$$
$$\bar{\mathcal{A}}(x,\xi) := \frac{\tilde{\mathcal{A}}(x, M\xi)}{|\tilde{\mu}|(B(0,1))}, \quad \bar{\mu} := \frac{\tilde{\mu}}{|\tilde{\mu}|(B(0,1))}.$$

Then we find that

$$\begin{aligned} -\operatorname{div} \bar{\mathcal{A}}(x, \nabla \bar{u}) &= \bar{\mu} \ \text{ on } B(0,1), \\ \operatorname{div} \bar{\mathcal{A}}(x, \nabla \bar{v}) &= 0 \ \text{ on } B(0,1) \end{aligned}$$

hold in the weak sense and $|\bar{\mu}|(B(0,1)) = 1$. Then by applying the result (3.6) in Step 1, one has

$$\begin{aligned} &\int_{B(0,1)} P\left(|\nabla \bar{u}(x_0+2Ry) - \nabla \bar{v}(x_0+2Ry)|\right) \mathrm{d}y \\ = \ &\int_{B(0,1)} P\left(\frac{|\nabla \tilde{u}(x_0+2Ry) - \nabla \tilde{v}(x_0+2Ry)|}{M} \right) \mathrm{d}y \ \le \ C. \end{aligned} \tag{3.9}$$

Considering (3.9) on $B(x_0,2R)$, then

$$\int_{B(0,1)} P\left(|\nabla \bar{u}(x_0+2Ry) - \nabla \bar{v}(x_0+2Ry)|\right) \mathrm{d}y = \frac{1}{(2R)^n} \int_{B(x_0,2R)} P\left(\frac{|\nabla u(x) - \nabla v(x)|}{M} \right) \mathrm{d}x,$$

we apply Lemma 2.5 with λ replaced by M to obtain

$$\begin{aligned} &\fint_{B(x_0,2R)} P\left(|\nabla u(x) - \nabla v(x)|\right) \mathrm{d}x \\ \le \ & C_1 M^{1+\varepsilon_0} \fint_{B(x_0,2R)} P\left(\frac{|\nabla u(x) - \nabla v(x)|}{M} \right) \mathrm{d}x \ \le \ C\, \frac{|\mu|(B(x_0,2R))}{(2R)^{\theta_0}}, \end{aligned} \tag{3.10}$$

where C depends on n, ν, and C_1. From (3.10), we complete the proof of Proposition 3.2. □

Via a classical approach, we have the following estimate inspired by [10].

Lemma 3.3. *Let* $v \in W^{1,P}(\Omega)$ *be a weak solution to the Dirichlet problem* (3.1) *under the assumptions* (A1)*, let* P *be an N-function satisfying* $\Delta_2(P, \widetilde{P}) < \infty$*, Assumption 2.4 and* (2.12)*. There exist constants* $\alpha \in (0,1]$ *and* $C_4 \equiv C_4(n,\nu,L) \ge 1$ *such that the estimate*

$$\fint_{B(x_0,\rho)} P(|\nabla v|)\, \mathrm{d}x \le C_4 \left(\frac{\rho}{R}\right)^{-1+\alpha} \fint_{B(x_0,R)} P(|\nabla v|)\, \mathrm{d}x \tag{3.11}$$

holds whenever $B(x_0,\rho) \subset B(x_0,R) \subset \Omega$.

We note that the estimate (3.11) is tenable by imitating the proof for the p-Laplacian equations.

Lemma 3.4. *Let $u \in W^{1,P}(\Omega)$ be a weak solution to* (1.1) *under the assumptions* $(\mathrm{A1})_2$. *Let P be an N-function satisfying $\Delta_2(P, \widetilde{P}) < \infty$, Assumption 2.4, and* (2.12). *Then there exist constants $\ell \equiv \ell(n, \nu, L, C_{\Delta_2}) \geq 1$ and $C \equiv C(n, \nu, L, C_{\Delta_2}, C_1) \geq 1$ such that*

$$\fint_{B(x_0,\rho)} P(|\nabla u|)\,\mathrm{d}x \leq \ell \left(\frac{\rho}{R}\right)^{-1+\alpha} \fint_{B(x_0,R)} P(|\nabla u|)\,\mathrm{d}x + C\left(\frac{R}{\rho}\right)^n \frac{|\mu|(B(x_0,2R))}{(2R)^{\theta_0}}$$

holds whenever $B(x_0,\rho) \subset B(x_0,R) \subset \Omega$.

Proof. From the triangle inequality and (2.2), we see that

$$\begin{aligned}\fint_{B(x_0,\rho)} P(|\nabla u|)\,\mathrm{d}x &\leq C_{\Delta_2}\left(\fint_{B(x_0,\rho)} P(|\nabla u - \nabla v|)\,\mathrm{d}x + \fint_{B(x_0,\rho)} P(|\nabla v|)\,\mathrm{d}x\right)\\ &\leq C_{\Delta_2}\left(\frac{R}{\rho}\right)^n \fint_{B(x_0,R)} P(|\nabla u - \nabla v|)\,\mathrm{d}x + C_{\Delta_2}\fint_{B(x_0,\rho)} P(|\nabla v|)\,\mathrm{d}x.\end{aligned}$$

By applying (3.11), one carries out

$$\begin{aligned}\fint_{B(x_0,\rho)} P(|\nabla u|)\,\mathrm{d}x &\leq C_{\Delta_2}\left(\frac{R}{\rho}\right)^n \fint_{B(x_0,R)} P(|\nabla u - \nabla v|)\,\mathrm{d}x\\ &\quad + C_{\Delta_2} C_4 \left(\frac{\rho}{R}\right)^{-1+\alpha} \fint_{B(x_0,R)} P(|\nabla v|)\,\mathrm{d}x. \end{aligned} \tag{3.12}$$

Using the triangle inequality $|\nabla v| \leq |\nabla u - \nabla v| + |\nabla u|$ again, the inequality (3.12) leads to

$$\begin{aligned}\fint_{B(x_0,\rho)} P(|\nabla u|)\,\mathrm{d}x &\leq C_{\Delta_2}\left[\left(\frac{R}{\rho}\right)^n + C_{\Delta_2}C_4\left(\frac{\rho}{R}\right)^{-1+\alpha}\right] \fint_{B(x_0,R)} P\left(|\nabla u - \nabla v|\right)\,\mathrm{d}x\\ &\quad + \ell\left(\frac{\rho}{R}\right)^{-1+\alpha}\fint_{B(x_0,R)} P(|\nabla u|)\,\mathrm{d}x\end{aligned}$$

with $\ell = C_{\Delta_2}^2 C_4$. It follows from (3.3) that

$$\begin{aligned}\fint_{B(x_0,\rho)} P(|\nabla u|)\,\mathrm{d}x &\leq C\left[\left(\frac{R}{\rho}\right)^n + C_{\Delta_2}C_4\left(\frac{\rho}{R}\right)^{-1+\alpha}\right] \frac{|\mu|(B(x_0,2R))}{(2R)^{\theta_0}}\\ &\quad + \ell\left(\frac{\rho}{R}\right)^{-1+\alpha}\fint_{B(x_0,R)} P(|\nabla u|)\,\mathrm{d}x.\end{aligned}$$

Notice that $\rho \leq R$ with

$$C_{\Delta_2}C_4\left(\frac{\rho}{R}\right)^{-1+\alpha} < C\left(\frac{R}{\rho}\right)^n,$$

we complete the proof of Lemma 3.4. □

We also define $w \in v + W_0^{1,P}(B(x_0,R))$ as the unique solution to the homogeneous Dirichlet problem with frozen coefficients

$$\begin{cases} \operatorname{div}\mathcal{A}(x_0,\nabla w) = 0 & \text{in } B(x_0,R),\\ w = v & \text{on } \partial B(x_0,R). \end{cases} \tag{3.13}$$

We have the following decay estimate.

Lemma 3.5. *Let $w \in W^{1,P}(\Omega)$ be a weak solution to* (3.13) *under the assumption* (A1). *Then there exist constants $\tilde{\alpha} \in (0,1]$ and $C \geq 1$, both depending on n, ν, L, such that*

$$\fint_{B(x_0,\rho)} P\left(|\nabla w - (\nabla w)_{B(x_0,\rho)}|\right) \mathrm{d}x \leq C\left(\frac{\rho}{R}\right)^{\tilde{\alpha}} \fint_{B(x_0,R)} P\left(|\nabla w - (\nabla w)_{B(x_0,R)}|\right) \mathrm{d}x \tag{3.14}$$

holds whenever $B(x_0,\rho) \subset B(x_0,R) \subset \Omega$.

Notice that the conclusion (3.14) is inspired by [32].

Lemma 3.6. *Under the assumptions* (A1) *and* (A2) *of Theorem 1.1, with v as in* (3.1) *and w as in* (3.13)*, there exists a constant $C \equiv C(n, \nu, L)$ such that*

$$\fint_{B(x_0,R)} P(|\nabla v - \nabla w|)\,\mathrm{d}x \leq CK\omega(R) \fint_{B(x_0,R)} P(|\nabla v| + s)\,\mathrm{d}x \tag{3.15}$$

for $B(x_0,R) \subset \Omega$, where K and $\omega(R)$ are given in the assumption (A1).

Proof. We test Eq $(3.13)_1$ with $v-w$. Since both v and w are weak solutions, then the assumption $(\mathrm{A1})_2$ gives us that

$$\begin{aligned} C\nu^{-1} \int_{B(x_0,R)} P(|\nabla v - \nabla w|)\,\mathrm{d}x &\leq \int_{B(x_0,R)} \langle \mathcal{A}(x_0,\nabla v) - \mathcal{A}(x_0,\nabla w), \nabla v - \nabla w\rangle\,\mathrm{d}x \\ &= \int_{B(x_0,R)} \langle \mathcal{A}(x_0,\nabla v) - \mathcal{A}(x,\nabla v), \nabla v - \nabla w\rangle\,\mathrm{d}x. \end{aligned}$$

By using $(\mathrm{A1})_3$ with $|x - x_0| \leq R$ and Young's inequality (2.9), we derive that

$$\begin{aligned} C\nu^{-1} \int_{B(x_0,R)} P(|\nabla v - \nabla w|)\,\mathrm{d}x &\leq K\omega(R) \int_{B(x_0,R)} \frac{P\left((|\nabla v|^2 + s^2)^{\frac{1}{2}}\right)}{(|\nabla v|^2 + s^2)^{\frac{1}{2}}} |\nabla v - \nabla w|\,\mathrm{d}x \\ &\leq \varepsilon K\omega(R) \int_{B(x_0,R)} P(|\nabla v - \nabla w|)\,\mathrm{d}x \\ &\quad + C_\varepsilon K\omega(R) \int_{B(x_0,R)} \widetilde{P}\left[\frac{P\left((|\nabla v|^2 + s^2)^{\frac{1}{2}}\right)}{(|\nabla v|^2 + s^2)^{\frac{1}{2}}}\right] \mathrm{d}x. \end{aligned} \tag{3.16}$$

Finally, Lemma 3.6 is proved by using the assumptions (A2), (3.16), and (2.3). □

Lemma 3.7. *Assume that $u \in W^{1,P}(\Omega)$ is a weak solution to* (1.1) *satisfying* (A1), (A2)*, and P is an N-function with $\Delta_2(P,\widetilde{P}) < \infty$, and satisfies Assumption 2.4,* (2.12). *Let w be defined in* (3.13)*, and μ be a Radon measure that satisfies* (M). *There exists a constant $C \equiv C(n, \nu, L, s, C_{\Delta_2}, C_1)$ such that*

$$\fint_{B(x_0,R)} P(|\nabla u - \nabla w|)\,\mathrm{d}x \leq C\,\frac{|\mu|(B(x_0,2R))}{(2R)^{\theta_0}} + CK\omega(R) \fint_{B(x_0,2R)} P(|\nabla u| + s)\,\mathrm{d}x. \tag{3.17}$$

The key to the proof of Lemma 3.7 is the triangle inequality as follows:

$$P(|\nabla u - \nabla w|) \leq C_{\Delta_2}\left[P\left(|\nabla u - \nabla v|\right) + P\left(|\nabla v - \nabla w|\right)\right]$$

with (3.3), (3.15), and (3.2).

Corollary 3.8. *Let $u \in W^{1,P}(\Omega)$ be a weak solution to* (1.1) *under the assumptions* (A1), (A2). *Let P be an N-function satisfying $\Delta_2(P, \widetilde{P}) < \infty$, Assumption 2.4, and* (2.12). *Then there exists a constant $C \equiv C(n, \nu, L, C_{\Delta_2}, C_1) \geq 1$ such that*

$$\fint_{B(x_0,\rho)} P(|\nabla u|)\,\mathrm{d}x \leq C \fint_{B(x_0,R)} P(|\nabla u|)\,\mathrm{d}x + C\left(\frac{R}{\rho}\right)^n \left[\frac{|\mu|(B(x_0,2R))}{(2R)^{\theta_0}} + K\omega(R)\fint_{B(x_0,2R)} P(|\nabla u|)\,\mathrm{d}x\right]$$

holds whenever $B(x_0,\rho) \subset B(x_0,R) \subset B(x_0,2R) \subset \Omega$.

Our goal is to derive an oscillation decay estimate of ∇u. Based on Lemmas 3.5 and 3.7, we first involve the corresponding oscillation decay estimate (3.14) of ∇w, and then compare ∇u and ∇w by (3.17). We note that Lemma 2.10 and the triangle inequality play an essential role in the following lemma.

Lemma 3.9. *Let u be a weak solution to* (1.1) *under the assumptions* (A1), (A2), (M), $\Delta_2(P, \widetilde{P}) < \infty$, *Assumption 2.4, and* (2.12). *Then there exists $C \equiv C(n, \nu, L, s, C_{\Delta_2}, C_1) > 0$ such that*

$$\begin{aligned}\fint_{B(x_0,\rho)} P\left(|\nabla u - (\nabla u)_{B(x_0,\rho)}|\right)\mathrm{d}x \leq\ & C\left(\frac{\rho}{R}\right)^{\tilde{\alpha}} \fint_{B(x_0,2R)} P\left(|\nabla u - (\nabla u)_{B(x_0,2R)}|\right)\mathrm{d}x \\ & + CK\left(\frac{R}{\rho}\right)^n \omega(R) \fint_{B(x_0,2R)} P\left(|\nabla u| + s\right)\mathrm{d}x \\ & + C\left(\frac{R}{\rho}\right)^n \frac{|\mu|(B(x_0,2R))}{(2R)^{\theta_0}}\end{aligned}$$

for $B(x_0,\rho) \subset B(x_0,2R) \subset \Omega$. Here the constant $\tilde{\alpha}$ is introduced in Lemma 3.5.

4. Caccioppoli-type inequality and maximal estimate

In this section, we use the estimate established in Proposition 3.2 to derive the Caccioppoli-type inequality and the maximal estimate. First, the following Caccioppoli-type inequality gives a connection between ∇u and u.

Proposition 4.1. (Caccioppoli-type inequality) *Let $u \in W^{1,P}(\Omega)$ be a weak solution to* (1.1) *with measurable coefficients and satisfy* $(A1)_1$, $(A1)_2$. *Let μ be a Radon measure with* (M). *Suppose P is an N-function satisfying Assumption 2.4. Then there exists a constant $C \equiv C(n, \nu, L, C_1, C_{\Delta_2})$ such that*

$$\fint_{B(x_0,R)} P(|\nabla u|)\,\mathrm{d}x \leq C \fint_{B(x_0,2R)} P\left(\frac{|u - (u)_{B(x_0,2R)}|}{R}\right)\mathrm{d}x + C\,\frac{|\mu|(B(x_0,2R))}{(2R)^{\theta_0}}, \tag{4.1}$$

where $B(x_0,2R) \subset \Omega$.

Proof. We may assume that $(u)_{B(x_0,2R)} = 0$ as if u solves (1.1) also $u - (u)_{B(x_0,2R)}$ dose. Let $\eta \in C_0^\infty(B(x_0,2R))$ such that $0 \leq \eta \leq 1$, and

$$\begin{cases} \eta = 1, & \text{in } B(x_0,R), \\ |\nabla\eta| \leq \dfrac{1}{R}, & \text{in } B(x_0,2R) \setminus B(x_0,R), \\ \eta = 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Let $v \in u + W_0^{1,P}(B(x_0, 2R))$ be the weak solution to (3.1). We choose a test function $\varphi := v\eta$ to (3.1), and obtain

$$\int_{B(x_0,2R)} \langle \mathcal{A}(x, \nabla v), \nabla\varphi \rangle \, \mathrm{d}x = 0.$$

It is clear that

$$\int_{B(x_0,2R)} \langle \mathcal{A}(x, \nabla v), \eta\nabla v \rangle \, \mathrm{d}x = -\int_{B(x_0,2R)} \langle \mathcal{A}(x, \nabla v), v\nabla\eta \rangle \, \mathrm{d}x. \tag{4.3}$$

By Lemma 2.6 and (4.2), we deduce that

$$\int_{B(x_0,R)} P(|\nabla v|) \, \mathrm{d}x \; \le \; C \int_{B(x_0,R)} \langle \mathcal{A}(x, \nabla v), \nabla v \rangle \, \mathrm{d}x \; \le \; C \int_{B(x_0,2R)} \langle \mathcal{A}(x, \nabla v), \eta\nabla v \rangle \, \mathrm{d}x. \tag{4.4}$$

By (4.3), (4.4), Cauchy-Schwartz inequality, and Lemma 2.6, one has

$$\int_{B(x_0,R)} P(|\nabla v|) \, \mathrm{d}x \le \int_{B(x_0,2R)} P'(|\nabla v|) \cdot |v| \; |\nabla\eta| \; \mathrm{d}x.$$

According to Young's inequality (2.10), we derive that for $\varepsilon > 0$, there exists $\widetilde{C}_\varepsilon$ such that

$$\int_{B(x_0,R)} P(|\nabla v|) \, \mathrm{d}x \le \varepsilon \int_{B(x_0,2R)} \widetilde{P}\left(P'(|\nabla v|)\right) \, \mathrm{d}x + \widetilde{C}_\varepsilon \int_{B(x_0,2R)} P\left(|v| \, |\nabla\eta|\right) \, \mathrm{d}x.$$

By (2.3) and Lemma 2.13, we deduce that

$$\int_{B(x_0,R)} P(|\nabla v|) \, \mathrm{d}x \le \widetilde{C}_\varepsilon \int_{B(x_0,2R)\setminus B(x_0,R)} P\left(\frac{|v|}{R}\right) \mathrm{d}x.$$

Dividing by $|B(x_0, R)|$, one gives

$$\fint_{B(x_0,R)} P(|\nabla v|) \, \mathrm{d}x \le 2^n \, \widetilde{C}_\varepsilon \fint_{B(x_0,2R)} P\left(\frac{|v|}{R}\right) \mathrm{d}x. \tag{4.5}$$

Applying the triangle inequality with $P(|\nabla u|) \le C_{\Delta_2} \left(P(|\nabla u - \nabla v|) + P(|\nabla v|)\right)$ and (4.5), one has

$$\fint_{B(x_0,R)} P(|\nabla u|) \, \mathrm{d}x \; \le \; 2^n C_{\Delta_2} \fint_{B(x_0,2R)} P(|\nabla u - \nabla v|) \, \mathrm{d}x \; + \; 2^n \, C_{\Delta_2} \widetilde{C}_\varepsilon \fint_{B(x_0,2R)} P\left(\frac{|v|}{R}\right) \mathrm{d}x. \tag{4.6}$$

In order to estimate the last term of (4.6), we use the triangle inequality again to obtain

$$\fint_{B(x_0,2R)} P\left(\frac{|v|}{R}\right) \mathrm{d}x \le C_{\Delta_2} \fint_{B(x_0,2R)} P\left(\frac{|u - v|}{R}\right) \mathrm{d}x + C_{\Delta_2} \fint_{B(x_0,2R)} P\left(\frac{|u|}{R}\right) \mathrm{d}x. \tag{4.7}$$

Lemma 2.11 and the classical Hölder's inequality give us that

$$\begin{aligned}
\fint_{B(x_0,2R)} P\left(\frac{|u - v|}{R}\right) \mathrm{d}x \; &\le \; C\left(\fint_{B(x_0,2R)} P^{\theta_1}(|\nabla u - \nabla v|) \, \mathrm{d}x\right)^{\frac{1}{\theta_1}} \\
&\le \; C \fint_{B(x_0,2R)} P(|\nabla u - \nabla v|) \, \mathrm{d}x.
\end{aligned} \tag{4.8}$$

Thus we combine (3.3) and (4.6)–(4.8), and conclude that

$$\fint_{B(x_0,R)} P(|\nabla u|) \, \mathrm{d}x \le C \, \frac{|\mu|(B(x_0, 2R))}{(2R)^{\theta_0}} + C \fint_{B(x_0,2R)} P\left(\frac{|u|}{R}\right) \mathrm{d}x.$$

This establishes the Caccioppoli-type inequality (4.1). □

Based on the definition of maximal functions in Section 2.4 and the control estimate in Lemma 3.4, we present the following pointwise estimate involving the maximal functions.

Proposition 4.2. (Maximal estimate) *Let* $u \in W^{1,P}(\Omega)$ *be a weak solution to* (1.1) *under* (A1), (A2). *Let* P *be an N-function with* $\Delta_2(P, \widetilde{P}) < \infty$*, Assumption 2.4, and* (2.12)*. Let Radon measure* μ *satisfy* (M)*. Then there exists a constant* $C \equiv C(n,\ \nu,\ L,\ \theta_0,\ \omega(\cdot),\ C_{\Delta_2},\ C_1)$ *such that*

$$\begin{aligned} & \widetilde{M}^{\#,P}_{1+\varepsilon_0-\alpha,R}(u)(x_0) + \left[M^{P^{\theta_1}}_{(1+\varepsilon_0-\alpha)\theta_1,R}(\nabla u)(x_0) \right]^{\frac{1}{\theta_1}} \\ \leq \ & C\, R^{1+\varepsilon_0}\, \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x_0, 2R) + C, R^{1+\varepsilon_0-\alpha} \fint_{B(x_0,R)} P(|\nabla u| + s)\, \mathrm{d}x. \end{aligned} \tag{4.9}$$

Here $\mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x_0, 2R)$ *is a Riesz potential that is introduced in* (1.4)*. In* (4.9)*, the constants* ε_0*,* $\alpha \in (0, 1]$*,* θ_1 *and* θ_0 *are given in Lemmas 2.5, 3.3, 2.11, and* (1.3)*, respectively.*

Proof. The key of the proof is to consider the radii R satisfying that $R \leq R_0$, where the quantity $R_0 > 0$ is in dependence of the data n, ν, L, α, and $\omega(\cdot)$. More precisely, by (A2), we shall choose R_0 so that

$$\omega(R_0) \leq \delta,$$

where δ will be a small quantity that will be reduced at several stages, as a decreasing function of the quantities n, ν, L, and also α. The proof of Proposition 4.2 is accomplished through two steps, to which the following content is devoted.

By (2.15) and (2.13) with $\theta = 1 + \varepsilon_0 - \alpha$ and $\tau = (1 + \varepsilon_0 - \alpha)\theta_1$, there holds

$$\widetilde{M}^{\#,P}_{1+\varepsilon_0-\alpha,R}(u)(x) \leq C \left[M^{P^{\theta_1}}_{(1+\varepsilon_0-\alpha)\theta_1,R}(\nabla u)(x) \right]^{\frac{1}{\theta_1}}. \tag{4.10}$$

By using Hölder's inequality, we obtain

$$\begin{aligned} \left[M^{P^{\theta_1}}_{(1+\varepsilon_0-\alpha)\theta_1,R}(\nabla u)(x) \right]^{\frac{1}{\theta_1}} &= \sup_{0<r\leq R} \left(r^{(1+\varepsilon_0-\alpha)\theta_1} \fint_{B(x_0,r)} P^{\theta_1}(|\nabla u|)\, \mathrm{d}x \right)^{\frac{1}{\theta_1}} \\ &\leq \sup_{0<r\leq R} \left(r^{1+\varepsilon_0-\alpha} \fint_{B(x_0,r)} P(|\nabla u|)\, \mathrm{d}x \right) \\ &= M^{P}_{1+\varepsilon_0-\alpha,R}(\nabla u)(x_0). \end{aligned} \tag{4.11}$$

Then the inequality (4.9) will follow if we are able to show that

$$M^{P}_{1+\varepsilon_0-\alpha,R}(\nabla u)(x_0) \leq C\, R^{1+\varepsilon_0}\, \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x_0, 2R) + C\, R^{1+\varepsilon_0-\alpha} \fint_{B(x_0,R)} P(|\nabla u| + s)\, \mathrm{d}x. \tag{4.12}$$

Step 1. The case for small radii $R \leq R_0$.

We take $0 < \rho \leq r/2 \leq r \leq R$, and adopt the estimate in Lemma 3.4 with two radii ρ and $r/2$. There exists a constant $C_5 \equiv C_5(n,\ \nu,\ L,\ C_{\Delta_2},\ C_1)$ such that

$$\fint_{B(x_0,\rho)} P(|\nabla u|)\, \mathrm{d}x \leq C_5 \left(\frac{\rho}{r}\right)^{-1+\alpha} \fint_{B(x_0,r)} P(|\nabla u|)\, \mathrm{d}x + C \left(\frac{r}{\rho}\right)^{n} \frac{|\mu|(B(x_0, r))}{r^{\theta_0}}. \tag{4.13}$$

Multiplying both sides of (4.13) by $\rho^{1+\varepsilon_0-\alpha}$, and taking $S = r/\rho$, it follows that

$$\begin{aligned}\rho^{1+\varepsilon_0-\alpha} \fint_{B(x_0,\rho)} P(|\nabla u|)\, \mathrm{d}x \quad &\leq \quad C_5\, S^{-\varepsilon_0}\, r^{1+\varepsilon_0-\alpha} \fint_{B(x_0,r)} P(|\nabla u|)\, \mathrm{d}x \\ &\quad + C\, S^{n+\alpha-\varepsilon_0-1}\, r^{1+\varepsilon_0}\, \frac{|\mu|(B(x_0,r))}{r^{\theta_0+\alpha}}\end{aligned}$$

for $\rho \leq r/2 \leq R/2$. We choose the constant $S \geq 2$ large enough, which satisfies that

$$\frac{C_5}{S^{\varepsilon_0}} \leq \frac{1}{2},$$

and take the supremum with $0 < r \leq R$ such that the following estimate holds

$$\begin{aligned}\sup_{0<r\leq R}\left(\rho^{1+\varepsilon_0-\alpha} \fint_{B(x_0,\rho)} P(|\nabla u|)\, \mathrm{d}x\right) \quad &\leq \quad \frac{1}{2} \sup_{0<r\leq R}\left(r^{1+\varepsilon_0-\alpha} \fint_{B(x_0,r)} P(|\nabla u|)\, \mathrm{d}x\right) \\ &\quad + C\, R^{1+\varepsilon_0} \sup_{0<r\leq R} \frac{|\mu|(B(x_0,r))}{r^{\theta_0+\alpha}},\end{aligned} \tag{4.14}$$

where $0 < r \leq R$ is equivalent to $0 < \rho \leq R/S$. By (2.13) and (4.14), we obtain

$$\sup_{\rho\leq R/S}\left(\rho^{1+\varepsilon_0-\alpha} \fint_{B(x_0,\rho)} P(|\nabla u|)\, \mathrm{d}x\right) \leq \frac{1}{2} M^P_{1+\varepsilon_0-\alpha,\, R}(\nabla u)(x_0) + C\, R^{1+\varepsilon_0} \sup_{0<r\leq R} \frac{|\mu|(B(x_0,r))}{r^{\theta_0+\alpha}} \tag{4.15}$$

with a constant C depending on n, ν, L, C_{Δ_2}, C_1, S, and α.

On the other hand, we notice that

$$\sup_{R/S\leq\rho\leq R}\left(\rho^{1+\varepsilon_0-\alpha} \fint_{B(x_0,\rho)} P(|\nabla u|)\, \mathrm{d}x\right) \leq C\, S^n\, R^{1+\varepsilon_0-\alpha} \fint_{B(x_0,R)} P(|\nabla u| + s)\, \mathrm{d}x. \tag{4.16}$$

Recalling the constant S, and putting (4.15) and (4.16) together , we obtain the following:

$$\begin{aligned}M^P_{1+\varepsilon_0-\alpha,\, R}(\nabla u)(x_0) \quad &\leq \quad \frac{1}{2} M^P_{1+\varepsilon_0-\alpha,\, R}(\nabla u)(x_0) + C\, R^{1+\varepsilon_0} \sup_{0<r\leq R} \frac{|\mu|(B(x_0,r))}{r^{\theta_0+\alpha}} \\ &\quad + C\, R^{1+\varepsilon_0-\alpha} \fint_{B(x_0,R)} P(|\nabla u| + s)\, \mathrm{d}x.\end{aligned} \tag{4.17}$$

The definition of the supremum shows that for any $\varepsilon > 0$, there is $r \in (0, R]$ such that

$$\sup_{0<r\leq R} \frac{|\mu|(B(x_0,r))}{r^{\theta_0+\alpha}} \leq \frac{|\mu|(B(x_0,r))}{r^{\theta_0+\alpha}} + \varepsilon. \tag{4.18}$$

This leads to

$$\frac{|\mu|(B(x_0,r))}{r^{\theta_0+\alpha}} = \frac{|\mu|(B(x_0,r))}{r^{\theta_0+\alpha}} \frac{1}{\ln 2} \int_r^{2r} \frac{\mathrm{d}\rho}{\rho} \leq \frac{2^{\theta_0+\alpha}}{\ln 2} \int_r^{2r} \frac{|\mu|(B(x,\rho))}{\rho^{\theta_0+\alpha}} \frac{\mathrm{d}\rho}{\rho}. \tag{4.19}$$

Since ε is arbitrary, and $0 < r < 2r \leq 2R$, the preceding estimates (4.18) and (4.19) show that there exists a constant $C \equiv C(n, \theta_0)$ such that

$$\sup_{0<r\leq R} \frac{|\mu|(B(x_0,r))}{r^{\theta_0+\alpha}} \leq C \int_0^{2R} \frac{|\mu|(B(x_0,\rho))}{\rho^{\theta_0+\alpha}} \frac{\mathrm{d}\rho}{\rho} = C\, \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x_0, 2R). \tag{4.20}$$

Combining (4.17) and (4.20), we deduce the desired estimate (4.12) with $R \le R_0$.

Step 2. Removing the condition $R \le R_0$.

Our goal is to prove (4.12) without the restriction $R \le R_0$. Taking $R > R_0$ and recalling Definition (2.13), it is clear that

$$\begin{aligned} & M^{P}_{1+\varepsilon_0-\alpha,R}(\nabla u)(x_0) \\ = & \sup_{0<r\le R}\left(r^{1+\varepsilon_0-\alpha}\fint_{B(x_0,r)} P(|\nabla u|)\,\mathrm{d}x\right) \\ \le & \sup_{0<r\le R_0}\left(r^{1+\varepsilon_0-\alpha}\fint_{B(x_0,r)} P(|\nabla u|)\,\mathrm{d}x\right) + \sup_{R_0<r\le R}\left(r^{1+\varepsilon_0-\alpha}\fint_{B(x_0,r)} P(|\nabla u|)\,\mathrm{d}x\right) \\ \le & M^{P}_{1+\varepsilon_0-\alpha,R_0}(\nabla u)(x_0) + \left(\frac{R}{R_0}\right)^n R^{1+\varepsilon_0-\alpha}\fint_{B(x_0,R)} P(|\nabla u|+s)\,\mathrm{d}x. \end{aligned} \tag{4.21}$$

We apply (4.12) with radius R_0, i.e.,

$$\begin{aligned} M^{P}_{1+\varepsilon_0-\alpha,R_0}(\nabla u)(x_0) \le\ & C R_0^{1+\varepsilon_0}\,\mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x_0,2R_0) \\ & + C R_0^{1+\varepsilon_0-\alpha}\fint_{B(x_0,R_0)} P(|\nabla u|+s)\,\mathrm{d}x. \end{aligned} \tag{4.22}$$

By the definition of Riesz potential (1.4), one has

$$\begin{aligned} R_0^{1+\varepsilon_0}\,\mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x_0,2R_0) &= R_0^{1+\varepsilon_0}\int_0^{2R_0}\frac{|\mu|(B(x_0,\rho))}{\rho^{\theta_0+\alpha}}\frac{\mathrm{d}\rho}{\rho} \\ \le R^{1+\varepsilon_0}\int_0^{2R}\frac{|\mu|(B(x_0,\rho))}{\rho^{\theta_0+\alpha}}\frac{\mathrm{d}\rho}{\rho} &= R^{1+\varepsilon_0}\,\mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x_0,2R). \end{aligned} \tag{4.23}$$

It is apparent to enlarge the integral by

$$R_0^{1+\varepsilon_0-\alpha}\fint_{B(x_0,R_0)} P(|\nabla u|+s)\,\mathrm{d}x \le \left(\frac{R}{R_0}\right)^n R^{1+\varepsilon_0-\alpha}\fint_{B(x_0,R)} P(|\nabla u|+s)\,\mathrm{d}x. \tag{4.24}$$

By using (4.21)–(4.24), we derive

$$M^{P}_{1+\varepsilon_0-\alpha,R}(\nabla u)(x_0) \le R^{1+\varepsilon_0}\,\mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x_0,2R) + \left(\frac{R}{R_0}\right)^n R^{1+\varepsilon_0-\alpha}\fint_{B(x_0,R)} P(|\nabla u|+s)\,\mathrm{d}x.$$

Since Ω is bounded, then (4.12) holds.

Combining (4.10)–(4.12), we obtain (4.9), which means the proof of Proposition 4.2 is completed. □

5. Proofs of Theorems 1.1–1.3

By establishing the preceding technical tools and lemmas, we are in a position to present the proofs of the main theorems. We first have the following proof.

Proof of Theorem 1.1. We set a sequence of balls $\{B_i\}_{i=0}^{\infty}$ by

$$B_i := B(x_0, R_i) = B\left(x_0, \frac{R}{(2\Lambda)^i}\right), \tag{5.1}$$

where $2\Lambda > 1$ will be chosen later. It is clear that $B_{i+1} \subset B_i$ for every $i \geq 0$. We set two sequences $\{K_i\}_{i=0}^{\infty}$ and $\{k_i\}_{i=0}^{\infty}$ by

$$K_i := \fint_{B_i} P\left(\left|\nabla u - (\nabla u)_{B_i}\right|\right) \mathrm{d}x, \quad k_i := P\left(\left|(\nabla u)_{B_i}\right|\right) + P(s). \tag{5.2}$$

We also introduce $\widetilde{k_0}$ by

$$\widetilde{k_0} := \fint_{B(x_0,R)} [P(|\nabla u|) + P(s)]\ \mathrm{d}x. \tag{5.3}$$

By (2.10), it is obvious that

$$k_0 = P\left(\left|(\nabla u)_{B_0}\right|\right) + P(s) = P\left(\left|\fint_{B(x_0,R)} \nabla u \,\mathrm{d}x\right|\right) + P(s) \leq \widetilde{k_0}, \tag{5.4}$$

as well as

$$K_0 = \fint_{B_0} P\left(\left|\nabla u - (\nabla u)_{B_0}\right|\right) \mathrm{d}x \leq C \fint_{B(x_0,R)} P(|\nabla u|) \,\mathrm{d}x \leq \widetilde{k_0}. \tag{5.5}$$

Step 1. An estimate of the summation of K_i.

An application of Lemma 3.9 with $B(x_0, \rho) \equiv B\left(x_0, \frac{R}{2\Lambda}\right) \subset B(x_0, R)$ shows that

$$\begin{aligned}
&\fint_{B\left(x_0, \frac{R}{2\Lambda}\right)} P\left(\left|\nabla u - (\nabla u)_{B\left(x_0, \frac{R}{2\Lambda}\right)}\right|\right) \mathrm{d}x \\
\leq\ & C\left(\frac{1}{2\Lambda}\right)^{\tilde{\alpha}} \fint_{B(x_0,R)} P\left(\left|\nabla u - (\nabla u)_{B(x_0,R)}\right|\right) \mathrm{d}x \\
&+ C(2\Lambda)^n \frac{|\mu|(B(x_0,2R))}{(2R)^{\theta_0}} + C(2\Lambda)^n K\omega(R) \fint_{B(x_0,R)} P(|\nabla u| + s) \,\mathrm{d}x,
\end{aligned}$$

where the constants C depend on $n, \nu, L, s, C_{\Delta_2}$, and C_1. Using (5.1), we choose $\Lambda \equiv \Lambda(n, \nu, L, s, C_{\Delta_2}, C_1) > 1$ large enough such that

$$C\left(\frac{1}{2\Lambda}\right)^{\tilde{\alpha}} \leq \frac{1}{4},$$

where $\tilde{\alpha} \in (0, 1]$ is given in Lemma 3.5. By (2.2), it is clear that

$$\fint_{B(x_0,R)} P(|\nabla u| + s) \,\mathrm{d}x \leq C\left[\fint_{B(x_0,R)} P\left(\left|\nabla u - (\nabla u)_{B(x_0,R)}\right|\right) \mathrm{d}x + P\left(\left|(\nabla u)_{B(x_0,R)}\right|\right) + P(s)\right].$$

Hence there exists a constant $C_6 > 0$ depending on $n, \nu, L, K, s, C_{\Delta_2}$, and C_1 such that

$$
\begin{aligned}
&\fint_{B\left(x_0,\frac{R}{2\Lambda}\right)} P\left(\left|\nabla u-(\nabla u)_{B\left(x_0,\frac{R}{2\Lambda}\right)}\right|\right)\,\mathrm{d}x\\
\le\ &\left(\frac{1}{4}+C_6\,\omega(R)\right)\fint_{B(x_0,R)} P\left(\left|\nabla u-(\nabla u)_{B(x_0,R)}\right|\right)\,\mathrm{d}x\\
&+C\,\frac{|\mu|(B(x_0,2R))}{(2R)^{\theta_0}}+C_6\,\omega(R)\left[P\left(\left|(\nabla u)_{B(x_0,R)}\right|\right)+P(s)\right].
\end{aligned}
\tag{5.6}
$$

Using (A2), we take $\widetilde{R}$ small enough to obtain

$$
C_6\,\omega\left(\widetilde{R}\right)\le\frac{1}{4}.
$$

It follows that if $R\le\widetilde{R}$, then all $R_i\le\widetilde{R}$. Applying the estimate (5.6) with $R\equiv R_{i-1}$, and noting that $\omega(\cdot)$ is non-decreasing, it yields that

$$
\begin{aligned}
&\fint_{B_i} P\left(\left|\nabla u-(\nabla u)_{B_i}\right|\right)\,\mathrm{d}x\\
\le\ &\frac{1}{2}\fint_{B_{i-1}} P\left(\left|\nabla u-(\nabla u)_{B_{i-1}}\right|\right)\,\mathrm{d}x+C\,\frac{|\mu|(2B_{i-1})}{(2R_{i-1})^{\theta_0}}+C\omega(R_{i-1})\left[P\left(\left|(\nabla u)_{B_{i-1}}\right|\right)+P(s)\right],
\end{aligned}
$$

which can be simplified as

$$
K_i\ \le\ \frac{1}{2}K_{i-1}\ +C\,\frac{|\mu|(2B_{i-1})}{(2R_{i-1})^{\theta_0}}\ +C\omega(R_{i-1})\,k_{i-1}.
\tag{5.7}
$$

Via a summation, one deduces that

$$
\sum_{i=1}^{m}K_i\ \le\ \frac{1}{2}\sum_{i=0}^{m-1}K_i\ +\ C\sum_{i=0}^{m-1}\frac{|\mu|(2B_i)}{(2R_i)^{\theta_0}}\ +\ C\sum_{i=0}^{m-1}\omega(R_i)\,k_i
$$

for $C\equiv C(n,\,\nu,\,L,\,K,\,s,\,C_{\Delta_2},\,C_1)$ and for every integer m. This implies that

$$
\sum_{i=1}^{m}K_i\ \le\ K_0\ +\ 2C\sum_{i=0}^{m-1}\frac{|\mu|(2B_i)}{(2R_i)^{\theta_0}}\ +\ 2C\sum_{i=0}^{m-1}\omega(R_i)\,k_i
\tag{5.8}
$$

holds for every $m\in\mathbb{N}$.

Step 2. An estimate of k_{m+1}.

Using (5.2), we have

$$
k_{m+1}\ :=\ \sum_{i=0}^{m}(k_{i+1}-k_i)+k_0\ \le\ \sum_{i=0}^{m}\left|P\left(\left|(\nabla u)_{B_{i+1}}\right|\right)-P\left(\left|(\nabla u)_{B_i}\right|\right)\right|+k_0.
\tag{5.9}
$$

By using Lemma 2.9 with $x=\left|(\nabla u)_{B_{i+1}}\right|$, $y=\left|(\nabla u)_{B_i}\right|$, we estimate the difference as

$$
\left|P\left(\left|(\nabla u)_{B_{i+1}}\right|\right)-P\left(\left|(\nabla u)_{B_i}\right|\right)\right|\le\varepsilon\,C_{\Delta_2}P\left(\left|(\nabla u)_{B_i}\right|\right)+CP\left(\left|\,\left|(\nabla u)_{B_{i+1}}\right|-\left|(\nabla u)_{B_i}\right|\,\right|\right).
$$

Considering the triangle inequality $||x|-|y|| \le |x-y|$ and Lemma 2.10, we have

$$\begin{aligned} P\left(\left|\left|(\nabla u)_{B_{i+1}}\right|-\left|(\nabla u)_{B_i}\right|\right|\right) &\le CP\left(\fint_{B_{i+1}} \left|\nabla u-(\nabla u)_{B_i}\right| \,\mathrm{d}x\right) \\ &\le C(2\Lambda)^n \fint_{B_i} P\left(\left|\nabla u-(\nabla u)_{B_i}\right|\right) \mathrm{d}x. \end{aligned}$$

Hence there exists a constant $C \equiv C(n,\ \nu,\ L,\ s,\ C_{\Delta_2},\ C_1)$ such that

$$\left| P\left(\left|(\nabla u)_{B_{i+1}}\right|\right) - P\left(\left|(\nabla u)_{B_i}\right|\right)\right| \le \varepsilon C_{\Delta_2} P\left(\left|(\nabla u)_{B_i}\right|\right) + C \fint_{B_i} P\left(\left|\nabla u-(\nabla u)_{B_i}\right|\right) \mathrm{d}x. \tag{5.10}$$

For each i, we choose $\varepsilon = \varepsilon(i, n, C_{\Delta_2}, \Lambda)$ small enough such that

$$\varepsilon C_{\Delta_2} \le \frac{1}{(2\Lambda)^{i(n+1)}}.$$

Then we have

$$\begin{aligned} \varepsilon C_{\Delta_2} P\left(\left|(\nabla u)_{B_i}\right|\right) &\le \varepsilon C_{\Delta_2} k_i \le \varepsilon C_{\Delta_2} \fint_{B_i} \left[P\left(|(\nabla u)|\right)+P(s)\right] \mathrm{d}x \\ &\le \frac{1}{(2\Lambda)^{i(n+1)}} \left[(2\Lambda)^i\right]^n \fint_{B(x_0,R)} \left[P\left(|(\nabla u)|\right)+P(s)\right] \mathrm{d}x \\ &= \frac{1}{(2\Lambda)^i} \widetilde{k}_0. \end{aligned} \tag{5.11}$$

Here we use the fact that the sum of geometric series is finite, i.e.,

$$\sum_{i=0}^{m} \left(\frac{1}{2\Lambda}\right)^i \le \sum_{i=0}^{\infty} \left(\frac{1}{2\Lambda}\right)^i = \frac{1}{1-\frac{1}{2\Lambda}}. \tag{5.12}$$

By combining (5.9)–(5.12), and (5.4), there exists a constant C such that

$$k_{m+1} \le C \sum_{i=0}^{m} \left[K_i + \frac{1}{(2\Lambda)^i} \widetilde{k}_0\right] + k_0 \le C \sum_{i=0}^{m} K_i + C \widetilde{k}_0. \tag{5.13}$$

Making use of (5.13), (5.8), and (5.5), one derives that for every integer $m \ge 1$, there holds

$$\begin{aligned} k_{m+1} &\le C\left[K_0 + \sum_{i=0}^{m-1} \frac{|\mu|(2B_i)}{(2R_i)^{\theta_0}} + \sum_{i=0}^{m-1} \omega(R_i)\, k_i + \widetilde{k}_0\right] \\ &\le C\left[\widetilde{k}_0 + \sum_{i=0}^{m-1} \frac{|\mu|(2B_i)}{(2R_i)^{\theta_0}} + \sum_{i=0}^{m-1} \omega(R_i)\, k_i\right]. \end{aligned} \tag{5.14}$$

For the second term on the right side of (5.14), one has

$$\sum_{i=0}^{m-1} \frac{|\mu|(2B_i)}{(2R_i)^{\theta_0}} \le \sum_{i=0}^{\infty} \frac{|\mu|(2B_i)}{(2R_i)^{\theta_0}} \le \frac{|\mu|(B(x_0,2R))}{(2R)^{\theta_0}} + \sum_{i=0}^{\infty} \frac{|\mu|(2B_{i+1})}{(2R_{i+1})^{\theta_0}}.$$

Using $\Lambda > 1$, the method adopted in (4.20), and (1.4), we obtain that

$$\begin{aligned}\sum_{i=0}^{m-1}\frac{|\mu|(2B_i)}{(2R_i)^{\theta_0}} &\leq \frac{2^{\theta_0}}{\ln 2}\int_{2R}^{4R}\frac{|\mu|(B(x_0,\rho))}{\rho^{\theta_0}}\frac{\mathrm{d}\rho}{\rho}+\frac{(2\Lambda)^{\theta_0}}{\ln(2\Lambda)}\sum_{i=0}^{\infty}\int_{2R_{i+1}}^{2R_i}\frac{|\mu|(B(x_0,\rho))}{\rho^{\theta_0}}\frac{\mathrm{d}\rho}{\rho}\\ &\leq C\,\mathbf{I}^{\mu}_{n-\theta_0}(x_0,4R) \end{aligned}\tag{5.15}$$

holds with a constant C depending on n and θ_0. Inserting (5.15) in (5.14), we obtain the following inequality:

$$k_{m+1}\leq C\left(\widetilde{k}_0+\mathbf{I}^{\mu}_{n-\theta_0}(x_0,4R)\right)+C\sum_{i=0}^{m-1}\omega(R_i)\,k_i. \tag{5.16}$$

Step 3. An induction approach.

By setting

$$J:=\widetilde{k}_0+\mathbf{I}^{\mu}_{n-\theta_0}(x_0,4R)=\fint_{B(x_0,R)}[P(|\nabla u|)+P(s)]\;\mathrm{d}x+\mathbf{I}^{\mu}_{n-\theta_0}(x_0,4R),$$

we shall use the mathematical induction to prove

$$k_{m+1}\leq C\,J. \tag{5.17}$$

Initial Step. If $m=-1$, then by (5.4), we see that (5.17) is trivial. For the case $m=0$, (5.17) holds by using (5.13).

Inductive Step. Assuming that (5.17) is valid for any $\tilde{m}<m$, we shall prove it for $m+1$. By (5.16), we have

$$k_{m+1}\;\leq\;C\,J+C\sum_{i=0}^{m-1}\omega(R_i)\,k_i\;\leq\;C\,J+C\,J\sum_{i=0}^{m-1}\omega(R_i).$$

Due to the fact that $\omega(\cdot)$ is non-decreasing, we estimate

$$\begin{aligned}\sum_{i=0}^{m-1}\omega(R_i) &\leq \omega(R_0)+\sum_{i=0}^{\infty}\omega(R_{i+1})\\ &\leq \frac{1}{\ln 2}\int_R^{2R}\omega(\rho)\frac{\mathrm{d}\rho}{\rho}+\sum_{i=0}^{\infty}\omega(R_{i+1})\\ &\leq \frac{1}{\ln 2}\int_R^{2R}\omega(\rho)\frac{\mathrm{d}\rho}{\rho}+\frac{1}{\ln(2\Lambda)}\sum_{i=0}^{\infty}\int_{R_{i+1}}^{R_i}\omega(\rho)\frac{\mathrm{d}\rho}{\rho}\\ &\leq \left(\frac{1}{\ln 2}+\frac{1}{\ln(2\Lambda)}\right)\int_0^{2R}\omega(\rho)\frac{\mathrm{d}\rho}{\rho}.\end{aligned}$$

Considering the fact that $\Lambda > 1$ and the definition of $d(\cdot)$ in (A3), we have

$$\sum_{i=0}^{m-1}\omega(R_i)\leq\frac{2\,d(2R)}{\ln 2}. \tag{5.18}$$

By applying (5.18), we complete the proof of the inequality (5.17).

For every Lebesgue point x_0 of $P(|(\nabla u)|)$, we let $m \to \infty$, and show that

$$\begin{aligned} P(|\nabla u(x_0)|) + P(s) &= \lim_{m\to\infty} k_{m+1} \\ &\le C \fint_{B(x_0,R)} [P(|\nabla u|) + P(s)]\, \mathrm{d}x + C\, \mathbf{I}^{\mu}_{n-\theta_0}(x_0, 4R), \end{aligned}$$

where C depends on n, ν, L, K, s, θ_0, C_{Δ_2} and C_1. Therefore, (1.6) has been proved. □

Via a similar approach to the previous proof, we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. We introduce a sequence of concentric balls $\{\widetilde{B}_i\}_{i=0}^{\infty}$ by

$$\widetilde{B}_i := B(x_0, r_i) = B\left(x_0, \frac{r}{(2H)^i}\right),$$

where $H > 1$ is a constant determined later, and $r \le R < 1$. Hence $\widetilde{B}_{i+1} \subset B\left(x_0, \frac{r_i}{2}\right) \subset \widetilde{B}_i$ for every $i \ge 0$. We define

$$A_i := r_i^{1+\varepsilon_0} \fint_{\widetilde{B}_i} P\left(\frac{\left|u - (u)_{\widetilde{B}_i}\right|}{r_i}\right) \mathrm{d}x, \quad \text{and} \quad a_i := P\left(\left|(u)_{\widetilde{B}_i}\right|\right).$$

We also introduce $\widetilde{a}_0$ by

$$\widetilde{a}_0 := r^{1+\varepsilon_0} \fint_{B(x_0,r)} P\left(\frac{|u|}{r}\right) \mathrm{d}x.$$

By (2.10) and (2.5), it is obvious that

$$a_0 = P\left(\left|\fint_{\widetilde{B}_0} u\, \mathrm{d}x\right|\right) \le \fint_{B(x_0,r)} P(|u|)\, \mathrm{d}x \le C_1\, r^{1+\varepsilon_0} \fint_{B(x_0,r)} P\left(\frac{|u|}{r}\right) \mathrm{d}x = C\widetilde{a}_0, \tag{5.19}$$

as well as

$$\begin{aligned} A_0 &= r^{1+\varepsilon_0} \fint_{\widetilde{B}_0} P\left(\frac{\left|u - (u)_{\widetilde{B}_0}\right|}{r}\right) \mathrm{d}x \\ &\le C_{\Delta_2}\, r^{1+\varepsilon_0} \left[\fint_{\widetilde{B}_0} P\left(\frac{|u|}{r}\right) \mathrm{d}x + P\left(\frac{\left|(u)_{\widetilde{B}_0}\right|}{r}\right)\right] \le C\widetilde{a}_0. \end{aligned} \tag{5.20}$$

One applies Lemma 2.11 and Hölder's inequality to obtain

$$A_{i+1} \le C r_{i+1}^{1+\varepsilon_0} \left[\fint_{\widetilde{B}_{i+1}} P^{\theta_1}(|\nabla u|)\, \mathrm{d}x\right]^{\frac{1}{\theta_1}} \le C\, r_{i+1}^{1+\varepsilon_0} \fint_{\widetilde{B}_{i+1}} P(|\nabla u|)\, \mathrm{d}x.$$

Applying Lemma 3.4 with $\rho \equiv r_{i+1}$, $R \equiv \frac{r_i}{2}$, one has

$$A_{i+1} \le C\left[r_i\, r_{i+1}^{\varepsilon_0} \left(\frac{2r_{i+1}}{r_i}\right)^{\alpha} \fint_{B\left(x_0, \frac{r_i}{2}\right)} P(|\nabla u|)\, \mathrm{d}x + r_{i+1}^{1+\varepsilon_0} \left(\frac{r_i}{2r_{i+1}}\right)^{n} \frac{|\mu|(\widetilde{B}_i)}{r_i^{\theta_0}}\right], \tag{5.21}$$

where the constant C depends on n, ν, L, C_{Δ_2}, and C_1. By Caccioppoli-type inequality (4.1) and Definition 2.3, we obtain

$$\fint_{B\left(x_0,\frac{r_i}{2}\right)} P(|\nabla u|)\,\mathrm{d}x \le C \fint_{\widetilde{B}_i} P\left(\frac{\left|u-(u)_{\widetilde{B}_i}\right|}{r_i}\right)\mathrm{d}x + C\,\frac{|\mu|(\widetilde{B}_i)}{r_i^{\theta_0}}. \tag{5.22}$$

Combining (5.21) and (5.22) with $r_{i+1} \le r_i$, it follows that

$$\begin{aligned} A_{i+1} &\le C\left\{\left(\frac{1}{H}\right)^{\alpha} r_i^{1+\varepsilon_0} \fint_{\widetilde{B}_i} P\left(\frac{\left|u-(u)_{\widetilde{B}_i}\right|}{r_i}\right)\mathrm{d}x + \left[r_i^{1+\varepsilon_0}\left(\frac{1}{H}\right)^{\alpha} + r_{i+1}^{1+\varepsilon_0} H^n\right]\frac{|\mu|(\widetilde{B}_i)}{r_i^{\theta_0}}\right\} \\ &\le C_7\left(\frac{1}{H}\right)^{\alpha} A_i + C\,\frac{|\mu|(\widetilde{B}_i)}{r_i^{\theta_0}}, \end{aligned}$$

where C_7 depends on n, ν, L, C_{Δ_2}, C_1, and R. By choosing $H \equiv H(n, \nu, L, C_{\Delta_2}, C_1, R)$ large enough, one has

$$\left(\frac{1}{H}\right)^{\alpha} \le \frac{1}{2\,C_7},$$

which implies immediately that

$$A_{i+1} \le \frac{1}{2} A_i + C\,\frac{|\mu|(\widetilde{B}_i)}{r_i^{\theta_0}}.$$

We consider a summation with respect to i from 0 to $m-1$, and deduce that

$$\sum_{i=1}^{m} A_i \le A_0 + 2\,C \sum_{i=0}^{m-1} \frac{|\mu|(\widetilde{B}_i)}{r_i^{\theta_0}}. \tag{5.23}$$

For every non-negative integer m, one writes

$$a_{m+1} := \sum_{i=0}^{m}(a_{i+1}-a_i) + a_0 \le \sum_{i=0}^{m}\left| P\left(\left|(u)_{\widetilde{B}_{i+1}}\right|\right) - P\left(\left|(u)_{\widetilde{B}_i}\right|\right)\right| + a_0.$$

Adopting a similar approach as in the proof of (5.10), we have

$$\begin{aligned} &\left| P\left(\left|(u)_{\widetilde{B}_{i+1}}\right|\right) - P\left(\left|(u)_{\widetilde{B}_i}\right|\right)\right| \\ \le\;& \varepsilon\, C_{\Delta_2} P\left(\left|(u)_{\widetilde{B}_i}\right|\right) + C \fint_{\widetilde{B}_i} P\left(\left|u-(u)_{\widetilde{B}_i}\right|\right)\mathrm{d}x \\ \le\;& \varepsilon\, C_{\Delta_2} P\left(\left|(u)_{\widetilde{B}_i}\right|\right) + C\, r_i^{1+\varepsilon_0} \fint_{\widetilde{B}_i} P\left(\frac{\left|u-(u)_{\widetilde{B}_i}\right|}{r_i}\right)\mathrm{d}x \\ =\;& \varepsilon\, C_{\Delta_2}\, a_i + C\, A_i, \end{aligned}$$

and choose $\varepsilon = (2H)^{-i(n+1)}$ sufficiently small. With the help of (5.19), it follows that there exists a constant $C \equiv C(n, \nu, L, C_{\Delta_2}, C_1, R)$ such that

$$a_{m+1} \le C \sum_{i=0}^{m}\left[\frac{1}{(2H)^i}\,\widetilde{a}_0 + A_i\right] + a_0 \le C \sum_{i=0}^{m} A_i + C\,\widetilde{a}_0. \tag{5.24}$$

Analogous to (5.15), we obtain

$$\sum_{i=0}^{m-1} \frac{|\mu|(\widetilde{B}_i)}{r_i^{\theta_0}} \le r^{\alpha} \sum_{i=0}^{m-1} \frac{|\mu|(\widetilde{B}_i)}{r_i^{\theta_0+\alpha}} \le C\, r^{\alpha}\, \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x_0, 2r). \tag{5.25}$$

Applying (5.23)–(5.25) and (5.20), one gets that for every integer $m \ge 1$, there holds

$$\begin{aligned} a_{m+1} &\le C\left[A_0 + \widetilde{a}_0 + \sum_{i=0}^{m-1} \frac{|\mu|(\widetilde{B}_i)}{r_i^{\theta_0}}\right] \\ &\le C\, r^{1+\varepsilon_0} \fint_{B(x_0,r)} P\left(\frac{|u|}{r}\right) \mathrm{d}x + C\, r^{\alpha}\, \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x_0, 2r). \end{aligned}$$

By the dominated convergence theorem, for every Lebesgue point $x_0 \in P(|u|)$, there holds

$$P(|u(x_0)|) = \lim_{m\to\infty} a_{m+1} \le C\, r^{1+\varepsilon_0} \fint_{B(x_0,r)} P\left(\frac{|u|}{r}\right) \mathrm{d}x + C\, r^{\alpha}\, \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x_0, 2r), \tag{5.26}$$

where constant C depends on n, ν, L, θ_0, C_{Δ_2}, C_1, and diam(Ω). □

By establishing (5.26), we are ready to prove the result of Theorem 1.3.

Proof of Theorem 1.3. For any real number g, we observe that if u is a weak solution to (1.1), then $u - g$ is still a solution to (1.1). Let $B(x_0, 2R) \subset \Omega$. We consider $x, y \in B\left(x_0, \frac{R}{2}\right)$ satisfying that $r := |x - y| < \frac{R}{4}$. By (5.26), it follows that

$$\begin{aligned} P\left(|u(x) - g|\right) &\le C\, r^{1+\varepsilon_0} \fint_{B(x,r)} P\left(\frac{|u-g|}{r}\right) \mathrm{d}\zeta + C\, r^{\alpha}\, \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x, 2r), \\ P\left(|u(y) - g|\right) &\le C\, r^{1+\varepsilon_0} \fint_{B(y,r)} P\left(\frac{|u-g|}{r}\right) \mathrm{d}\zeta + C\, r^{\alpha}\, \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(y, 2r). \end{aligned}$$

By (1.4), one has the following monotone property:

$$\mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x, 2r) \le \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x, 2R), \text{ and } \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(y, 2r) \le \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(y, 2R).$$

Via (2.2) and a direct computation, one has

$$\begin{aligned} P\left(|u(x) - u(y)|\right) &\le C_{\Delta_2}\left[P\left(|u(x) - g|\right) + P\left(|u(y) - g|\right)\right] \\ &\le C\, r^{1+\varepsilon_0}\left[\fint_{B(x,r)} P\left(\frac{|u-g|}{r}\right) \mathrm{d}\zeta + \fint_{B(y,r)} P\left(\frac{|u-g|}{r}\right) \mathrm{d}\zeta\right] \\ &\quad + C\, r^{\alpha}\left[\mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x, 2R) + \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(y, 2R)\right]. \end{aligned} \tag{5.27}$$

We take $g := (u)_{B(x,2r)}$, and observe that $B(x, r) \cup B(y, r) \subset B(x, 2r) \subset B\left(x, \frac{R}{2}\right)$. Using Definition 2.3, we deduce that

$$\fint_{B(x,r)} P\left(\frac{|u-g|}{r}\right) \mathrm{d}\zeta + \fint_{B(y,r)} P\left(\frac{|u-g|}{r}\right) \mathrm{d}\zeta \le C\, C_{\Delta_2} \fint_{B(x,2r)} P\left(\frac{|u - (u)_{B(x,2r)}|}{2r}\right) \mathrm{d}\zeta.$$

Since $2r < \frac{R}{2}$, then (2.14) gives us that

$$\fint_{B(x,2r)} P\left(\frac{|u-(u)_{B(x,2r)}|}{2r}\right) \mathrm{d}\zeta \leq (2r)^{-(1+\varepsilon_0-\alpha)} \cdot \widetilde{M}^{\#,P}_{1+\varepsilon_0-\alpha,\,\frac{R}{2}}(u)(x).$$

Then it follows from Proposition 4.2 that

$$\begin{aligned} & \fint_{B(x,r)} P\left(\frac{|u-g|}{r}\right) \mathrm{d}\zeta + \fint_{B(y,r)} P\left(\frac{|u-g|}{r}\right) \mathrm{d}\zeta \\ \leq\ & C r^{-(1+\varepsilon_0-\alpha)} R^{1+\varepsilon_0} \left\{ \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x,2R) + R^{-\alpha}\left[\fint_{B\left(x,\frac{R}{2}\right)} P(|\nabla u|)\,\mathrm{d}\zeta + P(s)\right]\right\}. \end{aligned} \tag{5.28}$$

To estimate the last integral, we use Caccioppoli-type inequality (4.1) and (4.19) to obtain that

$$\begin{aligned} \fint_{B\left(x,\frac{R}{2}\right)} P(|\nabla u|)\,\mathrm{d}\zeta &\leq C\,C_{\Delta_2} \fint_{B(x,R)} P\left(\frac{|u|}{R}\right) \mathrm{d}\zeta + C\,R^{\alpha}\,\frac{|\mu|(B(x,R))}{R^{\theta_0+\alpha}} \\ &\leq C \fint_{B(x,R)} P\left(\frac{|u|}{R}\right) \mathrm{d}\zeta + C\,R^{\alpha}\,\mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x,2R). \end{aligned} \tag{5.29}$$

Substituting (5.29) into (5.28) and considering $0 \leq \alpha < 1$, it yields that

$$\begin{aligned} & \fint_{B(x,r)} P\left(\frac{|u-g|}{r}\right) \mathrm{d}\zeta + \fint_{B(y,r)} P\left(\frac{|u-g|}{r}\right) \mathrm{d}\zeta \\ \leq\ & C r^{-(1+\varepsilon_0-\alpha)} R^{1+\varepsilon_0} \left\{ \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x,2R) + R^{-\alpha}\left[\fint_{B(x,R)} P\left(\frac{|u|}{R}\right) \mathrm{d}\zeta + P(s)\right]\right\}. \end{aligned} \tag{5.30}$$

Combining (5.27)–(5.30) together with $B(x,R) \subset B(x_0,2R)$, there is a constant C such that

$$\begin{aligned} & P\left(|u(x)-u(y)|\right) \\ \leq\ & C\,r^{\alpha}\left[\mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x,2R) + \mathbf{I}^{\mu}_{n-\theta_0-\alpha}(y,2R)\right] + C\,r^{\alpha}\,R^{1+\varepsilon_0}\,\mathbf{I}^{\mu}_{n-\theta_0-\alpha}(x,2R) \\ & + C\,r^{\alpha} R^{1+\varepsilon_0-\alpha}\left[\fint_{B(x_0,2R)} P\left(\frac{|u|}{R}\right) \mathrm{d}\zeta + P(s)\right], \end{aligned} \tag{5.31}$$

where the constant C depends on n, ν, L, θ_0, $\omega(\cdot)$, C_{Δ_2}, C_1, and $\mathrm{diam}(\Omega)$. Since $R < 1$ and $\alpha \in [0,1)$, then the estimate (5.31) is the desired interior Hölder estimate of Theorem 1.3. □

6. Conclusions

In this work, we establish pointwise potential estimates of weak solutions to a class of elliptic equations in divergence form with measure data. Our primary result is to employ the Riesz potential to prove the pointwise estimates of the solutions. The key innovation of this paper manifests in the proof of Proposition 3.2, which enables the relationship between measure data and the Riesz potential in the Sobolev-Orlicz spaces. Furthermore, we obtain Hölder continuity estimates for the solutions by establishing the Caccioppoli-type inequality and the maximal estimate. This systematic approach extends the potential estimates of regularity for nonlinear elliptic equations in the existing literature.

Author contributions

Zhaoyue Sui: Conceptualization, methodology, writing–original draft preparation; Feng Zhou: Supervision, funding acquisition, project administration, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the anonymous referee for their careful reading and valuable comments. The authors are supported by the National Natural Science Foundation of China (NNSF Grant No. 12001333) and the Shandong Provincial Natural Science Foundation (Grant No. ZR2020QA005).

Conflict of interest

The authors declare no conflicts of interest regarding the publication of this article.

References

1. T. Kilpeläinen, J. Malý, Degenerate elliptic equations with measure data and nonlinear potentials, *Ann. Scuola Norm. Sci.*, **19** (1992), 591–613. https://doi.org/10.1016/0022-1236(89)90005-0
2. T. Kilpeläinen, J. Malý, The wiener test and potential estimates for quasilinear elliptic equations, *Acta Math.*, **172** (1994), 137–161. https://doi.org/10.1007/BF02392793
3. N. Trudinger, X. Wang, On the week continuity of elliptic operators and applications to potential theory, *Am. J. Math.*, **124** (2002), 369–410. https://doi.org/10.1353/AJM.2002.0012
4. F. Duzaar, G. Mingione, Gradient estimates in non-linear potential theory, *Rend. Lincei. Mat. Appl.*, **20** (2009), 179–190. https://doi.org/10.4171/RLM/540
5. L. Hedberg, T. Wolff, Thin sets in nonlinear potential theory, *Ann. I. Fourier*, **33** (1983), 161–187. https://doi.org/10.5802/aif.944
6. T. Kuusi, G. Mingione, Linear potentials in nonlinear potential theory, *Arch. Ration. Mech. Anal.*, **207** (2013), 215–246. https://doi.org/10.1007/s00205-012-0562-z
7. G. Mingione, Gradient potential estimates, *J. Eur. Math. Soc.*, **13** (2011), 459–486. https://doi.org/10.4171/JEMS/258
8. P. Baroni, Riesz potential estimates for a general class of quasilinear equations, *Calc. Var.*, **53** (2015), 803–846. https://doi.org/10.1007/s00526-014-0768-z
9. T. Kuusi, G. Mingione, Riesz potentials and nonlinear parabolic equations, *Arch. Ration. Mech. Anal.*, **212** (2014), 727–780. https://doi.org/10.1007/s00205-013-0695-8

10. T. Kuusi, G. Mingione, Universal potential estimates, *J. Funct. Anal.*, **262** (2021), 4205–4269. https://doi.org/10.1016/j.jfa.2012.02.018
11. F. Yao, M. Zheng, Gradient estimates via the Wolff potentials for a class of quasilinear elliptic equations, *J. Math. Anal. Appl.*, **452** (2017), 926–940. https://doi.org/10.1016/j.jmaa.2017.03.037
12. A. Benyaiche, I. Khlifi, Wolff potential estimates for supersolutions of equations with generalized Orlicz growth, *Potential Anal.*, **58** (2023), 761–783. https://doi.org/10.1007/s11118-021-09958-5
13. V. Bögelein, J. Habermann, Gradient estimates via non-standard potentials and continuity, *Ann. Fenn. Mathematici*, **35** (2010), 741–678. https://doi.org/10.5186/aasfm.2010.3541
14. I. Chlebicka, F. Giannetti, A. Zatorska-Goldstein, Wolff potentials and local behavior of solutions to elliptic problems with Orlicz growth and measure data, *Adv. Calc. Var.*, **17** (2024), 1293–1321. https://doi.org/10.1515/acv-2023-0005
15. I. Chlebicka, Y. Youn, A. Zatorska-Goldstein, Wolff potentials and measure data vectorial problems with Orlicz growth, *Calc. Var.*, **62** (2023), 64. https://doi.org/10.1007/s00526-022-02402-5
16. I. Chlebicka, Y. Youn, A. Zatorska-Goldstein, Measure data systems with Orlicz growth, *Ann. Mat.*, **204** (2025), 407–426. https://doi.org/10.1007/s10231-024-01489-1
17. T. Gkikas, Quasilinear elliptic equations involving measure valued absorption terms and measure data, *J. Anal. Math.*, **153** (2024), 555–594. https://doi.org/10.1007/s11854-023-0321-0
18. P. Tran, N. Nguyen, A global fractional Caccioppoli-Type estimate for solutions to nonlinear elliptic problems with measure data, *Stud. Math.*, **263** (2022), 323–338. https://doi.org/10.4064/sm201121-12-3
19. H. Cheng, F. Zhou, Besov estimates for sub-elliptic equations in the Heisenberg group, *Adv. Pure Math.*, **14** (2024), 744–758. https://doi.org/10.4236/apm.2024.149039
20. T. Gkikas, Nonlinear nonlocal equations involving subcritical or power nonlinearities and measure data, *Math. Eng.*, **6** (2024), 45–80. https://doi.org/10.3934/mine.2024003
21. F. Duzaar, G. Mingione, Gradient continuity estimates, *Calc. Var. Partial Differ. Equ.*, **39** (2010), 379–418. https://doi.org/10.1007/s00526-010-0314-6
22. J. Musielak, *Orlicz spaces and modular spaces*, Berlin, Heidelberg: Springer, 1983. https://doi.org/10.1007/BFb0072210
23. L. Diening, F. Ettwein, Fractional estimates for non-differentiable elliptic systems with general growth, *Forum Math.*, **20** (2008), 523–556. https://doi.org/10.1515/FORUM.2008.027
24. G. Barletta, Existence and regularity results for nonlinear elliptic equations in Orlicz spaces, *Nonlinear Differ. Equ. Appl.*, **31** (2024), 29. https://doi.org/10.1007/s00030-024-00922-x
25. A. Cianchi, A sharp embedding theorem for Orlicz-Sobolev spaces, *Indiana Univ. Math. J.*, **45** (1996), 39–66. https://doi.org/10.1512/IUMJ.1996.45.1958
26. L. Diening, P. Kaplicky, S. Schwarzacher, Campanato estimates for the generalized Stokes system, *Ann. Mat. Pura Appl.*, **193** (2014), 1779–1794. https://doi.org/10.1007/s10231-013-0355-5
27. L. Diening, P. Kaplicky, S. Schwarzacher, BMO estimates for the p-Laplacian, *Nonlinear Anal. Theor.*, **75** (2012), 637–650. https://doi.org/10.1016/j.na.2011.08.065
28. R. A. Adams, J. J. F. Fournier, *Sobolev spaces*, 2 Eds., New York: Academic Press, 2003.

29. E. Giusti, *Direct methods in the calculus of variations*, World Scientific, 2003. https://doi.org/10.1142/5002
30. B. Simon, *Harmonic analysis: A comprehensive course in analysis, Part 3*, American Mathematical Society, 2015.
31. J. Kinnunen, *Sobolev spaces*, Aalto University, 2025.
32. F. Duzaar, G. Mingione, Gradient estimates via non-linear potentials, *Am. J. Math.*, **133** (2009), 1093–1149. https://doi.org/10.1353/AJM.2011.0023