AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(4): 7906–7919.
DOI: 10.3934/math.2025363
Received: 02 February 2025
Revised: 05 March 2025
Accepted: 14 March 2025
Published: 07 April 2025

Research article

Numerical investigation based on the Chebyshev-HPM for Riccati/Logistic differential equations

M. M. Khader[1], **A. M. Shloof**[2,*] **and Halema Ali Hamead**[3]

[1] Department of Mathematics and Statistics, College of Science, Imam Mohammad, Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia

[2] Department of Mathematics, Faculty of Science, University of Zintan, AlZintan, Libya

[3] Department of Mathematics, Faculty of Education, University of Aljufra, Aljufra, Libya

* **Correspondence:** Email: aml.shloof@uoz.edu.ly.

Abstract: We give the approximate solution of the Riccati/Logistic differential equations (RDE/LDE). The suggested approach depends on the homotopy perturbation method developed with the Chebyshev series (CHPM). A study of the convergence analysis of CHPM is presented. The residual error function is calculated and used as a basic criterion in evaluating the accuracy and efficiency of the given numerical technique. We use the exact solution and the Runge-Kutta method of fourth order for comparison with the results of the method used. Through these results, we can confirm that the applied method is an easy and effective tool for the numerical simulation of such models. Illustrative models are given to confirm the validity and usefulness of the proposed procedure.

Keywords: Riccati/Logistic differential equation; homotopy perturbation method; Chebyshev expansion; Convergence analysis; RK4 method
Mathematics Subject Classification: 26A33, 34A08, 65N06, 65N12

1. Introduction

Financial mathematics is one of the most important branches of mathematics, which includes applications to problems of diffusion, random processes, and others. One of the most important of these problems is the Riccati differential equation [1], the basic theories of which were presented in [2]. This type of equation has many applications in engineering sciences, in addition to important and traditional applications such as stochastic perception theory, network structure, optimal control, and elastic stability. From this standpoint, many classical numerical techniques have been used, including, but not limited to, the Euler's forward method, Runge-Kutta methods, HAM [3], and variational iteration method [4], the unconditional stable method [5], and others.

In 1838, the so-called logistic differential equation model was introduced by Pierre Verhulst [6]. These models have been used to describe the chaotic behavior and periodic multiplication of some well-known dynamic systems. One of the most popular models of this type of equation is the population model, which contains many variables [7]. In medicine, there is another common use of the logistic curve, where tumor growth is modeled using the logistic differential equation, in which the tumor size at time t is expressed as $N(t)$. Finally, ecology is an important field in which this logistic differential equation appears.

Due to the problems of accuracy and convergence faced by most numerical methods, a large number of researchers have used these semi-analytical methods (SAM) to investigate such problems and gain deeper insights into their complexities. The advantage of the SAM as the preferred method for finding analytical approximations to complex problems containing highly nonlinear terms [8] lies in the challenges associated with obtaining exact solutions using traditional analytical techniques. Among these methods, which have attracted the attention of many scientists and have found application in solving a wide range of complex problems, is the HPM in particular. An improved version of this method called the new HPM, was presented in 2010 in the research paper [9] and was then applied to obtain approximate solutions to the quadratic RDE. This approach has become a popular choice in many studies ([10, 11]) due to its simplicity and ease of mathematical calculation. This distinction and simplicity can be achieved by defining the first approximation as a power series and then setting all iterations to zero except for the initial iteration. That is, the power series takes a central place in the method, which in turn facilitates obtaining approximate solutions to nonlinear equations in the form of the Taylor expansion. Based on what was mentioned above about the importance of using power series, whether Taylor expansion or others, and in being more effective in improving the accuracy and convergence in the analytical approximation solutions, Chebyshev series, which depends on orthogonal Chebyshev polynomials, were used to benefit from the properties of these functions in good approximation of functions, as well as their distinction in the faster convergence rate than other functions such as Taylor series, for example, but not limited to [12]. Finally, in [13], a new collocation method is presented to evaluate the elliptic PDEs (including the Poisson equation, Helmholtz type equation, and transient heat conduction equation) efficiently and accurately. In this method, the problem domain under study is divided into a series of small overlapping subdomains, and in each subdomain, the Chebyshev local collocation method is utilized, where the unknown functions at each node can be estimated using a linear combination of the unknowns at nearby nodes.

Due to these important properties of handling nonlinear equations and providing very accurate approximations, the transformed Chebyshev series has been widely applied in many academic studies, we mention, for example, but not limited to point equations of motion in their linear/nonlinear form [14], space-variable approximation of the Burger equation [15], the initial and BVPs associated with the fractional heat equation [16], and others ([17, 18]).

The present research focuses on developing semi-analytical methods to overcome the challenges inherent in these methods, taking advantage of the pivotal role played by the Chebyshev series, as shown in the historical context mentioned above, as well as addressing the time-consuming nature of numerical methods, mitigating these challenges.

The major objective of this manuscript is to apply this method called Chebyshev-Homotopy Perturbation Method (CHPM) by combining the effective Chebyshev series (CS) with the new homotopy perturbation approach, to use a new accurate, and efficient analytical approximation approach used for the first time to address the present problems, the RDE and LDE. This new method was compared with some existing methods to confirm its efficiency and high accuracy, and perfect agreement was obtained.

2. Basic concepts of the CHPM

The novel approach mainly depends on the use of the Chebyshev series in the new HPM. Here, we mention some basic assumption of the Chebyshev series, the new homotopy perturbation algorithm, and then discuss our new algorithm for finding approximate solutions to the RDE and LDE.

2.1. Chebyshev series

If we have a continuous function $q(t)$ in $[a, b]$, then it can be rewritten in terms of the CS of the first kind as follows [19, 20]:

$$q(t) = \sum_{k=0}^{\infty}{}' \, c_k T_k\left(\frac{2t-b-a}{b-a}\right), \tag{2.1}$$

where the $'$ sign indicates that the coefficient of $T_0(t)$ must be reduced by half, $T_k(t) = \cos(k\cos^{-1}(t))$ where

$$c_k = \frac{2}{\pi}\int_{-1}^{1}(1-t^2)^{-0.5}\, q(0.5(b+a+(b-a)t))T_k(t)dt.$$

By the following recurrence relation, we can obtain Chebyshev polynomials of the first kind given on $[-1, 1]$:

$$T_s(t) = 2t\,T_{s-1}(t) - T_{s-2}(t), \qquad s = 2, 3, ..., \qquad T_0(t) = 1, \quad T_1(t) = t. \tag{2.2}$$

Furthermore, these functions can be expressed analytically using the finite sum of powers of t as follows:

$$T_k(t) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor}(-1)^i\, 2^{k-2i-1}\frac{k}{k-i}\binom{k-i}{i} t^{k-2i}. \tag{2.3}$$

If the domain of the problem under study is $[0, 1]$, we need to derive the so-called Chebyshev transformed polynomials $\mathbb{T}_k(t)$ of the first kind, using an appropriate linear transformation via the recurrence relation (2.2) as follows:

$$\mathbb{T}_k(t) = 2(2t-1)\,\mathbb{T}_{k-1}(t) - \mathbb{T}_{k-2}(t), \qquad k = 2, 3, ..., \qquad \mathbb{T}_0(t) = 1, \quad \mathbb{T}_1(t) = 2t-1. \tag{2.4}$$

2.2. The algorithm of the CHPM

To clarify the novel methodology, we shall outline the following steps for its application:

Step 1: Suppose the nonlinear ODE as follows [10, 11]:

$$L[p(x)] + R[p(x)] + N[p(x)] + s(x) = 0, \qquad x \in [a, b], \tag{2.5}$$

where N is the nonlinear term, and $s(x)$ is a given source term.

Step 2: Employ the principle of homotopy to get:

$$H(p,\ell) = L[p(x)] - p^*(x) + \ell\, p^*(x) + \ell(R[p(x)] + N[p(x)] + s(x)) = 0, \tag{2.6}$$

where $p^*(x)$ is an initial solution to the proposed problem & $0 \le \ell \le 1$ is an embedding parameter. From (2.6), we can find the following:

$$L[p(x)] = p^*(x) - \ell\, p^*(x) - \ell(R[p(x)] + N[p(x)] + s(x)). \tag{2.7}$$

Step 3: Operate by the operator $L^{-1} = \int_0^x (.)dx$ to Eq (2.7), and rearrange it to obtain $p(x)$ as follows:

$$p(x) = p(0) + L^{-1}[p^*(x)] - \ell\, L^{-1}[p^*(x)] - \ell\, L^{-1}[R[p(x)] + N[p(x)] + s(x)]. \tag{2.8}$$

Step 4: Assume that

$$p(x) = \sum_{k=0}^{\infty} \ell^k\, u_k(x), \qquad p^*(x) = \sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(x),$$

then, by substituting in (2.8), we can obtain the following:

$$\begin{aligned} \sum_{k=0}^{\infty} \ell^k\, u_k(x) = p(0) + L^{-1}\left[\sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(x)\right] - \ell\, L^{-1}\left[\sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(x)\right] \\ - \ell\, L^{-1}\left[R\Big[\sum_{k=0}^{\infty} \ell^k\, u_k(x)\Big] + N\Big[\sum_{k=0}^{\infty} \ell^k\, u_k(x)\Big] + s(x)\right]. \end{aligned} \tag{2.9}$$

Step 5: Compare the coefficients of ℓ^k, $k = 0, 1, 2, ..$ on both sides of Eq (2.9), which leads to obtain:

$$\begin{aligned} \ell^0 &: \quad u_0(x) = p(0) + L^{-1}\Big[\sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(x)\Big], \\ \ell^1 &: \quad u_1(x) = -L^{-1}\Big[\sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(x)\Big] - L^{-1}[R(u_0) + N(u_0) + s(x)], \\ \ell^2 &: \quad u_2(x) = -L^{-1}[R(u_0, u_1) + N(u_0, u_1)], \cdots, \\ \ell^j &: \quad u_j(x) = -L^{-1}[R(u_0, u_1, \cdots, u_{j-1}) + N(u_0, u_1, \cdots, u_{j-1})], \qquad j = 3, 4, \cdots. \end{aligned} \tag{2.10}$$

Step 6: Assume that $u_1(x) = 0$.

Step 7: The values of the Chebyshev's coefficients c_k, $k = 0, 1, 2, .., m$ can be determined by collocating the equation $u_1(x) = 0$ (the second equation in the system (2.10) after substituting $u_0(x)$ from the first equation in the same system) at $m + 1$ points x_j, $j = 0, 1, 2, ..., m$. The best choice for these nodes is the roots of the polynomial equation $\mathbb{T}_{m+1}(x) = 0$. This leads us to obtain a set of algebraic equations in Chebyshev's coefficients, and by solving these equations, we obtain these coefficients.

Therefore, the analytical solution can be formulated as follows:

$$p_m(x) = u_0(x) = p(0) + L^{-1}\left[\sum_{k=0}^{m} c_k\, \mathbb{T}_k(x)\right]. \tag{2.11}$$

2.3. Convergent analysis of CHPM

Here, we give the convergence analysis of the following numerical solution obtained utilizing the CHPM:

$$p_m(x) = \sum_{j=0}^{m} u_j(x) = \sum_{j=0}^{m} c_j\, \mathbb{T}_j(x). \tag{2.12}$$

Theorem 1. *[21] Consider a Hilbert space $\mathbb{H}$ & define the operator $\mathbb{B} : \mathbb{H} \to \mathbb{H}$. Then the approximate solutions $p_m(x)$ presented in (2.12) converge to $p(x)$ as an exact solution of the original differential equation (2.5) if there is a constant $\kappa \in [0, 1)$, and $\|u_{m+1}\| = \kappa\, \|u_m\|$, $\forall\ m = 0, 1, 2, \ldots$.*

Proof. We begin by showing the generated sequence of solutions $\{p_m(x)\}_{m=0}^{\infty}$ is a Cauchy sequence, as follows:

$$\|p_{m+1} - p_m\| = \|u_{m+1}\| \leq \kappa\, \|u_m\| \leq \kappa^2\, \|u_{m-1}\| \leq \cdots \leq \kappa^{m+1}\, \|u_0\|.$$

Now, for $m, n \in \mathbb{N}$, $m \geq n$:

$$\begin{aligned}\|p_m - p_n\| &= \|(p_m - p_{m-1}) + (p_{m-1} - p_{m-2}) + \cdots + (p_{n+1} - p_n)\|\\ &\leq \|(p_m - p_{m-1})\| + \|(p_{m-1} - p_{m-2})| + \cdots + \|(p_{n+1} - p_n)\|\\ &\leq \kappa^m\, \|u_0\| + \kappa^{m-1}\, \|u_0\| + \cdots + \kappa^{n+1}\, \|u_0\|\\ &\leq (\kappa^{n+1} + \kappa^{n+2} + \cdots + \kappa^m)\, \|u_0\| = \kappa^{n+1}\|u_0\|\frac{1 - \kappa^{m-n}}{1 - \kappa}.\end{aligned}$$

Since $\kappa << 1$, we can see that:

$$\lim_{n,m\to\infty} \|p_m - p_n\| = 0,$$

which means that the sequence $\{p_m(x)\}_{m=0}^{\infty}$ is a Cauchy sequence in $\mathbb{H}$, i.e.,

$$\lim_{m\to\infty} p_m(x) = p(x).$$

□

3. Numerical implementation

3.1. Utilizing CHPM on RDE

We consider the RDE [22, 23] as follows:

$$\dot{\psi}(t) = 1 - \psi^2(t), \qquad \psi(0) = \hat{\psi}, \tag{3.1}$$

the exact solution is defined as $\psi(t) = (\sigma\, e^{2t} - 1)(\sigma\, e^{2t} + 1)^{-1}$, where $\sigma = (1 + \hat{\psi})/(1 - \hat{\psi})$.

The CHPM technique was used to solve the current problem (3.1). The primary phases of the new technique are given below:

Step 1: Applying the homotopy property on Eq (3.1), we find:

$$\dot{\psi}(t) + \ell\, \psi^*(t) - \psi^*(t) - \ell\left(1 - \psi^2(t)\right) = 0. \tag{3.2}$$

Step 2: Taking $L^{-1} = \int_0^t (.)dt$, for both sides of Eq (3.2) yields:

$$\psi(t) = \psi(0) - \ell\, L^{-1}(\psi^*(t)) + L^{-1}(\psi^*(t)) + \ell\, L^{-1}\Big(1 - \psi^2(t)\Big). \tag{3.3}$$

Step 3: Assuming that

$$\psi(t) = \sum_{k=0}^{\infty} \ell^k\, \psi_k(t), \qquad \psi^*(t) = \sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(t),$$

then, by substituting in (3.3), we can obtain the following:

$$\begin{aligned} \sum_{k=0}^{\infty} \ell^k\, \psi_k(t) &= \hat{\psi} - \ell\, L^{-1}\Big(\sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(t)\Big) + L^{-1}\Big(\sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(t)\Big) \\ &\quad + \ell\, L^{-1}\Big(1 - \Big(\sum_{k=0}^{\infty} \ell^k\, \psi_k(t)\Big)^2\Big). \end{aligned} \tag{3.4}$$

Step 4: Equaling the terms for the equation (3.4), which have the same powers of ℓ:

$$\begin{aligned} &\ell^0 :\ \psi_0(t) = \hat{\psi} + L^{-1}\Big(\sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(t)\Big), \\ &\ell^1 :\ \psi_1(t) = -L^{-1}\Big(\sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(t)\Big) + L^{-1}\Big(1 - \psi_0^2(t)\Big), \\ &\ell^2 :\ \psi_2(t) = L^{-1}\Big(-2\psi_0(t)\psi_1(t)\Big), \end{aligned} \tag{3.5}$$

and so on.

Step 5: Assume that $\psi_1(t) = 0$.

Step 6: The values of Chebyshev's coefficients c_k, $k = 0, 1, 2, .., m$ can be determined by collocating the equation $\psi_1(t) = 0$ (the second equation in the system (3.5) after substituting $\psi_0(t)$ from the first equation in the same system) at $m + 1$ points t_j, $j = 0, 1, 2, ..., m$. The best choice for these nodes is the roots of the polynomial equation $\mathbb{T}_{m+1}(t) = 0$. This leads us to obtain a set of algebraic equations in c_k, and by solving these equations we get these coefficients.

Therefore, the analytical approximate solution of $\psi(t)$ becomes as follows:

$$\psi(t) = \psi_0(t) = \hat{\psi} + L^{-1}\Big(\sum_{k=0}^{m} c_k\, \mathbb{T}_k(t)\Big). \tag{3.6}$$

It should be noted that in step 6, the relations (Derivation, Multiplication, and Integration) that were mentioned earlier are applied.

Figures 1–3 present the approximate solution of the model (3.1) on [0, 1], with varying values of $\hat{\psi}$.

(1) Figure 1 presents the approximate & exact solutions implementing the CHPM at $m = 9$ and $\hat{\psi} = 0$.
(2) Figure 2 gives the numerical solution with various values of initial solutions $\hat{\psi} = 0.25$, 0.5, 0.75, 1.0, and $m = 9$.

(3) Figure 3 presents the REF with $\hat{\psi} = 0.5$ and $m = 9$.

These figures, and the excellent agreement between the given approximate solution & exact solution allow us to conclude that the approach can be implemented successfully to solve the given model.

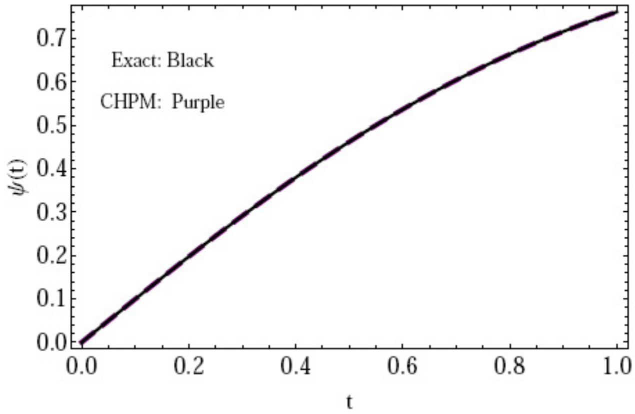

Figure 1. A comparison between the approximate and exact solutions using CHPM.

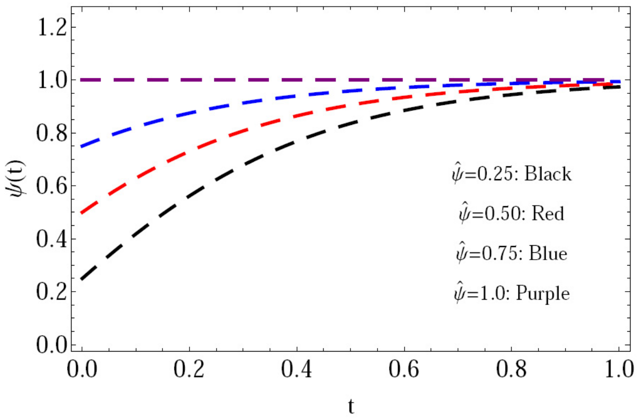

Figure 2. The approximate solution versus $\hat{\psi}$.

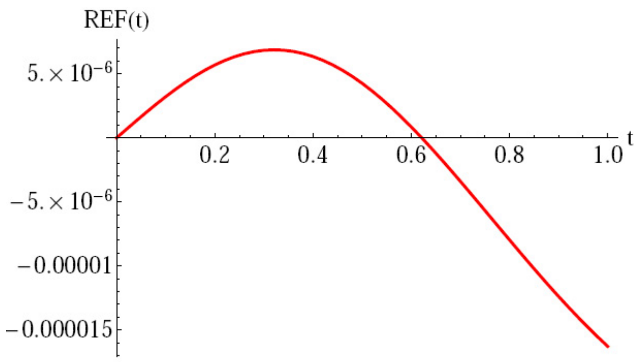

Figure 3. The REF of the solution at $\hat{\psi} = 0.5$.

To validate the numerical solutions of the RDE at $\hat{\psi} = 0.5$ with different values of the approximation order m, a comparison of the absolute error (AE) is presented in Table 1 for the CHPM and Runge-Kutta method of fourth order (RK4M) for the same model. This comparison demonstrates how thorough the approach suggested in this article is suitable for solving the model under study.

Table 1. Comparison of the AE for numerical solutions of the RDE by the CHPM and RK4M.

	AE of the present method		AE of the RK4M	
t	$m = 8$	$m = 12$	$h = 0.2$	$h = 0.1$
0.0	5.753951E-04	9.753615E-05	4.654031E-04	1.752147E-06
0.2	3.654162E-05	5.950140E-06	1.852174E-05	2.852014E-07
0.4	7.015975E-04	4.125047E-05	0.975369E-04	4.620287E-06
0.6	6.025874E-04	5.014785E-07	3.652014E-04	0.650054E-07
0.8	7.320514E-03	3.952581E-06	2.792145E-05	1.654852E-06
1.0	2.250170E-05	7.650287E-07	1.654792E-04	9.753951E-07

3.2. Implementation CHPM on LDE

In this example, we consider the following LDE [23, 24]:

$$\dot{\theta}(t) = \beta\,\theta(t)(1 - \theta(t)), \qquad \theta(0) = \hat{\theta}, \qquad \beta > 0. \tag{3.7}$$

Where

$$\theta(t) = \hat{\theta}\left((1 - \hat{\theta})e^{-\beta t} + \hat{\theta}\right)^{-1}$$

is the corresponding exact solution.

The existence and uniqueness of (3.7) can be found in details at [25].

The CHPM was used to solve the current problem (3.7). The primary phases of the new technique are given below:

Step 1: Applying the homotopy property on Eq (3.7), we get:

$$\dot{\theta}(t) + \ell\,\theta^*(t) - \theta^*(t) - \ell\left(\beta\,\theta(t)(1 - \theta(t))\right) = 0. \tag{3.8}$$

Step 2: Taking $L^{-1} = \int_0^t (.)dt$, for both sides of the Eq (3.8) yields:

$$\theta(t) = \theta(0) - \ell\,L^{-1}(\theta^*(t)) + L^{-1}(\theta^*(t)) + \ell\,L^{-1}\left(\beta\,\theta(t)(1 - \theta(t))\right). \tag{3.9}$$

Step 3: Assuming that

$$\theta(t) = \sum_{k=0}^{\infty} \ell^k\,\theta_k(t), \qquad \theta^*(t) = \sum_{k=0}^{\infty} d_k\,\mathbb{T}_k(t),$$

then, by substituting in (3.9), we can get the following:

$$\begin{aligned}\sum_{k=0}^{\infty} \ell^k\,\theta_k(t) = \hat{\theta} + L^{-1}\Big(\sum_{k=0}^{\infty} d_k\,\mathbb{T}_k(t)\Big) - \ell\,L^{-1}\Big(\sum_{k=0}^{\infty} d_k\,\mathbb{T}_k(t)\Big)\\ + \ell\,L^{-1}\Big(\beta\Big(\sum_{k=0}^{\infty} \ell^k\,\theta_k(t)\Big)\Big(1 - \sum_{k=0}^{\infty} \ell^k\,\theta_k(t)\Big)\Big).\end{aligned} \tag{3.10}$$

Step 4: Equaling the terms for the equation (3.10) with $\ell^k,\ k = 0, 1, ...$:

$$\begin{aligned}
\ell^0 &: \ \theta_0(t) = \hat{\theta} + L^{-1}\Big(\sum_{k=0}^{\infty} d_k\, \mathbb{T}_k(t)\Big),\\
\ell^1 &: \ \theta_1(t) = -L^{-1}\Big(\sum_{k=0}^{\infty} d_k\, \mathbb{T}_k(t)\Big) + L^{-1}\Big(\beta\, \theta_0(t)(1 - \theta_0(t))\Big),\\
\ell^2 &: \ \theta_2(t) = L^{-1}\Big(\beta\, \theta_1(t)(1 - 2\theta_0(t))\Big),
\end{aligned} \tag{3.11}$$

and so on.

Step 5: Assume that $\theta_1(t) = 0$.

Step 6: The values of Chebyshev's coefficients $d_k,\ k = 0, 1, 2, .., m$ can be determined by collocating the equation $\psi_1(t) = 0$ (The second equation in the system (3.11) after substituting on $\theta_0(t)$ from the first equation in the same system) at $m + 1$ points $t_j,\ j = 0, 1, 2, ..., m$. The best choice for these nodes is the roots of the polynomial equation $\mathbb{T}_{m+1}(t) = 0$. This leads us to obtain a set of algebraic equations in d_k, and by solving these equations we get these coefficients.

Therefore, the analytical approximate solution of $\theta(t)$ becomes as follows:

$$\theta_m(t) = \theta_0(t) = \hat{\theta} + L^{-1}\Big(\sum_{k=0}^{m} d_k\, \mathbb{T}_k(t)\Big). \tag{3.12}$$

It should be noted that in step 6, the relations (Derivation, Multiplication, and Integration) that were mentioned earlier are applied.

The approximate solution of the Eq (3.7) in [0, 1] is presented in Figures 4–7 with distinct values of initial solutions $\hat{\theta}$; and various values of the parameter β.

(1) Figure 4 presents the approximate & exact solutions using the CHPM at $\hat{\theta} = 0.25$, with $m = 10$ and $\beta = 0.5$.
(2) Figure 5 plots the approximate solution at various quantities of $\hat{\theta} = 0.25,\ 0.5,\ 0.75,\ 1.0$ and $\beta = 0.5,\ m = 9$.
(3) Figure 6 displays the impact of the parameter β on the approximate solution with distinct values of $\beta = 0.4,\ 0.8,\ 1.2,\ 1.4$, at $m = 10,\ \hat{\theta} = 0.5$.
(4) Figure 7 presents the REF with $\beta = 0.75$, and $\hat{\theta} = 0.5,\ m = 14$.

These figures, and the excellent agreement between the given approximate solution and exact solution allow us to draw the conclusion that the approach can be implemented successfully to solve the given model. Also, the given approximate solutions depend on the changes in the initial solutions, and β.

To validate the numerical solutions of the LDE at $\hat{\theta} = 0.25$ and $\beta = 0.5$ with different values of the approximation order m, a comparison of the AE is presented in Table 2 for the CHPM and RK4M for the same model. This comparison demonstrates how thorough the approach suggested in this article is suitable for solving the model under investigation.

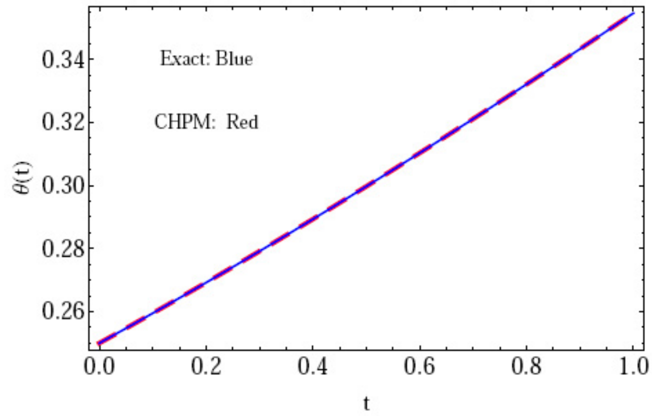

Figure 4. The approximate and exact solutions using CHPM.

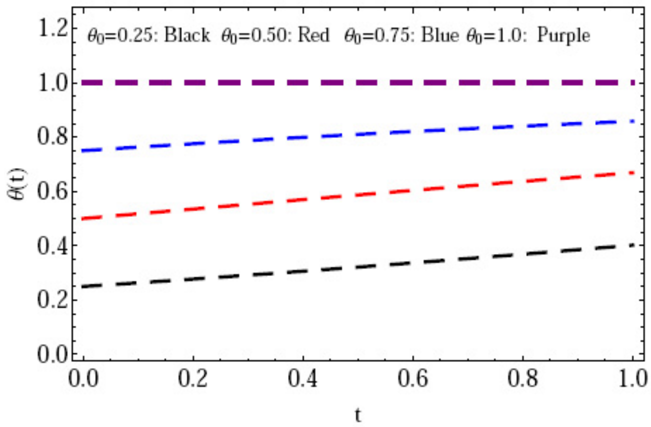

Figure 5. The approximate solution with distinct $\hat{\theta}$.

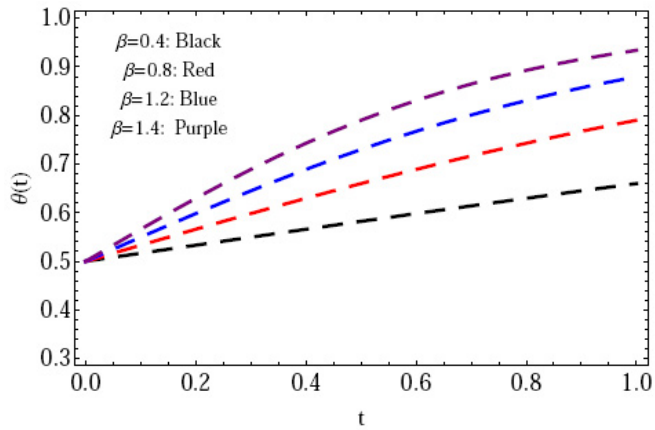

Figure 6. The approximate solution versus β.

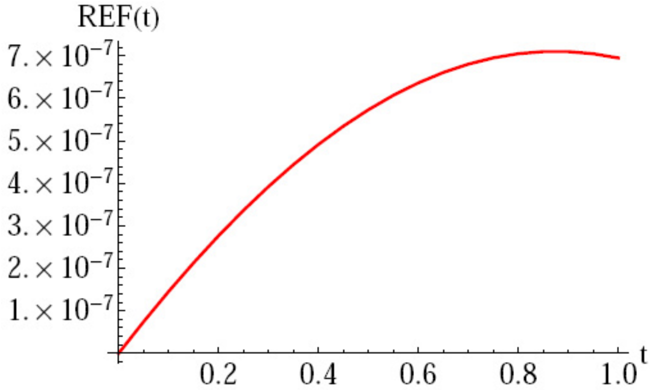

Figure 7. The REF of the solution at $\beta = 0.75$.

Table 2. Comparison of the AE for approximate solutions of the LDE by the CHPM and RK4M.

	AE of the present method		AE of the RK4M	
t	$m = 8$	$m = 12$	$h = 0.2$	$h = 0.1$
0.0	0.123654E-04	4.258123E-05	0.772014E-04	7.951753E-06
0.2	5.014785E-05	4.750281E-06	4.201478E-05	0.852014E-07
0.4	4.654258E-04	3.925880E-05	2.014725E-04	1.620287E-06
0.6	3.963741E-04	8.258004E-07	1.654782E-04	4.654782E-07
0.8	5.320514E-03	8.756542E-06	3.325201E-05	3.654712E-06
1.0	5.250170E-05	8.985214E-07	0.652587E-04	0.654258E-07

4. Conclusions

In this research, the CHPM is applied to obtain the numerical solutions for two important models, the RDE and LDE with different initial solutions, and some parameters. By comparing the numerical solutions, exact solution, and RK4M of each of the proposed models, we were able to draw the conclusion that the approximate solutions presented by applying the given procedure are in excellent agreement with the real solution. Also, through the resulting numerical results, we found to a large extent how effective this technique is in solving the problems under study and highlighted the validity and potential of the proposed technique. Finally, this view of analytical and numerical solutions of dynamic variables was due to their being present and effectively influencing various models and fields of applied mathematics. For future work, we can try to provide a theoretical study of the convergence order measurement, with an expanded focus on the study of convergence and stability of the given technique. Also, the technique will be applied to more complex models or systems of nonlinear ordinary or partial differential equations to expand the scope of application of the proposed technique.

Author contributions

MM Khader: Conceptualization, Formal analysis, Investigation, Methodology, Resources, Software, Validation, Writing-review & editing; AM Shloof: Conceptualization, Data curation, Investigation, Methodology, Resources, Software, Supervision, Writing-review & editing; Halema Ali Hamead: Data curation, Formal analysis, Investigation, Methodology, Visualization, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2503).

Conflict of interest

Authors do not have any conflicts of interest.

References

1. I. Lasiecka, R. Triggiani, *Differential and algebraic Riccati equations with application to boundary/point control problems: continuous theory and approximation theory*, Berlin: Springer, 1991.

2. W. T. Reid, *Riccati differential equations, mathematics in science and engineering*, New York: Academic Press, 1972.

3. Y. Tan, S. Abbasbandy, Homotopy analysis method for quadratic Riccati differential equation, *Commun. Nonlinear Sci. Numer. Simul.*, **13** (2008), 539–546. https://doi.org/10.1016/j.cnsns.2006.06.006

4. H. Jafari, H. Tajadodi, He's variational iteration method for solving fractional Riccati differential equation, *Int. J. Differ. Equ.*, **2010** (2010), 764738. https://doi.org/10.1155/2010/764738

5. F. Dubois, A. Saidi, Unconditionally stable scheme for Riccati equation, *ESAIM Proc.*, **8** (2000), 39–52. https://doi.org/10.1051/proc:2000003

6. H. Pastijn, Chaotic growth with the logistic model of P.-F. verhulst, In: *Understanding complex systems*, Berlin: Springer, 2006. https://doi.org/10.1007/3-540-32023-7_1

7. M. Ausloos, *The Logistic map and the route to chaos: from the beginnings to modern applications XVI*, 2006.

8. T. A. J. Al-Griffi, A. S. J. Al-Saif, Yang transform-homotopy perturbation method for solving a non-Newtonian viscoelastic fluid flow on the turbine disk, *ZAMM*, **2022** (2022), e202100116. https://doi.org/10.1002/zamm.202100116

9. H. Aminikhah, M. Hemmatnezhad, An efficient method for quadratic Riccati differential equation, *Commun. Nonlinear Sci. Numer. Simul.*, **15** (2010), 835–839. https://doi.org/10.1016/j.cnsns.2009.05.009

10. M. S. Abdul-Wahab, A. J. A. Al-Saif, Chebyshev-homotopy perturbation method for studying the flow and heat transfer of a Non-Newtonian fluid flow on the turbine disk, *Basrah Res. Sci.*, **50** (2024), 150. https://doi.org/10.56714/bjrs.50.1.13

11. K. Pal, V. G. Gupta, H. Singh, V. Pawar, Enlightenment of heat diffusion using new homotopy perturbation method, *J. Appl. Sci. Eng.*, **27** (2023), 2213–2216. http://doi.org/10.6180/jase.202403_27(3).0007

12. J. Wu, Y. Zhang, L. Chen, Z. Luo, A Chebyshev interval method for nonlinear dynamic systems under uncertainty, *Math. Model.*, **37** (2013), 4578–4591. https://doi.org/10.1016/j.apm.2012.09.073

13. F. Wang, Q. Zhao, Z. Chen, C. M. Fan, Localized Chebyshev collocation method for solving elliptic partial differential equations in arbitrary 2D domains, *Appl. Math. Comput.*, **397** (2021), 125903. https://doi.org/10.1016/j.amc.2020.125903

14. Y. M. Hamada, A new accurate numerical method based on shifted Chebyshev series for nuclear reactor dynamical systems, *Sci. Technol. Nucl. Install.*, **15** (2018), 7105245. https://doi.org/10.1155/2018/7105245

15. M. Izadi, S. Y*ü*zba, D. Baleanu, Taylor-Chebyshev approximation technique to solve the 1D and 2D nonlinear Burger's equations, *Math. Sci.*, **16** (2022), 459–471. https://doi.org/10.1007/s40096-021-00433-1

16. M. M. Khader, M. Adel, Modeling and numerical simulation for covering the fractional Covid-19 model using spectral collocation-optimization algorithms, *Fractal Fract.*, **6** (2022), 363. https://doi.org/10.3390/fractalfract6070363

17. M. M. Khader, Numerical study for unsteady Casson fluid flow with heat flux using a spectral collocation method, *Indian J. Phys.*, **96** (2021), 777–786. https://doi.org/10.1007/s12648-021-02025-0

18. A. M. Shloof, N. Senu, A. Ahmadian, M. N. L. Nik, S. Soheil, Solving fractal-fractional differential equations using an operational matrix of derivatives via Hilfer fractal-fractional derivative sense, *Appl. Numer. Math.*, **178** (2022), 386–403. https://doi.org/10.1016/j.apnum.2022.02.006

19. M. H. Mudde, Chebyshev approximation, Master's thesis/essay, *University of Groningen, Netherlands, Faculty of Science and Engineering*, 2017.

20. H. C. Thacher, Conversion of a power to a series of Chebyshev polynomials, *Commun. ACM*, **7** (1964), 181–182. https://doi.org/10.1145/363958.363998

21. M. S. Abdul-Wahab, A. J. A. Al-Saif, Studying the effects of electro-osmotic and several parameters on blood flow in stenotic arteries using CAGHPM, *PDE Appl. Math.*, **11** (2024), 100767. https://doi.org/10.1016/j.padiff.2024.100767

22. M. M. Khader, Numerical treatment for solving fractional Riccati differential equation, *J. Egyptian Math. Soc.*, **21** (2013), 32–37. https://doi.org/10.1016/j.joems.2012.09.005

23. M. M. Khader, M. Adel, Numerical approach for solving the Riccati and Logistic equations via QLM-rational Legendre collocation method, *Comp. Appl. Math.*, **39** (2020), 166. https://doi.org/10.1007/s40314-020-01207-6

24. M. M. Khader, Numerical treatment for solving fractional Logistic differential equation, *Differ. Equ. Dyn. Syst.*, **24** (2016), 99–107. https://doi.org/10.1007/s12591-014-0207-9

25. A. M. A. El-Sayed, A. E. M. El-Mesiry, H. A. A. El-Saka, On the fractional-order Logistic equation, *Appl. Math. Lett.*, **20** (2007), 817–823. https://doi.org/10.1016/j.aml.2006.08.013