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Research article

Bäcklund transformation and soliton solutions for a generalized Wadati-Konno-Ichikawa equation

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Abstract: By employing a reciprocal transformation, we relate the generalized Wadati–Konno–Ichikawa (gWKI) equation to an associated gWKI (agWKI) equation. Utilizing the Darboux transformation of the agWKI equation and Bianchi's theorem of permutability, we derive the N-Bäcklund transformation for the gWKI equation. As an application, we obtain some soliton solutions for the gWKI equation, including smooth solitons, bursting solitons, and loop-type solitons. Furthermore, we explore the interactions between two solitons.

Keywords: generalized WKI equation; Bäcklund transformation; soliton solutions
Mathematics Subject Classification: 37K10, 35C08, 37K40

1. Introduction

The inverse scattering method (ISM) is a highly effective tool for solving initial value problems in integrable systems [1]. With the aim of generalizing the ISM to cover more integrable systems, Wadati, Konno, and Ichikawa proposed two integrable equations, commonly referred to as the WKI equations [2]. They feature highly nonlinear terms exhibiting saturation effects, and can describe nonlinear transverse oscillations of elastic beams under tension [3,4]. The first WKI equation is

$$\begin{aligned} q_t &= i\left(\frac{q}{\sqrt{1-rq}}\right)_{xx}, \\ r_t &= -i\left(\frac{r}{\sqrt{1-rq}}\right)_{xx}, \end{aligned} \tag{1.1}$$

which, if taking $r = \pm q*$, reduces to

$$q_t = i\left(\frac{q}{\sqrt{1 \pm |q|^2}}\right)_{xx}. \tag{1.2}$$

The derivation of the WKI equation from the motion of curves in Euclidean geometry E^3 was given by Qu and Zhang [5]. Shimizu, and Wadati [6] and Choudhury [7] investigated the ISM of the WKI equation under vanishing and nonvanishing boundary conditions, respectively. They obtained implicit N-soliton formulas but only analyzed one soliton solution, which includes smooth soliton and bursting soliton. Moreover, it has been demonstrated that the WKI spectral problem is linked to the Ablowiz–Kaup–Newell–Segur spectral problem through reciprocal transformation (RT) and gauge transformation [8, 9]. This connection enables researchers to construct a Bäcklund transformation (BT) for the WKI equation [10–13]. Later on, numerous studies have been dedicated to obtaining solutions for the WKI equation using various techniques, such as nonlinearization [14], the algebro-geometric method [15], Darboux transformation (DT) [16], and the Riemann–Hilbert method [17–20]. In addition, some integrable generalizations of the WKI equation have been proposed [21–24]. One interesting gWKI equation recently given by Geng et al. [24] is

$$\begin{aligned} u_t &= \frac{\alpha}{2}\left(\frac{u}{\sqrt{1+uv}}\right)_{xx} + \alpha\left(\frac{u}{\sqrt{1+uv}}\right)_x, \\ v_t &= -\frac{\alpha}{2}\left(\frac{v}{\sqrt{1+uv}}\right)_{xx} + \alpha\left(\frac{v}{\sqrt{1+uv}}\right)_x, \end{aligned} \tag{1.3}$$

where u, and v are two potentials and α is a nonzero constant. Let $x \to ix$, $\alpha = -2i$, and $v = \pm u^*$, the gWKI equation (1.3) reduces to

$$u_t = i\left(\frac{u}{\sqrt{1 \pm |u|^2}}\right)_{xx} - 2\left(\frac{u}{\sqrt{1 \pm |u|^2}}\right)_x.$$

The authors also established infinitely many nonlocal conservation laws of the gWKI equation. However, as far as we know, there are currently no relevant results regarding the solution to this equation.

The BT first emerged in the 1880s when Bäcklund investigated the geometry of surfaces with constant negative curvature [25]. He discovered that certain transformations could map one surface of constant curvature to another, preserving specific geometric properties. This work showed that solutions to the sine-Gordon equation, which describes surfaces of constant negative curvature, could be generated from known ones using a set of transformations, now known as the BT. Subsequently, Bianchi identified that this transformation possesses the permutability property, which leads to the derivation of a nonlinear superposition formula [26]. These significant characteristics enable researchers to recognize the extensive applications of BT in integrable systems [27–30].

However, for the Camassa–Holm and WKI type equations, their exceptional spectral problems make the direct construction of their DTs and BTs through conventional methods employed in classical integrable systems infeasible [31]. In 2017, Rasin and Schiff [32] discussed the BT of the Camassa–Holm equation and discovered that it involves not only the dependent variables but also independent ones, which is different from the classical integrable systems such as the Korteweg–de Vries and sine-Gordon equations. Subsequently, this framework was successfully extended to construct BTs for the modified CH, Degasperis–Procesi, short pulse, and generalized Harry–Dym equations [33–37] etc.

The present paper focuses on the BT for the gWKI equation (1.3) and is organized as follows. In Section 2, using the first conservation law of the gWKI equation, we introduce an RT that connects the gWKI equation with an agWKI equation. We then construct the DT for the agWKI equation. This, together with the inverse of the RT, allows us to obtain the BT of the gWKI equation. Moreover, with the help of Bianchi's theorem of permutability, we derive the N-BT for the gWKI equation, expressed in terms of determinants. In Section 3, applying the N-BT, we derive an explicit one-soliton solution for the gWKI equation under nonzero background, which includes smooth solitons, bursting solitons, and loop-type solitons. Furthermore, the interactions between two solitons are discussed. Finally, some conclusions are given in Section 4.

2. N-Bäcklund transformation

In this section, we shall construct the N-BT for the gWKI equation (1.3) with the help of RT and DT. According to [24], the gWKI equation admits a Lax pair

$$\begin{aligned} &\Psi_x = F\Psi, \quad F = \begin{pmatrix} \lambda & (1+\lambda)u \\ \lambda v & -\lambda \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \\ &\Psi_t = G\Psi, \quad G = \begin{pmatrix} \lambda(1+\lambda)m\alpha & \alpha(1+\lambda)(\tilde{c}+\lambda mu) \\ \alpha\lambda(\tilde{b}+\lambda mv) & -\lambda(1+\lambda)m\alpha \end{pmatrix}, \end{aligned} \tag{2.1}$$

where λ is a constant spectral parameter and

$$m = \frac{1}{\sqrt{1+uv}}, \quad \tilde{c} = mu + \frac{1}{2}(mu)_x, \quad \tilde{b} = mv - \frac{1}{2}(mv)_x.$$

The compatibility condition of the Lax pair (2.1), $F_t - G_x + [F, G] = 0$, yields the gWKI equation (1.3). Using the Lax pair, one may compute an infinite number of local conservation laws in which the first one is

$$(m^{-1})_t = \frac{\alpha}{4}[m^2(vu_x - uv_x + 2uv)]_x.$$

It enables us to define an RT

$$dy = m^{-1}dx + \frac{\alpha m^2}{4}(vu_x - uv_x + 2uv)dt, \quad d\tau = dt, \tag{2.2}$$

which infers

$$\frac{\partial}{\partial x} = m^{-1}\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \rho\frac{\partial}{\partial y}, \quad \rho = \frac{\alpha m}{4}(vu_y - uv_y + 2muv). \tag{2.3}$$

Applying the RT (2.2), the Lax pair (2.1) is converted to

$$\begin{aligned} &\Psi_y = U\Psi, \quad U = \begin{pmatrix} \lambda m & (1+\lambda)mu \\ \lambda mv & -\lambda m \end{pmatrix}, \\ &\Psi_\tau = V\Psi, \quad V = \begin{pmatrix} \lambda(1+\lambda)m\alpha - \lambda m\rho & \alpha m(1+\lambda)(c+\lambda u) \\ \alpha m\lambda(b+\lambda v) & -\lambda(1+\lambda)m\alpha + \lambda m\rho \end{pmatrix}, \end{aligned} \tag{2.4}$$

where

$$c = u + \frac{m}{2}u_y - \frac{m^2}{2}u^2v, \quad b = v - \frac{m}{2}v_y - \frac{m^2}{2}v^2u.$$

A direct calculation shows that the compatibility condition of the Lax pair (2.4) gives rise to an agWKI equation

$$\begin{aligned} u_\tau &= \frac{\alpha}{2m^2}(mu)_{yy} + \frac{\alpha}{2m}(m^{-1})_y(mu)_y + \frac{\alpha}{m}(mu)_y - \rho u_y, \\ v_\tau &= -\frac{\alpha}{2m^2}(mv)_{yy} - \frac{\alpha}{2m}(m^{-1})_y(mv)_y + \frac{\alpha}{m}(mv)_y - \rho v_y, \end{aligned} \tag{2.5}$$

which is exactly that the equation obtained from the gWKI equation (1.3) by the RT (2.2).

Now we set out to construct the DT for the agWKI equation. To this end, let us introduce a gauge transformation

$$\Psi_{[1]} = T\Psi, \quad T = \lambda \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, \tag{2.6}$$

such that

$$\Psi_{[1],y} = U_{[1]}\Psi_{[1]}, \quad \Psi_{[1],\tau} = V_{[1]}\Psi_{[1]}. \tag{2.7}$$

Here a_i, b_i, c_i, and d_i ($i = 0, 1$) are functions of y and τ. Matrices $U_{[1]}$ and $V_{[1]}$ have the same form as U and V, with u, v, and m replaced by $u_{[1]}$, $v_{[1]}$, and $m_{[1]}$, respectively. Clearly, equation (2.7) is equivalent to

$$T_y + TU - U_{[1]}T = 0, \quad T_\tau + TV - V_{[1]}T = 0. \tag{2.8}$$

Substituting T given by (2.6) into (2.8), we get

$$T = T(\lambda, \lambda_1, a_1) = \lambda \begin{pmatrix} a_1 & 1 \\ 1 & (1+\lambda_1)a_1^{-1} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2.9}$$

$$u_{[1]} = \frac{a_{1,y} + mv}{m - (\ln|a_1|)_y}, \quad v_{[1]} = \frac{(1+\lambda_1)(a_1^{-1})_y + mu}{m - (\ln|a_1|)_y}, \tag{2.10}$$

in which λ_1 is an arbitrary constant excluding 0 and -1, and a_1 is determined by the following differential system

$$\begin{aligned} a_{1,y} &= \frac{m}{\lambda_1}(ua_1^2 - (1+\lambda_1)v - 2a_1), \\ a_{1,\tau} &= \frac{\alpha}{\lambda_1}[(1+\lambda_1)a_{1,y} + \frac{m^2a_1^2}{2}(u_y - mu^2v) \\ &\quad + \frac{m^2}{2}(1+\lambda_1)(v_y + mv^2u) + 2m\alpha^{-1}a_1]. \end{aligned} \tag{2.11}$$

Moreover, let $(g_1, g_2)^T$ be a solution of the Eq. (2.4) with $\lambda = \lambda_1^{-1}$. Solving the algebraic system $T(\lambda_1^{-1}, \lambda_1, a_1)(g_1, g_2)^T = 0$, we get $a_1 = -(1+\lambda_1)\frac{g_2}{g_1}$, which precisely satisfies the differential system (2.11).

Noticing the RT (2.2), we find that the BT for the gWKI equation involves not only the potentials u and v, but also the independent variable x. Through the RT (2.2), we get

$$dx_{[1]} - dx = (m_{[1]} - m)dy - (m_{[1]}\rho_{[1]} - m\rho)d\tau. \tag{2.12}$$

Taking Eqs (2.8)–(2.11) into account, we have

$$m_{[1]} - m = -\frac{a_{1,y}}{a_1}, \quad m_{[1]}\rho_{[1]} - m\rho = \frac{a_{1,\tau}}{a_1},$$

which, together with Eq (2.12), implies

$$dx_{[1]} - dx = -\frac{a_{1,y}}{a_1}dy - \frac{a_{1,\tau}}{a_1}d\tau = -d(\ln|a_1|).$$

Integrating both sides of the above equation and taking the integration constant as zero, we have

$$x_{[1]} = x - \ln|a_1|. \tag{2.13}$$

Given these results, we can establish the BT for the gWKI equation.

Proposition 1. *Suppose λ_1 is an arbitrary constant excluding* 0 *and* -1, *and* $(h_1, h_2)^T$ *is a solution to the equation* (2.1) *with* $\lambda = \lambda_1^{-1}$. *Then the BT for the gWKI equation* (1.3) *reads*

$$\begin{aligned} x_{[1]} &= x - \ln|a_1|, \quad t_{[1]} = t, \\ u_{[1]} &= \frac{a_{1,x} + v}{1 - (\ln|a_1|)_x}, \\ v_{[1]} &= \frac{(1+\lambda_1)(a_1^{-1})_x + u}{1 - (\ln|a_1|)_x}, \end{aligned} \tag{2.14}$$

where $a_1 = -(1+\lambda_1)\frac{h_2}{h_1} = \frac{1}{u}(1 - \lambda_1\frac{h_{1,x}}{h_1})$ *satisfies differential system*

$$\begin{aligned} a_{1,x} &= \frac{1}{\lambda_1}(ua_1^2 - (1+\lambda_1)v - 2a_1), \\ a_{1,t} &= \frac{\alpha m^3}{4\lambda_1}[(u_x v - uv_x)(ua_1^2 - (1+\lambda_1)v) \\ &\quad + 2a_1^2u_x + 2(1+\lambda_1)v_x + 4(1+\lambda_1)m^{-2}a_{1,x}]. \end{aligned} \tag{2.15}$$

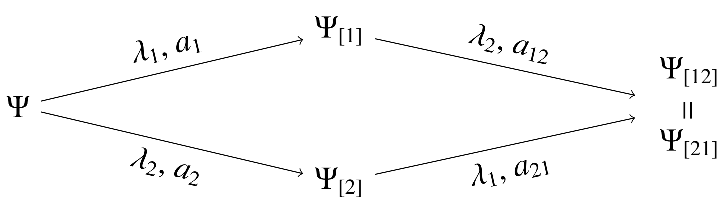
Next, we turn to establishing 2-BT for the gWKI equation using Bianchi's theorem of permutability. Consider two sets of gauge transformations

$$\Psi_{[1]} = T(\lambda, \lambda_1, a_1)\Psi, \qquad \Psi_{[12]} = T(\lambda, \lambda_2, a_{12})\Psi_{[1]},$$

and

$$\Psi_{[2]} = T(\lambda, \lambda_2, a_2)\Psi, \qquad \Psi_{[21]} = T(\lambda, \lambda_1, a_{21})\Psi_{[2]},$$

where a_1, a_2, a_{12}, and a_{21} are four auxiliary variables and λ_1, and λ_2 are two constant parameters excluding 0 and -1. According to Bianchi's theorem of permutability, which is shown in the following diagram,



we have $\Psi_{[12]} = \Psi_{[21]}$. It infers

$$T(\lambda, \lambda_2, a_{12})T(\lambda, \lambda_1, a_1) = T(\lambda, \lambda_1, a_{21})T(\lambda, \lambda_2, a_2),$$

which gives rise to

$$\begin{aligned} &a_1 a_{12} = a_2 a_{21}, \\ &a_{12} + (1+\lambda_1)a_1^{-1} = a_{21} + (1+\lambda_2)a_2^{-1}, \\ &a_1 + (1+\lambda_2)a_{12}^{-1} = a_2 + (1+\lambda_1)a_{21}^{-1}, \\ &a_{12} = \frac{(1+\lambda_2)a_1 - (1+\lambda_1)a_2}{a_1(a_2 - a_1)}, \\ &a_{21} = \frac{(1+\lambda_2)a_1 - (1+\lambda_1)a_2}{a_2(a_2 - a_1)}. \end{aligned} \tag{2.16}$$

Here a_2 satisfies differential system (2.15) with λ_1 replaced by λ_2.

Proposition 2. *The 2-BT for the gWKI equation* (1.3) *is given by*

$$\begin{aligned} &x_{[12]} = x - \ln|a_1 a_{12}|, \quad t_{[12]} = t, \\ &u_{[12]} = u_{[21]} = \frac{a_{12,x} + (1+\lambda_1)(a_1^{-1})_x + u}{1 - (\ln|a_1 a_{12}|)_x}, \\ &v_{[12]} = v_{[21]} = \frac{(1+\lambda_2)(a_{12}^{-1})_x + a_{1,x} + v}{1 - (\ln|a_1 a_{12}|)_x}, \end{aligned} \tag{2.17}$$

where a_{12} is defined by (2.16).

Proof. By Proposition 1, we have

$$\begin{aligned} x_{[12]} &= x_{[1]} - \ln|a_{12}| = x - \ln|a_1 a_{12}|, \\ x_{[21]} &= x_{[2]} - \ln|a_{21}| = x - \ln|a_2 a_{21}|, \end{aligned}$$

which, together with the first identity in (2.16), implies $x_{[12]} = x_{[21]}$. Next, we prove only the second identity in (2.17), since the third one can be verified similarly. Again, by Proposition 1, we get

$$\begin{aligned} u_{[12]} &= \frac{a_{12,x_{[1]}} + v_{[1]}}{1 - (\ln|a_{12}|)_{x_{[1]}}} = \frac{(1 - (\ln|a_1|)_x)a_{12,x_{[1]}} + (1+\lambda_1)(a_1^{-1})_x + u}{(1 - (\ln|a_1|)_x)(1 - (\ln|a_{12}|)_{x_{[1]}})} \\ &= \frac{a_{12,x} + (1+\lambda_1)(a_1^{-1})_x + u}{1 - (\ln|a_1 a_{12}|)_x}, \\ u_{[21]} &= \frac{a_{21,x_{[2]}} + v_{[2]}}{1 - (\ln|a_{21}|)_{x_{[2]}}} = \frac{(1 - (\ln|a_2|)_x)a_{21,x_{[2]}} + (1+\lambda_2)(a_2^{-1})_x + u}{(1 - (\ln|a_2|)_x)(1 - (\ln|a_{21}|)_{x_{[2]}})} \\ &= \frac{a_{21,x} + (1+\lambda_2)(a_2^{-1})_x + u}{1 - (\ln|a_2 a_{21}|)_x}, \end{aligned}$$

where the following derivative relations

$$\frac{\partial}{\partial x_{[i]}} = \frac{1}{1 - (\ln|a_i|)_x}\frac{\partial}{\partial x}, \quad i = 1, 2,$$

have been used.

Inserting (2.16) into the above equation, we have

$$u_{[12]} = u_{[21]} = \frac{\Theta_1}{\Theta_2}$$

with

$$\begin{aligned}\Theta_1 &= ((1+\lambda_2)a_1 - (1+\lambda_1)a_2)[(a_1-a_2)^2 u - (\lambda_1-\lambda_2)(a_{1,x}-a_{2,x})],\\ \Theta_2 &= (a_1-a_2)[a_2((1+\lambda_1)a_2 + (\lambda_2-\lambda_1)a_{1,x}) + (1+\lambda_2)a_1^2\\ &\quad + a_1((\lambda_1-\lambda_2)a_{2,x} - (2+\lambda_1+\lambda_2)a_2)].\end{aligned}$$

□

Our task now is to construct the N-BT for the gWKI equation. In order to provide a uniform expression, we introduce a new variable

$$\omega_{i_1 i_2\cdots i_N} = a_{i_1} a_{i_1 i_2}\cdots a_{i_1 i_2\cdots i_N} = \prod_{k=1}^{N} a_{i_1 i_2\cdots i_k}, \quad i_k \in \mathbb{N}^+,\ i_j \neq i_k\ (j\neq k). \tag{2.18}$$

It infers that

$$\begin{aligned}&\omega_{i_1} = a_{i_1}, \qquad \omega_{12} = a_1 a_{12} = \frac{(1+\lambda_2)\omega_1 - (1+\lambda_1)\omega_2}{\omega_2-\omega_1} = \frac{\Delta_2}{\Omega_2},\\ &\omega_{123} = a_1 a_{12} a_{123} = \omega_1 \frac{(1+\lambda_3)\omega_{12} - (1+\lambda_2)\omega_{13}}{\omega_{13}-\omega_{12}} = \frac{\Delta_3}{\Omega_3},\\ &\omega_{1234} = a_1 a_{12} a_{123} a_{1234} = \omega_{12} \frac{(1+\lambda_4)\omega_{123} - (1+\lambda_3)\omega_{124}}{\omega_{124}-\omega_{123}} = \frac{\Delta_4}{\Omega_4},\end{aligned} \tag{2.19}$$

where

$$\Delta_2 = \begin{vmatrix} a_1 & 1+\lambda_1 \\ a_2 & 1+\lambda_2 \end{vmatrix}, \qquad \Omega_2 = \begin{vmatrix} 1 & a_1 \\ 1 & a_2 \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} a_1 & 1+\lambda_1 & (1+\lambda_1)a_1 \\ a_2 & 1+\lambda_2 & (1+\lambda_2)a_2 \\ a_3 & 1+\lambda_3 & (1+\lambda_3)a_3 \end{vmatrix}, \qquad \Omega_3 = \begin{vmatrix} 1 & a_1 & 1+\lambda_1 \\ 1 & a_2 & 1+\lambda_2 \\ 1 & a_3 & 1+\lambda_3 \end{vmatrix},$$

$$\Delta_4 = \begin{vmatrix} a_1 & 1+\lambda_1 & (1+\lambda_1)a_1 & (1+\lambda_1)^2 \\ a_2 & 1+\lambda_2 & (1+\lambda_2)a_2 & (1+\lambda_2)^2 \\ a_3 & 1+\lambda_3 & (1+\lambda_3)a_3 & (1+\lambda_3)^2 \\ a_4 & 1+\lambda_4 & (1+\lambda_4)a_4 & (1+\lambda_4)^2 \end{vmatrix}, \qquad \Omega_4 = \begin{vmatrix} 1 & a_1 & 1+\lambda_1 & (1+\lambda_1)a_1 \\ 1 & a_2 & 1+\lambda_2 & (1+\lambda_2)a_2 \\ 1 & a_3 & 1+\lambda_3 & (1+\lambda_3)a_3 \\ 1 & a_4 & 1+\lambda_4 & (1+\lambda_4)a_4 \end{vmatrix}.$$

Consequently, it is easy to check that the gWKI equation admits 3-BT

$$\begin{aligned}&x_{[123]} = x - \ln|\omega_{123}|, \quad t_{[123]} = t,\\ &u_{[123]} = \frac{a_{123,x} + (1+\lambda_2)(a_{12}^{-1})_x + a_{1,x} + v}{1-(\ln|\omega_{123}|)_x},\\ &v_{[123]} = \frac{(1+\lambda_3)(a_{123}^{-1})_x + a_{12,x} + (1+\lambda_1)(a_1^{-1})_x + u}{1-(\ln|\omega_{123}|)_x},\end{aligned} \tag{2.20}$$

and 4-BT

$$
\begin{aligned}
x_{[1234]} &= x - \ln|\omega_{1234}|, \quad t_{[1234]} = t,\\
u_{[1234]} &= \frac{a_{1234,x} + (1+\lambda_3)(a_{123}^{-1})_x + a_{12,x} + (1+\lambda_1)(a_1^{-1})_x + u}{1 - (\ln|\omega_{1234}|)_x},\\
v_{[1234]} &= \frac{(1+\lambda_4)(a_{1234}^{-1})_x + a_{123,x} + (1+\lambda_2)(a_{12}^{-1})_x + a_{1,x} + v}{1 - (\ln|\omega_{1234}|)_x},
\end{aligned}
\tag{2.21}
$$

where

$$
a_{123} = \frac{\omega_{123}}{\omega_{12}} = \frac{\Delta_3\Omega_2}{\Delta_2\Omega_3}, \qquad a_{1234} = \frac{\omega_{1234}}{\omega_{123}} = \frac{\Delta_4\Omega_3}{\Delta_3\Omega_4}. \tag{2.22}
$$

By observing the expressions in (2.19), we introduce the following determinants:

$$
\begin{aligned}
\Omega_{2n+2} &= \begin{vmatrix}
1 & a_1 & 1+\lambda_1 & (1+\lambda_1)a_1 & \cdots & (1+\lambda_1)^n & (1+\lambda_1)^n a_1\\
1 & a_2 & 1+\lambda_2 & (1+\lambda_2)a_2 & \cdots & (1+\lambda_2)^n & (1+\lambda_2)^n a_2\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
1 & a_{2n+1} & 1+\lambda_{2n+1} & (1+\lambda_{2n+1})a_{2n+1} & \cdots & (1+\lambda_{2n+1})^n & (1+\lambda_{2n+1})^n a_{2n+1}\\
1 & a_{2n+2} & 1+\lambda_{2n+2} & (1+\lambda_{2n+2})a_{2n+2} & \cdots & (1+\lambda_{2n+2})^n & (1+\lambda_{2n+2})^n a_{2n+2}
\end{vmatrix},\\
\Omega_{2n+1} &= \Omega_{2n+2}\begin{bmatrix} 2n+2\\ 2n+2\end{bmatrix}, \quad \Delta_{2n+1} = \Omega_{2n+2}\begin{bmatrix} 2n+2\\ 1\end{bmatrix}, \quad n \in \mathbb{N}.
\end{aligned}
\tag{2.23}
$$

Here $H\begin{bmatrix} i_1 & i_2 & \cdots & i_k\\ j_1 & j_2 & \cdots & j_k\end{bmatrix}$ denotes the determinant with the i_1-th, i_2-th, $\cdots$, i_k-th rows and j_1-th, j_2-th, $\cdots$, j_k-th columns removed from the determinant H.

Proposition 3. *The expression for $\omega_{12\cdots N}$ in terms of determinants Δ_N and Ω_N is given by*

$$
\omega_{12\cdots N} = \frac{\Delta_N}{\Omega_N}, \quad N \in \mathbb{N}^+. \tag{2.24}
$$

Proof. We verify (2.24) by mathematical induction. Obviously, in view of (2.19), (2.24) holds for $N = 1, 2, 3, 4$. Assuming (2.24) holds for N, we aim to verify it for $N+1$. Using the Bianchi's theorem of permutability, we obtain

$$
\omega_{12\cdots N+1} = \omega_{12\cdots N-1}\frac{(1+\lambda_{N+1})\omega_{12\cdots N} - (1+\lambda_N)\omega_{12\cdots N-1,N+1}}{\omega_{12\cdots N-1,N+1} - \omega_{12\cdots N}}. \tag{2.25}
$$

The use of inductive assumption leads to

$$
\omega_{12\cdots N} = \frac{\Delta_N}{\Omega_N}, \quad \omega_{12\cdots N-1,N+1} = \frac{\Delta_{N+1}\begin{bmatrix} N\\ N+1\end{bmatrix}}{\Omega_{N+1}\begin{bmatrix} N\\ N+1\end{bmatrix}}. \tag{2.26}
$$

Substituting (2.26) into (2.25) yields

$$
\omega_{12\cdots N+1} = \frac{\Delta_{N-1}}{\Omega_{N-1}}\frac{(1+\lambda_{N+1})\Delta_N\Omega_{N+1}\begin{bmatrix} N\\ N+1\end{bmatrix} - (1+\lambda_N)\Omega_N\Delta_{N+1}\begin{bmatrix} N\\ N+1\end{bmatrix}}{\Omega_N\Delta_{N+1}\begin{bmatrix} N\\ N+1\end{bmatrix} - \Delta_N\Omega_{N+1}\begin{bmatrix} N\\ N+1\end{bmatrix}}. \tag{2.27}
$$

Before proceeding further, let us list two identities

$$(1+\lambda_{N+1})\Omega_{N+1}\begin{bmatrix}N\\N+1\end{bmatrix}=\frac{1}{\prod_{j=1}^{N-1}(1+\lambda_j)}\Delta_{N+1}\begin{bmatrix}N\\1\end{bmatrix},$$

$$(1+\lambda_N)\Omega_N=\frac{1}{\prod_{j=1}^{N-1}(1+\lambda_j)}\Delta_{N+1}\begin{bmatrix}N+1\\1\end{bmatrix},$$

and the Jacobi identity

$$HH\begin{bmatrix}i_1 & i_2\\ j_1 & j_2\end{bmatrix}=H\begin{bmatrix}i_1\\j_1\end{bmatrix}H\begin{bmatrix}i_2\\j_2\end{bmatrix}-H\begin{bmatrix}i_1\\j_2\end{bmatrix}H\begin{bmatrix}i_2\\j_1\end{bmatrix},$$

which will be used in the subsequent analysis. Thus, we compute

$$\begin{aligned}
&(1+\lambda_{N+1})\Delta_N\Omega_{N+1}\begin{bmatrix}N\\N+1\end{bmatrix}-(1+\lambda_N)\Omega_N\Delta_{N+1}\begin{bmatrix}N\\N+1\end{bmatrix}\\
&=\frac{1}{\prod_{j=1}^{N-1}(1+\lambda_j)}\Delta_N\Delta_{N+1}\begin{bmatrix}N\\1\end{bmatrix}-\frac{1}{\prod_{j=1}^{N-1}(1+\lambda_j)}\Delta_{N+1}\begin{bmatrix}N+1\\1\end{bmatrix}\Delta_{N+1}\begin{bmatrix}N\\N+1\end{bmatrix}\\
&=\frac{1}{\prod_{j=1}^{N-1}(1+\lambda_j)}(\Delta_N\Delta_{N+1}\begin{bmatrix}N\\1\end{bmatrix}-\Delta_{N+1}\begin{bmatrix}N+1\\1\end{bmatrix}\Delta_{N+1}\begin{bmatrix}N\\N+1\end{bmatrix})\\
&=\frac{1}{\prod_{j=1}^{N-1}(1+\lambda_j)}\Delta_{N+1}\Delta_{N+1}\begin{bmatrix}N & N+1\\1 & N+1\end{bmatrix}=\Delta_{N+1}\frac{1}{\prod_{j=1}^{N-1}(1+\lambda_j)}\Delta_N\begin{bmatrix}N\\1\end{bmatrix}\\
&=\Delta_{N+1}\Omega_{N-1},
\end{aligned}$$

and

$$\begin{aligned}
&\Omega_N\Delta_{N+1}\begin{bmatrix}N\\N+1\end{bmatrix}-\Delta_N\Omega_{N+1}\begin{bmatrix}N\\N+1\end{bmatrix}\\
&=\Omega_{N+1}\begin{bmatrix}N\\1\end{bmatrix}\Omega_{N+1}\begin{bmatrix}N+1\\N+1\end{bmatrix}-\Omega_{N+1}\begin{bmatrix}N+1\\1\end{bmatrix}\Omega_{N+1}\begin{bmatrix}N\\N+1\end{bmatrix}\\
&=\Omega_{N+1}\Omega_{N+1}\begin{bmatrix}N & N+1\\1 & N+1\end{bmatrix}=\Omega_{N+1}\Omega_N\begin{bmatrix}N\\1\end{bmatrix}\\
&=\Omega_{N+1}\Delta_{N-1}.
\end{aligned}$$

Inserting the above results into (2.27), we have

$$\omega_{12\cdots N+1}=\frac{\Delta_{N-1}}{\Omega_{N-1}}\frac{\Delta_{N+1}\Omega_{N-1}}{\Delta_{N-1}\Omega_{N+1}}=\frac{\Delta_{N+1}}{\Omega_{N+1}}.$$

It follows that (2.24) holds for $N+1$, which completes the proof. □

Theorem 1. *Suppose that $\lambda_1, \cdots, \lambda_N$ are different constant spectral parameters excluding* 0 *and* -1*; thus, the N-BT for the gWKI equation is given by*

$$\begin{cases} x_{[12\cdots N]} = x - \ln|\omega_{12\cdots N}|, \\ u_{[12\cdots N]} = \frac{\sum\limits_{k=1}^{n}[a_{12\cdots 2k,x}+(1+\lambda_{2k-1})(a_{12\cdots 2k-1}^{-1})_x]+u}{1-(\ln|\omega_{12\cdots N}|)_x}, \\ v_{[12\cdots N]} = \frac{\sum\limits_{k=1}^{n}[(1+\lambda_{2k})(a_{12\cdots 2k}^{-1})_x+a_{12\cdots 2k-1,x}]+v}{1-(\ln|\omega_{12\cdots N}|)_x}, \end{cases} \quad N = 2n,\ n \in \mathbb{N}^+ \tag{2.28}$$

and

$$\begin{cases} x_{[12\cdots N]} = x - \ln|\omega_{12\cdots N}|, \\ u_{[12\cdots N]} = \frac{\sum\limits_{k=1}^{n}[a_{12\cdots 2k-1,x}+(1+\lambda_{2k-2})(a_{12\cdots 2k-2}^{-1})_x]+v}{1-(\ln|\omega_{12\cdots N}|)_x}, \\ v_{[12\cdots N]} = \frac{\sum\limits_{k=1}^{n}[(1+\lambda_{2k-1})(a_{12\cdots 2k-1}^{-1})_x+a_{12\cdots 2k-2,x}]+u}{1-(\ln|\omega_{12\cdots N}|)_x}, \end{cases} \quad N = 2n-1,\ n \in \mathbb{N}^+ \tag{2.29}$$

where $a_0 = 0, \lambda_0 = -1, w_{12\cdots N} = \frac{\Delta_N}{\Omega_N}, a_{12\cdots N} = \frac{\Delta_N \Omega_{N-1}}{\Omega_N \Delta_{N-1}}$, Ω_N *and* Δ_N *are given by* (2.23).

Proof. Here we only prove (2.28) by mathematical induction since the proof of (2.29) is similar. According to Proposition 2, it is clear that (2.28) holds for $N = 2$. Assuming (2.28) holds for $N = 2n$, we need to verify it for $N = 2n + 2$. Again by Proposition 2 and the Bianchi's theorem of permutability, we have

$$\begin{aligned} x_{[12\cdots N+2]} &= x_{[12\cdots N]} - \ln|a_{12\cdots N+1}a_{12\cdots N+2}|, \\ u_{[12\cdots N+2]} &= \frac{a_{12\cdots N+2,x_{[12\cdots N]}} + (1+\lambda_{N+1})(a_{12\cdots N+1})^{-1}_{x_{[12\cdots N]}} + u_{[12\cdots N]}}{1 - (\ln|\frac{\omega_{12\cdots N+2}}{\omega_{12\cdots N}}|)_{x_{[12\cdots N]}}}, \\ v_{[12\cdots N+2]} &= \frac{(1+\lambda_{N+2})(a_{12\cdots N+2})^{-1}_{x_{[12\cdots N]}} + a_{12\cdots N+1,x_{[12\cdots N]}} + v_{[12\cdots N]}}{1 - (\ln|\frac{\omega_{12\cdots N+2}}{\omega_{12\cdots N}}|)_{x_{[12\cdots N]}}}. \end{aligned}$$

Applying inductive assumption and the following derivative relation

$$\frac{\partial}{\partial x_{[12\cdots N]}} = \frac{1}{1-(\ln|\omega_{12\cdots N}|)_x}\frac{\partial}{\partial x},$$

we obtain

$$\begin{aligned} x_{[12\cdots N+2]} &= x - \ln|\omega_{12\cdots N}| - \ln|a_{12\cdots N+1}a_{12\cdots N+2}| = x - \ln|\omega_{12\cdots N+2}|, \\ u_{[12\cdots N+2]} &= \frac{1}{1-(\ln|\omega_{12\cdots N+2}|)_x}[a_{12\cdots N+2,x} + (1+\lambda_{N+1})(a_{12\cdots N+1}^{-1})_x \\ &\quad + \sum_{k=1}^{n}(a_{12\cdots 2k,x} + (1+\lambda_{2k-1})(a_{12\cdots 2k-1}^{-1})_x) + u] \\ &= \frac{\sum\limits_{k=1}^{n+1}[a_{12\cdots 2k,x} + (1+\lambda_{2k-1})(a_{12\cdots 2k-1}^{-1})_x] + u}{1-(\ln|\omega_{12\cdots N+2}|)_x}, \end{aligned}$$

and

$$v_{[12\cdots N+2]} = \frac{1}{1-(\ln|\omega_{12\cdots N+2}|)_x}[(1+\lambda_{N+2})(a^{-1}_{12\cdots N+2})_x + a_{12\cdots N+1,x} + \sum_{k=1}^{n}((1+\lambda_{2k})(a^{-1}_{12\cdots 2k})_x + a_{12\cdots 2k-1,x}) + v], = \frac{\sum_{k=1}^{n+1}[(1+\lambda_{2k})(a^{-1}_{12\cdots 2k})_x + a_{12\cdots 2k-1,x}] + v}{1-(\ln|\omega_{12\cdots N+2}|)_x}.$$

It implies (2.28) is true for $N = 2n + 2$, which completes the proof. □

3. Soliton solutions of the gWKI equation

In this section, we shall apply the N-BT obtained in Section 2 to discuss soliton solutions for the gWKI equation. To this end, we rewrite the Lax pair (2.1) as a scalar form

$$\begin{aligned} \psi_{1,xx} &= \frac{u_x}{u}\psi_{1,x} + [\lambda^2 + \lambda(1+\lambda)uv - \frac{u_x}{u}\lambda]\psi_1, \\ \psi_{1,t} &= -\frac{\alpha\lambda}{2u}(mu)_x\psi_1 + [\alpha(1+\lambda)m + \frac{\alpha}{2u}(mu)_x]\psi_{1,x}. \end{aligned} \tag{3.1}$$

Let f_1 be a solution to the linear system (3.1) with $u = u_0 \neq 0$, $v = v_0 \neq 0$, and $\lambda = \lambda_1^{-1}$, i.e.

$$\begin{aligned} f_{1,xx} &= \left(\frac{1+(1+\lambda_1)u_0v_0}{\lambda_1^2}\right)f_1, \\ f_{1,t} &= \frac{\alpha}{\sqrt{1+u_0v_0}}(1+\lambda_1^{-1})f_{1,x}. \end{aligned}$$

It is easy to see that under the following conditions: either $\lambda_1 > -1$ with $u_0v_0 > -\frac{1}{1+\lambda_1}$, and $u_0v_0 > -1$, or $\lambda_1 < -1$ with $-1 < u_0v_0 < -\frac{1}{1+\lambda_1}$, we obtain a real solution

$$f_1 = e^{\xi_1} + \delta_1 e^{-\xi_1} = (1+\delta_1)\cosh\xi_1 + (1-\delta_1)\sinh\xi_1,$$

where $\xi_1 = \frac{\beta_1}{\lambda_1}[x + \frac{\alpha}{\sqrt{1+u_0v_0}}(1+\lambda_1^{-1})t] + x_{01}$, $\beta_1 = \sqrt{1+(1+\lambda_1)u_0v_0}$, $\delta_1 = \pm 1$ and x_{01} is an arbitrary constant. By Proposition 1, we have

$$a_1 = \frac{1}{u_0}(1-\lambda_1\frac{f_{1,x}}{f_1}) = \frac{1}{u_0}[1-\beta_1\frac{(1+\delta_1)\sinh\xi_1 + (1-\delta_1)\cosh\xi_1}{(1+\delta_1)\cosh\xi_1 + (1-\delta_1)\sinh\xi_1}].$$

If $\delta_1 = 1$, then

$$a_1 = \frac{1}{u_0}(1-\beta_1\tanh\xi_1). \tag{3.2}$$

Substituting (3.2) into (2.14), we obtain a tanh-type one-soliton solution

$$\begin{aligned} x_{[1]} &= x - \ln|1-\beta_1\tanh\xi_1| + \ln|u_0|, \qquad t_{[1]} = t, \\ u_{[1]} &= -\frac{(\tanh^2\xi_1 + \lambda_1\beta_1^{-2}v_0u_0 - 1)(1-\beta_1\tanh\xi_1)}{u_0(\tanh^2\xi_1 + \lambda_1\beta_1^{-1}\tanh\xi_1 - \lambda_1\beta_1^{-2} - 1)}, \\ v_{[1]} &= \frac{u_0(\tanh^2\xi_1 + 2\lambda_1\beta_1^{-1}\tanh\xi_1 - \lambda_1\beta_1^{-2} - \lambda_1 - 1)}{(\tanh^2\xi_1 + \lambda_1\beta_1^{-1}\tanh\xi_1 - \lambda_1\beta_1^{-2} - 1)(1-\beta_1\tanh\xi_1)}. \end{aligned} \tag{3.3}$$

By analyzing the expressions of $u_{[1]}$ and $v_{[1]}$ in (3.3), we find that they both have no poles under the following conditions: i) $-2 \leq \lambda_1 < -1$ and $\frac{-(2+\lambda_1)^2}{4(1+\lambda_1)} < u_0 v_0 < -\frac{1}{1+\lambda_1}$; ii) $\lambda_1 < -2$ and $0 < u_0 v_0 < -\frac{1}{1+\lambda_1}$. Under these conditions, a direct calculation shows that $\frac{\partial x_{[1]}}{\partial x} > 0$. This implies that the map from $x_{[1]}$ to x is bijective, and $x_{[1]} \to \pm\infty$ as $x \to \pm\infty$. Therefore, in this case, (3.3) yields a smooth one-soliton solution.

Furthermore, when $-2 < \lambda_1 < -1$ and $0 < u_0 v_0 \leq \frac{-(2+\lambda_1)^2}{4(1+\lambda_1)}$, the solution (3.3) becomes singular. Specially, when $u_0 v_0 = \frac{-(2+\lambda_1)^2}{4(1+\lambda_1)}$, (3.3) describes a bursting soliton. In addition, when $-2 < \lambda_1 < -1$ and $0 < u_0 v_0 < \frac{-(2+\lambda_1)^2}{4(1+\lambda_1)}$, the map from $x_{[1]}$ to x becomes multivalued. As a result, (3.3) gives a loop-type solution. The smooth soliton, bursting soliton, and loop-type soliton are illustrated in Figures 1–3, respectively.

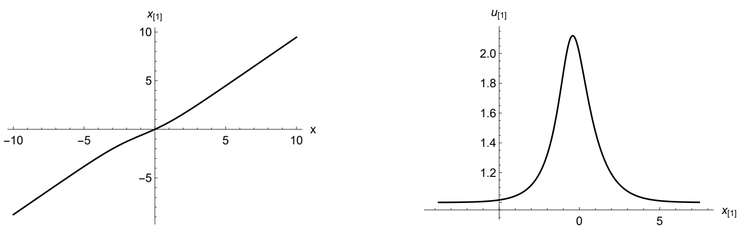


Figure 1. The profile of a smooth soliton with $u_0 = 1$, $v_0 = 1$, $\lambda_1 = -1.5$, $\alpha = 1$, and $x_{01} = 0$ when $t = 0$.

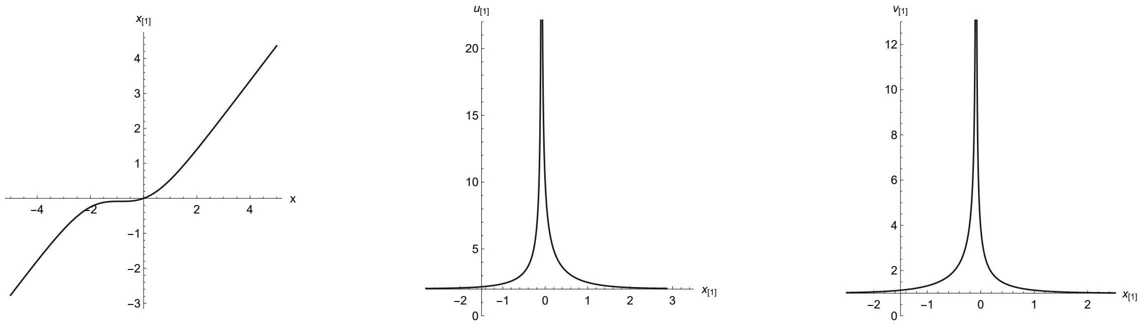


Figure 2. The profile of a bursting soliton with $u_0 = 1$, $v_0 = 2.025$, $\lambda_1 = -1.1$, $\alpha = 1$, and $x_{01} = 0$ when $t = 0$.

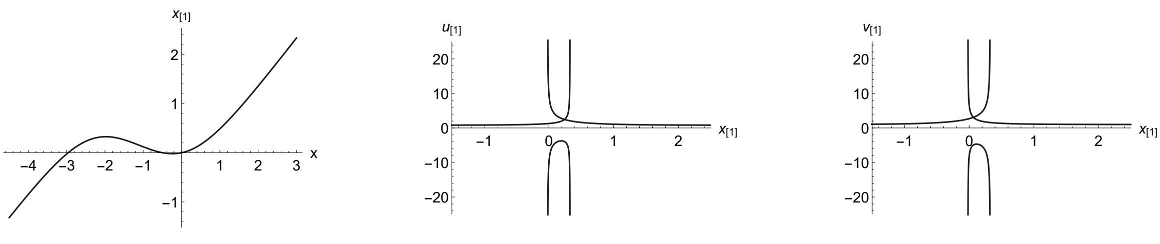


Figure 3. The profile of a loop-type soliton with $u_0 = 1$, $v_0 = 0.8$, $\lambda_1 = -1.1$, $\alpha = 1$, and $x_{01} = 0$ when $t = 0$.

Reamrk: 1. The profiles of $u_{[1]}$ and $v_{[1]}$ in Figure 3 are different from those of standard loop solitons [35, 37] and the WKI loop solitons [17, 18]. However, the profiles of the absolute values of

$u_{[1]}$ and $v_{[1]}$ correspond to the WKI loop solitons. Therefore, we refer to $u_{[1]}$ and $v_{[1]}$ in Figure 3 as loop-type solitons. 2. If $\delta_1 = -1$, then $a_1 = \frac{1}{u_0}(1-\beta_1 \coth\xi_1)$, which yields a coth-type soliton solution. However, this solution is always singular, so we do not analyze it further here.

Furthermore, applying the 2-BT given in Proposition 2, we get a two-soliton solution for the gWKI equation

$$\begin{aligned}
x_{[12]} &= x - \ln\left|\frac{(1+\lambda_2)(1-\beta_1\tanh\xi_1) - (1+\lambda_1)(1-\beta_2\coth\xi_2)}{\beta_1\tanh\xi_1 - \beta_2\coth\xi_2}\right|, \quad t_{[12]} = t,\\
u_{[12]} &= \frac{a_{12,x} + (1+\lambda_1)(a_1^{-1})_x + u}{1-(\ln|a_1a_{12}|)_x} = \frac{\Xi_1}{\Xi_2},\\
v_{[12]} &= \frac{(1+\lambda_2)(a_{12}^{-1})_x + a_{1,x} + v}{1-(\ln|a_1a_{12}|)_x} = \frac{\Xi_3}{\Xi_4},
\end{aligned}$$

where

$$\begin{aligned}
\Xi_1 &= ((1+\lambda_2)a_1 - (1+\lambda_1)a_2)[(a_1-a_2)^2u - (\lambda_1-\lambda_2)(a_{1,x}-a_{2,x})],\\
\Xi_2 &= (a_1-a_2)[a_2((1+\lambda_1)a_2 + (\lambda_2-\lambda_1)a_{1,x})\\
&\quad + (1+\lambda_2)a_1^2 + a_1((\lambda_1-\lambda_2)a_{2,x} - (2+\lambda_1+\lambda_2)a_2)],\\
\Xi_3 &= (a_1-a_2)[(1+\lambda_1)a_2^2((1+\lambda_1)v - 2(1+\lambda_1)(1+\lambda_2)a_1a_2v\\
&\quad + (\lambda_1-\lambda_2)a_{1,x}) + (1+\lambda_2)a_1^2((1+\lambda_2)v + (\lambda_2-\lambda_1)a_{2,x})],\\
\Xi_4 &= ((1+\lambda_2)a_1 - (1+\lambda_1)a_2)[(1+\lambda_2)a_1^2 + a_2((1+\lambda_1)a_2\\
&\quad + (\lambda_2-\lambda_1)a_{1,x}) + a_1((\lambda_1-\lambda_2)a_{2,x} - (2+\lambda_1+\lambda_2)a_2)],
\end{aligned}$$

and $u = u_0 \neq 0$, $v = v_0 \neq 0$, $a_1 = \frac{1}{u_0}(1-\beta_1\tanh\xi_1)$, $a_2 = \frac{1}{u_0}(1-\beta_2\coth\xi_2)$, $\xi_2 = \frac{\beta_2}{\lambda_2}[x + \frac{\alpha}{\sqrt{1+u_0v_0}}(1 + \lambda_2^{-1})t] + x_{02}, \beta_2 = \sqrt{1+(1+\lambda_2)u_0v_0}$.

In fact, the above two-soliton solution is formed by the superposition of a tanh-type one-soliton and a coth-type one-soliton. By analyzing the parameter values, as in the case of the one-soliton, we present some interesting interactions between two solitons, including two smooth solitons, two loop-type solitons, smooth and bursting solitons, smooth and loop-type solitons, and loop-type and bursting solitons. Their corresponding figures are provided in Figures 4–8, respectively. As can be observed from the figures, the two solitons maintain identical structure before and after the collision. Notably, when two different types of solitons collide, they temporarily transform into the same type of soliton and subsequently revert to their original forms. For instance, a smooth soliton and a loop-type soliton become two smooth solitons during collision but return to a smooth soliton and a loop-type soliton after the collision (see Figure 6).

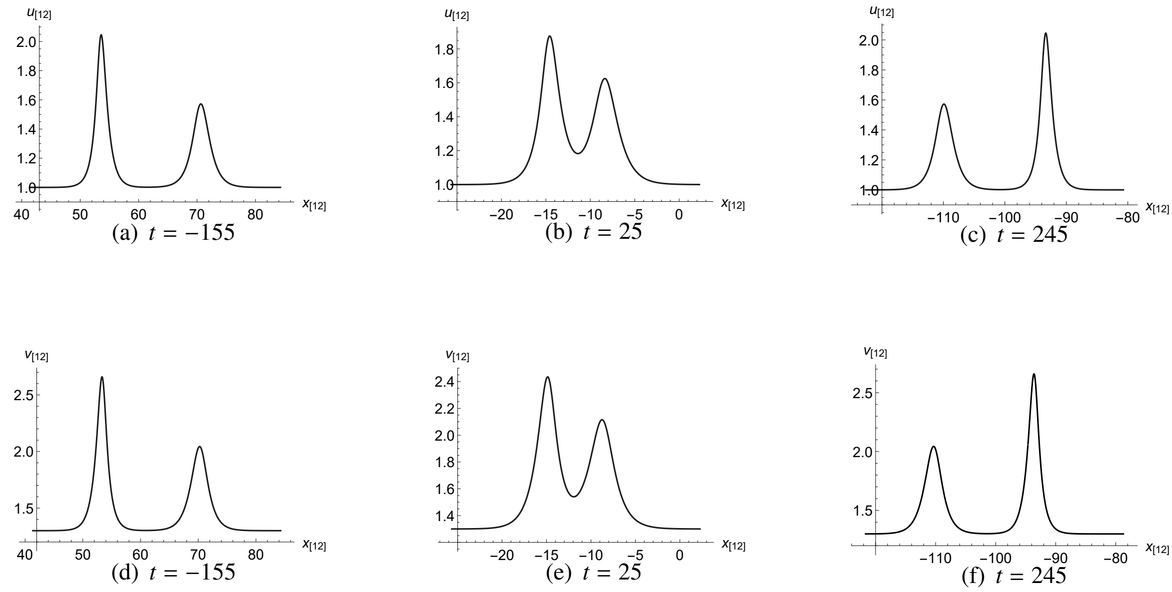


Figure 4. The interaction between two smooth solitons with $u_0 = 1$, $v_0 = 1.3$, $\lambda_1 = -1.5$, $\lambda_2 = -1.4$, $\alpha = 2$, $x_{01} = 0$, $x_{02} = 3$.

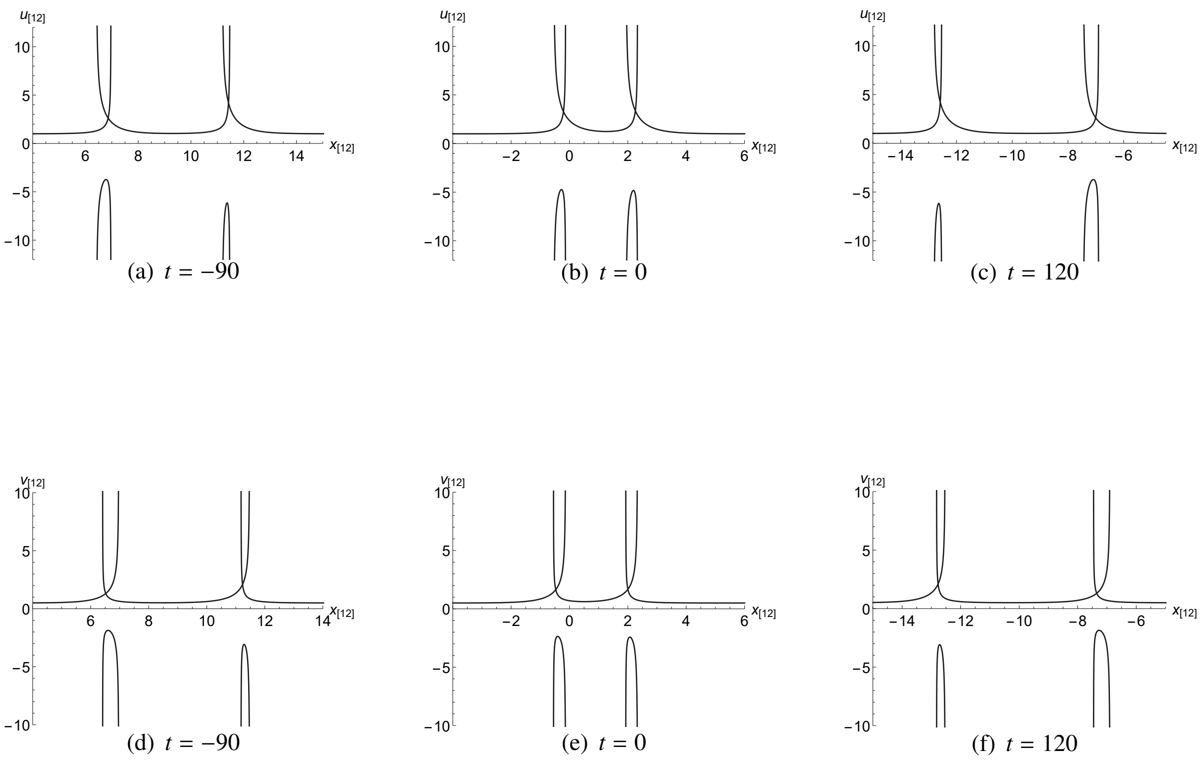


Figure 5. The interaction between two loop-type solitons with $u_0 = 1$, $v_0 = 0.7$, $\lambda_1 = -1.15$, $\lambda_2 = -1.1$, $\alpha = 1$, $x_{01} = x_{02} = 0$.

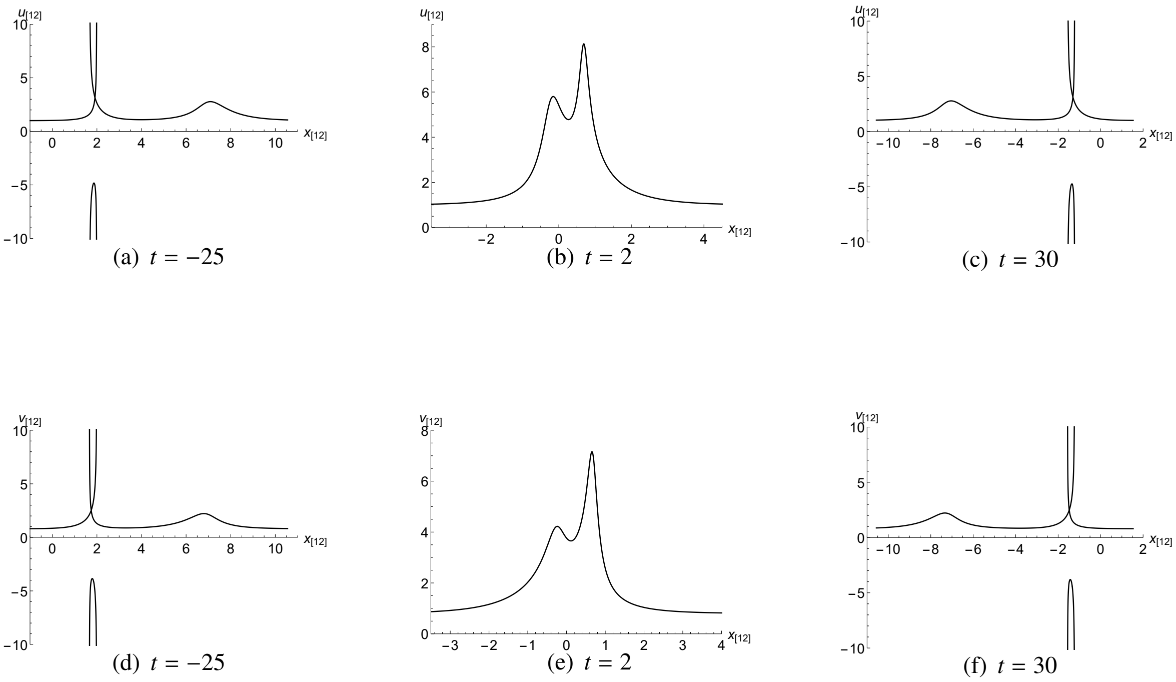


Figure 6. The interaction between a smooth soliton and a loop-type soliton with $u_0 = 1$, $v_0 = 0.8$, $\lambda_1 = -1.5$, $\lambda_2 = -1.1$, $\alpha = 1$, $x_{01} = x_{02} = 0$.

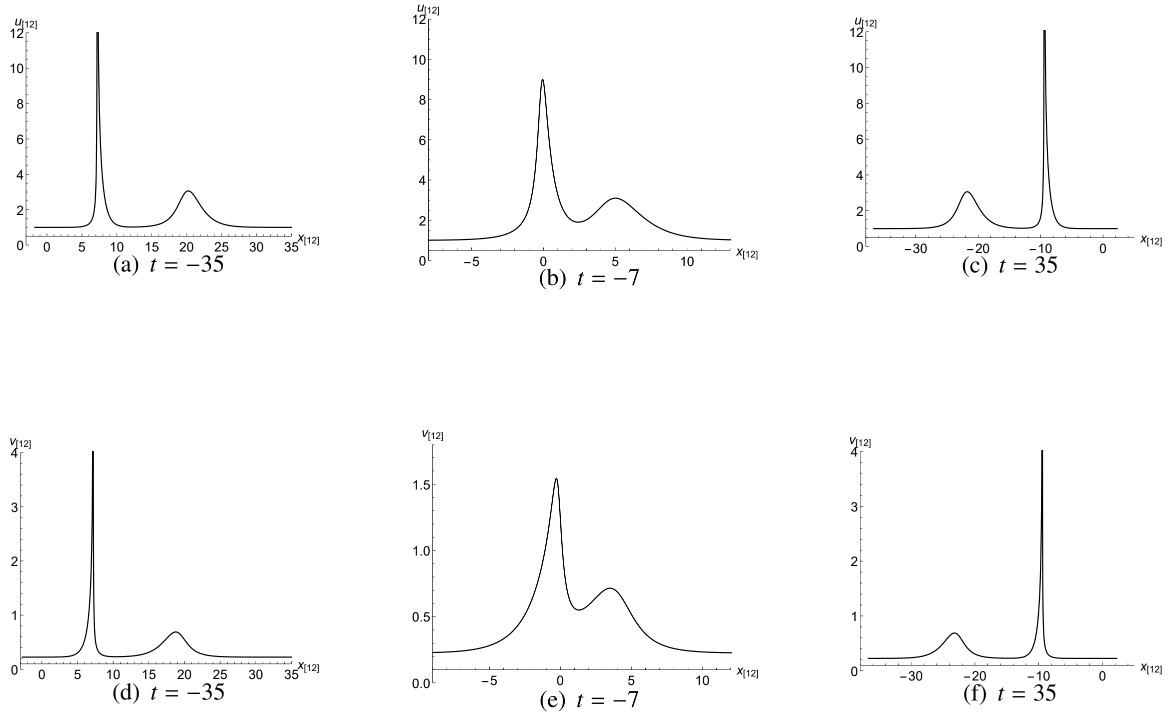
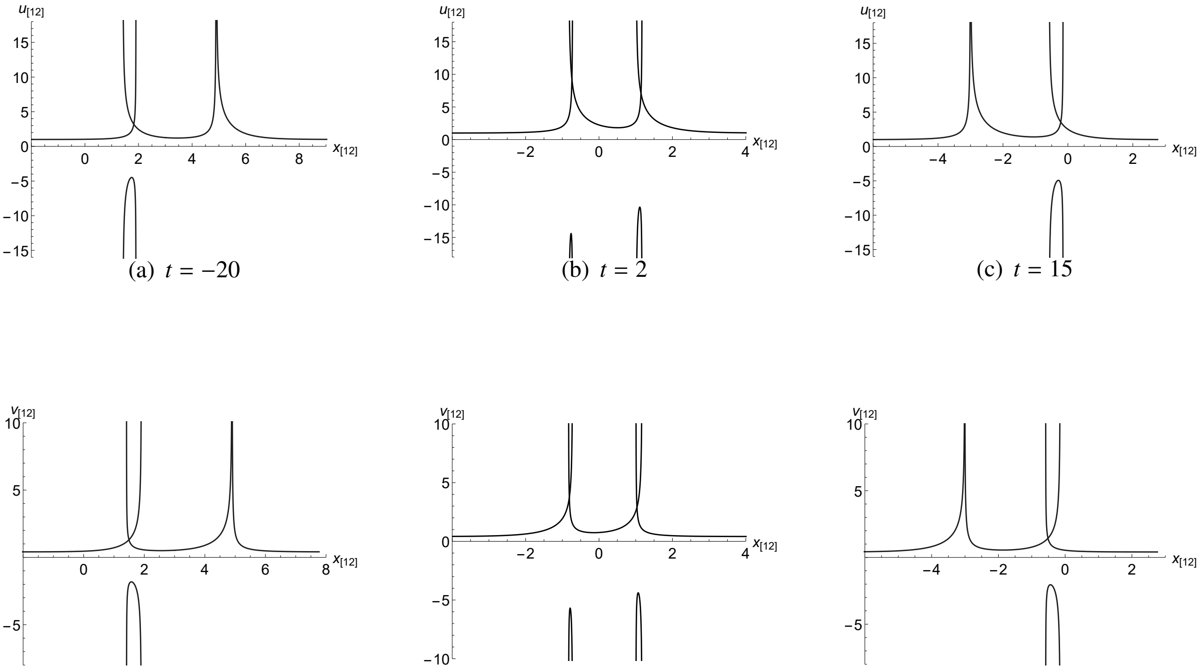


Figure 7. The interaction between a smooth soliton and a bursting soliton with $u_0 = 1$, $v_0 = 0.225$, $\lambda_1 = -2.5$, $\lambda_2 = -1.4$, $\alpha = 1$, $x_{01} = x_{02} = 0$.



(a) $t = -20$ (b) $t = 2$ (c) $t = 15$

(d) $t = -20$ (e) $t = 2$ (f) $t = 15$

Figure 8. The interaction between a loop-type soliton and a bursting soliton with $u_0 = 1$, $v_0 = \frac{49}{120}$, $\lambda_1 = -1.3$, $\lambda_2 = -1.12$, $\alpha = 1$, $x_{01} = x_{02} = 0$.

4. Conclusions

In this paper, using RT and DT, we have successfully constructed an N-BT for the gWKI equation (1.3), whose expression is explicit and free of integrals. As an application of the N-BT, we obtain some solutions for the gWKI equation, including smooth, bursting, and loop-type solitons (see Figures 1–3). We observe that the loop-type soliton of the gWKI equation in this case differs from that of the WKI equation [17, 18]. Moreover, when investigating the interactions between two solitons (see Figures 4–8), an interesting phenomenon emerges: two distinct types of solitons temporarily transform into the same type during collision and subsequently revert to their original forms afterward (see Figures 6–8).

Future research may explore the following directions. Firstly, investigation into the discretization, and symmetries of the gWKI equation could help reveal the underlying mathematical structures. Secondly, further study on the physical applications of the obtained solutions could yield significant practical insights. Additionally, the integration of numerical simulations with analytical results may provide a deeper understanding of the stability and dynamics of the gWKI equation.

Author contributions

Chenglu Zhu: Methodology, formal analysis, software, writing–original draft preparation. Lihua Wu: Conceptualization, methodology, supervision, writing–review and editing. All authors read and approved the final manuscript for publication.

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Conflict of interest

The authors declare that they have no competing interests.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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