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Research article

Non-global nonlinear mixed skew Jordan Lie triple derivations on prime $*$-rings

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Abstract: Let $\mathcal{R}$ be a 2-torsion free unital prime $*$-ring containing a nontrivial symmetric idempotent and $Z_c(\mathcal{R})$ be the anti-symmetric center of $\mathcal{R}$. We prove that if a map $\varphi : \mathcal{R} \to \mathcal{R}$ satisfies $\varphi([A \bullet B, C]) = [\varphi(A) \bullet B, C] + [A \bullet \varphi(B), C] + [A \bullet B, \varphi(C)]$ for any $A, B, C \in \mathcal{R}$ with $ABC^* = 0$, then there exists an additive $*$-derivation Θ of $\mathcal{R}$ and a nonlinear map $g : \mathcal{R} \to Z_c(\mathcal{R})$ such that $\varphi(A) = \Theta(A) + g(A)$ for any $A \in \mathcal{R}$.

Keywords: mixed skew Jordan Lie triple derivation; additive $*$-derivation; prime $*$-ring
Mathematics Subject Classification: 16W10, 16W25

1. Introduction

Let $\mathcal{A}$ be an associative $*$-algebra. For $A, B \in \mathcal{A}$, denote by $[A, B] = AB - BA$, $[A, B]_* = AB - BA^*$ and $A \bullet B = AB + BA^*$ the Lie product, skew Lie product and skew Jordan product of A and B, respectively. A map $\delta : \mathcal{A} \to \mathcal{A}$ is called an additive derivation if $\delta(A + B) = \delta(A) + \delta(B)$ and $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$. Moreover, δ is called an additive $*$-derivation if it is an additive derivation and satisfies $\delta(A^*) = \delta(A)^*$ for all $A \in \mathcal{A}$. A map $\varphi : \mathcal{A} \to \mathcal{A}$ (without the additivity assumption) is called a nonlinear Lie derivation (resp. nonlinear Lie triple derivation) if $\varphi([A, B]) = [\varphi(A), B] + [A, \varphi(B)]$ for all $A, B \in \mathcal{A}$ (resp. $\varphi([[A, B], C]) = [[\varphi(A), B], C] + [[A, \varphi(B)], C] + [[A, B], \varphi(C)]$ for all $A, B, C \in \mathcal{A}$). In the past years, nonlinear Lie derivations and Lie triple derivations on various algebras have been studied. Chen and Zhang [1], Yu and Zhang [11], and Yang [9] gave the structure of nonlinear Lie derivations on upper triangular matrices, triangular algebras, and incidence algebras, respectively. Ji, Liu, and Zhao [4] proved that every nonlinear Lie triple derivation on triangular algebras can be expressed as the sum of an additive derivation and a center value map vanishing on Lie triple products. A map $\varphi : \mathcal{A} \to \mathcal{A}$ (without the additivity assumption) is called a nonlinear skew Jordan derivation (resp. nonlinear skew Jordan triple derivation) if $\varphi(A \bullet B) = \varphi(A) \bullet B + A \bullet \varphi(B)$ for all $A, B \in \mathcal{A}$ (resp. $\varphi(A \bullet B \bullet C) = \varphi(A) \bullet B \bullet C + A \bullet \varphi(B) \bullet C + A \bullet B \bullet \varphi(C)$

for all $A, B, C \in \mathcal{A}$). Nonlinear skew Jordan derivations and skew Jordan triple derivations on various algebras also have been studied by some authors. Taghavi, Rohi, and Darvish [8], and Zhang [12] proved every nonlinear skew Jordan derivation on factor von Neumann algebras is an additive $*$-derivation, respectively. Darvish et al. [2] proved that every nonlinear skew Jordan triple derivation on prime $*$-algebras is an additive $*$-derivation under some assumption. Recently, derivations (nonlinear maps) corresponding to (preserving) the new products of the mixture of (skew) Lie product and skew Jordan product have attracted the attention of several authors. Li and Zhang [6] studied nonlinear mixed Jordan triple $*$-derivations on factor von Neumann algebras. Yang and Zhang [10] characterized nonlinear maps preserving the second mixed Lie triple products on factor von Neumann algebras. Zhou, Yang, and Zhang [15] proved that any map φ from a unital prime $*$-algebra $\mathcal{A}$ to itself satisfying $\varphi([[A, B]_*, C]) = [[\varphi(A), B]_*, C] + [[A, \varphi(B)]_*, C] + [[A, B]_*, \varphi(C)]$ for all $A, B, C \in \mathcal{A}$ is an additive $*$-derivation. Let $\mathcal{A}$ and $\mathcal{B}$ be two factors with $\dim\mathcal{A} > 4$. Zhao, Li, and Chen [14] characterized the concrete structure of any bijective map $\varphi : \mathcal{A} \to \mathcal{B}$ satisfying

$$\varphi([A \bullet B, C]) = [\varphi(A) \bullet \varphi(B), \varphi(C)]$$

for all $A, B, C \in \mathcal{A}$.

The conditions under which linear (nonlinear) Lie (triple) derivations of operator algebras can be completely determined by the action on some proper subsets of these operator algebras were considered. Let $\mathcal{M}$ be a factor with $\dim\mathcal{M} > 1$. Liu [7] investigated any linear map $\varphi : \mathcal{M} \to \mathcal{M}$ satisfying $\varphi([[A, B], C]) = [[\varphi(A), B], C] + [[A, \varphi(B)], C] + [[A, B], \varphi(C)]$ for any $A, B, C \in \mathcal{A}$ with $AB = 0$ (resp. $AB = P$, where P is a fixed non-trivial projection of $\mathcal{M}$). Zhao and Hao [13] proved that if a nonlinear map φ from a finite von Neumann algebra $\mathcal{M}$ with no central summands of type I_1 to itself satisfies $\delta([[A, B], C]) = [[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$ for any $A, B, C \in \mathcal{M}$ with $ABC = 0$, then $\delta = d + \tau$, where d is an additive derivation from $\mathcal{M}$ into itself and τ is a nonlinear map from $\mathcal{M}$ into its center such that $\tau([[A, B], C]) = 0$ with $ABC = 0$. In sequel, we introduce the notion of non-global nonlinear mixed skew Jordan Lie triple derivation. Let $F : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a map and Q be a proper subset of $\mathcal{A}$. If φ satisfies

$$\varphi([A \bullet B, C]) = [\varphi(A) \bullet B, C] + [A \bullet \varphi(B), C] + [A \bullet B, \varphi(C)]$$

for any $A, B, C \in \mathcal{A}$ with $F(A, B, C) \in Q$, then φ is called a non-global nonlinear mixed skew Jordan Lie triple derivation.

A ring $\mathcal{R}$ is called a $*$-ring if there is an additive map $* : \mathcal{R} \to \mathcal{R}$ satisfying $(AB)^* = B^*A^*$ and $(A^*)^* = A$ for all $A, B \in \mathcal{R}$. $\mathcal{R}$ is called prime when for $A, B \in \mathcal{R}$, if $A\mathcal{R}B = \{0\}$, then $A = 0$ or $B = 0$. Let $Z(\mathcal{R})$ be the center of $\mathcal{R}$ and $Z_c(\mathcal{R}) = \{A \in Z(\mathcal{R}) : A^* = -A\}$ be the anti-symmetric center of $\mathcal{R}$. Motivated by the above-mentioned works, we will study the concrete structure of a kind of non-global nonlinear mixed skew Jordan Lie triple derivations φ on prime $*$-rings $\mathcal{R}$ satisfying $\varphi([A \bullet B, C]) = [\varphi(A) \bullet B, C] + [A \bullet \varphi(B), C] + [A \bullet B, \varphi(C)]$ for any $A, B, C \in \mathcal{R}$ with $ABC^* = 0$.

2. Main results

The main result is the following theorem:

Theorem 2.1. *Let $\mathcal{R}$ be a 2-torsion free unital prime $*$-ring containing a nontrivial symmetric* *idempotent. If a map* $\varphi : \mathcal{R} \to \mathcal{R}$ *satisfies*

$$\varphi([A \bullet B, C]) = [\varphi(A) \bullet B, C] + [A \bullet \varphi(B), C] + [A \bullet B, \varphi(C)]$$

for any $A, B, C \in \mathcal{R}$ with $ABC^* = 0$, then there exists an additive $*$-derivation Θ of $\mathcal{R}$ and a nonlinear map $g : \mathcal{R} \to Z_c(\mathcal{R})$ such that

$$\varphi(A) = \Theta(A) + g(A)$$

for any $A \in \mathcal{R}$.

Let $P \in \mathcal{R}$ be the nontrivial symmetric idempotent. Write $P_1 = P$, $P_2 = I - P_1$, $\mathcal{R}_{ij} = P_i \mathcal{R} P_j$ $(i, j = 1, 2)$, then $\mathcal{R}$=$\mathcal{R}_{11}$+$\mathcal{R}_{12}$+$\mathcal{R}_{21}$+$\mathcal{R}_{22}$. For every $A \in \mathcal{R}$, $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} \in \mathcal{R}_{ij}$ $(i, j = 1, 2)$.

Lemma 2.1. *For every* $A_{ii} \in \mathcal{R}_{ii}, B_{ij} \in \mathcal{R}_{ij}, C_{ji} \in \mathcal{R}_{ji}$ $(1 \leq i \neq j \leq 2)$*, we have*

$$\varphi(A_{ii} + B_{ij} + C_{ji}) = \varphi(A_{ii}) + \varphi(B_{ij}) + \varphi(C_{ji}).$$

Proof: Clearly, $\varphi(0) = 0$. Let

$$T = \varphi(A_{ii} + B_{ij} + C_{ji}) - \varphi(A_{ii}) - \varphi(B_{ij}) - \varphi(C_{ji}).$$

Next, we show that $T = 0$. For any $X_{ij} \in \mathcal{R}_{ij}$ with $1 \leq i \neq j \leq 2$, since

$$X_{ij}(A_{ii} + B_{ij} + C_{ji})P_j^* = X_{ij}A_{ii}P_j^* = X_{ij}B_{ij}P_j^* = X_{ij}C_{ji}P_j^* = 0,$$

and

$$[X_{ij} \bullet A_{ii}, P_j] = [X_{ij} \bullet C_{ji}, P_j] = 0.$$

We have

$$\begin{aligned}
&[\varphi(X_{ij}) \bullet (A_{ii} + B_{ij} + C_{ji}), P_j] + [X_{ij} \bullet \varphi(A_{ii} + B_{ij} + C_{ji}), P_j] \\
&\quad + [X_{ij} \bullet (A_{ii} + B_{ij} + C_{ji}), \varphi(P_j)] \\
&=\varphi([X_{ij} \bullet (A_{ii} + B_{ij} + C_{ji}), P_j]) \\
&=\varphi([X_{ij} \bullet A_{ii}, P_j]) + \varphi([X_{ij} \bullet B_{ij}, P_j]) + \varphi([X_{ij} \bullet C_{ji}, P_j]) \\
&=[\varphi(X_{ij}) \bullet (A_{ii} + B_{ij} + C_{ji}), P_j] + [X_{ij} \bullet (\varphi(A_{ii}) + \varphi(B_{ij}) + \varphi(C_{ji})), P_j] \\
&\quad + [X_{ij} \bullet (A_{ii} + B_{ij} + C_{ji}), \varphi(P_j)].
\end{aligned}$$

This implies that

$$[X_{ij} \bullet T, P_j] = 0. \tag{2.1}$$

Multiplying Eq (2.1) by P_j from the right, we have $X_{ij}TP_j = 0$. Hence, $T_{jj} = 0$ by the primeness of $\mathcal{R}$.

From

$$(A_{ii} + B_{ij} + C_{ji})X_{ij}P_i^* = A_{ii}X_{ij}P_i^* = B_{ij}X_{ij}P_i^* = C_{ji}X_{ij}P_i^* = 0,$$

and

$$[B_{ij} \bullet X_{ij}, P_i] = [C_{ji} \bullet X_{ij}, P_i] = 0.$$

We have

$$\begin{aligned}&[\varphi(A_{ii} + B_{ij} + C_{ji}) \bullet X_{ij}, P_i] + [(A_{ii} + B_{ij} + C_{ji}) \bullet \varphi(X_{ij}), P_i]\\&\quad + [(A_{ii} + B_{ij} + C_{ji}) \bullet X_{ij}, \varphi(P_i)]\\&=\varphi([(A_{ii} + B_{ij} + C_{ji}) \bullet X_{ij}, P_i])\\&=\varphi([A_{ii} \bullet X_{ij}, P_i]) + \varphi([B_{ij} \bullet X_{ij}, P_i]) + \varphi([C_{ji} \bullet X_{ij}, P_i])\\&=[(\varphi(A_{ii}) + \varphi(B_{ij}) + \varphi(C_{ji})) \bullet X_{ij}, P_i] + [(A_{ii} + B_{ij} + C_{ji}) \bullet \varphi(X_{ij}), P_i]\\&\quad + [(A_{ii} + B_{ij} + C_{ji}) \bullet X_{ij}, \varphi(P_i)].\end{aligned}$$

It follows that

$$[T \bullet X_{ij}, P_i] = 0. \tag{2.2}$$

Multiplying Eq (2.2) by P_j from the right and by the fact that $T_{jj} = 0$, we have $P_iTX_{ij} = 0$. Thus, $T_{ii} = 0$.

Since

$$(A_{ii} + B_{ij} + C_{ji})P_jP_i^* = A_{ii}P_jP_i^* = B_{ij}P_jP_i^* = C_{ji}P_jP_i^* = 0,$$

and

$$[A_{ii} \bullet P_j, P_i] = [C_{ji} \bullet P_j, P_i] = 0.$$

We have

$$\begin{aligned}&[\varphi(A_{ii} + B_{ij} + C_{ji}) \bullet P_j, P_i] + [(A_{ii} + B_{ij} + C_{ji}) \bullet \varphi(P_j), P_i]\\&\quad + [(A_{ii} + B_{ij} + C_{ji}) \bullet P_j, \varphi(P_i)]\\&=\varphi([(A_{ii} + B_{ij} + C_{ji}) \bullet P_j, P_i])\\&=\varphi([A_{ii} \bullet P_j, P_i]) + \varphi([B_{ij} \bullet P_j, P_i]) + \varphi([C_{ji} \bullet P_j, P_i])\\&=[(\varphi(A_{ii}) + \varphi(B_{ij}) + \varphi(C_{ji})) \bullet P_j, P_i] + [(A_{ii} + B_{ij} + C_{ji}) \bullet \varphi(P_j), P_i]\\&\quad + [(A_{ii} + B_{ij} + C_{ji}) \bullet P_j, \varphi(P_i)].\end{aligned}$$

Then,

$$[T \bullet P_j, P_i] = 0. \tag{2.3}$$

Multiplying Eq (2.3) by P_i from the left, we obtain $T_{ij} = 0$.

From

$$(A_{ii} + B_{ij} + C_{ji})P_iP_j^* = A_{ii}P_iP_j^* = B_{ij}P_iP_j^* = C_{ji}P_iP_j^* = 0,$$

and

$$[A_{ii} \bullet P_i, P_j] = [B_{ij} \bullet P_i, P_j] = 0.$$

We have

$$\begin{aligned}&[\varphi(A_{ii} + B_{ij} + C_{ji}) \bullet P_i, P_j] + [(A_{ii} + B_{ij} + C_{ji}) \bullet \varphi(P_i), P_j]\\&\quad + [(A_{ii} + B_{ij} + C_{ji}) \bullet P_i, \varphi(P_j)]\\&=\varphi([(A_{ii} + B_{ij} + C_{ji}) \bullet P_i, P_j])\\&=\varphi([A_{ii} \bullet P_i, P_j]) + \varphi([B_{ij} \bullet P_i, P_j]) + \varphi([C_{ji} \bullet P_i, P_j])\end{aligned}$$

$$=[(\varphi(A_{ii})+\varphi(B_{ij})+\varphi(C_{ji}))\bullet P_i,P_j]+[(A_{ii}+B_{ij}+C_{ji})\bullet\varphi(P_i),P_j]$$
$$+[(A_{ii}+B_{ij}+C_{ji})\bullet P_i,\varphi(P_j)].$$

It follows that

$$[T\bullet P_i,P_j]=0. \tag{2.4}$$

Multiplying Eq (2.4) by P_j from the left, we obtain $T_{ji}=0$. Therefore, $T=0$.

Lemma 2.2. *For every* $A_{ij},B_{ij}\in\mathcal{R}_{ij}$ $(1\le i\ne j\le 2)$*, we have*

$$\varphi(A_{ij}+B_{ij})=\varphi(A_{ij})+\varphi(B_{ij}).$$

Proof: Since $(P_i-A_{ij})(P_j-B_{ij})P_i^*=0$,

$$P_iP_jP_i^*=P_i(-B_{ij})P_i^*=(-A_{ij})P_jP_i^*=(-A_{ij})(-B_{ij})P_i^*=0,$$

and

$$[(P_i-A_{ij})\bullet(P_j-B_{ij}),P_i]=-A_{ij}^*+A_{ij}+B_{ij}.$$

We have from Lemma 2.1 that

$$\begin{aligned}
&\varphi(-A_{ij}^*)+\varphi(A_{ij}+B_{ij})\\
&=\varphi([(P_i-A_{ij})\bullet(P_j-B_{ij}),P_i])\\
&=[\varphi(P_i-A_{ij})\bullet(P_j-B_{ij}),P_i]+[(P_i-A_{ij})\bullet\varphi(P_j-B_{ij}),P_i]\\
&\quad+[(P_i-A_{ij})\bullet(P_j-B_{ij}),\varphi(P_i)]\\
&=[(\varphi(P_i)+\varphi(-A_{ij}))\bullet(P_j-B_{ij}),P_i]\\
&\quad+[(P_i-A_{ij})\bullet(\varphi(P_j)+\varphi(-B_{ij})),P_i]\\
&\quad+[(P_i-A_{ij})\bullet(P_j-B_{ij}),\varphi(P_i)]\\
&=\varphi([P_i\bullet P_j,P_i])+\varphi([P_i\bullet(-B_{ij}),P_i])+\varphi([(-A_{ij})\bullet P_j,P_i])\\
&\quad+\varphi([(-A_{ij})\bullet(-B_{ij}),P_i])\\
&=\varphi(B_{ij})+\varphi(-A_{ij}^*+A_{ij})\\
&=\varphi(-A_{ij}^*)+\varphi(A_{ij})+\varphi(B_{ij}).
\end{aligned}$$

Then, $\varphi(A_{ij}+B_{ij})=\varphi(A_{ij})+\varphi(B_{ij})$.

Lemma 2.3. *For every* $A_{ii},B_{ii}\in\mathcal{R}_{ii}$ $(i=1,2)$*, we have*

$$\varphi(A_{ii}+B_{ii})=\varphi(A_{ii})+\varphi(B_{ii}).$$

Proof: Let

$$T=\varphi(A_{ii}+B_{ii})-\varphi(A_{ii})-\varphi(B_{ii}).$$

Since

$$P_j(A_{ii}+B_{ii})P_j^*=P_jA_{ii}P_j^*=P_jB_{ii}P_j^*=0,$$

and

$$[P_j\bullet A_{ii},P_j]=0.$$

We have

$$\begin{aligned}&[\varphi(P_j)\bullet(A_{ii}+B_{ii}),P_j]+[P_j\bullet\varphi(A_{ii}+B_{ii}),P_j]+[P_j\bullet(A_{ii}+B_{ii}),\varphi(P_j)]\\&=\varphi([P_j\bullet(A_{ii}+B_{ii}),P_j])\\&=\varphi([P_j\bullet A_{ii},P_j])+\varphi([P_j\bullet B_{ii},P_j])\\&=[\varphi(P_j)\bullet(A_{ii}+B_{ii}),P_j]+[P_j\bullet(\varphi(A_{ii})+\varphi(B_{ii})),P_j]\\&\quad+[P_j\bullet(A_{ii}+B_{ii}),\varphi(P_j)].\end{aligned}$$

This gives that

$$[P_j\bullet T,P_j]=0. \tag{2.5}$$

Multiplying Eq (2.5) by P_i from the left, by P_i from the right, respectively, we obtain $T_{ij}=0$, $T_{ji}=0$, respectively.

For any $X_{ij}\in\mathcal{R}_{ij}$ with $1\le i\ne j\le 2$, since

$$X_{ij}(A_{ii}+B_{ii})P_j^*=X_{ij}A_{ii}P_j^*=X_{ij}B_{ii}P_j^*=0,$$

and

$$[X_{ij}\bullet A_{ii},P_j]=0.$$

We have

$$\begin{aligned}&[\varphi(X_{ij})\bullet(A_{ii}+B_{ii}),P_j]+[X_{ij}\bullet\varphi(A_{ii}+B_{ii}),P_j]+[X_{ij}\bullet(A_{ii}+B_{ii}),\varphi(P_j)]\\&=\varphi([X_{ij}\bullet(A_{ii}+B_{ii}),P_j])\\&=\varphi([X_{ij}\bullet A_{ii},P_j])+\varphi([X_{ij}\bullet B_{ii},P_j])\\&=[\varphi(X_{ij})\bullet(A_{ii}+B_{ii}),P_j]+[X_{ij}\bullet(\varphi(A_{ii})+\varphi(B_{ii})),P_j]\\&\quad+[X_{ij}\bullet(A_{ii}+B_{ii}),\varphi(P_j)].\end{aligned}$$

It follows that

$$[X_{ij}\bullet T,P_j]=0. \tag{2.6}$$

Multiplying Eq (2.6) by P_i from the left, we have $X_{ij}TP_j=0$, and so $T_{jj}=0$.
For any $X_{ji}\in\mathcal{R}_{ji}$ with $1\le i\ne j\le 2$, since

$$X_{ji}(A_{ii}+B_{ii})P_j^*=X_{ji}A_{ii}P_j^*=X_{ji}B_{ii}P_j^*=0.$$

We have from Lemmas 2.1 and 2.2 that

$$\begin{aligned}&[\varphi(X_{ji})\bullet(A_{ii}+B_{ii}),P_j]+[X_{ji}\bullet\varphi(A_{ii}+B_{ii}),P_j]+[X_{ji}\bullet(A_{ii}+B_{ii}),\varphi(P_j)]\\&=\varphi([X_{ji}\bullet(A_{ii}+B_{ii}),P_j])\\&=\varphi(A_{ii}X_{ji}^*+B_{ii}X_{ji}^*-X_{ji}A_{ii}-X_{ji}B_{ii})\\&=\varphi(A_{ii}X_{ji}^*-X_{ji}A_{ii})+\varphi(B_{ii}X_{ji}^*-X_{ji}B_{ii})\\&=\varphi([X_{ji}\bullet A_{ii},P_j])+\varphi([X_{ji}\bullet B_{ii},P_j])\\&=[\varphi(X_{ji})\bullet(A_{ii}+B_{ii}),P_j]+[X_{ji}\bullet(\varphi(A_{ii})+\varphi(B_{ii})),P_j]\end{aligned}$$

$$+ [X_{ji} \bullet (A_{ii} + B_{ii}), \varphi(P_j)],$$

which gives that

$$[X_{ji} \bullet T, P_j] = 0. \tag{2.7}$$

Multiplying Eq (2.7) by P_i from the right, we have $X_{ji}TP_i = 0$. Hence, $T_{ii} = 0$.

Lemma 2.4. (a) $\varphi(P_i)^* = \varphi(P_i)$ $(i = 1, 2)$;

(b) $P_i\varphi(P_i)P_j = -P_i\varphi(P_j)P_j$ $(1 \leq i \neq j \leq 2)$.

Proof: (a) Let $1 \leq i \neq j \leq 2$. Since $P_jP_iP_j^* = 0$ and $[P_j \bullet P_i, P_j] = 0$, we have

$$\begin{aligned} 0 &= \varphi([P_j \bullet P_i, P_j]) \\ &= [\varphi(P_j) \bullet P_i, P_j] + [P_j \bullet \varphi(P_i), P_j] + [P_j \bullet P_i, \varphi(P_j)] \\ &= P_i\varphi(P_j)^*P_j - P_j\varphi(P_j)P_i + \varphi(P_i)P_j - P_j\varphi(P_i). \end{aligned} \tag{2.8}$$

Multiplying Eq (2.8) by P_i from the left and by P_j from the right, we obtain

$$P_i\varphi(P_j)^*P_j = -P_i\varphi(P_i)P_j. \tag{2.9}$$

It follows that

$$P_j\varphi(P_j)P_i = -P_j\varphi(P_i)^*P_i. \tag{2.10}$$

Multiplying Eq (2.8) by P_j from the left and by P_i from the right, we obtain

$$P_j\varphi(P_j)P_i = -P_j\varphi(P_i)P_i. \tag{2.11}$$

Comparing Eqs (2.10) and (2.11), we obtain

$$P_j\varphi(P_i)^*P_i = P_j\varphi(P_i)P_i. \tag{2.12}$$

Since $P_jP_jP_i^* = 0$ and $[P_j \bullet P_j, P_i] = 0$, we have

$$\begin{aligned} 0 &= \varphi([P_j \bullet P_j, P_i]) \\ &= [\varphi(P_j) \bullet P_j, P_i] + [P_j \bullet \varphi(P_j), P_i] + [P_j \bullet P_j, \varphi(P_i)] \\ &= P_j\varphi(P_j)^*P_i - 2P_i\varphi(P_j)P_j + P_j\varphi(P_j)P_i + 2P_j\varphi(P_i) - 2\varphi(P_i)P_j. \end{aligned} \tag{2.13}$$

Multiplying Eq (2.13) by P_i from the left and by P_j from the right, we have

$$2(P_i\varphi(P_j)P_j + P_i\varphi(P_i)P_j) = 0. \tag{2.14}$$

Since $\mathcal{R}$ is 2-torsion free, we have from Eq (2.14) that

$$P_i\varphi(P_j)P_j = -P_i\varphi(P_i)P_j. \tag{2.15}$$

From $P_iP_iP_j^* = 0$ and $[P_i \bullet P_i, P_j] = 0$, we have

$$\begin{aligned} 0 &= \varphi([P_i \bullet P_i, P_j]) \\ &= [\varphi(P_i) \bullet P_i, P_j] + [P_i \bullet \varphi(P_i), P_j] + [P_i \bullet P_i, \varphi(P_j)] \end{aligned}$$

$$= P_i\varphi(P_i)^*P_j - 2P_j\varphi(P_i)P_i + P_i\varphi(P_i)P_j + 2P_i\varphi(P_j) - 2\varphi(P_j)P_i. \tag{2.16}$$

Multiplying Eq (2.16) by P_i from the left and by P_j from the right, then by Eq. (2.15), we have

$$P_i\varphi(P_i)^*P_j = P_i\varphi(P_i)P_j. \tag{2.17}$$

For any $X_{ij} \in \mathcal{R}_{ij}$ with $1 \le i \ne j \le 2$, we have from $X_{ij}P_iP_i^* = 0$ and $[X_{ij} \bullet P_i, P_i] = 0$ that

$$\begin{aligned} 0 =& \varphi([X_{ij} \bullet P_i, P_i]) \\ =& [\varphi(X_{ij}) \bullet P_i, P_i] + [X_{ij} \bullet \varphi(P_i), P_i] + [X_{ij} \bullet P_i, \varphi(P_i)] \\ =& \varphi(X_{ij})P_i + P_i\varphi(X_{ij})^*P_i - P_i\varphi(X_{ij})P_i - P_i\varphi(X_{ij})^* + X_{ij}\varphi(P_i)P_i \\ &+ \varphi(P_i)X_{ij}^* - X_{ij}\varphi(P_i) - P_i\varphi(P_i)X_{ij}^*. \end{aligned} \tag{2.18}$$

Multiplying Eq (2.18) by P_i from the left and by P_j from the right, we have

$$P_i\varphi(X_{ij})^*P_j = -X_{ij}\varphi(P_i)P_j. \tag{2.19}$$

Multiplying Eq (2.18) by P_j from the left and by P_i from the right, we have $P_j\varphi(X_{ij})P_i + P_j\varphi(P_i)X_{ij}^* = 0$. Then,

$$P_i\varphi(X_{ij})^*P_j = -X_{ij}\varphi(P_i)^*P_j. \tag{2.20}$$

Comparing Eqs (2.19) and (2.20), we obtain $X_{ij}(P_j\varphi(P_i)P_j - P_j\varphi(P_i)^*P_j) = 0$. It follows that

$$P_j\varphi(P_i)^*P_j = P_j\varphi(P_i)P_j. \tag{2.21}$$

From $P_iP_iX_{ij}^* = 0$ and $[P_i \bullet P_i, X_{ij}] = 2X_{ij}$, we have

$$\begin{aligned} \varphi(2X_{ij}) =& \varphi([P_i \bullet P_i, X_{ij}]) \\ =& [\varphi(P_i) \bullet P_i, X_{ij}] + [P_i \bullet \varphi(P_i), X_{ij}] + [P_i \bullet P_i, \varphi(X_{ij})] \\ =& \varphi(P_i)X_{ij} + P_i\varphi(P_i)^*X_{ij} - X_{ij}\varphi(P_i)P_i + P_i\varphi(P_i)X_{ij} + \varphi(P_i)X_{ij} \\ &- X_{ij}\varphi(P_i)P_i + 2P_i\varphi(X_{ij}) - 2\varphi(X_{ij})P_i. \end{aligned} \tag{2.22}$$

Multiplying Eq (2.22) by P_i from the left and by P_j from the right, we have

$$P_i\varphi(2X_{ij})P_j = 3P_i\varphi(P_i)X_{ij} + P_i\varphi(P_i)^*X_{ij} + 2P_i\varphi(X_{ij})P_j. \tag{2.23}$$

It follows from Eq (2.23) and Lemma 2.2 that $3P_i\varphi(P_i)X_{ij} + P_i\varphi(P_i)^*X_{ij} = 0$, and so

$$3P_i\varphi(P_i)P_i + P_i\varphi(P_i)^*P_i = 0. \tag{2.24}$$

For any $X_{ji} \in \mathcal{R}_{ji}$ with $1 \le i \ne j \le 2$, since $P_iX_{ji}P_i^* = 0$ and $[P_i \bullet X_{ji}, P_i] = X_{ji}$, we have

$$\begin{aligned} \varphi(X_{ji}) =& \varphi([P_i \bullet X_{ji}, P_i]) \\ =& [\varphi(P_i) \bullet X_{ji}, P_i] + [P_i \bullet \varphi(X_{ji}), P_i] + [P_i \bullet X_{ji}, \varphi(P_i)] \\ =& X_{ji}\varphi(P_i)^*P_i - P_i\varphi(P_i)X_{ji} + \varphi(X_{ji})P_i \\ &- P_i\varphi(X_{ji}) + X_{ji}\varphi(P_i). \end{aligned} \tag{2.25}$$

Multiplying Eq (2.25) by P_j from the left and by P_i from the right, we have $X_{ji}(P_i\varphi(P_i)^*P_i + P_i\varphi(P_i)P_i) = 0$. Then,

$$P_i\varphi(P_i)^*P_i = -P_i\varphi(P_i)P_i. \tag{2.26}$$

By Eqs (2.24) and (2.26), we obtain that $2P_i\varphi(P_i)P_i = 0$. Then by $\mathcal{R}$ is 2-torsion free, we obtain

$$P_i\varphi(P_i)^*P_i = P_i\varphi(P_i)P_i. \tag{2.27}$$

From Eqs (2.12), (2.17), (2.21), and (2.27), we can see that $\varphi(P_i)^* = \varphi(P_i)$.

(b) It follows from Eq (2.9) and (a) that (b) holds.

Lemma 2.5. $P_j\varphi(P_i)P_j = 0$ $(1 \le i \ne j \le 2)$.

Proof: For any $X_{ij} \in \mathcal{R}_{ij}$ with $1 \le i \ne j \le 2$, since $P_jX_{ij}P_j^* = 0$ and $[P_j \bullet X_{ij}, P_j] = X_{ij}$, we have

$$\begin{aligned}\varphi(X_{ij}) =&\varphi([P_j \bullet X_{ij}, P_j])\\ =&[\varphi(P_j) \bullet X_{ij}, P_j] + [P_j \bullet \varphi(X_{ij}), P_j] + [P_j \bullet X_{ij}, \varphi(P_j)]\\ =&X_{ij}\varphi(P_j)^*P_j - P_j\varphi(P_j)X_{ij} + \varphi(X_{ij})P_j\\ &- P_j\varphi(X_{ij}) + X_{ij}\varphi(P_j).\end{aligned} \tag{2.28}$$

Multiplying Eq (2.28) by P_j from the left and by P_i from the right, we obtain $2P_j\varphi(X_{ij})P_i = 0$, and so,

$$P_j\varphi(X_{ij})P_i = 0. \tag{2.29}$$

Combining Eqs (2.19) and (2.29), we have $X_{ij}\varphi(P_i)P_j = 0$. It follows that $P_j\varphi(P_i)P_j = 0$.

Lemma 2.6. $P_i\varphi(P_i)P_i = 0$ $(i = 1, 2)$.

Proof: For any $X_{ij} \in \mathcal{R}_{ij}$ with $1 \le i \ne j \le 2$, since

$$P_iX_{ij}P_i^* = 0, [P_i \bullet X_{ij}, P_i] = -X_{ij},$$

we have from Lemma 2.4 that

$$\begin{aligned}\varphi(-X_{ij}) =&\varphi([P_i \bullet X_{ij}, P_i])\\ =&[\varphi(P_i) \bullet X_{ij}, P_i] + [P_i \bullet \varphi(X_{ij}), P_i] + [P_i \bullet X_{ij}, \varphi(P_i)]\\ =&X_{ij}\varphi(P_i)P_i - P_i\varphi(P_i)X_{ij} - X_{ij}\varphi(P_i) + \varphi(X_{ij})P_i - P_i\varphi(X_{ij})\\ &+ X_{ij}\varphi(P_i) - \varphi(P_i)X_{ij}.\end{aligned} \tag{2.30}$$

Multiplying Eq (2.30) by P_i from the left and by P_j from the right, we have

$$P_i\varphi(-X_{ij})P_j = -2P_i\varphi(P_i)X_{ij} - P_i\varphi(X_{ij})P_j. \tag{2.31}$$

It follows from Eq (2.31) and Lemma 2.2 that $2P_i\varphi(P_i)X_{ij} = 0$, and so $P_i\varphi(P_i)X_{ij} = 0$. Hence, $P_i\varphi(P_i)P_i = 0$.

Let $T = P_1\varphi(P_1)P_2 - P_2\varphi(P_1)P_1$. Then, $T^* = -T$ by Lemma 2.4. We define a map $\Phi : \mathcal{R} \to \mathcal{R}$ by

$$\Phi(A) = \varphi(A) - [A, T]$$

for all $A \in \mathcal{R}$.

Remark 2.1. *It is easy to check that* Φ *also satisfies*

$$\Phi([A \bullet B, C]) = [\Phi(A) \bullet B, C] + [A \bullet \Phi(B), C] + [A \bullet B, \Phi(C)]$$

for any $A, B, C \in \mathcal{R}$ with $ABC^* = 0$. By Lemmas 2.1–2.6, it follows that

(a) For every $A_{ii} \in \mathcal{R}_{ii}, B_{ij} \in \mathcal{R}_{ij}, C_{ji} \in \mathcal{R}_{ji}$ $(1 \le i \ne j \le 2)$, we have

$$\Phi(A_{ii} + B_{ij} + C_{ji}) = \Phi(A_{ii}) + \Phi(B_{ij}) + \Phi(C_{ji}).$$

(b) For every $A_{ij}, B_{ij} \in \mathcal{R}_{ij}$ $(1 \le i \ne j \le 2)$, we have

$$\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).$$

(c) For every $A_{ii}, B_{ii} \in \mathcal{R}_{ii}$ $(i = 1, 2)$, we have

$$\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).$$

(d) $\Phi(P_i) = 0$ $(i = 1, 2)$.

Lemma 2.7. $\Phi(\mathcal{R}_{ij}) \subseteq \mathcal{R}_{ij}$ $(1 \le i \ne j \le 2)$.

Proof: Let $A_{ij} \in \mathcal{R}_{ij}$ with $1 \le i \ne j \le 2$. From $A_{ij}P_iP_i^* = 0, [A_{ij} \bullet P_i, P_i] = 0$ and $\Phi(P_i) = 0$, we have

$$\begin{aligned} 0 =&\Phi([A_{ij} \bullet P_i, P_i]) \\ =&[\Phi(A_{ij}) \bullet P_i, P_i] \\ =&\Phi(A_{ij})P_i + P_i\Phi(A_{ij})^*P_i - P_i\Phi(A_{ij})P_i - P_i\Phi(A_{ij})^*. \end{aligned} \tag{2.32}$$

Multiplying Eq (2.32) by P_j from the left, we obtain $P_j\Phi(A_{ij})P_i = 0$.
Since $P_j(-A_{ij})P_i^* = 0, [P_j \bullet (-A_{ij}), P_i] = A_{ij}$ and $\Phi(P_i) = \Phi(P_j) = 0$, we have

$$\begin{aligned} \Phi(A_{ij}) =&\Phi([P_j \bullet (-A_{ij}), P_i]) \\ =&[P_j \bullet \Phi(-A_{ij}), P_i] \\ =&P_j\Phi(-A_{ij})P_i - P_i\Phi(-A_{ij})P_j. \end{aligned} \tag{2.33}$$

Multiplying Eq (2.33) by P_i from both sides, by P_j from both sides, respectively, we have $P_i\Phi(A_{ij})P_i = 0$, $P_j\Phi(A_{ij})P_j = 0$, respectively. Therefore, $\Phi(\mathcal{R}_{ij}) \subseteq \mathcal{R}_{ij}$.

Lemma 2.8. $\Phi(\mathcal{R}_{ii}) \subseteq \mathcal{R}_{ii}$ $(i = 1, 2)$.

Proof: It follows from $P_iA_{ii}P_j^* = 0, [P_i \bullet A_{ii}, P_j] = 0$ and $\Phi(P_i) = \Phi(P_j) = 0$ that

$$0 = \Phi([P_i \bullet A_{ii}, P_j]) = [P_i \bullet \Phi(A_{ii}), P_j] = P_i\Phi(A_{ii})P_j - P_j\Phi(A_{ii})P_i. \tag{2.34}$$

Multiplying Eq (2.34) by P_i from the left, by P_j from the left, respectively, we have $P_i\Phi(A_{ii})P_j = 0$, $P_j\Phi(A_{ii})P_i = 0$, respectively.

For any $X_{ij} \in \mathcal{R}_{ij}$ with $1 \le i \ne j \le 2$, since $X_{ij}A_{ii}P_j^* = 0, [X_{ij} \bullet A_{ii}, P_j] = 0$ and $\Phi(P_j) = 0$, we have

$$\begin{aligned} 0 =&\Phi([X_{ij} \bullet A_{ii}, P_j]) \\ =&[\Phi(X_{ij}) \bullet A_{ii}, P_j] + [X_{ij} \bullet \Phi(A_{ii}), P_j] \\ =&A_{ii}\Phi(X_{ij})^*P_j - P_j\Phi(X_{ij})A_{ii} + X_{ij}\Phi(A_{ii})P_j - P_j\Phi(A_{ii})X_{ij}^*. \end{aligned} \tag{2.35}$$

Multiplying Eq (2.35) by P_i from the left, and by Lemma 2.7, we have

$$X_{ij}\Phi(A_{ii})P_j = -A_{ii}(P_j\Phi(X_{ij})P_i)^* = 0.$$

Then, $P_j\Phi(A_{ii})P_j = 0$. Hence, $\Phi(\mathcal{R}_{ii}) \subseteq \mathcal{R}_{ii}$.

Lemma 2.9. *For every $A_{ii} \in \mathcal{R}_{ii}, B_{ij} \in \mathcal{R}_{ij}, C_{ji} \in \mathcal{R}_{ji}, D_{jj} \in \mathcal{R}_{jj}$ $(1 \le i \ne j \le 2)$, we have*

(a) $\Phi(A_{ii} + B_{ij} + D_{jj}) = \Phi(A_{ii}) + \Phi(B_{ij}) + \Phi(D_{jj})$;

(b) $\Phi(A_{ii} + C_{ji} + D_{jj}) = \Phi(A_{ii}) + \Phi(C_{ji}) + \Phi(D_{jj})$.

Proof: (a) Let

$$T = \Phi(A_{ii} + B_{ij} + D_{jj}) - \Phi(A_{ii}) - \Phi(B_{ij}) - \Phi(D_{jj}).$$

Since

$$P_j(A_{ii} + B_{ij} + D_{jj})P_i^* = P_jA_{ii}P_i^* = P_jB_{ij}P_i^* = P_jD_{jj}P_i^* = 0,$$

and

$$[P_j \bullet A_{ii}, P_i] = [P_j \bullet D_{jj}, P_i] = 0, \Phi(P_i) = \Phi(P_j) = 0.$$

We have

$$\begin{aligned}
&[P_j\bullet\Phi(A_{ii} + B_{ij} + D_{jj}), P_i]\\
&\quad=\Phi([P_j \bullet (A_{ii} + B_{ij} + D_{jj}), P_i])\\
&\quad=\Phi([P_j \bullet A_{ii}, P_i]) + \Phi([P_j \bullet B_{ij}, P_i]) + \Phi([P_j \bullet D_{jj}, P_i])\\
&\quad=[P_j \bullet \Phi(A_{ii}), P_i] + [P_j \bullet \Phi(B_{ij}), P_i] + [P_j \bullet \Phi(D_{jj}), P_i]\\
&\quad=[P_j \bullet (\Phi(A_{ii}) + \Phi(B_{ij}) + \Phi(D_{jj})), P_i].
\end{aligned}$$

This implies that

$$[P_j \bullet T, P_i] = 0. \tag{2.36}$$

Multiplying Eq (2.36) by P_i from the left, by P_j from the left, respectively, we have $T_{ij} = 0, T_{ji} = 0$, respectively.

For any $X_{ij} \in \mathcal{R}_{ij}$ with $1 \le i \ne j \le 2$, from

$$X_{ij}(A_{ii} + B_{ij} + D_{jj})P_i^* = X_{ij}A_{ii}P_i^* = X_{ij}B_{ij}P_i^* = X_{ij}D_{jj}P_i^* = 0,$$

and

$$[X_{ij} \bullet A_{ii}, P_i] = [X_{ij} \bullet B_{ij}, P_i] = 0, \Phi(P_i) = 0.$$

We have

$$\begin{aligned}
&[\Phi(X_{ij}) \bullet (A_{ii} + B_{ij} + D_{jj}), P_i] + [X_{ij} \bullet \Phi(A_{ii} + B_{ij} + D_{jj}), P_i]\\
&\quad=\Phi([X_{ij} \bullet (A_{ii} + B_{ij} + D_{jj}), P_i])\\
&\quad=\Phi([X_{ij} \bullet A_{ii}, P_i]) + \Phi([X_{ij} \bullet B_{ij}, P_i]) + \Phi([X_{ij} \bullet D_{jj}, P_i])\\
&\quad=[\Phi(X_{ij}) \bullet (A_{ii} + B_{ij} + D_{jj}), P_i] + [X_{ij} \bullet (\Phi(A_{ii}) + \Phi(B_{ij}) + \Phi(D_{jj})), P_i],
\end{aligned}$$

which implies that

$$[X_{ij} \bullet T, P_i] = 0. \tag{2.37}$$

Multiplying Eq (2.37) by P_j from the right, we obtain $X_{ij}TP_j = 0$, and so $T_{jj} = 0$.

For any $X_{ji} \in \mathcal{R}_{ji}$ with $1 \le i \ne j \le 2$, since

$$(A_{ii} + B_{ij} + D_{jj})X_{ji}P_j^* = A_{ii}X_{ji}P_j^* = B_{ij}X_{ji}P_j^* = D_{jj}X_{ji}P_j^* = 0,$$

and

$$[B_{ij} \bullet X_{ji}, P_j] = 0, [(A_{ii} + D_{jj}) \bullet X_{ji}, P_j] = -X_{ji}A_{ii}^* - D_{jj}X_{ji}, \Phi(P_j) = 0.$$

We have from Remark 2.1 (b) that

$$\begin{aligned}
&[\Phi(A_{ii} + B_{ij} + D_{jj}) \bullet X_{ji}, P_j] + [(A_{ii} + B_{ij} + D_{jj}) \bullet \Phi(X_{ji}), P_j] \\
&=\Phi([(A_{ii} + B_{ij} + D_{jj}) \bullet X_{ji}, P_j]) \\
&=\Phi([(A_{ii} + D_{jj}) \bullet X_{ji}, P_j]) + \Phi([B_{ij} \bullet X_{ji}, P_j]) \\
&=\Phi(-X_{ji}A_{ii}^*) + \Phi(-D_{jj}X_{ji}) + \Phi([B_{ij} \bullet X_{ji}, P_j]) \\
&=\Phi([A_{ii} \bullet X_{ji}, P_j]) + \Phi([B_{ij} \bullet X_{ji}, P_j]) + \Phi([D_{jj} \bullet X_{ji}, P_j]) \\
&=[(\Phi(A_{ii}) + \Phi(B_{ij}) + \Phi(D_{jj})) \bullet X_{ji}, P_j] + [(A_{ii} + B_{ij} + D_{jj}) \bullet \Phi(X_{ji}), P_j].
\end{aligned}$$

It follows that

$$[T \bullet X_{ji}, P_j] = 0. \tag{2.38}$$

Multiplying Eq (2.38) by P_i from the right and by the fact that $T_{jj} = 0$, we obtain $X_{ji}T^*P_i = 0$ and so $T_{ii} = 0$. Hence, $T = 0$.

(b) Similarly, we can show that (b) holds.

Lemma 2.10. *For $A \in \mathcal{R}$, there exists a map $g : \mathcal{R} \to Z_c(\mathcal{R})$ such that*

$$\Phi(A) - g(A) = \Phi(A_{ii}) + \Phi(A_{ij}) + \Phi(A_{ji}) + \Phi(A_{jj}).$$

Proof: For $A \in \mathcal{R}$, write $A = \sum_{i,j=1}^{2} A_{ij}$. Let

$$T = \Phi(A_{ii} + A_{ij} + A_{ji} + A_{jj}) - \Phi(A_{ii}) - \Phi(A_{ij}) - \Phi(A_{ji}) - \Phi(A_{jj}). \tag{2.39}$$

For any $X_{ij} \in \mathcal{R}_{ij}$ with $1 \leq i \neq j \leq 2$, since

$$P_jX_{ij}(A_{ii} + A_{ij} + A_{ji} + A_{jj})^* = 0,$$

$$P_jX_{ij}A_{ii}^* = P_jX_{ij}A_{ij}^* = P_jX_{ij}A_{ji}^* = P_jX_{ij}A_{jj}^* = 0, \Phi(P_j) = 0,$$

and

$$[P_j \bullet X_{ij}, A_{ii} + A_{ij} + A_{ji} + A_{jj}] = X_{ij}A_{ji} + X_{ij}A_{jj} - A_{ii}X_{ij} - A_{ji}X_{ij}.$$

We have from Lemma 2.9 and Remark 2.1 (b) that

$$\begin{aligned}
&[P_j \bullet \Phi(X_{ij}), A_{ii} + A_{ij} + A_{ji} + A_{jj}] + [P_j \bullet X_{ij}, \Phi(A_{ii} + A_{ij} + A_{ji} + A_{jj})] \\
&=\Phi([P_j \bullet X_{ij}, A_{ii} + A_{ij} + A_{ji} + A_{jj}]) \\
&=\Phi(X_{ij}A_{ji}) + \Phi(X_{ij}A_{jj}) + \Phi(-A_{ii}X_{ij}) + \Phi(-A_{ji}X_{ij}) \\
&=\Phi(-A_{ii}X_{ij}) + \Phi(-A_{ji}X_{ij} + X_{ij}A_{ji}) + \Phi(X_{ij}A_{jj}) \\
&=\Phi([P_j \bullet X_{ij}, A_{ii}]) + \Phi([P_j \bullet X_{ij}, A_{ij}]) + \Phi([P_j \bullet X_{ij}, A_{ji}]) \\
&\quad + \Phi([P_j \bullet X_{ij}, A_{jj}]) \\
&=[P_j \bullet \Phi(X_{ij}), A_{ii} + A_{ij} + A_{ji} + A_{jj}] \\
&\quad + [P_j \bullet X_{ij}, \Phi(A_{ii}) + \Phi(A_{ij}) + \Phi(A_{ji}) + \Phi(A_{jj})].
\end{aligned}$$

It follows that

$$[P_j \bullet X_{ij}, T] = 0. \tag{2.40}$$

Multiplying Eq (2.40) by P_i from the right, we obtain $X_{ij}TP_i = 0$. Then, $T_{ji} = 0$. Multiplying Eq (2.40) by P_i from the left and by P_j from the right, we have $P_iTX_{ij} = X_{ij}TP_j$. It follows from [5, Lemma 1.6] that $T_{ii} + T_{jj} \in Z(\mathcal{R})$. Similarly, we can show that $T_{ij} = 0$.

Since $(A_{ii} + A_{ij} + A_{ji} + A_{jj})P_iX_{ij}^* = 0$,

$$A_{ii}P_iX_{ij}^* = A_{ij}P_iX_{ij}^* = A_{ji}P_iX_{ij}^* = A_{jj}P_iX_{ij}^* = 0, \Phi(P_i) = 0,$$

and

$$[(A_{ii} + A_{ij} + A_{ji} + A_{jj}) \bullet P_i, X_{ij}] = A_{ii}X_{ij} + A_{ji}X_{ij} + A_{ii}^*X_{ij} - X_{ij}A_{ji}.$$

We have from Lemma 2.9 that

$$\begin{aligned}
&[\Phi(A_{ii} + A_{ij} + A_{ji} + A_{jj}) \bullet P_i, X_{ij}] + [(A_{ii} + A_{ij} + A_{ji} + A_{jj}) \bullet P_i, \Phi(X_{ij})]\\
&=\Phi([(A_{ii} + A_{ij} + A_{ji} + A_{jj}) \bullet P_i, X_{ij}])\\
&=\Phi(A_{ii}X_{ij} + A_{ii}^*X_{ij}) + \Phi(A_{ji}X_{ij} - X_{ij}A_{ji})\\
&=\Phi([A_{ii} \bullet P_i, X_{ij}]) + \Phi([A_{ij} \bullet P_i, X_{ij}]) + \Phi([A_{ji} \bullet P_i, X_{ij}])\\
&\quad + \Phi([A_{jj} \bullet P_i, X_{ij}])\\
&=[(\Phi(A_{ii}) + \Phi(A_{ij}) + \Phi(A_{ji}) + \Phi(A_{jj}) \bullet P_i, X_{ij}]\\
&\quad + [(A_{ii} + A_{ij} + A_{ji} + A_{jj}) \bullet P_i, \Phi(X_{ij})],
\end{aligned}$$

which implies that

$$[T \bullet P_i, X_{ij}] = 0. \tag{2.41}$$

Multiplying Eq (2.41) by P_i from the left and by P_j from the right, we have $P_iTX_{ij} + P_iT^*X_{ij} = 0$, and so $T_{ii} = -T_{ii}^*$. Similarly, $T_{jj} = -T_{jj}^*$. It follows that $T_{ii} + T_{jj} \in Z_c(\mathcal{R})$ by $T_{ii} + T_{jj} \in Z(\mathcal{R})$. Define a map $g : \mathcal{R} \to Z_c(\mathcal{R})$ by

$$g(A) = T_{ii} + T_{jj}. \tag{2.42}$$

Combining Eqs (2.39) and (2.42), we can obtain the desired result.

Lemma 2.11. *For every $A_{ii}, B_{ii} \in \mathcal{R}_{ii}, A_{ij}, B_{ij} \in \mathcal{R}_{ij}, B_{ji} \in \mathcal{R}_{ji}, B_{jj} \in \mathcal{R}_{jj}$ $(1 \le i \ne j \le 2)$, we have*

(a) $\Phi(A_{ii}B_{ij}) = \Phi(A_{ii})B_{ij} + A_{ii}\Phi(B_{ij})$;

(b) $\Phi(A_{ii}B_{ii}) = \Phi(A_{ii})B_{ii} + A_{ii}\Phi(B_{ii})$;

(c) $\Phi(A_{ij}B_{ji}) = \Phi(A_{ij})B_{ji} + A_{ij}\Phi(B_{ji})$;

(d) $\Phi(A_{ij}B_{jj}) = \Phi(A_{ij})B_{jj} + A_{ij}\Phi(B_{jj})$.

Proof: (a) Since

$$A_{ii}B_{ij}(-P_i)^* = 0, [A_{ii} \bullet B_{ij}, -P_i] = A_{ii}B_{ij},$$

we have from Remark 2.1 (c), (d), and Lemmas 2.7 and 2.8 that

$$\begin{aligned}
\Phi(A_{ii}B_{ij}) =&\Phi([A_{ii} \bullet B_{ij}, -P_i])\\
=&[\Phi(A_{ii}) \bullet B_{ij}, -P_i] + [A_{ii} \bullet \Phi(B_{ij}), -P_i]\\
=&\Phi(A_{ii})B_{ij} + A_{ii}\Phi(B_{ij}).
\end{aligned}$$

(b) For any $X_{ij} \in \mathcal{R}_{ij}$ with $1 \leq i \neq j \leq 2$, it follows from (a) that

$$\begin{aligned}
&\Phi(A_{ii}B_{ii})X_{ij} + A_{ii}B_{ii}\Phi(X_{ij}) \\
&=\Phi(A_{ii}B_{ii}X_{ij}) \\
&=\Phi(A_{ii})B_{ii}X_{ij} + A_{ii}\Phi(B_{ii}X_{ij}) \\
&=\Phi(A_{ii})B_{ii}X_{ij} + A_{ii}\Phi(B_{ii})X_{ij} + A_{ii}B_{ii}\Phi(X_{ij}).
\end{aligned}$$

This yields that $(\Phi(A_{ii}B_{ii}) - \Phi(A_{ii})B_{ii} - A_{ii}\Phi(B_{ii}))X_{ij} = 0$. Hence, (b) holds by Lemma 2.8.

(c) For any $X_{ij} \in \mathcal{R}_{ij}$ with $1 \leq i \neq j \leq 2$, since $A_{ij}B_{ji}X_{ij}^* = 0$ and $[A_{ij} \bullet B_{ji}, X_{ij}] = A_{ij}B_{ji}X_{ij}$, we have from (a) and Lemma 2.7 that

$$\begin{aligned}
&\Phi(A_{ij}B_{ji})X_{ij} + A_{ij}B_{ji}\Phi(X_{ij}) \\
&=\Phi(A_{ij}B_{ji}X_{ij}) \\
&=\Phi([A_{ij} \bullet B_{ji}, X_{ij}]) \\
&=[\Phi(A_{ij}) \bullet B_{ji}, X_{ij}] + [A_{ij} \bullet \Phi(B_{ji}), X_{ij}] + [A_{ij} \bullet B_{ji}, \Phi(X_{ij})] \\
&=\Phi(A_{ij})B_{ji}X_{ij} + A_{ij}\Phi(B_{ji})X_{ij} + A_{ij}B_{ji}\Phi(X_{ij}).
\end{aligned}$$

Then, $(\Phi(A_{ij}B_{ji}) - \Phi(A_{ij})B_{ji} - A_{ij}\Phi(B_{ji}))X_{ij} = 0$, and so (c) holds by Lemmas 2.7 and 2.8.

(d) For any $X_{ji} \in \mathcal{R}_{ji}$ with $1 \leq i \neq j \leq 2$, we have from (a) and (c) that

$$\begin{aligned}
&\Phi(A_{ij}B_{jj})X_{ji} + A_{ij}B_{jj}\Phi(X_{ji}) \\
&=\Phi(A_{ij}B_{jj}X_{ji}) \\
&=\Phi(A_{ij})B_{jj}X_{ji} + A_{ij}\Phi(B_{jj}X_{ji}) \\
&=\Phi(A_{ij})B_{jj}X_{ji} + A_{ij}\Phi(B_{jj})X_{ji} + A_{ij}B_{jj}\Phi(X_{ji}).
\end{aligned}$$

It follows that $(\Phi(A_{ij}B_{jj}) - \Phi(A_{ij})B_{jj} - A_{ij}\Phi(B_{jj}))X_{ji} = 0$. Then, (d) holds by Lemmas 2.7 and 2.8.

Lemma 2.12. *For every $A_{ij} \in \mathcal{R}_{ij}$ $(i, j = 1, 2)$, we have*

$$\Phi(A_{ij}^*) = \Phi(A_{ij})^*.$$

Proof: Let $1 \leq i \neq j \leq 2$. Since $A_{ij}P_j(P_i)^* = 0, [A_{ij} \bullet P_j, P_i] = A_{ij}^* - A_{ij}$, we have from Remark 2.1 (a), (b), (d), and Lemma 2.7 that

$$\Phi(A_{ij}^*) - \Phi(A_{ij}) = \Phi([A_{ij} \bullet P_j, P_i]) = [\Phi(A_{ij}) \bullet P_j, P_i] = \Phi(A_{ij})^* - \Phi(A_{ij}).$$

Hence, $\Phi(A_{ij}^*) = \Phi(A_{ij})^*$.

For any $X_{ij} \in \mathcal{R}_{ij}$ with $1 \leq i \neq j \leq 2$, since $A_{ii}P_iX_{ij}^* = 0, [A_{ii} \bullet P_i, X_{ij}] = A_{ii}X_{ij} + A_{ii}^*X_{ij}$, we have from Remark 2.1 (a), (b), and Lemmas 2.7, 2.8, and 2.11 that

$$\begin{aligned}
&\Phi(A_{ii})X_{ij} + A_{ii}\Phi(X_{ij}) + \Phi(A_{ii}^*)X_{ij} + A_{ii}^*\Phi(X_{ij}) \\
&=\Phi(A_{ii}X_{ij}) + \Phi(A_{ii}^*X_{ij}) \\
&=\Phi([A_{ii} \bullet P_i, X_{ij}]) \\
&=[\Phi(A_{ii}) \bullet P_i, X_{ij}] + [A_{ii} \bullet P_i, \Phi(X_{ij})]
\end{aligned}$$

$$=\Phi(A_{ii})X_{ij} + \Phi(A_{ii})^*X_{ij} + A_{ii}\Phi(X_{ij}) + A_{ii}^*\Phi(X_{ij}).$$

It follows that $(\Phi(A_{ii}^*) - \Phi(A_{ii})^*)X_{ij} = 0$. Then, $\Phi(A_{ii}^*) = \Phi(A_{ii})^*$ by Lemma 2.8.
Proof of Theorem 2.1: We define a map $\Psi : \mathcal{R} \to \mathcal{R}$ by

$$\Psi(A) = \Phi(A) - g(A)$$

for all $A \in \mathcal{R}$. For every $A, B \in \mathcal{R}$, let $A = \sum_{i,j=1}^{2} A_{ij}$, $B = \sum_{i,j=1}^{2} B_{ij}$. From Remark 2.1 (b), (c), and Lemma 2.10, we have

$$\begin{aligned}\Psi(A+B) =&\Phi(A+B) - g(A+B)\\ =&\Phi(P_i(A+B)P_i) + \Phi(P_i(A+B)P_j) + \Phi(P_j(A+B)P_i)\\ &+ \Phi(P_j(A+B)P_j) + g(A+B) - g(A+B)\\ =&\Phi(A_{ii}) + \Phi(A_{ij}) + \Phi(A_{ji}) + \Phi(A_{jj}) + \Phi(B_{ii}) + \Phi(B_{ij})\\ &+ \Phi(B_{ji}) + \Phi(B_{jj})\\ =&\Phi(A) - g(A) + \Phi(B) - g(B)\\ =&\Psi(A) + \Psi(B).\end{aligned}$$

Hence, Ψ is additive on $\mathcal{R}$. Next, it follows from Remark 2.1 (b), (c), and Lemmas 2.7, 2.8, 2.10, and 2.11 that

$$\begin{aligned}\Psi(AB) =&\Phi(AB) - g(AB)\\ =&\Phi(P_i(AB)P_i) + \Phi(P_i(AB)P_j) + \Phi(P_j(AB)P_i) + \Phi(P_j(AB)P_j)\\ &+ g(AB) - g(AB)\\ =&\Phi(A_{ii}B_{ii} + A_{ij}B_{ji}) + \Phi(A_{ii}B_{ij} + A_{ij}B_{jj}) + \Phi(A_{ji}B_{ii} + A_{jj}B_{ji})\\ &+ \Phi(A_{ji}B_{ij} + A_{jj}B_{jj})\\ =&\Phi(A_{ii}B_{ii}) + \Phi(A_{ij}B_{ji}) + \Phi(A_{ii}B_{ij}) + \Phi(A_{ij}B_{jj}) + \Phi(A_{ji}B_{ii})\\ &+ \Phi(A_{jj}B_{ji}) + \Phi(A_{ji}B_{ij}) + \Phi(A_{jj}B_{jj})\\ =&(\Phi(A_{ii}) + \Phi(A_{ij}) + \Phi(A_{ji}) + \Phi(A_{jj}))(B_{ii} + B_{ij} + B_{ji} + B_{jj})\\ &+ (A_{ii} + A_{ij} + A_{ji} + A_{jj})(\Phi(B_{ii}) + \Phi(B_{ij}) + \Phi(B_{ji}) + \Phi(B_{jj}))\\ =&(\Phi(A) - g(A))B + A(\Phi(B) - g(B))\\ =&\Psi(A)B + A\Psi(B).\end{aligned}$$

It follows that Ψ is an additive derivation on $\mathcal{R}$. Moreover, we have from Lemmas 2.10 and 2.12 that

$$\begin{aligned}\Psi(A)^* &= (\Phi(A) - g(A))^*\\ &= (\Phi(A_{ii}) + \Phi(A_{ij}) + \Phi(A_{ji}) + \Phi(A_{jj}) + g(A))^* - g(A)^*\\ &= \Phi(A_{ii})^* + \Phi(A_{ij})^* + \Phi(A_{ji})^* + \Phi(A_{jj})^*\\ &= \Phi(A_{ii}^*) + \Phi(A_{ij}^*) + \Phi(A_{ji}^*) + \Phi(A_{jj}^*)\\ &= \Phi(A^*) - g(A^*)\\ &= \Psi(A^*).\end{aligned}$$

Consequently, Ψ is an additive $*$-derivation on $\mathcal{R}$. By the definitions of Φ and Ψ, we can see that $\varphi(A) = \Theta(A) + g(A)$, where $\Theta(A) = \Psi(A) + [A, T]$ is an additive $*$-derivation.

As applications of Theorem 2.1, we have the following corollaries.

Corollary 2.1. *Let $\mathcal{A}$ be a factor von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$ with dim$\mathcal{A}$>1. If a map $\varphi : \mathcal{A} \to \mathcal{A}$ satisfies $\varphi([A \bullet B, C]) = [\varphi(A) \bullet B, C] + [A \bullet \varphi(B), C] + [A \bullet B, \varphi(C)]$ for any $A, B, C \in \mathcal{A}$ with $ABC^* = 0$, then there exists an additive $*$-derivation Θ of $\mathcal{A}$ and a nonlinear map $g : \mathcal{A} \to \mathrm{i}\mathbb{R}I$ such that $\varphi(A) = \Theta(A) + g(A)$ for any $A \in \mathcal{A}$.*

Corollary 2.2. *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all linear bounded operators on $\mathcal{H}$. If a map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies $\varphi([A \bullet B, C]) = [\varphi(A) \bullet B, C] + [A \bullet \varphi(B), C] + [A \bullet B, \varphi(C)]$ for any $A, B, C \in \mathcal{B}(\mathcal{H})$ with $ABC^* = 0$, then there exists $T \in \mathcal{B}(\mathcal{H})$ satisfying $T + T^* = 0$ and a nonlinear map $g : \mathcal{B}(\mathcal{H}) \to \mathrm{i}\mathbb{R}I$ such that*

$$\varphi(A) = AT - TA + g(A)$$

for any $A \in \mathcal{B}(\mathcal{H})$.

Proof: It follows from Theorem 2.1 that there exists an additive $*$-derivation Θ of $\mathcal{B}(\mathcal{H})$ and a nonlinear map $g : \mathcal{B}(\mathcal{H}) \to \mathrm{i}\mathbb{R}I$ such that $\varphi(A) = \Theta(A) + g(A)$ for any $A \in \mathcal{B}(\mathcal{H})$. By the result of [3], Θ is linear and, so it is inner. Hence, there exists $S \in \mathcal{B}(\mathcal{H})$ such that $\Theta(A) = AS - SA$ for any $A \in \mathcal{B}(\mathcal{H})$. Therefore,

$$A^*S - SA^* = \Theta(A^*) = \Theta(A)^* = S^*A^* - A^*S^*$$

for all $A \in \mathcal{B}(\mathcal{H})$. From this, we can see that $S + S^* = \lambda I$ for some $\lambda \in \mathbb{R}$. Write $T = S - \frac{1}{2}\lambda I$ and so $T + T^* = 0$. Hence $\varphi(A) = AT - TA + g(A)$ for any $A \in \mathcal{B}(\mathcal{H})$.

Corollary 2.3. *Let $\mathcal{A}$ be a standard operator algebra on an infinite dimensional complex Hilbert space $\mathcal{H}$ with dim$\mathcal{A}$>1, which is closed under the adjoint operation and contains a nontrivial projection. If a map $\varphi : \mathcal{A} \to \mathcal{A}$ satisfies $\varphi([A \bullet B, C]) = [\varphi(A) \bullet B, C] + [A \bullet \varphi(B), C] + [A \bullet B, \varphi(C)]$ for any $A, B, C \in \mathcal{A}$ with $ABC^* = 0$, then there exists an additive $*$-derivation Θ of $\mathcal{A}$ and a nonlinear map $g : \mathcal{A} \to \mathrm{i}\mathbb{R}I$ such that $\varphi(A) = \Theta(A) + g(A)$ for any $A \in \mathcal{A}$.*

3. Conclusions

In this paper, we characterized the structure of a specific type of non-global nonlinear mixed skew Jordan Lie triple derivations on prime $*$-rings. Moreover, we applied the above result to factor von Neumann algebras and standard operator algebras.

Author contributions

Fenhong Li: Writing–original draft, writing–review & editing; Liang Kong: Writing–original draft, writing–review & editing, funding acquisition; Chao Li: Writing–original draft, writing–review & editing. All authors are contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article

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Conflict of interest

The authors declare that there are no conflicts of interest.

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