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Research article

The L_p-Petty projection inequality for s-concave functions

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Abstract: We study the "functionalization" of the L_p-projection body and two related important inequalities in geometry. On the class of s-concave functions, a general function counterpart of the L_p-projection body is introduced. In addition, the L_p-Petty projection inequality and the L_p isoperimetric inequality are established on this class. Finally, we show that the L_p-Petty projection inequality strengthens the L_p isoperimetric inequality on the same class.

Keywords: s-concave function; L_p-projection body; L_p-Petty projection inequality; L_p isoperimetric inequality
Mathematics Subject Classification: 26D10, 52A40

1. Introduction

The isoperimetric inequality is a very classical problem in mathematics, which has been well known since Ancient Greece. It was first stated on the plane that the disk has the largest area among all domains enclosed by the closed curve of fixed perimeter. In the 19th century, Steiner first used the symmetrization to directly explain the existence of the solution to the isoperimetric problem, and it was not until 1870 that a rigorous and complete mathematical proof was given by Weierstrass with the variational method of analysis. Later, Hurwitz made use of the Fourier series to give a purely analytic proof. Nowadays, the isoperimetric inequality has been extended to higher-dimensional Euclidean spaces, some special manifolds, and other spaces.

Theorem 1.1. *Let $C \subset \mathbb{R}^n$ be a bounded closed domain. Suppose the surface area S of C is fixed, then its volume V is maximized when C is an Euclidean ball (we call it a ball for brevity), i.e.,*

$$S(C)^n \geq n^n \omega_n V(C)^{n-1},$$

with equality if and only if C is a ball, where ω_n denotes the volume of the unit ball.

Many important mathematical ideas and methods occurred in the study of the isoperimetric inequality and have been applied to a great many mathematical disciplines. Among these, Osserman [15] gave its diverse implications and applications in analysis and differential geometry. Chaville [7] described the effects of the isoperimetric inequality in analysis. And even in physics, the isoperimetric inequality was used to study the principle of least action [16].

For convex bodies (compact subsets that are convex and have nonempty interiors) in $\mathbb{R}^n$, the isoperimetric inequality can be derived from the Brunn-Minkowski inequality. Later, Lutwak introduced the Firey p-Minkowski addition and generalized the classical Brunn-Minkowski theory. The following L_p isoperimetric inequality is obtained from the L_p Brunn-Minkowski inequality; see [18]. When $p = 1$, we obtain the classical isoperimetric inequality.

Theorem 1.2. *Let $1 \leq p < \infty$ and M be a convex body in $\mathbb{R}^n$. Then*

$$S_p(M)^n \geq n^n \omega_n^p V(M)^{n-p},$$

with equality if and only if M is a ball.

Another important topic is the Petty projection inequality, which is closely related to the projection body introduced by Minkowski in the last century. It indicates that the polar projection body of the ellipsoid has the largest volume among all convex bodies with fixed volume. It is important due to the fact that it is also a version of the affine isoperimetric inequality and thus further strengthens the classical isoperimetric inequality. In addition, Zhang [20] established the inverse inequality. Subsequently, Zhang [21] removed the assumption of convexity and gave the Petty projection inequality on compact sets with smooth boundaries. Soon after, Wang [19] generalized it to sets of finite perimeter. Moreover, Lutwak [13] deduced an inequality relating the volume of a convex body to the power mean of its luminosity function based on the Petty projection inequality.

In 2000, the projection body and the Petty projection inequality were generalized to the L_p cases by Lutwak, Yang, and Zhang [12]. When $p = 1$, the following inequality is equivalent to the classical Petty projection inequality.

Theorem 1.3. *Let $1 \leq p < \infty$ and M be a convex body in $\mathbb{R}^n$. Then*

$$V(M)^{\frac{n-p}{p}} V(\Pi_p^* M) \leq \omega_n^{\frac{n}{p}},$$

with equality if and only if M is an ellipsoid centered at the origin. Here, $\Pi_p^* M$ is the polar body of L_p-projection body $\Pi_p M$.

Recently, there has been a new trend towards the functionalization of geometry, which has received widespread attention in the fields of geometry and analysis. Artstein-Avidan, Klartag, and Milman [1] provided a connection between convex bodies and s-concave functions, which made the functionalization possible. For $\varphi \in \mathrm{Conc}_s(\mathbb{R}^n)$ (see Section 2 for definitions), they defined the convex body $K_s(\varphi)$ in $\mathbb{R}^n \times \mathbb{R}^s$ by

$$K_s(\varphi) = \left\{(z', \tilde{z}) \in \mathbb{R}^n \times \mathbb{R}^s : z' \in \mathrm{supp}(\varphi), \|\tilde{z}\| \leq \varphi^{\frac{1}{s}}(z')\right\},$$

where $\|\cdot\|$ denotes the Euclidean norm and s is a positive integer throughout the paper. Thereafter, they further obtained a general functional form of the Blaschke-Santaló inequality based on the link above. Moreover, Milman and Rotem [14] studied some important inequalities on mixed integrals of s-concave functions. See [2,4–6,10,17] for the corresponding geometric inequalities on the s-concave functions and more research between them.

More recently, Fang and Zhou [9] defined the projection body $\Pi^{(s)}\varphi$ of s-concave functions by $\Pi^{(s)}\varphi = \Pi K_s(\varphi)$, for $\varphi \in \mathrm{Conc}_s(\mathbb{R}^n)$. They also established the Petty projection inequality for s-concave functions.

Theorem 1.4. *Let* $z' = (z_1, \cdots, z_n)$*,* $\tilde{z} = (z_{n+1}, \cdots, z_{n+s})$ *and* $\varphi \in \mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$*. Then for all* $(z', \tilde{z}) \in \partial K_s(\varphi)$ *with* $z' \in \mathrm{int}(\mathrm{supp}(\varphi))$*,*

$$\int_{\mathbb{S}^{n+s-1}} \Big[\int_{\mathbb{R}^{n+s-1}} |\langle \theta, (\varphi^{\frac{1}{s}} \nabla \varphi^{\frac{1}{s}}, -\tilde{z}) \rangle| \frac{dz_1 \cdots dz_{n+s-1}}{|z_{n+s}|} \Big]^{-(n+s)} d\theta \leq M_{n,s} \Big(\int_{\mathbb{R}^n} \varphi dz' \Big)^{1-n-s},$$

where $M_{n,s} = (n+s)\omega_s \left(\frac{\omega_{n+s}}{\omega_{n+s-1}\omega_s} \right)^{n+s}$*. Equality holds if and only if* $\varphi = (a + \langle b, z' \rangle - \langle Cz', z' \rangle)_+^{\frac{s}{2}}$*, where* $a > 0$*,* $b \in \mathbb{R}^n$*, and* C *is a positive definite matrix. Here* $\alpha_+ = \max\{\alpha, 0\}$ *for* $\alpha \in \mathbb{R}$*,* $\nabla\varphi$ *denotes the gradient of* φ*,* $\langle \cdot, \cdot \rangle$ *denotes the inner product in Euclidean space* $\mathbb{R}^n$*, and see Section 2 for definitions of* $\mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$*.*

Besides, Fang and Zhou [9] established the isoperimetric inequality for s-concave functions.

Theorem 1.5. *Let* $\varphi \in \mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$*. Then*

$$\int_{\mathbb{R}^n} \varphi^{1-\frac{1}{s}} (1 + \|\nabla \varphi^{\frac{1}{s}}\|^2)^{\frac{1}{2}} dz' \geq N_{n,s} \Big(\int_{\mathbb{R}^n} \varphi dz' \Big)^{1-\frac{1}{n+s}},$$

where $N_{n,s} = \frac{n+s}{s} (\frac{\omega_{n+s}}{\omega_s})^{\frac{1}{n+s}}$*. Equality holds if and only if* $\varphi = (d - \|z' - e\|^2)_+^{\frac{s}{2}}$*, where* $d > 0$ *and* $e \in \mathbb{R}^n$*.*

In this paper, we aim to establish analogues of Theorems 1.2 and 1.3 for s-concave functions. First, we give a general functional analogue of the L_p-projection body. Let $1 \leq p < \infty$ and $\varphi \in \mathrm{Conc}_s(\mathbb{R}^n)$. We define its L_p-projection body $\Pi_p^{(s)}\varphi$ as

$$\Pi_p^{(s)}\varphi = \Pi_p K_s(\varphi).$$

Note that when $p = 1$, it is the projection body $\Pi^{(s)}\varphi$ of s-concave functions defined in [9], and further, the projection body on the class of log-concave functions (the case as $s \to \infty$) has also been defined in [8]. Moreover, this new functional L_p-projection body corresponds to the L_p-projection body of the $n + s$ dimensional convex body associated with φ. Thus, $\Pi_p^{(s)}(\varphi)$ has affine invariance, continuity, and many other properties similar to the L_p-projection body of convex bodies.

Next, we obtain a functional analogue of the L_p-Petty projection inequality using analytic methods in convex geometry.

Theorem 1.6. *Let* $1 \leq p < \infty$*,* $z' = (z_1, \cdots, z_n)$*,* $\tilde{z} = (z_{n+1}, \cdots, z_{n+s})$ *and* $\varphi \in \mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$*. Then for all* $(z', \tilde{z}) \in \partial K_s(\varphi)$ *with* $z' \in \mathrm{int}(\mathrm{supp}(\varphi))$*,*

$$\int_{\mathbb{S}^{n+s-1}} \Big[\int_{\mathbb{R}^{n+s-1}} |\langle \theta, (\varphi^{\frac{1}{s}} \nabla \varphi^{\frac{1}{s}}, -\tilde{z}) \rangle|^p (\langle \varphi^{\frac{1}{s}} \nabla \varphi^{\frac{1}{s}}, z' \rangle - \varphi^{\frac{2}{s}})^{1-p} \frac{dz_1 \cdots dz_{n+s-1}}{|z_{n+s}|} \Big]^{-\frac{n+s}{p}} d\theta \leq K_{n,s} \Big(\int_{\mathbb{R}^n} \varphi dz' \Big)^{1-\frac{n+s}{p}},$$

where $K_{n,s} = [(n+s)\omega_s]^{1-\frac{n+s}{p}} \left(\frac{2}{c_{n+s-2,p}} \right)^{\frac{n+s}{p}}$*. Equality holds if and only if* $\varphi = (t - \langle Dz', z' \rangle)_+^{\frac{s}{2}}$*, where* $t > 0$ *and* D *is a positive definite matrix.*

In addition, we acquire the L_p isoperimetric inequality for s-concave functions.

Theorem 1.7. *Let* $1 \le p < \infty$, $\varphi \in \mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$ *and* $z' = (z_1, \cdots, z_n)$. *Then*

$$\int_{\mathbb{R}^n} (\langle \nabla \varphi^{\frac{1}{s}}, z' \rangle - \varphi^{\frac{1}{s}})^{1-p} (1 + \|\nabla \varphi^{\frac{1}{s}}\|^2)^{\frac{p}{2}} \varphi^{1-\frac{1}{s}} dz' \ge L_{n,s} \Big(\int_{\mathbb{R}^n} \varphi dz' \Big)^{1-\frac{p}{n+s}},$$

where $L_{n,s} = \frac{n+s}{s} (\frac{\omega_{n+s}}{\omega_s})^{\frac{p}{n+s}}$. *Equality holds if and only if* $\varphi = (r - \|z' - d\|^2)_+^{\frac{s}{2}}$, *where* $r > 0$ *and* $d \in \mathbb{R}^n$.

In fact, Theorems 1.6 and 1.7 are the L_p extensions of Theorems 1.4 and 1.5, respectively. We remark that, although both our results and the results in [9] use geometric inequalities and their relation with s-concave functions, combining with tools in the L_p Brunn-Minkowski theory for $p \ge 1$, our results include the results in [9] as special cases when $p = 1$.

Moreover, our results generalize the geometric structures on $\mathcal{K}^n$ in the following sense. Let $K \in \mathcal{K}^n$ and $\varphi = (1 - \|z'\|_K)_+^s$, $z' \in \mathbb{R}^n$. For each $t \in (0, 1]$ and $z_0 \in t\mathbb{S}^{s-1}$, it is clear that

$$K_s(\varphi) \supset \left\{ (z', z_0) : z' \in \mathbb{R}^n, t = \|z_0\| \le \varphi^{\frac{1}{s}}(z') = (1 - \|z'\|_K)_+ \right\} = ((1-t)K, z_0),$$

which is a dilate of K. In other words,

$$K_s(\varphi) = \left\{ (1-t)K \times (t\mathbb{S}^{s-1}) : t \in (0, 1] \right\}.$$

Based on Theorems 1.6 and 1.7, we make a further investigation of the connection between the L_p-Petty projection inequality and the L_p isoperimetric inequality on the class of s-concave functions in Section 3.

2. Preliminaries

In this section, we collect basics regarding convex bodies and s-concave functions, most of which can be found in [18].

Let $\mathcal{K}^n$ denote the set of all convex bodies in $\mathbb{R}^n$, and let $\mathcal{K}_o^n$ be the subset of $\mathcal{K}^n$ whose elements contain the origin in their interiors. Let ∂M be the boundary of $M \in \mathcal{K}^n$. The unit ball and the unit sphere in $\mathbb{R}^d$ are denoted respectively by $B_2^d = \{z \in \mathbb{R}^d : \|z\| \le 1\}$ and $\mathbb{S}^{d-1} = \{z \in \mathbb{R}^d : \|z\| = 1\}$.

For $M \in \mathcal{K}^n$, define its support function $h_M = h(M, \cdot) : \mathbb{R}^n \to \mathbb{R}$ by

$$h(M, z) = \max\{\langle z, y \rangle : y \in M\}, \quad z \in \mathbb{R}^n.$$

Let $M \in \mathcal{K}_o^n$. Define its radial function $\rho_M = \rho(M, \cdot) : \mathbb{R}^n \setminus \{o\} \to \mathbb{R}$ by

$$\rho(M, z) = \max\{t \ge 0 : tz \in M\}, \quad z \in \mathbb{R}^n \setminus \{o\}.$$

And define the polar body M^* of M by

$$M^* = \{z \in \mathbb{R}^n : \langle z, y \rangle \le 1 \text{ for all } y \in M\}.$$

Based on the definition above, there are (see e.g., [18, Lemma 1.7.13])

$$h_{M^*} = \rho_M^{-1} \quad \text{and} \quad \rho_{M^*} = h_M^{-1}.$$

For a convex body $M \in \mathcal{K}_o^n$, the n-dimensional volume of M can be expressed as

$$V_n(M) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_M(\theta)^n d\theta.$$

In the case that $M = B_2^d$, we have $V_d(B_2^d) = \omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})}$, where $\Gamma(\cdot)$ is the Gamma function.

Let $M \in \mathcal{K}^n$ and $1 \leq p < \infty$. The support function of the L_p-projection body, $\Pi_p M$, of M is defined by Lutwak, Yang, and Zhang [12] as

$$h(\Pi_p M, \eta)^p = \frac{1}{n\omega_n c_{n-2,p}} \int_{\mathbb{S}^{n-1}} |\langle \eta, \theta\rangle|^p \, dS_p(M, \theta), \quad \eta \in \mathbb{S}^{n-1},$$

where $c_{n,p} = \frac{\omega_{n+p}}{\omega_2\omega_n\omega_{p-1}}$ and $S_p(M,\cdot)$ denotes the p-surface area measure of M. In particular, when $p = 1$, $S_1(M,\cdot)$ corresponds to the well-known surface area measure $S(M,\cdot)$ of M. Indeed, the measure $S_p(M,\cdot)$, is absolutely continuous with respect to $S(M,\cdot)$ and the Radon-Nikodym derivative is

$$\frac{dS_p(M,\cdot)}{dS(M,\cdot)} = h(M,\cdot)^{1-p}.$$

Let $N_M(z)$ be the outer unit normal vector at $z \in \partial M$, which exists a.e. on ∂M. For $\theta = N_M(z) \in \mathbb{S}^{n-1}$, we have

$$\int_{\mathbb{S}^{n-1}} dS_p(M,\theta) = \int_{\mathbb{S}^{n-1}} h(M,\theta)^{1-p} dS(M,\theta) = \int_{\partial M} \langle N_M(z), z\rangle^{1-p} \, d\mu_M(z), \tag{2.1}$$

where $\mu_M(\cdot)$ is the $(n-1)$-dimensional Hausdorff measure on ∂M. Hence, the support function $h(\Pi_p M, \eta)$ has the following form:

$$h(\Pi_p M, \eta)^p = \frac{1}{n\omega_n c_{n-2,p}} \int_{\partial M} |\langle \eta, N_M(z)\rangle|^p \, \langle N_M(z), z\rangle^{1-p} \, d\mu_M(z), \quad \eta \in \mathbb{S}^{n-1}.$$

We call a function $\varphi : \mathbb{R}^n \to [0,\infty)$ s-concave if $\mathrm{supp}(\varphi) = \mathrm{cl}\{z \in \mathbb{R}^n : \varphi(z) > 0\}$ is a convex body, φ is upper semi-continuous, and $\varphi^{\frac{1}{s}}$ is concave on $\mathrm{supp}(\varphi)$. Let $\mathrm{Conc}_s(\mathbb{R}^n)$ denote the class of all s-concave functions, and $\mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$ denote its subset of functions that are twice continuously differentiable in the interior of their supports. It is clear that as s tends to infinity, the class of s-concave functions converges to the class of log-concave functions in the sense of uniform convergence of compact sets, and as s tends to 0, it converges to the class of indicator functions of convex sets in the same sense as above. See [1–3] for more information.

Next, we give basics related to $K_s(\varphi)$. For $\varphi \in \mathrm{Conc}_s(\mathbb{R}^n)$, by Fubini's theorem, we have

$$V_{n+s}(K_s(\varphi)) = \omega_s \int_{\mathbb{R}^n} \varphi(z')dz'. \tag{2.2}$$

On the other hand, as a special case, setting

$$\psi_s(z') = (1 - \|z'\|^2)_+^{\frac{s}{2}}, \quad z' \in \mathbb{R}^n,$$

gives $K_s(\psi_s) = B_2^{n+s}$, due to the definition of $K_s(\varphi)$. As for its boundary, it again follows from the definition that

$$\partial K_s(\varphi) = \left\{(z', \tilde{z}) \in \mathbb{R}^n \times \mathbb{R}^s : \|\tilde{z}\| = \varphi^{\frac{1}{s}}(z')\right\}.$$

Thus, we have the following mapping:

$$(z', z_{n+1}, \cdots, z_{n+s-1}) \mapsto (z', z_{n+1}, \cdots, z_{n+s-1}, \pm z_{n+s}),$$

where $z' = (z_1, \cdots, z_n) \in \mathbb{R}^n$, and

$$z_{n+s} = \left(\varphi^{\frac{2}{s}}(z') - \sum_{i=n+1}^{n+s-1} z_i^2 \right)^{\frac{1}{2}}.$$

Since $\partial K_s(\varphi)$ is symmetric, it suffices to consider the case that $z_{n+s} > 0$ in the calculation. Hence, the surface area element of $\partial K_s(\varphi)$ is given by [2],

$$d\mu_{K_s(\varphi)} = \frac{\varphi^{\frac{1}{s}}(1 + \|\nabla \varphi^{\frac{1}{s}}\|^2)^{\frac{1}{2}}}{|z_{n+s}|} dz_1 \cdots dz_{n+s-1}. \tag{2.3}$$

Moreover, we need the following lemma.

Lemma 2.1. *[2] Let $\varphi \in \mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$, $z' = (z_1, \cdots, z_n)$ and $\tilde{z} = (z_{n+1}, \cdots, z_{n+s})$. Then for all $(z', \tilde{z}) \in \partial K_s(\varphi)$ with $z' \in \mathrm{int}(\mathrm{supp}(\varphi))$,*

$$N_{K_s(\varphi)}(z', \tilde{z}) = \frac{(\varphi^{\frac{1}{s}} \nabla \varphi^{\frac{1}{s}}, -\tilde{z})}{\varphi^{\frac{1}{s}}(1 + \|\nabla \varphi^{\frac{1}{s}}\|^2)^{\frac{1}{2}}}.$$

Here φ is evaluated at z'.

3. Main results and proofs

We first introduce the L_p-projection body for s-concave functions, using its correspondence with convex bodies.

Definition 3.1. *Let $1 \leq p < \infty$ and $\varphi \in \mathrm{Conc}_s(\mathbb{R}^n)$. Define the L_p-projection body $\Pi_p^{(s)}\varphi$ of φ by*

$$h_{\Pi_p^{(s)}\varphi}(y)^p = h_{\Pi_p K_s(\varphi)}(y)^p = \frac{1}{\lambda_{n,s,p}} \int_{\partial K_s(\varphi)} |\langle y, N_{K_s(\varphi)}(z) \rangle|^p \langle z, N_{K_s(\varphi)}(z) \rangle^{1-p} d\mu_{K_s(\varphi)}(z),$$

for $y \in \mathbb{R}^n \times \mathbb{R}^s$, where $z = (z_1, z_2, \cdots, z_{n+s}) \in \partial K_s(\varphi)$ and $\lambda_{n,s,p} = (n+s)\omega_{n+s} c_{n+s-2,p}$.

Notice that $h_{\Pi_p^{(s)}\varphi}$ is defined by the support function of the convex body $\Pi_p K_s(\varphi)$; hence, it is convex. Therefore, its continuity on compact subsets of $\mathbb{R}^{n+s}$ follows (see e.g., [18, Theorem 1.5.3]).

In addition, we use its following form to facilitate later calculations.

Lemma 3.1. *Let $1 \leq p < \infty$ and $\varphi \in \mathrm{Conc}_s^{(2)}(\mathbb{R}^n)$. For each $\theta \in \mathbb{S}^{n+s-1}$, we have*

$$h_{\Pi_p^{(s)}\varphi}(\theta)^p = \frac{2}{\lambda_{n,s,p}} \int_{\mathbb{R}^{n+s-1}} |\langle \theta, (\varphi^{\frac{1}{s}} \nabla \varphi^{\frac{1}{s}}, -\tilde{z}) \rangle|^p (\langle \varphi^{\frac{1}{s}} \nabla \varphi^{\frac{1}{s}}, z' \rangle - \varphi^{\frac{2}{s}})^{1-p} \frac{dz_1 \cdots dz_{n+s-1}}{|z_{n+s}|},$$

where $z' = (z_1, \cdots, z_n) \in \mathbb{R}^n$ and $\tilde{z} = (z_{n+1}, \cdots, z_{n+s}) \in \mathbb{R}^s$.

Proof. Let $\theta \in \mathbb{S}^{n+s-1}$, $z' = (z_1, \cdots, z_n)$, $\tilde{z} = (z_{n+1}, \cdots, z_{n+s})$ such that $z = (z', \tilde{z}) \in (\mathbb{R}^n \times \mathbb{R}^s) \cap \partial K_s(\varphi)$. And denote $\tilde{\partial} K_s(\varphi) = \left\{ z \in \partial K_s(\varphi) : z' \in \mathrm{int}(\mathrm{supp}(\varphi)) \right\}$. Since $\partial K_s(\varphi) \setminus \tilde{\partial} K_s(\varphi)$ has measure zero, by Definition 3.1, Lemma 2.1, and (2.3), we obtain

$$\begin{aligned} h_{\Pi_p^{(s)}\varphi}(\theta)^p &= h_{\Pi_p K_s(\varphi)}(\theta)^p \\ &= \frac{1}{\lambda_{n,s,p}} \int_{\partial K_s(\varphi)} |\langle \theta, N_{K_s(\varphi)}(z) \rangle|^p \langle z, N_{K_s(\varphi)}(z) \rangle^{1-p} d\mu_{K_s(\varphi)}(z) \\ &= \frac{1}{\lambda_{n,s,p}} \int_{\tilde{\partial} K_s(\varphi)} \left| \left\langle \theta, \frac{(\varphi^{\frac{1}{s}} \nabla \varphi^{\frac{1}{s}}, -\tilde{z})}{\varphi^{\frac{1}{s}} (1 + \|\nabla \varphi^{\frac{1}{s}}\|^2)^{\frac{1}{2}}} \right\rangle \right|^p \left(\frac{\langle \nabla \varphi^{\frac{1}{s}}, z' \rangle - \varphi^{\frac{1}{s}}}{(1 + \|\nabla \varphi^{\frac{1}{s}}\|^2)^{\frac{1}{2}}} \right)^{1-p} d\mu_{K_s(\varphi)}(z) \\ &= \frac{2}{\lambda_{n,s,p}} \int_{\mathbb{R}^{n+s-1}} |\langle \theta, (\varphi^{\frac{1}{s}} \nabla \varphi^{\frac{1}{s}}, -\tilde{z}) \rangle|^p (\langle \varphi^{\frac{1}{s}} \nabla \varphi^{\frac{1}{s}}, z' \rangle - \varphi^{\frac{2}{s}})^{1-p} \frac{dz_1 \cdots dz_{n+s-1}}{|z_{n+s}|}. \end{aligned}$$

The equation above follows from the symmetry of $\partial K_s(\varphi)$. Here φ is evaluated at $z' = (z_1, \cdots, z_n) \in \mathbb{R}^n$. □

Next, in light of Lemma 3.1 and the L_p-Petty projection inequality on convex bodies, we give the proof of Theorem 1.6.

Proof of Theorem 1.6. Let $\Pi_p^{(s),*}(\varphi)$ be the polar body of $\Pi_p^{(s)}(\varphi)$. By Lemma 3.1, Theorem 1.3, and (2.2), we obtain

$$\begin{aligned} & V_{n+s}(\Pi_p^{(s),*}(\varphi)) \\ &= \frac{1}{n+s} \int_{\mathbb{S}^{n+s-1}} h_{\Pi_p^{(s)}(\varphi)}(\theta)^{-(n+s)} d\theta \\ &= \alpha_{n,s,p} \int_{\mathbb{S}^{n+s-1}} \left[\int_{\mathbb{R}^{n+s-1}} |\langle \theta, (\varphi^{\frac{1}{s}} \nabla \varphi^{\frac{1}{s}}, -\tilde{z}) \rangle|^p (\langle \varphi^{\frac{1}{s}} \nabla \varphi^{\frac{1}{s}}, z' \rangle - \varphi^{\frac{2}{s}})^{1-p} \frac{dz_1 \cdots dz_{n+s-1}}{|z_{n+s}|} \right]^{-\frac{n+s}{p}} d\theta \\ &\le \omega_{n+s}^{\frac{n+s}{p}} \omega_s^{1-\frac{n+s}{p}} \left(\int_{\mathbb{R}^n} \varphi dz' \right)^{1-\frac{n+s}{p}}, \end{aligned}$$

where $\alpha_{n,s,p} = \frac{1}{n+s} \left(\frac{\lambda_{n,s,p}}{2} \right)^{\frac{n+s}{p}}$. This gives the desired L_p-Petty projection inequality for s-concave functions.

From the proof above, it is clear that the equality holds if and only if $K_s(\varphi)$ is an ellipsoid with the origin as its center due to Theorem 1.3. It is further equivalent to $\varphi = (t - \langle Dz', z' \rangle)_+^{\frac{s}{2}}$, where $t > 0$ and D is a positive definite matrix, since $K_s(\varphi)$ clearly corresponds to an ellipsoid by the definition in this case. □

Now, we prove the L_p isoperimetric inequality on the class of s-concave functions.

Proof of Theorem 1.7. Using (2.1) and the similar treatment as in the proof of Lemma 3.1, we obtain

$$\begin{aligned} S_p(K_s(\varphi)) &= \int_{\mathbb{S}^{n+s-1}} dS_p(K_s(\varphi), \eta) \\ &= \int_{\partial K_s(\varphi)} \langle N_{K_s(\varphi)}(z), z \rangle^{1-p} d\mu_{K_s(\varphi)}(z) \end{aligned}$$

$$= 2\int_{\mathbb{R}^{n+s-1}} \Big(\frac{\langle \nabla\varphi^{\frac{1}{s}}, z'\rangle - \varphi^{\frac{1}{s}}}{(1+\|\nabla\varphi^{\frac{1}{s}}\|^2)^{\frac{1}{2}}}\Big)^{1-p} \cdot \frac{\varphi^{\frac{1}{s}}(1+\|\nabla\varphi^{\frac{1}{s}}\|^2)^{\frac{1}{2}}}{|z_{n+s}|} dz_1\cdots dz_{n+s-1}$$
$$= 2\int_{\mathbb{R}^{n+s-1}} (\langle \nabla\varphi^{\frac{1}{s}}, z'\rangle - \varphi^{\frac{1}{s}})^{1-p}(1+\|\nabla\varphi^{\frac{1}{s}}\|^2)^{\frac{p}{2}}\varphi^{\frac{1}{s}}\frac{dz_1\cdots dz_{n+s-1}}{|z_{n+s}|}.$$

Applying (3.3) in [9], we have

$$\int_{\mathbb{R}^{s-1}} \frac{dz_{n+1}\cdots dz_{n+s-1}}{|z_{n+s}|} = \frac{1}{2}s\varphi^{1-\frac{2}{s}}V_s(B_2^s) = \varphi^{1-\frac{2}{s}}\frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})},$$

and then

$$S_p(K_s(\varphi)) = \frac{2\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}\int_{\mathbb{R}^n} (\langle \nabla\varphi^{\frac{1}{s}}, z'\rangle - \varphi^{\frac{1}{s}})^{1-p}(1+\|\nabla\varphi^{\frac{1}{s}}\|^2)^{\frac{p}{2}}\varphi^{1-\frac{1}{s}}dz'.$$

Therefore, Theorem 1.2 and (2.2) imply

$$\frac{2\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}\int_{\mathbb{R}^n} (\langle \nabla\varphi^{\frac{1}{s}}, z'\rangle - \varphi^{\frac{1}{s}})^{1-p}(1+\|\nabla\varphi^{\frac{1}{s}}\|^2)^{\frac{p}{2}}\varphi^{1-\frac{1}{s}}dz' \geq \Big[(n+s)^{n+s}\omega_{n+s}^p\omega_s^{n+s-p}\Big(\int_{\mathbb{R}^n}\varphi dz'\Big)^{n+s-p}\Big]^{\frac{1}{n+s}},$$

and Theorem 1.7 follows. The equality holds if and only if $K_s(\varphi)$ is a ball, which is equivalent to $\varphi = (r - \|z' - d\|^2)_+^{\frac{s}{2}}$, where $r > 0$ and $d \in \mathbb{R}^n$. □

On convex bodies, Lutwak [11] pointed out that the classical isoperimetric inequality can be deduced from the Petty projection inequality. After our verification, we found a corresponding conclusion on s-concave functions, as shown in the next result.

Theorem 3.1. *The L_p-Petty projection inequality for s-concave functions*

$$\int_{\mathbb{S}^{n+s-1}}\Big[\int_{\mathbb{R}^{n+s-1}} |\langle\theta, (\varphi^{\frac{1}{s}}\nabla\varphi^{\frac{1}{s}}, -\tilde{z})\rangle|^p(\langle\varphi^{\frac{1}{s}}\nabla\varphi^{\frac{1}{s}}, z'\rangle - \varphi^{\frac{2}{s}})^{1-p}\frac{dz_1\cdots dz_{n+s-1}}{|z_{n+s}|}\Big]^{-\frac{n+s}{p}} d\theta \leq K_{n,s}\Big(\int_{\mathbb{R}^n}\varphi dz'\Big)^{1-\frac{n+s}{p}},$$

strengthens the L_p isoperimetric inequality for s-concave functions

$$\int_{\mathbb{R}^n} (\langle \nabla\varphi^{\frac{1}{s}}, z'\rangle - \varphi^{\frac{1}{s}})^{1-p}(1+\|\nabla\varphi^{\frac{1}{s}}\|^2)^{\frac{p}{2}}\varphi^{1-\frac{1}{s}}dz' \geq L_{n,s}\left(\int_{\mathbb{R}^n}\varphi dz'\right)^{1-\frac{p}{n+s}}.$$

Proof. First, notice that $\|(\varphi^{\frac{1}{s}}\nabla\varphi^{\frac{1}{s}}, -\tilde{z})\| = \varphi^{\frac{1}{s}}(1+\|\nabla\varphi^{\frac{1}{s}}\|^2)^{\frac{1}{2}}$, and that, for $y \in \mathbb{R}^{n+s}$, $\eta = \frac{y}{\|y\|}$,

$$\int_{\mathbb{S}^{n+s-1}} |\langle\theta, y\rangle|^p d\theta = \|y\|^p\int_{\mathbb{S}^{n+s-1}} |\langle\theta, \eta\rangle|^p d\theta = \|y\|^p(n+s)\omega_{n+s}c_{n+s-2,p}.$$

Next, we use Jensen's inequality, Fubini's theorem, and the results above to obtain

$$\left(\beta_{n,s}\int_{\mathbb{S}^{n+s-1}}\left[\int_{\mathbb{R}^{n+s-1}} |\langle\theta, (\varphi^{\frac{1}{s}}\nabla\varphi^{\frac{1}{s}}, -\tilde{z})\rangle|^p(\langle\varphi^{\frac{1}{s}}\nabla\varphi^{\frac{1}{s}}, z'\rangle - \varphi^{\frac{2}{s}})^{1-p}\frac{dz_1\cdots dz_{n+s-1}}{|z_{n+s}|}\right]^{-\frac{n+s}{p}} d\theta\right)^{-\frac{p}{n+s}}$$
$$\leq \beta_{n,s}\int_{\mathbb{S}^{n+s-1}}\int_{\mathbb{R}^{n+s-1}} |\langle\theta, (\varphi^{\frac{1}{s}}\nabla\varphi^{\frac{1}{s}}, -\tilde{z})\rangle|^p(\langle\varphi^{\frac{1}{s}}\nabla\varphi^{\frac{1}{s}}, z'\rangle - \varphi^{\frac{2}{s}})^{1-p}\frac{dz_1\cdots dz_{n+s-1}}{|z_{n+s}|}d\theta$$

$$= c_{n+s-2,p} \int_{\mathbb{R}^{n+s-1}} (\langle \nabla \varphi^{\frac{1}{s}}, z' \rangle - \varphi^{\frac{1}{s}})^{1-p} (1 + \|\nabla \varphi^{\frac{1}{s}}\|^2)^{\frac{p}{2}} \varphi^{\frac{1}{s}} \frac{dz_1 \cdots dz_{n+s-1}}{|z_{n+s}|}$$
$$= \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} c_{n+s-2,p} \int_{\mathbb{R}^n} (\langle \nabla \varphi^{\frac{1}{s}}, z' \rangle - \varphi^{\frac{1}{s}})^{1-p} (1 + \|\nabla \varphi^{\frac{1}{s}}\|^2)^{\frac{p}{2}} \varphi^{1-\frac{1}{s}} dz'.$$

It follows that

$$\int_{\mathbb{R}^n} (\langle \nabla \varphi^{\frac{1}{s}}, z' \rangle - \varphi^{\frac{1}{s}})^{1-p} (1 + \|\nabla \varphi^{\frac{1}{s}}\|^2)^{\frac{p}{2}} \varphi^{1-\frac{1}{s}} dz' \geq \left[\left(\int_{\mathbb{R}^n} \varphi dz' \right)^{1-\frac{n+s}{p}} \right]^{-\frac{p}{n+s}} \left(\frac{1}{\beta_{n,s} K_{n,s}} \right)^{\frac{p}{n+s}} \frac{\Gamma(\frac{s}{2})}{c_{n+s-2,p} \pi^{\frac{s}{2}}}$$
$$= L_{n,s} \left(\int_{\mathbb{R}^n} \varphi dz' \right)^{1-\frac{p}{n+s}},$$

where $\beta_{n,s} = \frac{1}{(n+s)\omega_{n+s}}$. Therefore, we obtain the desired result. □

4. Conclusions

This work gives a clear picture of the correspondence between concepts and inequalities in convex geometry and those in analysis on s-concave functions. It has important applications in integral geometry, differential geometry, image analysis, and so on. It is among the leading focuses in the field of convex geometry. Based on the mapping K_s that maps from an s-concave function to a convex body, defined by Artstein-Avidan, Klartag, and Milman, and the work by Fang and Zhou in the case when $p = 1$, in this work, on the class of s-concave functions, a general function counterpart of the L_p-projection body is introduced. In addition, the L_p-Petty projection inequality and the L_p isoperimetric inequality are established on this class. Finally, we show that the L_p-Petty projection inequality strengthens the L_p isoperimetric inequality on the same class. We believe this work serves as an important bridge between finite-dimensional convex geometry and infinite-dimensional functional analysis and thus contributes to the advances of the functionalization of convex geometry.

Author contributions

Tian Gao and Dan Ma: Conceptualization, investigation, writing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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