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Theory article

Another Meir-Keeler-type nonlinear contractions

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Abstract: This paper deals with the concept of α-$\hat{v}$-A-B-C-Meir-Keeler type nonlinear contractions, a new class mappings within the of modular extended b-metric spaces. We establish common unique fixed-point theorems that generalize, unify, and extend several key results in modular b-metric and modular extended b-metric spaces. These theorems bridge the gap between classical and contemporary fixed-point theories, showcasing broader applicability in nonlinear analysis. To ensure clarity and practical relevance, a detailed example is presented, further validating the theoretical findings. This work provides some level of understanding of the space under investigation and sets the stage for future developments in this evolving domain.

Keywords: existence; uniqueness; metric modular; b-metric space; α-$\hat{v}$-A-B-C Meir-Keeler; nonlinear contractions
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1. Introduction

A major development in fixed-point theory was achieved by Meir and Keeler [32], who established a pivotal theorem that extends the renowned Banach contraction mapping principle.

Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that, for each $\epsilon > 0$, $\exists$ a $\delta(\epsilon) > 0$ such that

$$\epsilon \le d(x, y) < \epsilon + \delta(\epsilon) \quad \Longrightarrow \quad d(Tx, Ty) < \epsilon$$

for all $x, y \in X$. Then the fixed point of T in X is unique.

Meir and Keeler's fixed-point method [32] has become a fundamental topic in fixed-point theory. It has inspired extensive research, with numerous authors contributing new ideas and methods to further develop their work.

Maiti and Pal [31] introduced the following contraction mapping and provided it's validity.

For every $\epsilon > 0$, $\delta > 0$ exists such that

$$\epsilon \leq \max\{d(x,y), d(x,Ty), d(y,Ty)\} < \epsilon + \delta \implies d(Tx,Ty) < \epsilon.$$

However, other researchers [36, 39] enhanced the above result with the following condition:

$$\epsilon \leq \max\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{1}{2}(d(Sx,Ty) + d(Sy,Tx))\} < \epsilon + \delta$$
$$\implies d(Tx,Ty) < \epsilon, \textit{ where } S, T \textit{ are self mappings in metric space } (X,d).$$

In 1976, Jungck [25] demonstrated a shared fixed-point result for commute mappings, assuming that one of them is continuous. Subsequently, in 1982, Sessa [41] introduced the concept of weakly commuting pairs of self-mappings and established a fixed-point theorem in complete metric spaces. Sharma [42] later presented novel results for weakly commuting mappings in such spaces, extending several related findings. Kumam et al. [30] explored fixed point results under generalized contractive conditions in b-metric spaces, providing an example to highlight the practical implications of their work. Moreover, He et al. [23] examined the existence and uniqueness of fixed-points for weakly commutative mappings within the framework of complete multiplicative metric spaces. Alsulami et al. [7] introduced α-admissible and generalized α-admissible Meir-Keeler contractions in quasi-metric spaces and extended their findings to G-metric spaces, proving fixed-point theorems. Abtahi [2] provided a criterion for sequences in metric spaces to be Cauchy, simplifying proofs of fixed-point results for Meir-Keeler-type contractions. Canzoneri and Vetroa [11] studied asymptotic contractions of the integral Meir-Keeler-type and established a fixed-point theorem ensuring existence and uniqueness. Gholamian and Khanehgir [20] introduced generalized Meir-Keeler contractions in b-metric-like spaces, proving fixed-point theorems with illustrative examples. Gulyaz et al. [22] extended the above work to α-Meir-Keeler and generalized α-Meir-Keeler contractions in Branciari b-metric spaces, establishing the existence and uniqueness of fixed-points. Karapinar et al. [26] studied (α-ψ)-Meir-Keeler contractions in generalized b-metric spaces, while Barootkoob et al. [10] introduced (α-ψ-p)-Meir-Keeler contractions, extending fixed point results via w-distance and applying them to nonlinear Fredholm integral equations. Panthi [35] proved common fixed-point theorems for compatible mappings in metric and dislocated metric spaces, and Koti et al. [29] established fixed-point results using Gupta-Saxena's rational expression. Additional results are documented in [15, 16] and some other results on common fixed-point theory can be found in [17,18,33]. Further results in this area can also be found in [1, 9, 13]. Aksoya et al. [3] studied Meir-Keeler type contractions in modular metric spaces, proving fixed point theorems with examples. Further results on such contractions in partial Hausdorff and JS-metric spaces are found in [12, 27]. Aksoy et al. [4] extended the fixed-point results to a broader class of contractive and non-expansive mappings in modular metric spaces. Jleli et al. [24] introduced proximal quasi-contractions in modular spaces with the Fatou property, establishing the best proximity point theorems. Aksoy et al. [5] explored α-admissible contractions in b-metric spaces, proving the fixed-point results and applying them to differential equations. Arshad et al. [8] generalized Jleli et al.'s work using triangular α-orbital admissible mappings [38], while Alharbi et al. [6] combined α-orbital admissibility with simulation functions to establish fixed-point results in b-metric spaces. Gholidahneh et al. [21] introduced modular extended b-metric spaces and established fixed point theorems for α $\hat{v}$ Meir-Keeler contractions, extending their results to various settings, including graph structured and partially ordered spaces. They also linked fuzzy b metric spaces with modular extended b metric spaces and provided examples and applications to illustrate their findings. Modular extended b-metric spaces provide a powerful and flexible framework that extends classical metric spaces and enables the study of nonlinear contractions, multi-mapping systems, and function dependent fixed point problems. Their broad applicability, as seen in [21], makes them an essential tool in modern mathematical analysis. There are examples of extended b-modular metric spaces which are not classical, b-metric, modular, or modular b-metric spaces. Example 2.2 in [21] is an extended modular b-metric space which is not a classical metric or b-metric space.

We analyze whether the function introduced in [21, Example 2.2], namely

$$\hat{v}_\lambda(x, y) = \sinh(v_\lambda(x, y)),$$

defines a modular extended b-metric space and whether it satisfies the conditions of a classical metric or b-metric spaces.

1) Verification as an extended modular b-metric space

A modular extended b-metric space satisfies the following conditions:

- Non-negativity and identity of indiscernibles: Since $\sinh(t) \geq 0$ for all $t \geq 0$ and $\sinh(0) = 0$, we have
$$\hat{v}_\lambda(x, y) = 0 \iff v_\lambda(x, y) = 0 \iff x = y.$$
- Symmetry: Since $v_\lambda(x, y) = v_\lambda(y, x)$, we obtain
$$\hat{v}_\lambda(x, y) = \sinh(v_\lambda(x, y)) = \sinh(v_\lambda(y, x)) = \hat{v}_\lambda(y, x).$$
- Extended modular triangle inequality: Since v_λ satisfies the modular b-metric inequality
$$v_{\lambda+\mu}(x, y) \leq s(v_\lambda(x, z) + v_\mu(z, y)),$$
applying sinh gives
$$\hat{v}_{\lambda+\mu}(x, y) = \sinh(v_{\lambda+\mu}(x, y)) \leq \sinh\left(s(v_\lambda(x, z) + v_\mu(z, y))\right).$$
Defining $\Omega(t) = \sinh(st)$, which is strictly increasing, we obtain
$$\hat{v}_{\lambda+\mu}(x, y) \leq \Omega(\hat{v}_\lambda(x, z) + \hat{v}_\mu(z, y)).$$

Thus, $\hat{v}_\lambda$ satisfies the modular extended b-metric conditions.

2) Not a classical metric space

A classical metric satisfies the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z).$$

However, the function $\hat{\nu}_\lambda$ satisfies

$$\sinh(a + b) \neq \sinh(a) + \sinh(b),$$

which means that the standard triangle inequality does not hold. Hence, $\hat{\nu}_\lambda$ is not a classical metric space.

3) Not a b-metric space

A b-metric satisfies the inequality

$$d(x, z) \leq s(d(x, y) + d(y, z)).$$

However, for $\hat{\nu}_\lambda$, we have

$$\hat{\nu}_\lambda(x, z) \leq \sinh(s(\nu_\lambda(x, y) + \nu_\mu(y, z))),$$

which does not match the linear form of the b-metric inequality. Since sinh is non-linear, $\hat{\nu}_\lambda$ does not satisfy the b-metric condition.

- $\hat{\nu}_\lambda$ is an extended modular b-metric space, since it satisfies the extended modular triangle inequality.
- $\hat{\nu}_\lambda$ is not a classical metric space, as it violates the standard triangle inequality.
- $\hat{\nu}_\lambda$ is not a b-metric space, as it does not satisfy the linear b-metric triangle inequality.

Okeke et al. [34] further generalized these concepts by defining new types of α-$\hat{\nu}$-Meir-Keeler-type contractions in modular extended b-metric spaces, supported by examples that validated their results.

This paper builds on these works by presenting the concept of α-$\hat{\nu}$-A-B-C-Meir-Keeler-type nonlinear contractions, a new class of mappings within modular extended b-metric spaces. We establish common unique fixed-point theorems that generalize, unify, and extend several key results in modular b-metric and modular extended b-metric spaces. These theorems bridge the gap between classical and contemporary fixed-point theories, showcasing broader applicability in nonlinear analysis. An example is provided to support the findings.

The structure of this work is organized as follows: Initially, we review the fundamental definitions and results in Section 2. Subsequently, the core findings are introduced and examined in Section 3.

2. Preliminaries

Definition 1. *[30,41] Let f and g be mappings from a b-metric space (X, d) onto itself. The mappings f and g are called weakly commuting if $d(fgx, gfx) \leq d(fx, gx)$ for each $x \in X$.*

Definition 2. *[30] Let f and g be mappings from a b-metric space (X, d) onto itself. The mappings f and g are termed R-weakly commuting if a positive real number R exists such that $d(fgx, gfx) \leq Rd(fx, gx)$ for each $x \in X$.*

Definition 3. *[40] Let T be a self-mapping on X and $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$ be a function. Then T is called an α-admissible mapping if, for all $x, y \in X$,*

$$\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1.$$

Definition 4. *[28] Let $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$ be a mapping. Then the self-mapping $f : X \to X$ is said to be triangular α-admissible if*

(1) for all $x, y \in X$, $\alpha(x, y) \geq 1 \implies \alpha(fx, fy) \geq 1$;
(2) for all $x, y, z \in X$, $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1 \implies \alpha(x, y) \geq 1$.

Lemma 1. *[28] Let f be a triangular α-admissible mapping. Assume that $x_0 \in X$ exists such that $\alpha(x_0, fx_0) \geq 1$. Define a sequence $\{x_n\}_{n\geq1}$ as $x_n = f^n x_0$. Then $\alpha(x_m, x_n) \geq 1$ $\forall$ distinct $n, m \in \mathbb{N}$.*

Lemma 2. *[38] Let f be a triangular α-orbital admissible mapping. Assume that $x_1 \in X$ exists such that $\alpha(x_1, fx_1) \geq 1$. Define a sequence $\{x_n\}_{n\geq1}$ by $x_{n+1} = fx_n$. Then we have $\alpha(x_m, x_n) \geq 1$ $\forall$ distinct $n, m \in \mathbb{N}$.*

Definition 5. *[38] (a) Let $h : X \to X$ be a mapping and $\alpha : X \times X \to \mathbb{R}$ be a function. Then h is said to be α-orbital admissible if $\alpha(x, hx) \geq 1 \implies \alpha(hx, h^2x) \geq 1$.*
(b) Let $h : X \to X$ be a map and $\alpha : x \times X \to \mathbb{R}$ be a function. Then h is termed a triangular α-orbital admissible mapping if $\alpha(x, hx) \geq 1 \implies \alpha(hx, h^2x) \geq 1$, $\alpha(x, y) \geq 1$, and $\alpha(y, hy) \geq 1 \implies \alpha(x, hy) \geq 1$.

It is evident that every mapping that is α-admissible also qualifies as an α-orbital admissible mapping. Additionally, any triangular mapping that is α-admissible is inherently a triangular α-orbital admissible mapping as well. However, an instance of a triangular α-orbital admissible mapping exists that does not conform to the criteria of being triangular α-admissible. An example of this can be found in [38]. A b-metric space serves as a natural extension of both classical metric space by modifying the well-known triangle inequality to the form $d(x, z) \leq s(d(x, y) + d(y, z))$ for all points $x, y, z \in X$ and a fixed parameter $s \geq 1$. Recently, the concept of a b-metric space has been further extended to encompass p-metric spaces, as elaborated in [37]. However, the p above is not a partial metric space.

Definition 6. *[37] Let X be a non-empty set. A mapping $d : X \times X \to \mathbb{R}_+$ is called a p-metric if a strictly increasing continuous function $\Omega : [0, \infty) \to [0, \infty)$ exists with $t \leq \Omega(t)$ for all $t \in [0, \infty)$ such that, for all $x, y, z \in X$, the following conditions hold:*

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq \Omega(d(x, y) + d(y, z))$.

The pair (X, d) is called a p-metric space or an extended b-metric space.

Definition 7. *[14] Let X be a non-empty set. A mapping $\omega : (0, +\infty) \times X \times X \to \mathbb{R}_+ \cup \{\infty\}$ is called a metric modular on X if, for all $x, y, z \in X$ and $\lambda > 0$, the following conditions hold:*

(1) $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
(2) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$;
(3) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for al λ,$\mu > 0$.

The pair (X,ω) is called a modular metric space.

Definition 8. *[14] Let (X,ω) be a modular metric space. Fix $x_0 \in X$ and set*

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) = 0,\ as\ \lambda \to \infty\}$$

and

$$X^*_\omega = X^*_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) < \infty,\ for\ all\ \lambda > 0\}.$$

In this context, the sets X_ω and X^*_ω are referred to as modular spaces that are based on the point x_0.

Definition 9. *[19] Let X be a non-empty set and $s \geq 1$ be a real number. A mapping $\omega : (0,+\infty)\times X \times X \to \mathbb{R}_+ \cup \{\infty\}$ is called a modular b-metric on X if the following statements hold: for all $x,y,z \in X$:*

(1) $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
(2) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$;
(3) $\omega_{\lambda+\mu}(x,y) \leq s(\omega_\lambda(x,z) + \omega_\mu(z,y))$ for all $\lambda,\mu > 0$.

The pair (X,ω) is called a modular b-metric space.

In the paper, take $X_{\hat{v}} = X^*_{\hat{v}}(x_0) = \{x \in X : \hat{v}_\lambda(x,x_0) < \infty,\ for\ all\ \lambda > 0\}$.

Definition 10. *[21] Let X be a non-empty set. A mapping $\hat{v}_\lambda : (0,\infty)\times X \times X \to [0,\infty]$ is called a modular extended b-metric if a strictly increasing continuous function $\mathfrak{A} : [0,\infty) \to [0,\infty)$ with $\mathfrak{A}^{-1}(t) \leq t \leq \mathfrak{A}(t)$ for all $t \in [0,\infty)$ exists such that, for all $x,y,z \in X$, the following conditions hold:*

(1) $\hat{v}_\lambda(x,y) = 0$ if and only if $x = y$ for all $\lambda > 0$;
(2) $\hat{v}_\lambda(x,y) = \hat{v}_\lambda(y,x)$ for all $\lambda > 0$;
(3) $\hat{v}_{\lambda+\mu}(x,y) \leq \mathfrak{A}(\hat{v}_\lambda(x,z) + \hat{v}_\mu(z,y))$ for all $\lambda,\mu > 0$.

Then the pair $(X,\hat{v})$ is called a modular extended b-metric space.

The class of modular extended b-metric spaces is bigger than the class of b-metric spaces, since a b-metric space is modular extended b-metric space whenever $\mathfrak{A}(t) = st$ and a metric space if it is a modular extended b-metric space with $\mathfrak{A}(t) = t$.

Example 1. *[21] Consider X_ω to be a modular space equipped with a b-metric, where the coefficient satisfies $s \geq 1$ and $\hat{v}_\lambda(x,y) = \sinh(\omega_\lambda(x,y))$. Then $\hat{v}_\lambda$ is a modular extended b-metric space with $\mathfrak{A}(t) = \sinh(st)$ for all $t \geq 1$ and $\mathfrak{A}^{-1}(r) = \frac{1}{s}\sinh^{-1}(r)$ for all $r \geq 0$.*

Definition 11. *[21] Let $X_{\hat{v}}$ be a modular extended b-metric space. Then a sequence $\{x_n\}_{n\geq1} \subseteq X_{\hat{v}}$ is called*

(1) a $\hat{v}$-Cauchy sequence if, for all $\epsilon > 0$, $n(\epsilon) \in \mathbb{N}$ exists such that, for each $n,m \geq n(\epsilon)$ and $\lambda > 0$, $\hat{v}_\lambda(x_n,x_m) < \epsilon$;
(2) $\hat{v}$-convergent to $x^* \in X_{\hat{v}}$ if $\hat{v}_\lambda(x_n,x^*) \longrightarrow 0$ as $n \to \infty$ for all $\lambda > 0$;
(3) $\hat{v}$-complete if each $\hat{v}$-Cauchy sequence in $X_{\hat{v}}$ is $\hat{v}$-convergent and its limit is in $X_{\hat{v}}$.

Definition 12. *[34] Let $X_{\hat{v}}$ be a modular extended b-metric space, $T : X_{\hat{v}} \to X_{\hat{v}}$ be a mapping, and $x_0 \in X_{\hat{v}}$. T is said to be orbitally continuous at x_0 whenever $\lim_{k\to\infty} \hat{v}_\lambda(T^{n_k}u,x_0) = 0$ implies that $\lim_{k\to\infty} \hat{v}_\lambda(TT^{n_k}u,Tx_0) = 0$ whenever $u \in X_{\hat{v}}$ and $\{n_k\} \subseteq \mathbb{N}$ is a strictly increasing sequence of non-negative integer numbers.*

Definition 13. *Let $X_{\hat{v}}$ be a modular extended b-metric space, $T_1, T_2 : X_{\hat{v}} \to X_{\hat{v}}$ be mappings, and $x \in X_{\hat{v}}$. Then T_1, T_2 are called modular extended b-weakly commuting mappings in $X_{\hat{v}}$ if*

$$\hat{v}_\lambda(T_1T_2x, T_2T_1x) \leq \hat{v}_\lambda(T_2x, T_1x).$$

Example 2. *Consider the set $X = (\mathbb{R}_+ \setminus \{0\}) \cup \{\infty\}$ equipped with the modular extended b-metric defined by*

$$\hat{v}_\lambda(x, y) := \frac{1}{1+\lambda} \max_{x,y \in X_{\hat{v}}} \{\left\| x - y \right\|\},$$

which is complete on $X_{\hat{v}}$ for all $\lambda > 0$. Define the mappings $T_1, T_2 : (\mathbb{R}^+ \setminus \{0\}) \cup \{\infty\} \to (\mathbb{R}^+ \setminus \{0\}) \cup \{\infty\}$ as $T_1x = \log_{32} x^5$, $T_2x = \log_8 x^3$ for all $x \in (\mathbb{R}_+ \setminus \{0\}) \cup \{\infty\}$ and $\lambda > 0$, respectively. Then T_1 and T_2 are $\hat{v}$-weakly commuting mappings.

In fact, it suffices to show that T_1 and T_2 satisfy Definition 13. For all $\lambda > 0$ and $x \in (\mathbb{R}_+ \setminus \{0\})$, we show that

$$\hat{v}_\lambda(T_2T_4x, T_4T_2x) \leq \hat{v}_\lambda(T_4x, T_2x).$$

Using the above definition, we get,

$$\begin{aligned}
\frac{1}{1+\lambda} \max_{x \in X_{\hat{v}}} \{\|T_1T_2x - T_2T_1x\|\} &= \frac{1}{1+\lambda} \max_{x \in X_{\hat{v}}} \{\left\| \log_{32}(\log_8 x^3)^5 - \log_8(\log_{32} x^5)^3 \right\|\} \\
&= \frac{1}{1+\lambda} \max_{x \in X_{\hat{v}}} \{\left\| 5\log_{32}(\log_8 x^3) - 3\log_8(\log_{32} x^5) \right\|\} \\
&= \frac{1}{1+\lambda} \max_{x \in X_{\hat{v}}} \left\| \frac{\ln(\log_8 x^3)}{\ln(2)} - \frac{\ln(\log_{32} x^5)}{\ln(2)} \right\| \\
&= \frac{1}{\ln(2)} \frac{1}{1+\lambda} \max_{x \in X_{\hat{v}}} \left\| \ln(\log_8 x^3) - \ln(\log_{32} x^5) \right\| \\
&= \frac{1}{\ln(2)} \frac{1}{1+\lambda} \max_{x \in X_{\hat{v}}} \left\| \ln\left(\frac{\log_8 x^3}{\log_{32} x^5} \right) \right\| \\
&= \frac{1}{\ln(2)} \frac{1}{1+\lambda} \max_{x \in X_{\hat{v}}} \left\| \ln\left(\frac{3\log_8 x}{5\log_{32} x} \right) \right\| \\
&= 0.
\end{aligned}$$

Again, we have

$$\begin{aligned}
\hat{v}_\lambda(T_2x, T_1x) &= \frac{1}{1+\lambda} \max_{x \in X_{\hat{v}}} \{\|T_2x - T_1x\|\} \\
&= \frac{1}{1+\lambda} \max_{x \in X_{\hat{v}}} \{\left\| \log_8 x^3 - \log_{32} x^5 \right\|\} \\
&= \frac{1}{1+\lambda} \frac{1}{\ln(2)} \max_{x \in X_{\hat{v}}} \{\left\| \ln(x) - \ln(x) \right\|\} \\
&= 0.
\end{aligned}$$

Thus we have

$$\hat{v}_\lambda(T_1T_2x, T_2T_1x) \leq \hat{v}_\lambda(T_2x, T_1x),$$

which shows that T_1 and T_2 are weakly commuting.

We construct an example of the main result below related to Example 2 above.

Remark 1. *For all* $\lambda > 0$ *and* $x, y \in X_{\hat{\nu}}$, *we have*

$$\hat{\nu}_\lambda(T_1x, T_2y) \leq F_\lambda^A(T_1x, T_2y), \;\; \hat{\nu}_\lambda(T_1x, T_3y) \leq F_\lambda^A(T_1x, T_3y), \;\; \hat{\nu}_\lambda(T_2x, T_3y) \leq F_\lambda^A(T_2x, T_3y).$$

For the details of this remark, see [34, Remark 3.3].

The definition below is essential throughout this paper.

Definition 14. $T_i : X_{\hat{\nu}} \longrightarrow X_{\hat{\nu}}$ *be six continuous mappings. We say that* T_i *for* $i = 1, 2, \cdots, 6$ *follows an* α-$\hat{\nu}$-A-B-C *-Meir-Keeler-type contraction, and* $\{T_2, T_4\}$, $\{T_3, T_5\}$ *and* $\{T_1, T_6\}$ *are weakly commuting pairs of self-mappings such that* $T_3(X_{\hat{\nu}}) \subseteq T_4(X_{\hat{\nu}})$, $T_2(X_{\hat{\nu}}) \subseteq T_5(X_{\hat{\nu}})$, $T_1(X_{\hat{\nu}}) \subseteq T_6(X_{\hat{\nu}})$, *if a function* $\alpha : X_{\hat{\nu}} \times X_{\hat{\nu}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$ *and* $x_0 \in X_{\hat{\nu}}$ *exists such that* $\alpha(x_0, Tx_0) \geq 1$, *and, for each* $i = 1, \cdots, 6$, *the mapping* T_i *qualifies as triangular* α*-orbital admissible function for every* $\lambda, \epsilon, \delta > 0$ *satisfying the following conditions:*

$$\alpha(x, y)\hat{\nu}_\lambda(T_1x, T_2y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^A(T_1x, T_2y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{2.1}$$

$$\alpha(x, y)\hat{\nu}_\lambda(T_1x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^B(T_1x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{2.2}$$

$$\alpha(x, y)\hat{\nu}_\lambda(T_2x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^C(T_2x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{2.3}$$

where

$$F_\lambda^A(T_1x, T_2y) := a \max\{\hat{\nu}_\lambda(T_6x, T_4y), \hat{\nu}_\lambda(T_1x, T_6y), \hat{\nu}_\lambda(T_3x, T_4y), \hat{\nu}_\lambda(T_2x, T_5y)\}, \tag{2.4}$$

$$F_\lambda^B(T_1x, T_3y) := b \max\{\hat{\nu}_\lambda(T_5x, T_4y), \hat{\nu}_\lambda(T_2x, T_5y), \hat{\nu}_\lambda(T_1x, T_6^2x), \hat{\nu}_\lambda(T_3y, T_4^2y)\}, \tag{2.5}$$

$$F_\lambda^C(T_2x, T_3y) := c \max\{\hat{\nu}_\lambda(T_6x, T_2x), \hat{\nu}_\lambda(T_2x, T_5y), \hat{\nu}_\lambda(T_3x, T_4y), \hat{\nu}_\lambda(T_1x, T_6y)\}. \tag{2.6}$$

Remark 2. • *Following Definitions 10,* $\mathfrak{A}$ *is a strictly increasing continuous function* $\mathfrak{A} : [0, \infty) \to [0, \infty)$ *with* $\mathfrak{A}^{-1}(\epsilon) \leq \epsilon \leq \mathfrak{A}(\epsilon)$ *for all* $\epsilon \in [0, \infty)$.
• $F_\lambda^A(T_1x, T_2y)$, $F_\lambda^B(T_1x, T_3y)$, *and* $F_\lambda^C(T_2x, T_3y)$ *are functions containing some mixed-metric extended modular space at the* A^{th},B^{th}, *and* C^{th} *levels, respectively, for* $\lambda > 0$.
• *An example of Definition 14 will be given after the proof of Theorem 1 below.*

2.1. Relationships between Definition 14 and Definitions 1–12

Definition 14 introduces the concept of α-$\hat{\nu}$-A-B-C-Meir-Keeler-type contraction mappings in a modular extended b-metric space, where six self- mappings interact under certain admissibility and commutativity conditions. We analyze how it builds upon Definitions 1–12.

2.1.1. Foundations from Definitions 1–6 (basic metric and modular concepts)

Definitions 1 and 2 introduce weakly commuting mappings in b-metric spaces, which are crucial for Definition 14 because the contraction conditions require specific pairs of mappings $\{T_2, T_4\}, \{T_3, T_5\}, \{T_1, T_6\}$ to be weakly commuting.

- Definition 1 (weakly commuting mappings) states that two mappings f, g are weakly commuting if $d(fgx, gfx) \leq d(fx, gx)$. This ensures some level of interaction between the mappings, which is a necessary condition in Definition 14.

- Definition 2 (R-weakly commuting mappings) extends this concept by introducing a control parameter R, leading to $d(fgx, gfx) \leq Rd(fx, gx)$. This generalization helps establish the contraction conditions in Definition 14.
- Definition 3 (α-admissibility of a mapping) defines an auxiliary function $\alpha(x, y)$ that controls the behavior of a mapping T in relation to the fixed-point process. Definition 14 relies on this property, since the contraction conditions depend on *triangular* α-admissibility.
- Definition 4 (triangular α-admissible mappings) extends the concept of admissibility to triangular structures, ensuring that if $\alpha(x, z)$ and $\alpha(z, y)$ hold, then so does $\alpha(x, y)$. This property is crucial in Definition 14 for handling sequences within the modular extended b-metric space.
- Definitions 5 (orbital and triangular orbital admissibility) provides further generalizations that influence the structure of Definition 14. Since the mappings in Definition 14 must satisfy orbital admissibility, these preliminary definitions establish the conditions necessary for the contraction framework.

2.1.2. Modular extended b-metric space (Definitions 6–10)

The transition from metric and b-metric spaces to a modular extended b-metric space is key in Definition 14.

- Definition 6 (extended b metric space) introduces the concept of an extended b-distance function d, which satisfies $d(x, y) \leq \Omega(d(x, z) + d(z, y))$. This idea is extended in Definition 14 to work with six mappings instead of one, for a strictly increasing continuous function, $\Omega : [0, \infty) \to [0, \infty)$ with $t \leq \Omega(t)$ for every $t \in [0, \infty)$.
- Definition 7 (modular metric space) introduces the concept of a modular distance function ω, which satisfies $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$. This idea is extended in Definition 14 to work with six mappings instead of one.
- Definition 8 (modular spaces X_ω and X^*_ω) introduces two types of modular metric spaces based on growth conditions. These modular structures are embedded into Definition 14 to ensure the existence of fixed points in a well-defined modular space.
- Definition 9 (modular b-metric space) extends the modular metric concept to b-metric spaces, introducing a relaxation factor s similar to the classic b-metric condition. This generalization is needed in Definition 14 because the contraction inequalities involve a max operation, requiring a structure that supports modular-b-metric behavior.
- Definition 10 (modular extended b-metric space) is one of the most crucial precursors to Definition 14. It defines the function $\hat{\nu}_\lambda$ that satisfies: $\hat{\nu}_{\lambda+\mu}(x, y) \leq \mathfrak{A}(\hat{\nu}_\lambda(x, z) + \hat{\nu}_\mu(z, y))$, where $\mathfrak{A}$ is a strictly increasing function. Definition 14 applies this framework to analyze contractions under modular extended b-metric settings.

2.1.3. Properties of sequences in modular extended b-metric spaces (Definitions 11–12)

Definitions 11 and 12 introduce key sequence properties that Definition 14 relies upon to ensure convergence and the fixed points existence.

- Definition 11 ($\hat{\nu}$-Cauchy sequences and $\hat{\nu}$-convergence) formalizes when a sequence is Cauchy and convergent in modular extended b-metric spaces. In Definition 14, the iterative sequences

$$\xi_n = T_1 x_n = T_6 x_{n+1},$$

$$\xi_{n+1} = T_2 x_{n+1} = T_5 x_{n+2},$$
$$\xi_{n+2} = T_3 x_{n+2} = T_4 x_{n+3}, \tag{2.7}$$

must be $\hat{\nu}$-*Cauchy* to ensure the fixed points results hold.

- Definition 12 (orbital continuity in modular extended b-metric spaces) establishes the continuity conditions needed for taking the limits in Definition 14. Since the proof of Theorem 3 relies on passing those limits, these conditions ensure the stability of the mappings under iteration.

2.1.4. The role of Definition 14: Generalization and unification

Definition 14 unifies and extends all previous definitions by combining the following:

- Weakly commuting mappings (Definitions 1, 2, and 13).
- α-admissibility conditions (Definitions 3–5).
- Extended b-metric space and modular extended b-metric properties (Definitions 6–10).
- Convergence and continuity results (Definitions 11–12).

3. Main results

Now, we give the main results in this paper. We start with the following lemma.

Lemma 3. *Let $X_{\hat{\nu}}$ be a $\hat{\nu}$-regular $\hat{\nu}$-complete modular extended b-metric space and let $T_i : X_{\hat{\nu}} \longrightarrow X_{\hat{\nu}}$ be six orbitally continuous mappings satisfying the α-$\hat{\nu}$-A-B-C-Meir-Keeler-type contraction for $i = 1, 2, \cdots, 6$, $\{T_2, T_4\}$, $\{T_3, T_5\}$ and $\{T_1, T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\hat{\nu}}) \subseteq T_4(X_{\hat{\nu}})$, $T_2(X_{\hat{\nu}}) \subseteq T_5(X_{\hat{\nu}})$, $T_1(X_{\hat{\nu}}) \subseteq T_6(X_{\hat{\nu}})$. We then have a function $\alpha : X_{\hat{\nu}} \times X_{\hat{\nu}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$ and $x_0 \in X_{\hat{\nu}}$ such that $\alpha(x_0, x_1) \geq 1$, $\epsilon, \delta > 0$ and, for each $i = 1, \cdots, 6$, T_i remains a triangular α-orbital admissible mapping for every $\lambda > 0$, provided the following conditions are met:*

$$\alpha(x,y)\hat{\nu}_\lambda(T_1x, T_2y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^A_\lambda(T_1x, T_2y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.1}$$
$$\alpha(x,y)\hat{\nu}_\lambda(T_1x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^B_\lambda(T_1x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.2}$$
$$\alpha(x,y)\hat{\nu}_\lambda(T_2x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^C_\lambda(T_2x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.3}$$

where

$$F^A_\lambda(T_1x, T_2y) := a\max\{\hat{\nu}_\lambda(T_6x, T_4y), \hat{\nu}_\lambda(T_1x, T_6y), \hat{\nu}_\lambda(T_3x, T_4y), \hat{\nu}_\lambda(T_2x, T_5y)\}, \tag{3.4}$$
$$F^B_\lambda(T_1x, T_3y) := b\max\{\hat{\nu}_\lambda(T_5x, T_4y), \hat{\nu}_\lambda(T_2x, T_5y), \hat{\nu}_\lambda(T_1x, T_6^2x), \hat{\nu}_\lambda(T_3y, T_4^2y)\}, \tag{3.5}$$
$$F^C_\lambda(T_2x, T_3y) := c\max\{\hat{\nu}_\lambda(T_6x, T_2x), \hat{\nu}_\lambda(T_2x, T_5y), \hat{\nu}_\lambda(T_3x, T_4y), \hat{\nu}_\lambda(T_1x, T_6y)\}. \tag{3.6}$$

Suppose that $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ are sequences in $X_{\hat{\nu}}$ so that for x_n in $X_{\hat{\nu}}$, we choose x_{n+1} such that $\xi_n = T_1x_n = T_6x_{n+1}$; again for x_{n+1} in $X_{\hat{\nu}}$, we choose x_{n+2} such that $\xi_{n+1} = T_2x_{n+1} = T_5x_{n+2}$ and, for a point x_{n+2} in $X_{\hat{\nu}}$, we choose x_{n+3} such that $\xi_{n+2} = T_3x_{n+2} = T_4x_{n+3}$ for $n = 0, 1, 2, \cdots$. Then $\hat{\nu}_\lambda(\xi_n, \xi_m) = 0 \ \forall\ \lambda > 0$.

Proof. Suppose that $X_{\hat{\nu}}$ is empty. In that case, there is nothing to prove. We now assume that $X_{\hat{\nu}} \neq \emptyset$. Suppose that the mappings, T_i for $i = 1, \cdots, 6$ satisfy the inequalities in (3.1)–(3.6). Since x_0, x_1, and

x_2 are points in $X_{\hat{\nu}}$ and $T_1(X_{\hat{\nu}}) \subseteq T_6(X_{\hat{\nu}})$, we can find a point x_1 in $X_{\hat{\nu}}$ such that $\xi_0 = T_1x_0 = T_6x_1$. For $T_2(X_{\hat{\nu}}) \subseteq T_5(X_{\hat{\nu}})$, we can find a point $x_2 \in X_{\hat{\nu}}$ such that $\xi_1 = T_2x_1 = T_5x_2$ and for $T_3(X_{\hat{\nu}}) \subseteq T_4(X_{\hat{\nu}})$, we can find a point x_3 in $X_{\hat{\nu}}$ such that $\xi_2 = T_3x_2 = T_4x_3$. Now for all $\lambda > 0$, we induce on n, so that there are sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ within $X_{\hat{\nu}}$ that satisfy the following equations:

$$\begin{aligned} \xi_n &= T_1x_n = T_6x_{n+1}, \\ \xi_{n+1} &= T_2x_{n+1} = T_5x_{n+2}, \\ \xi_{n+2} &= T_3x_{n+2} = T_4x_{n+3}. \end{aligned} \tag{3.7}$$

If $n_0 \in \mathbb{N}$ exists such that $\xi_{n_0} = \xi_{n_0+1}$, $T_3(X_{\hat{\nu}}) \subseteq T_4(X_{\hat{\nu}})$, $T_2(X_{\hat{\nu}}) \subseteq T_5(X_{\hat{\nu}})$, $T_1(X_{\hat{\nu}}) \subseteq T_6(X_{\hat{\nu}})$ holds. In fact, if $m \in \mathbb{N}$ exists such that $\xi_{m+2} = \xi_{m+3}$, then $T_1u = T_6u$, where $u = x_{m+3}$. Therefore, the pair $\{T_1, T_6\}$ has a coincidence point $u \in X_{\hat{\nu}}$. If $\xi_m = \xi_{m+1}$, then $T_2u = T_4u$, where $u = x_{m+1}$. Therefore, the pair $\{T_2, T_4\}$ has a coincidence point $u \in X_{\hat{\nu}}$. If $\xi_{m+1} = \xi_{m+3}$, then $T_3u = T_5u$, where $u = x_{m+2}$. Thus, the pair $\{T_3, T_5\}$ has a coincidence point $u \in X_{\hat{\nu}}$. Again, if there is an $n_0 \in \mathbb{N}$ such that $\xi_{n_0} = \xi_{n_0+1} = \xi_{n_0+2}$, then $\xi_n = \xi_{n_0}$ for any $n \geq n_0$. This implies that $\{\xi_n\}$ is a modular $\hat{\nu}$ Cauchy sequence in $X_{\hat{\nu}}$. Actually, if $\eta \in \mathbb{N}$ exists such that (1) $\xi_\eta = \xi_{\eta+1} = \xi_{\eta+2}$, (2) $\xi_\eta \neq \xi_{\eta+1} = \xi_{\eta+2}$, (3) $\xi_\eta \neq \xi_{\eta+2} = \xi_{\eta+1}$, and (4) $\xi_\eta \neq \xi_{\eta+1} \neq \xi_{\eta+2}$ hold. In fact the Case (1) is easy, and Case (3) is similar to Case (2); then from inequality (3.1)–(3.6), we can set $x = \xi_{\eta+2}$ and $y = \xi_{\eta+3}$. Thus Case (2) stipulates that;

$$\alpha(\xi_{\eta+2}, \xi_{\eta+3})\hat{\nu}_\lambda(T_1\xi_{\eta+2}, T_2\xi_{\eta+3}) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^A_\lambda(T_1\xi_{\eta+2}, T_2\xi_{\eta+3})) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.8}$$

$$\alpha(\xi_{\eta+2}, \xi_{\eta+3})\hat{\nu}_\lambda(T_1\xi_{\eta+2}, T_3\xi_{\eta+3}) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^B_\lambda(T_1\xi_{\eta+2}, T_3\xi_{\eta+3})) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.9}$$

$$\alpha(\xi_{\eta+2}, \xi_{\eta+3})\hat{\nu}_\lambda(T_2\xi_{\eta+2}, T_3\xi_{\eta+3}) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^C_\lambda(T_2\xi_{\eta+2}, T_3\xi_{\eta+3})) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.10}$$

where

$$F^A_\lambda(T_1\xi_{\eta+2}, T_2\xi_{\eta+3}) := a\max\{\hat{\nu}_\lambda(T_6\xi_{\eta+2}, T_4\xi_{\eta+3}), \hat{\nu}_\lambda(T_1\xi_{\eta+2}, T_6\xi_{\eta+3}), \hat{\nu}_\lambda(T_3\xi_{\eta+2}, T_4\xi_{\eta+3}), \hat{\nu}_\lambda(T_2\xi_{\eta+2}, T_5\xi_{\eta+3})\}, \tag{3.11}$$

$$F^B_\lambda(T_1\xi_{\eta+2}, T_3\xi_{\eta+3}) := b\max\{\hat{\nu}_\lambda(T_5\xi_{\eta+2}, T_4\xi_{\eta+3}), \hat{\nu}_\lambda(T_2\xi_{\eta+2}, T_5\xi_{\eta+3}), \hat{\nu}_\lambda(T_1\xi_{\eta+2}, T_6^2\xi_{\eta+2}), \hat{\nu}_\lambda(T_3\xi_{\eta+3}, T_4^2\xi_{\eta+3})\}, \tag{3.12}$$

$$F^C_\lambda(T_2\xi_{\eta+2}, T_3\xi_{\eta+3}) := c\max\{\hat{\nu}_\lambda(T_6\xi_{\eta+2}, T_2\xi_{\eta+2}), \hat{\nu}_\lambda(T_2\xi_{\eta+2}, T_5\xi_{\eta+3}), \hat{\nu}_\lambda(T_3\xi_{\eta+2}, T_4\xi_{\eta+3}), \hat{\nu}_\lambda(T_1\xi_{\eta+2}, T_6\xi_{\eta+3})\}. \tag{3.13}$$

Using inequality (3.8), Eqs (3.11) and (3.7), we get

$$\hat{\nu}_\lambda(T_1\xi_{\eta+2}, T_2\xi_{\eta+3}) > 0, \ \ \hat{\nu}_\lambda(T_1\xi_{\eta+1}, T_3\xi_{\eta+2}) > 0, \ \ \hat{\nu}_\lambda(T_2\xi_\eta, T_3\xi_{\eta+1}) > 0$$

for all $\lambda > 0$. According to Lemma 2, given that T_i functions as triangular α-orbital admissible mapping, for each $i = 1, \cdots, 6$, we get $1 \leq \alpha(\eta, \eta+1)$ for all $\eta, \eta+1 \in \mathbb{N}$ with $\eta < \eta+1$.

Consider $\hat{\nu}_\lambda(T_1\xi_{\eta+2}, T_2\xi_{\eta+3}) > 0$ for all $\eta \in \mathbb{N} \cup \{0\}$. Since T_1 and T_2 are triangular α-orbital admissible mappings, it follows from $\hat{\nu}$-regularity that, for all $\lambda > 0$, $\hat{\nu}_\lambda(T_1\xi_{\eta+2}, T_2\xi_{\eta+3}) > 0$ and so, by Remark 1, we get

$$\mathfrak{A}^{-1}(F^A_\lambda(T_1\xi_{\eta+2}, T_2\xi_{\eta+3}) \geq \hat{\nu}_\lambda(T_1\xi_{\eta+2}, T_2\xi_{\eta+3}),$$

$$\mathfrak{A}^{-1}(F^B_\lambda(T_1\xi_{\eta+1}, T_3\xi_{\eta+2}) \geq \hat{\nu}_\lambda(T_1\xi_{\eta+1}, T_3\xi_{\eta+2}),$$

and

$$\mathfrak{A}^{-1}(F^{C}_{\lambda}(T_2\xi_{\eta}, T_3\xi_{\eta+1}) \geq \hat{v}_{\lambda}(T_2\xi_{\eta}, T_3\xi_{\eta+1}),$$

for all $\eta \in \mathbb{N} \cup \{0\}$. Therefore, we have

$$\begin{aligned}
\mathfrak{A}^{-1}(F^{A}_{\lambda}(T_1\xi_{\eta+2}, T_2\xi_{\eta+3})) =& a \max\{\hat{v}_{\lambda}(T_6\xi_{\eta+2}, T_4\xi_{\eta+3}), \hat{v}_{\lambda}(T_1\xi_{\eta+2}, T_6\xi_{\eta+3}),\\
& \hat{v}_{\lambda}(T_3\xi_{\eta+2}, T_4\xi_{\eta+3}), \hat{v}_{\lambda}(T_2\xi_{\eta+2}, T_5\xi_{\eta+3})\}\\
=& a \max\{\hat{v}_{\lambda}(T_1\xi_{\eta+1}, T_3\xi_{\eta+2}), \hat{v}_{\lambda}(T_1\xi_{\eta+2}, T_1\xi_{\eta+2}),\\
& \hat{v}_{\lambda}(T_3\xi_{\eta+2}, T_3\xi_{\eta+2}), \hat{v}_{\lambda}(T_2\xi_{\eta+2}, T_2\xi_{\eta+2})\}\\
=& a\hat{v}_{\lambda}(T_1\xi_{\eta+1}, T_3\xi_{\eta+2}). \qquad (3.14)
\end{aligned}$$

Again, from Eq (3.14) and for all $\lambda > 0$, according to Lemma 2, given that T_i functions as triangular α-orbital admissible mapping, for each $i = 1, \cdots, 6$, $1 \leq \alpha(\eta+1, \eta+2)$ for all $\eta+1, \eta+2 \in \mathbb{N}$ with $\eta+1 < \eta+2$, using inequality (3.2) and Eqs (3.5) and (3.7), we get

$$\begin{aligned}
\alpha(\xi_{\eta+1}, \xi_{\eta+2})\hat{v}_{\lambda}(T_1\xi_{\eta+1}, T_3\xi_{\eta+2}) =& \frac{b}{a} \max\{\hat{v}_{\lambda}(T_5\xi_{\eta+1}, T_4\xi_{\eta+2}), \hat{v}_{\lambda}(T_2\xi_{\eta+1}, T_5\xi_{\eta+2}),\\
& \hat{v}_{\lambda}(T_1\xi_{\eta+1}, T_6^2\xi_{\eta+1}), \hat{v}_{\lambda}(T_3\xi_{\eta+2}, T_4^2\xi_{\eta+2})\}\\
=& \frac{b}{a} \max\{\hat{v}_{\lambda}(T_2\xi_{\eta}, T_3\xi_{\eta+1}), \hat{v}_{\lambda}(T_2\xi_{\eta+1}, T_2\xi_{\eta+1}),\\
& \hat{v}_{\lambda}(T_1\xi_{\eta+1}, T_1\xi_{\eta+1}), \hat{v}_{\lambda}(T_3\xi_{\eta+2}, T_3\xi_{\eta+2})\}\\
=& \frac{b}{a}\hat{v}_{\lambda}(T_2\xi_{\eta}, T_3\xi_{\eta+1}). \qquad (3.15)
\end{aligned}$$

Finally, from Eq (3.15) and for all $\lambda > 0$, according to Lemma 2, given that T_i functions as triangular α-orbital admissible mapping, for each $i = 1, \cdots, 6$, $1 \leq \alpha(\eta, \eta+1)$ for all $\eta, \eta+1 \in \mathbb{N}$ with $\eta < \eta+1$, using inequality (3.3) and Eqs (3.6) and (3.7), we get

$$\begin{aligned}
\alpha(\xi_{\eta}, \xi_{\eta+1})\hat{v}_{\lambda}(T_2\xi_{\eta}, T_3\xi_{\eta+1}) =& \frac{ac}{b} \max\{\hat{v}_{\lambda}(T_6\xi_{\eta}, T_2\xi_{\eta}), \hat{v}_{\lambda}(T_2\xi_{\eta}, T_5\xi_{\eta+1}),\\
& \hat{v}_{\lambda}(T_3\xi_{\eta}, T_4\xi_{\eta+1}), \hat{v}_{\lambda}(T_1\xi_{\eta}, T_6\xi_{\eta+1})\}\\
=& \frac{ac}{b} \max\{\hat{v}_{\lambda}(T_1\xi_{\eta-1}, T_2\xi_{\eta}), \hat{v}_{\lambda}(T_2\xi_{\eta}, T_2\xi_{\eta}),\\
& \hat{v}_{\lambda}(T_3\xi_{\eta}, T_3\xi_{\eta}), \hat{v}_{\lambda}(T_1\xi_{\eta}, T_1\xi_{\eta})\}\\
=& \frac{ac}{b}\hat{v}_{\lambda}(T_1\xi_{\eta-1}, T_2\xi_{\eta})\\
=& \frac{ac}{b}\hat{v}_{\lambda}(\xi_{\eta-1}, \xi_{\eta}). \qquad (3.16)
\end{aligned}$$

So, it follows from (3.1)–(3.3) and (3.14)–(3.16) that, for all $\lambda > 0$,

$$\begin{aligned}
\alpha(\xi_{\eta+2}, \xi_{\eta+3})\hat{v}_{\lambda}(T_1\xi_{\eta+2}, T_2\xi_{\eta+3}) \leq& a\alpha(\xi_{\eta+1}, \xi_{\eta+2})\hat{v}_{\lambda}(T_1\xi_{\eta+1}, T_3\xi_{\eta+2})\\
\leq& \frac{b}{a}\alpha(\xi_{\eta}, \xi_{\eta+1})\hat{v}_{\lambda}(T_2\xi_{\eta}, T_3\xi_{\eta+1})\\
\leq& \frac{ac}{b}\alpha(\xi_{\eta}, \xi_{\eta+1})\hat{v}_{\lambda}(T_1\xi_{\eta-1}, T_2\xi_{\eta}), \qquad (3.17)
\end{aligned}$$

and hence, using Eq (3.7), we get

$$\begin{aligned}\alpha(\xi_{\eta+2},\xi_{\eta+3})\hat{v}_\lambda(\xi_{\eta+2},\xi_{\eta+3}) &\leq a\alpha(\xi_{\eta+1},\xi_{\eta+2})\hat{v}_\lambda(\xi_{\eta+1},\xi_{\eta+2})\\ &\leq \frac{b}{a}\alpha(\xi_\eta,\xi_{\eta+1})\hat{v}_\lambda(\xi_\eta,\xi_{\eta+1})\\ &\leq \frac{ac}{b}\alpha(\xi_\eta,\xi_{\eta+1})\hat{v}_\lambda(\xi_{\eta-1},\xi_\eta)\\ &= h\alpha(\xi_\eta,\xi_{\eta+1})\hat{v}_\lambda(\xi_{\eta-1},\xi_\eta). \end{aligned} \tag{3.18}$$

Therefore

$$\begin{aligned}\alpha(\xi_{\eta+2},\xi_{\eta+3})\hat{v}_\lambda(\xi_{\eta+2},\xi_{\eta+3}) &\leq a\alpha(\xi_{\eta+1},\xi_{\eta+2})\hat{v}_\lambda(\xi_{\eta+1},\xi_{\eta+2})\\ &\leq \frac{b}{a}\alpha(\xi_\eta,\xi_{\eta+1})\hat{v}_\lambda(\xi_\eta,\xi_{\eta+1})\\ &\leq \frac{ac}{b}\alpha(\xi_\eta,\xi_{\eta+1})\hat{v}_\lambda(\xi_{\eta-1},\xi_\eta)\\ &\vdots\\ &\leq h^{n+1}\alpha(\xi_0,\xi_1)\hat{v}_\lambda(\xi_{\eta-1},\xi_\eta)\\ &= h^{n+1}\alpha(\xi_0,\xi_1)\hat{v}_\lambda(\xi_0,\xi_1). \end{aligned} \tag{3.19}$$

Therefore, after some algebra and using condition (3) of Definition 10 we get

$$\hat{v}_\lambda(x_n,x_m) = 0 \ \forall\ \lambda > 0,\ n \geq m. \tag{3.20}$$

This is a modular extended $\hat{v}$-Cauchy sequence in $\hat{v}$-complete modular extended b-metric space.

Again, for $\xi_\eta \neq \xi_{\eta+1} \neq \xi_{\eta+2}$, take $x = \xi_\eta$ and $y = \xi_{\eta+1}$, from Case (4). We then have

$$\alpha(\xi_\eta,\xi_{\eta+1})\hat{v}_\lambda(T_1\xi_\eta,T_2\xi_{\eta+1}) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^A_\lambda(T_1\xi_\eta,T_2\xi_{\eta+1})) < \mathfrak{A}(\epsilon)+\mathfrak{A}(\delta(\epsilon)), \tag{3.21}$$

$$\alpha(\xi_\eta,\xi_{\eta+1})\hat{v}_\lambda(T_1\xi_\eta,T_3\xi_{\eta+1}) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^B_\lambda(T_1\xi_\eta,T_3\xi_{\eta+1})) < \mathfrak{A}(\epsilon)+\mathfrak{A}(\delta(\epsilon)), \tag{3.22}$$

$$\alpha(\xi_\eta,\xi_{\eta+1})\hat{v}_\lambda(T_2\xi_\eta,T_3\xi_{\eta+1}) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^C_\lambda(T_2\xi_\eta,T_3\xi_{\eta+1})) < \mathfrak{A}(\epsilon)+\mathfrak{A}(\delta(\epsilon)), \tag{3.23}$$

where

$$F^A_\lambda(T_1\xi_\eta,T_2\xi_{\eta+1}) := a\max\{\hat{v}_\lambda(T_6\xi_\eta,T_4\xi_{\eta+1}),\hat{v}_\lambda(T_1\xi_\eta,T_6\xi_{\eta+1}),\hat{v}_\lambda(T_3\xi_\eta,T_4\xi_{\eta+1}),\hat{v}_\lambda(T_2\xi_\eta,T_5\xi_{\eta+1})\}, \tag{3.24}$$

$$F^B_\lambda(T_1\xi_\eta,T_3\xi_{\eta+1}) := b\max\{\hat{v}_\lambda(T_5\xi_\eta,T_4\xi_{\eta+1}),\hat{v}_\lambda(T_2\xi_\eta,T_5\xi_{\eta+1}),\hat{v}_\lambda(T_1\xi_\eta,T_6^2\xi_\eta),\hat{v}_\lambda(T_3\xi_{\eta+1},T_4^2\xi_{\eta+1})\}, \tag{3.25}$$

$$F^C_\lambda(T_2\xi_\eta,T_3\xi_{\eta+1}) := c\max\{\hat{v}_\lambda(T_6\xi_\eta,T_2\xi_\eta),\hat{v}_\lambda(T_2\xi_\eta,T_5\xi_{\eta+1}),\hat{v}_\lambda(T_3\xi_\eta,T_4\xi_{\eta+1}),\hat{v}_\lambda(T_1\xi_\eta,T_6\xi_{\eta+1})\}. \tag{3.26}$$

Using inequality (3.21), Eqs (3.24) and (3.7), we get

$$\hat{v}_\lambda(T_1\xi_\eta,T_2\xi_{\eta+1}) > 0,\ \ \hat{v}_\lambda(T_1\xi_{\eta-1},T_3\xi_\eta) > 0,\ \ \hat{v}_\lambda(T_2\xi_{\eta-2},T_3\xi_{\eta-1}) > 0$$

for all $\lambda > 0$. According to Lemma 2, given that T_i functions as triangular α-orbital admissible mapping, for each $i = 1,\cdots,6$, we get $1 \leq \alpha(\eta,\eta+1)$ for all $\eta,\eta+1 \in \mathbb{N}$ with $\eta \neq \eta+1$.

Consider $\hat{v}_\lambda(T_1\xi_\eta, T_2\xi_{\eta+1}) > 0$ for all $\eta \in \mathbb{N}\cup\{0\}$. Since T_1 and T_2 are triangular α-orbital admissible mappings, it follows from $\hat{v}$-regularity that, for all $\lambda > 0$, $\hat{v}_\lambda(T_1\xi_\eta, T_2\xi_{\eta+1}) > 0$ and so, by Remark 1, we get

$$\mathfrak{A}^{-1}(F^A_\lambda(T_1\xi_\eta, T_2\xi_{\eta+1}) \geq \hat{v}_\lambda(T_1\xi_\eta, T_2\xi_{\eta+1}),$$
$$\mathfrak{A}^{-1}(F^B_\lambda(T_1\xi_{\eta-1}, T_3\xi_\eta) \geq \hat{v}_\lambda(T_1\xi_{\eta-1}, T_3\xi_\eta),$$

and

$$\mathfrak{A}^{-1}(F^C_\lambda(T_2\xi_{\eta-2}, T_3\xi_{\eta-1}) \geq \hat{v}_\lambda(T_2\xi_{\eta-2}, T_3\xi_{\eta-1}),$$

for all $\eta \in \mathbb{N}\cup\{0\}$. Therefore, we have

$$\begin{aligned}\mathfrak{A}^{-1}(F^A_\lambda(T_1\xi_\eta, T_2\xi_{\eta+1})) =& a\max\{\hat{v}_\lambda(T_6\xi_\eta, T_4\xi_{\eta+1}), \hat{v}_\lambda(T_1\xi_\eta, T_6\xi_{\eta+1}),\\ &\hat{v}_\lambda(T_3\xi_\eta, T_4\xi_{\eta+1}), \hat{v}_\lambda(T_2\xi_\eta, T_5\xi_{\eta+1})\}\\ =& a\max\{\hat{v}_\lambda(T_1\xi_{\eta-1}, T_3\xi_\eta), \hat{v}_\lambda(T_1\xi_\eta, T_1\xi_\eta),\\ &\hat{v}_\lambda(T_3\xi_\eta, T_3\xi_\eta), \hat{v}_\lambda(T_2\xi_\eta, T_2\xi_\eta)\}\\ =& a\hat{v}_\lambda(T_1\xi_{\eta-1}, T_3\xi_\eta). \end{aligned} \tag{3.27}$$

Again, from Eq (3.27) and for all $\lambda > 0$, and Lemma 2, given that T_i functions as triangular α-orbital admissible mapping, for each $i = 1, \cdots, 6$, $1 \leq \alpha(\eta-1, \eta)$ for all $\eta-1, \eta \in \mathbb{N}$ with $\eta - 1 \neq \eta$, using inequality (3.2) and Eqs (3.5) and (3.7), we get

$$\begin{aligned}\alpha(\xi_{\eta-1}, \xi_\eta)\hat{v}_\lambda(T_1\xi_{\eta-1}, T_3\xi_\eta) =& \frac{b}{a}\max\{\hat{v}_\lambda(T_5\xi_{\eta-1}, T_4\xi_\eta), \hat{v}_\lambda(T_2\xi_{\eta-1}, T_5\xi_\eta),\\ &\hat{v}_\lambda(T_1\xi_{\eta-1}, T_6^2\xi_\eta), \hat{v}_\lambda(T_3\xi_\eta, T_4^2\xi_\eta)\}\\ =& \frac{b}{a}\max\{\hat{v}_\lambda(T_2\xi_{\eta-2}, T_3\xi_{\eta-1}), \hat{v}_\lambda(T_2\xi_{\eta-1}, T_2\xi_{\eta-1}),\\ &\hat{v}_\lambda(T_1\xi_{\eta-1}, T_1\xi_{\eta-1}), \hat{v}_\lambda(T_3\xi_\eta, T_3\xi_\eta)\}\\ =& \frac{b}{a}\hat{v}_\lambda(T_2\xi_{\eta-2}, T_3\xi_{\eta-1}). \end{aligned} \tag{3.28}$$

Finally, from Eq (3.28) and for all $\lambda > 0$, and Lemma 2, given that T_i functions as triangular α-orbital admissible mapping, for each $i = 1, \cdots, 6$, $1 \leq \alpha(\eta-1, \eta-2)$ for all $\eta-1, \eta-2 \in \mathbb{N}$ with $\eta-1 \neq \eta-2$, using inequality (3.3) and Eqs (3.6) and (3.7), we get

$$\begin{aligned}\alpha(\xi_{\eta-2}, \xi_{\eta-1})\hat{v}_\lambda(T_2\xi_{\eta-2}, T_3\xi_{\eta-1}) =& \frac{ac}{b}\max\{\hat{v}_\lambda(T_6\xi_{\eta-2}, T_2\xi_{\eta-2}), \hat{v}_\lambda(T_2\xi_{\eta-2}, T_5\xi_{\eta-1}),\\ &\hat{v}_\lambda(T_3\xi_{\eta-2}, T_4\xi_{\eta-1}), \hat{v}_\lambda(T_1\xi_{\eta-2}, T_6\xi_{\eta-1})\}\\ =& \frac{ac}{b}\max\{\hat{v}_\lambda(T_1\xi_{\eta-3}, T_2\xi_{\eta-2}), \hat{v}_\lambda(T_2\xi_{\eta-2}, T_2\xi_{\eta-2}),\\ &\hat{v}_\lambda(T_3\xi_{\eta-2}, T_3\xi_{\eta-2}), \hat{v}_\lambda(T_1\xi_{\eta-2}, T_1\xi_{\eta-2})\}\\ =& \frac{ac}{b}\hat{v}_\lambda(T_1\xi_{\eta-3}, T_2\xi_{\eta-2})\\ =& \frac{ac}{b}\hat{v}_\lambda(\xi_{\eta-3}, \xi_{\eta-2}). \end{aligned} \tag{3.29}$$

So, it follows from (3.1)–(3.3) and (3.27)–(3.29) that, for all $\lambda > 0$,

$$\alpha(\xi_\eta, \xi_{\eta+1})\hat{v}_\lambda(T_1\xi_\eta, T_2\xi_{\eta+1}) \leq a\alpha(\xi_{\eta-1}, \xi_\eta)\hat{v}_\lambda(T_1\xi_{\eta-1}, T_3\xi_\eta)$$

$$\begin{aligned}&\leq \frac{b}{a}\alpha(\xi_{\eta-2},\xi_{\eta-1})\hat{v}_{\lambda}(T_2\xi_{\eta-2},T_3\xi_{\eta-1})\\&\leq \frac{ac}{b}\alpha(\xi_{\eta-3},\xi_{\eta-2})\hat{v}_{\lambda}(T_1\xi_{\eta-3},T_2\xi_{\eta-2}),\end{aligned} \tag{3.30}$$

Hence, using Eq (3.7), we get

$$\begin{aligned}\alpha(\xi_{\eta},\xi_{\eta+1})\hat{v}_{\lambda}(\xi_{\eta},\xi_{\eta+1}) &\leq a\alpha(\xi_{\eta-1},\xi_{\eta})\hat{v}_{\lambda}(\xi_{\eta-1},\xi_{\eta})\\&\leq \frac{b}{a}\alpha(\xi_{\eta-2},\xi_{\eta-1})\hat{v}_{\lambda}(\xi_{\eta-2},\xi_{\eta-1})\\&\leq \frac{ac}{b}\alpha(\xi_{\eta-3},\xi_{\eta-2})\hat{v}_{\lambda}(\xi_{\eta-3},\xi_{\eta-2})\\&= h\alpha(\xi_{\eta-3},\xi_{\eta-2})\hat{v}_{\lambda}(\xi_{\eta-3},\xi_{\eta-2}).\end{aligned} \tag{3.31}$$

Therefore

$$\begin{aligned}\alpha(\xi_{\eta},\xi_{\eta+1})\hat{v}_{\lambda}(\xi_{\eta},\xi_{\eta+1}) &\leq a\alpha(\xi_{\eta-1},\xi_{\eta})\hat{v}_{\lambda}(\xi_{\eta-1},\xi_{\eta})\\&\leq \frac{b}{a}\alpha(\xi_{\eta-2},\xi_{\eta-1})\hat{v}_{\lambda}(\xi_{\eta-2},\xi_{\eta-1})\\&\leq \frac{ac}{b}\alpha(\xi_{\eta-3},\xi_{\eta-2})\hat{v}_{\lambda}(\xi_{\eta-3},\xi_{\eta-2})\\&= h\alpha(\xi_{\eta-3},\xi_{\eta-2})\hat{v}_{\lambda}(\xi_{\eta-3},\xi_{\eta-2})\\&\vdots\\&\leq h^{n+1}\alpha(\xi_0,\xi_1)\hat{v}_{\lambda}(\xi_0,\xi_1).\end{aligned} \tag{3.32}$$

Therefore, after some algebra and condition (3) of Definition 10, we get the result. Hence,

$$\hat{v}_{\lambda}(\xi_n,\xi_m) = 0 \;\forall\; \lambda > 0,\; n \geq m. \tag{3.33}$$

□

Remark 3. *Suppose that u is the common fixed point of T_i for $i = 1,2,\cdots,6$ when either T_2 or T_4 is $\hat{v}$-continuous and the pair $\{T_2,T_4\}$ is weakly commuting. Again suppose that u is true for T_i, $i = 1,2,\cdots,6$ when T_1 is $\hat{v}$-continuous, it is also true when T_2 or T_6 is $\hat{v}$ continuous and the pair $\{T_1,T_6\}$ is weakly commuting. Furthermore, T_3 or T_5 is $\hat{v}$-continuous and the pair $\{T_3,T_5\}$ is weakly commuting.*

Theorem 1. *Suppose that Lemma 3 holds. Then T_i has a fixed point in $X_{\hat{v}}$. Moreover, if $\alpha(x^*,y^*) \geq 1$ for all $x^*,y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has a unique common fixed point in $\bigcap_{i=1} Fix(T_i)$ for each $i = 1,\cdots,6$.*

Proof. Suppose that $X_{\hat{v}}$ is empty, in which case, there is nothing to prove. We now assume that $X_{\hat{v}} \neq \emptyset$. Then a function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0,+\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$ and $x_0 \in X_{\hat{v}}$ exists such that $\alpha(x_0,x_1) \geq 1$, $\epsilon,\delta > 0$, and, for each $i = 1,\cdots,6$, T_i remain triangular α-orbital admissible mappings for every $\lambda > 0$. Therefore the mappings, T_i for $i = 1,\cdots,6$ satisfy the inequalities (3.1)–(3.6). Since x_0,x_1 and x_2 are points in $X_{\hat{v}}$ and $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$, we can find a point x_1 in $X_{\hat{v}}$ such that $\xi_0 = T_1x_0 = T_6x_1$. For $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, we can find a point $x_2 \in X_{\hat{v}}$ such that $\xi_1 = T_2x_1 = T_5x_2$ and for $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$; we can find a point x_3 in $X_{\hat{v}}$ such that $\xi_2 = T_3x_2 = T_4x_3$. Now for all $\lambda > 0$, we induce on n, so that there are sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ within $X_{\hat{v}}$ that satisfy the subsequent Eq (3.7).

If $n_0 \in \mathbb{N}$ exists such that $\xi_{n_0} = \xi_{n_0+1}$, $T_3(X_{\hat{\nu}}) \subseteq T_4(X_{\hat{\nu}})$, $T_2(X_{\hat{\nu}}) \subseteq T_5(X_{\hat{\nu}})$, $T_1(X_{\hat{\nu}}) \subseteq T_6(X_{\hat{\nu}})$ hold. In fact, if $m \in \mathbb{N}$ exists such that $\xi_{m+2} = \xi_{m+3}$, then $T_1u = T_6u$, where $u = x_{m+3}$. Therefore, the pair $\{T_1, T_6\}$ has a coincidence point $u \in X_{\hat{\nu}}$. If $\xi_m = \xi_{m+1}$, then $T_2u = T_4u$, where $u = x_{m+1}$. Therefore, the pair $\{T_2, T_4\}$ has a coincidence point $u \in X_{\hat{\nu}}$. If $\xi_{m+1} = \xi_{m+3}$, then $T_3u = T_5u$, where $u = x_{m+2}$. Thus, the pair $\{T_3, T_5\}$ has a coincidence point $u \in X_{\hat{\nu}}$. Again, if there is an $n_0 \in \mathbb{N}$ such that $\xi_{n_0} = \xi_{n_0+1} = \xi_{n_0+2}$, then $\xi_n = \xi_{n_0}$ for any $n \geq n_0$. This implies that $\{\xi_n\}$ is a modular $\hat{\nu}$ Cauchy sequence in $X_{\hat{\nu}}$. Actually, if $\eta \in \mathbb{N}$ exists such that (1) $\xi_\eta = \xi_{\eta+1} = \xi_{\eta+2}$, (2) $\xi_\eta \neq \xi_{\eta+1} = \xi_{\eta+2}$, (3) $\xi_\eta \neq \xi_{\eta+2} = \xi_{\eta+1}$, and (4) $\xi_\eta \neq \xi_{\eta+1} \neq \xi_{\eta+2}$ hold. In fact Case (1) is easy, and Case (3) is similar to Case (2); then from inequality (3.1)–(3.6), we get the result by setting $x = \xi_{\eta+2}$ and $y = \xi_{\eta+3}$. By Lemma 3, we conclude that

$$\hat{\nu}_\lambda(\xi_n, \xi_m) = 0 \ \forall \ \lambda > 0, \ n \geq m. \tag{3.34}$$

Consequently, Eq (3.34) indicates that $\{\xi_n\}_{n\in\mathbb{N}}$ forms a modular extended $\hat{\nu}$-Cauchy sequence within a $\hat{\nu}$-complete modular extended b-metric space. Therefore, a point $u \in X_{\hat{\nu}}$ exists such that ξ_n converges to u as n approaches infinity. Furthermore, since the sequences $\{T_1x_n\} = \{T_6x_{n+1}\}$, $\{T_2x_{n+1}\} = \{T_5x_{n+2}\}$, and $\{T_3x_{n-1}\} = \{T_4x_n\}$ for all $n \in \mathbb{N}$ are all subsequences of $\{\xi_n\}$, it follows that all subsequences of a convergent sequence converge to the same limit. Thus, we conclude that

$$\lim_{n\to\infty} T_1x_n = \lim_{n\to\infty} T_6x_{n+1} = \lim_{n\to\infty} \xi_n = u,$$

$$\lim_{n\to\infty} T_2x_{n+1} = \lim_{n\to\infty} T_5x_{n+2} = \lim_{n\to\infty} \xi_{n+1} = u,$$

$$\lim_{n\to\infty} T_3x_{n-1} = \lim_{n\to\infty} T_4x_n = \lim_{n\to\infty} \xi_{n+1} = u.$$

Since $\{T_2, T_4\}$ is weakly commuting mappings, we have, for all $\lambda > 0$,

$$\hat{\nu}_\lambda(T_2T_4x_{n+1}, T_4T_2x_{n+1}) \leq \hat{\nu}_\lambda(T_4x_{n+1}, T_2x_{n+1}). \tag{3.35}$$

Taking the limit of inequality (3.35) as $n \to \infty$ and noticing that T_2 or T_4 are $\hat{\nu}$-continuous mappings, we get

$$\hat{\nu}_\lambda(T_2T_4x_{n+1}, T_4T_2x_{n+1}) \leq \hat{\nu}_\lambda(T_4x_{n+1}, T_2x_{n+1}) \longrightarrow 0. \tag{3.36}$$

We know that T_4 is $\hat{\nu}$ continuous, then $T_4^2x_{n+1} \to T_4u$ as $n \to \infty$, $T_4T_2x_{n+1} \to T_4u$ as $n \to \infty$. But we can clearly see from inequality (3.35) that $T_2T_4x_{n+1} \to T_4u$ as $n \to \infty$. Since T_2, T_4 are weakly commuting mappings, for each $i = 1, \ldots, 6$, where T_i represents a triangular α-orbital admissible mapping. and $1 \leq \alpha(x_m, x_n)$ for all $n, m \in \mathbb{N}$ with $n < m$, it follows that, for all $\lambda > 0$,

$$\alpha(x_n, x_{n+1})\hat{\nu}_\lambda(T_2T_4x_{n+1}, T_4T_2x_{n+1}) \leq \alpha(x_n, x_{n+1})\hat{\nu}_\lambda(T_4x_{n+1}, T_2x_{n+1}),$$

and hence

$$\lim_{n\to\infty} \alpha(x_n, x_{n+1})\hat{\nu}_\lambda(T_2T_4x_{n+1}, T_4T_2x_{n+1}) \leq \lim_{n\to\infty} \alpha(x_n, x_{n+1})\hat{\nu}_\lambda(T_4x_{n+1}, T_2x_{n+1}).$$

Since T_2, T_4 are weakly commuting mappings and orbitally continuous, we have

$$\alpha(x_n, x_{n+1})\hat{\nu}_\lambda(\lim_{n\to\infty} T_2T_4x_{n+1}, \lim_{n\to\infty} T_4T_2x_{n+1})$$

$$\leq \alpha(x_n, x_{n+1})\hat{v}_\lambda(\lim_{n\to\infty} T_4 x_{n+1}, \lim_{n\to\infty} T_2 x_{n+1}),$$

such that

$$\alpha(x_n, x_{n+1})\hat{v}_\lambda(T_2 \lim_{n\to\infty} T_4 x_{n+1}, T_4 \lim_{n\to\infty} T_2 x_{n+1})$$
$$\leq \alpha(x_n, x_{n+1})\hat{v}_\lambda(\lim_{n\to\infty} T_4 x_{n+1}, \lim_{n\to\infty} T_2 x_{n+1}).$$

Therefore, we have

$$\hat{v}_\lambda(T_2 u, T_4 u) \leq \hat{v}_\lambda(u, u) = 0,$$

which implies that $T_2 u = T_4 u$ for all $\lambda > 0$.

Additionally, given that T_3 and T_5 are weakly commuting mappings, it follows that for every $\lambda > 0$,

$$\hat{v}_\lambda(T_3 T_5 x_{n+2}, T_5 T_3 x_{n+2}) \leq \hat{v}_\lambda(T_5 x_{n+2}, T_3 x_{n+2}). \tag{3.37}$$

Taking the limit of inequality (3.35) as $n \to \infty$ and noticing that T_3 or T_4 is a $\hat{v}$-continuous mapping, then we get

$$\hat{v}_\lambda(T_3 T_5 x_{n+2}, T_5 T_3 x_{n+2}) \leq \hat{v}_\lambda(T_5 x_{n+2}, T_3 x_{n+2}) \longrightarrow 0. \tag{3.38}$$

Since T_5 is $\hat{v}$ continuous, $T_5^2 x_{n+2} \to T_5 u$ as $n \to \infty$, and $T_5 T_3 x_{n+2} \to T_5 u$ as $n \to \infty$. But we can clearly see from inequality (3.37) that $T_3 T_5 x_{n+2} \to T_5 u$ as $n \to \infty$.

Now we know that $T_3,\ T_5$ are weakly commuting mappings and orbitally continuous, T_i remains a triangular α-orbital admissible mapping for each $i = 1, \cdots, 6$, and $1 \leq \alpha(x_m, x_n)$ for all $n, m \in \mathbb{N}$ with $n < m$, it follows that, for all $\lambda > 0$

$$\alpha(x_n, x_{n+2})\hat{v}_\lambda(\lim_{n\to\infty} T_3 T_5 x_{n+2}, \lim_{n\to\infty} T_5 T_3 x_{n+2})$$
$$\leq \alpha(x_n, x_{n+2})\hat{v}_\lambda(\lim_{n\to\infty} T_5 x_{n+2}, \lim_{n\to\infty} T_3 x_{n+2}),$$

thus,

$$\alpha(x_n, x_{n+2})\hat{v}_\lambda(T_3 \lim_{n\to\infty} T_5 x_{n+2}, T_5 \lim_{n\to\infty} T_3 x_{n+2})$$
$$\leq \alpha(x_n, x_{n+2})\hat{v}_\lambda(\lim_{n\to\infty} T_5 x_{n+2}, \lim_{n\to\infty} T_3 x_{n+2}).$$

Therefore, we have

$$\hat{v}_\lambda(T_3 u, T_5 u) \leq \hat{v}_\lambda(u, u) = 0,$$

which implies that $T_3 u = T_5 u$ for all $\lambda > 0$.

Lastly, since $T_1,\ T_6$ are weakly commuting mappings and orbitally continuous, T_i remains a triangular α-orbital admissible mapping for each $i = 1, \cdots, 6$, and $1 \leq \alpha(x_m, x_n)$ for all $n, m \in \mathbb{N}$ with $n < m$, it follow that, for all $\lambda > 0$

$$\alpha(x_n, x_{n+1})\hat{v}_\lambda(\lim_{n\to\infty} \sup T_1 T_6 x_n, \lim_{n\to\infty} T_6 T_2 x_n)$$
$$\leq \alpha(x_n, x_{n+1})\hat{v}_\lambda(\lim_{n\to\infty} \sup T_6 x_n, \lim_{n\to\infty} T_1 x_n),$$

and so

$$\alpha(x_n, x_{n+1})\hat{v}_\lambda(T_1 \lim_{n\to\infty} T_6x_n, T_6 \lim_{n\to\infty} T_1x_n)$$
$$\leq \alpha(x_n, x_{n+1})\hat{v}_\lambda(\lim_{n\to\infty} T_6x_n, \lim_{n\to\infty} T_1x_n).$$

Therefore, we have

$$\hat{v}_\lambda(T_1u, T_6u) \leq \hat{v}_\lambda(u, u) = 0,$$

which implies that $T_1u = T_6u$ for all $\lambda > 0$.

Now, we claim that, for all $u \in X_{\hat{v}}$, $T_1u = T_2u = T_3u$. If we suppose the contrary, then $T_1u \neq T_2u \neq T_3u$. So, for all $\lambda > 0$, the following cases emerge:

Case 1a. $T_1u \neq T_2u \implies \alpha(u, u)\hat{v}_\lambda(T_1u, T_2u) > 0$.

Case 1b. $T_1u \neq T_3u \implies \alpha(u, u)\hat{v}_\lambda(T_1u, T_3u) > 0$.

Case 1c. $T_2u \neq T_3u \implies \alpha(u, u)\hat{v}_\lambda(T_2u, T_3u) > 0$.

Indeed, since T_i remains a triangular α-orbital admissible mapping for each $i = 1, \cdots, 6$, and $\alpha(u, u) \neq 0$, it follows that, for all $\lambda > 0$

$$\alpha(x, y)\hat{v}_\lambda(T_1x, T_2y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^A_\lambda(T_1x, T_2y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)),$$
$$\alpha(x, y)\hat{v}_\lambda(T_1x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^B_\lambda(T_1x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)),$$
$$\alpha(x, y)\hat{v}_\lambda(T_2x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^C_\lambda(T_2x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)),$$

where

$$F^A_\lambda(T_1x, T_2y) := a \max\{\hat{v}_\lambda(T_6x, T_4y), \hat{v}_\lambda(T_1x, T_6y), \hat{v}_\lambda(T_3x, T_4y), \hat{v}_\lambda(T_2x, T_5y)\},$$
$$F^B_\lambda(T_1x, T_3y) := b \max\{\hat{v}_\lambda(T_5x, T_4y), \hat{v}_\lambda(T_2x, T_5y), \hat{v}_\lambda(T_1x, T_6^2x), \hat{v}_\lambda(T_3y, T_4^2y)\},$$
$$F^C_\lambda(T_2x, T_3y) := c \max\{\hat{v}_\lambda(T_6x, T_2x), \hat{v}_\lambda(T_2x, T_5y), \hat{v}_\lambda(T_3x, T_4y), \hat{v}_\lambda(T_1x, T_6y)\}.$$

Now, we consider Case 1a. Since $T_1u \neq T_2u$, for all $\lambda > 0$, we have $\alpha(u, u)\hat{v}_\lambda(T_1u, T_2u) > 0$ and so, from (3.28), (3.29), and (3.31), we have

$$\begin{aligned}\alpha(u, u)\hat{v}_\lambda(T_1u, T_2u) &\leq a\alpha(u, u)\hat{v}_\lambda(T_1u, T_3u)\\ &\leq \frac{b}{a}\alpha(u, u)\hat{v}_\lambda(T_2u, T_3u)\\ &\leq \frac{ac}{b}\alpha(u, u)\hat{v}_\lambda(T_1u, T_2u),\end{aligned} \tag{3.39}$$

which implies $\alpha(u, u) \neq 0$. Thus we have

$$\alpha(u, u)\hat{v}_\lambda(T_1u, T_2u) \leq \frac{ac}{b}\alpha(u, u)\hat{v}_\lambda(T_1u, T_2u),$$

which implies that

$$\alpha(u, u)\Big(1 - \frac{ac}{b}\Big)\hat{v}_\lambda(T_1u, T_2u) = 0 \implies T_1u = T_2u$$

for all $\lambda > 0$ since $\frac{ac}{b} < 1$ and $b \neq 0$.

For Case 1c, using Case 1a above, that is, $T_1u = T_2u$, it follows from (3.39) that

$$0 = \alpha(u, u)\hat{v}_\lambda(T_1u, T_1u) \leq a\alpha(u, u)\hat{v}_\lambda(T_2u, T_3u)$$

$$\leq \frac{b}{a}\alpha(u,u)\hat{v}_\lambda(T_2u,T_3u)$$
$$\leq \frac{ac}{b}\alpha(u,u)\hat{v}_\lambda(T_1u,T_1u). \tag{3.40}$$

So, by using the condition (1) of Definition 10,

$$0 \leq a\alpha(u,u)\hat{v}_\lambda(T_2u,T_3u) \leq \frac{b}{a}\alpha(u,u)\hat{v}_\lambda(T_2u,T_3u) \leq 0, \tag{3.41}$$

which implies that

$$\alpha(u,u)\frac{b}{a}\hat{v}_\lambda(T_2u,T_3u) = 0.$$

Hence $T_2u = T_3u$, since $\frac{b}{a} < 1$, $a^2 \neq 0$, and $\alpha(u,u) \neq 0$.

For Case 1b, using Cases 1a and 1c in (3.39), we get $0 < a\alpha(u,u)\hat{v}_\lambda(T_1u,T_3u) \leq 0$. Since $a \neq 0$ and $\alpha(u,u) \neq 0$, we have $T_1u = T_3u$.

So, in all the cases above, we have $T_1u = T_2u = T_3u$. However, since $T_1u = T_6u$, $T_2u = T_4u$ and $T_3u = T_5u$, it follows that

$$T_1u = T_2u = T_3u = T_4u = T_5u = T_6u,$$

which implies that $u \in X_{\hat{v}}$ is the coincidence point of T_i for each $i = 1, 2, \cdots, 6$.

We demonstrate that if a point is a fixed point of T_1, it is also a fixed point for T_2, T_3, T_4, T_5, and T_6. Assume that a point $p \in X_{\hat{v}}$ exists such that p satisfies $p = T_1p$. We assert that $p = T_2p = T_3p$. Indeed, suppose that this is not true. Then $p \neq T_2p$ and $p \neq T_3p$, and so we have the following cases for all $\lambda > 0$,

Case 1. $p \neq T_2p \implies T_1p \neq T_2p \implies \alpha(p,p)\hat{v}_\lambda(T_1p,T_2p) > 0$;

Case 2. $p \neq T_3p \implies T_1p \neq T_3p \implies \alpha(p,p)\hat{v}_\lambda(T_1p,T_3p) > 0$.

Indeed, note that $\alpha(p,p) \neq 0$ and T_i remain a triangular α-orbital admissible mapping for each $i = 1, \cdots, 6$ such that

$$\mathfrak{A}^{-1}(F^A_\lambda(T_1p,T_2p)) \geq \hat{v}_\lambda(T_1p,T_2p) > 0, \quad \mathfrak{A}^{-1}(F^B_\lambda(T_1p,T_3p)) \geq \hat{v}_\lambda(T_1p,T_3p) > 0.$$

For Case 1, it follows from (3.39) that

$$\begin{aligned}\alpha(p,p)\hat{v}_\lambda(T_1p,T_2p) &\leq a\alpha(p,p)\hat{v}_\lambda(T_1p,T_3p)\\ &\leq \frac{b}{a}\alpha(p,p)\hat{v}_\lambda(T_2p,T_3p)\\ &\leq \frac{ac}{b}\alpha(p,p)\hat{v}_\lambda(T_1p,T_2p).\end{aligned} \tag{3.42}$$

From $p \in Fix(T_1)$, we have

$$\begin{aligned}\alpha(T_1p,T_1p)\hat{v}_\lambda(p,T_2p) &\leq a\alpha(T_1p,T_1p)\hat{v}_\lambda(p,T_3p)\\ &\leq \frac{b}{a}\alpha(T_1p,T_1p)\hat{v}_\lambda(T_2p,T_3p)\\ &\leq \frac{ac}{b}\alpha(T_1p,T_1p)\hat{v}_\lambda(p,T_2p),\end{aligned} \tag{3.43}$$

which implies that $\alpha(T_1p, T_1p)(1-\frac{ac}{b})\hat{\nu}_\lambda(p, T_2p) \leq 0$, for all $\lambda > 0$ since $\alpha(T_1p, T_1p) \neq 0$ and $\frac{ac}{b} < 1$. This is a contradiction. Therefore, $p = T_2p$ and hence $p = T_1p = T_2p$.

For Case 2, if $p \neq T_3p$, then $T_1p \neq T_3p$ and therefore

$$\alpha(p, p)\hat{\nu}_\lambda(T_1p, T_3p) > 0$$

for all $\lambda > 0$. Thus, it follows from $p = T_1p = T_2p$, Case 1, and (3.39) that

$$\begin{aligned}\alpha(p, p)\hat{\nu}_\lambda(T_1p, T_2p) &\leq a\alpha(p, p)\hat{\nu}_\lambda(T_1p, T_3p)\\ &\leq \frac{b}{a}\alpha(p, p)\hat{\nu}_\lambda(T_2p, T_3p)\\ &\leq \frac{ac}{b}\alpha(p, p)\hat{\nu}_\lambda(T_1p, T_2p),\end{aligned} \tag{3.44}$$

which implies

$$\begin{aligned}0 \leq a\alpha(p, p)\hat{\nu}_\lambda(T_1p, T_3p) &\leq \frac{b}{a}\alpha(p, p)\hat{\nu}_\lambda(p, T_3p)\\ &\leq 0\end{aligned} \tag{3.45}$$

for all $\lambda > 0$. Since $b \neq 0$ and $\alpha(p, p) \neq 0$, we have

$$a\alpha(p, p)\frac{b}{a}\hat{\nu}_\lambda(T_1p, T_3p) = 0,$$

and so $\hat{\nu}_\lambda(T_1p, T_3p) = 0$; that is, $\hat{\nu}_\lambda(p, T_3p) = 0$. Therefore, $p = T_3p$ and so $p = T_1p = T_3p$. Hence, from Cases 1 and 2, $p = T_1p = T_2p = T_3p$.

Again, suppose that $p \in X_{\hat{\nu}}$ exists such that $p \in Fix(T_2)$, i.e., $p = T_2p$. We claim that $p = T_1p = T_3p$. Indeed, suppose that this is not true. Then $p \neq T_1p$ and $p \neq T_3p$, so we have the following cases for all $\lambda > 0$:

Case 3. $p \neq T_1p \implies T_2p \neq T_1p \implies \alpha(p, p)\hat{\nu}_\lambda(T_1p, T_2p) > 0$;

Case 4. $p \neq T_3p \implies T_2p \neq T_3p \implies \alpha(p, p)\hat{\nu}_\lambda(T_2p, T_3p) > 0$.

Indeed, note that $\alpha(p, p) \neq 0$ and T_i remains a triangular α-orbital admissible mapping for each $i = 1, \cdots, 6$ such that

$$\mathfrak{A}^{-1}(F^A_\lambda(T_1p, T_2p)) \geq \hat{\nu}_\lambda(T_1p, T_2p) > 0, \quad \mathfrak{A}^{-1}(F^C_\lambda(T_2p, T_3p)) \geq \hat{\nu}_\lambda(T_2p, T_3p) > 0.$$

For Case 3, it follows from (3.39) that

$$\begin{aligned}\alpha(p, p)\hat{\nu}_\lambda(T_1p, T_2p) &\leq a\alpha(p, p)\hat{\nu}_\lambda(T_1p, T_3p)\\ &\leq \frac{b}{a}\alpha(p, p)\hat{\nu}_\lambda(T_2p, T_3p)\\ &\leq \frac{ac}{b}\alpha(p, p)\hat{\nu}_\lambda(T_1p, T_2p).\end{aligned} \tag{3.46}$$

Since $p \in Fix(T_2)$, we have

$$\alpha(T_2p, T_2p)\hat{\nu}_\lambda(p, T_1p) \leq a\alpha(T_2p, T_2p)\hat{\nu}_\lambda(T_1p, T_3p)$$

$$
\begin{aligned}
&\leq \frac{b}{a}\alpha(T_2p, T_2p)\hat{v}_\lambda(p, T_3p) \\
&\leq \frac{ac}{b}\alpha(T_2p, T_2p)\hat{v}_\lambda(p, T_1p), \qquad (3.47)
\end{aligned}
$$

which implies that

$$\alpha(T_2p, T_2p)\Big(1 - \frac{ac}{b}\Big)\hat{v}_\lambda(p, T_1p) \leq 0$$

for all $\lambda > 0$, since $\alpha(T_2p, T_2p) \neq 0$ and $\frac{ac}{b} < 1$, which is a contradiction. Therefore, $p = T_1p$ and hence $p = T_1p = T_2p$.

For Case 4, if $p \neq T_3p$, then $T_2p \neq T_3p$ and therefore

$$\alpha(p, p)\hat{v}_\lambda(T_2p, T_3p) > 0$$

for all $\lambda > 0$. Thus it follows from $p = T_2p = T_1p$ and (3.39) that

$$
\begin{aligned}
\alpha(p, p)\hat{v}_\lambda(T_1p, T_2p) &\leq a\alpha(p, p)\hat{v}_\lambda(T_1p, T_3p) \\
&\leq \frac{b}{a}\alpha(p, p)\hat{v}_\lambda(T_2p, T_3p) \\
&\leq \frac{ac}{b}\alpha(p, p)\hat{v}_\lambda(T_1p, T_2p), \qquad (3.48)
\end{aligned}
$$

which implies

$$
\begin{aligned}
0 \leq a\alpha(p, p)\hat{v}_\lambda(p, T_3p) &\leq \frac{b}{a}\alpha(p, p)\hat{v}_\lambda(p, T_3p) \\
&\leq 0 \qquad (3.49)
\end{aligned}
$$

for all $\lambda > 0$. Since $b \neq 0$ and $\alpha(p, p) \neq 0$, we have

$$a\alpha(p, p)\frac{b}{a}\hat{v}_\lambda(p, T_3p) = 0,$$

and so $\hat{v}_\lambda(p, T_3p) = 0$; that is, $p = T_3p$. Therefore, $p = T_2p = T_3p$. Hence, from Cases 3 and 4, $p = T_1p = T_2p = T_3p$.

Lastly, suppose that $p \in X_{\hat{v}}$ exists such that $p \in Fix(T_3)$, i.e., $p = T_3p$. We claim that $p = T_1p = T_2p$. Indeed, suppose that this is not true. Then $p \neq T_1p$ and $p \neq T_2p$, and therefore we have the following cases for all $\lambda > 0$:

Case 5. $p \neq T_1p \implies T_3p \neq T_1p \implies \alpha(p, p)\hat{v}_\lambda(T_1p, T_3p) > 0$.

Case 6. $p \neq T_2p \implies T_3p \neq T_2p \implies \alpha(p, p)\hat{v}_\lambda(T_2p, T_3p) > 0$.

Indeed, note that $\alpha(p, p) \neq 0$ and T_i remains a triangular α-orbital admissible mapping for each $i = 1, \cdots, 6$ such that

$$\mathfrak{A}^{-1}(F^B_\lambda(T_1p, T_3p)) \geq \hat{v}_\lambda(T_1p, T_3p) > 0, \quad \mathfrak{A}^{-1}(F^C_\lambda(T_2p, T_3p)) \geq \hat{v}_\lambda(T_2p, T_3p) > 0.$$

For Case 5, if $p \neq T_1p$, $T_3p \neq T_1p$, and so

$$\alpha(p, p)\hat{v}_\lambda(T_1p, T_3p) > 0$$

for all $\lambda > 0$. Thus it follows from $p = T_3p$, (3.39), and Case 2 that

$$\begin{aligned}\alpha(p,p)\hat{v}_\lambda(T_1p,T_2p) &\leq a\alpha(p,p)\hat{v}_\lambda(T_1p,T_3p)\\ &\leq \frac{b}{a}\alpha(p,p)\hat{v}_\lambda(T_2p,T_3p)\\ &\leq \frac{ac}{b}\alpha(p,p)\hat{v}_\lambda(T_1p,T_2p),\end{aligned}\tag{3.50}$$

which implies

$$\alpha(p,p)\hat{v}_\lambda(T_1p,p) \leq \frac{ac}{b}\alpha(p,p)\hat{v}_\lambda(T_1p,p)\tag{3.51}$$

for all $\lambda > 0$. Since $b \neq 0$ and $\alpha(p,p) \neq 0$, we have

$$a\alpha(p,p)\left(1-\frac{ac}{b}\right)\hat{v}_\lambda(T_1p,p) = 0,$$

which implies $\hat{v}_\lambda(T_1p,p) = 0$; that is, $p = T_1p$. Therefore, $p = T_1p = T_2p = T_3p$.

For Case 6, if $p \neq T_2p$, then $T_2p \neq T_3p$, and therefore

$$\alpha(p,p)\hat{v}_\lambda(T_2p,T_3p) > 0$$

for all $\lambda > 0$. Thus, using $p = T_3p = T_1p$, it follows from (3.39) that

$$\begin{aligned}\alpha(p,p)\hat{v}_\lambda(T_1p,T_2p) &\leq a\alpha(p,p)\hat{v}_\lambda(T_1p,T_3p)\\ &\leq \frac{b}{a}\alpha(p,p)\hat{v}_\lambda(T_2p,T_3p)\\ &\leq \frac{ac}{b}\alpha(p,p)\hat{v}_\lambda(T_1p,T_2p),\end{aligned}\tag{3.52}$$

which implies

$$\alpha(p,p)\hat{v}_\lambda(T_1p,T_2p) \leq \frac{ac}{b}\alpha(p,p)\hat{v}_\lambda(T_1p,T_2p),$$

and so

$$\alpha(T_1p,T_1p)\hat{v}_\lambda(p,T_2p) \leq \frac{ac}{b}\alpha(T_1p,T_1p)\hat{v}_\lambda(p,T_2p)$$

for all $\lambda > 0$. Since $b \neq 0$ and $\alpha(T_1p,T_1p) \neq 0$, we have

$$\alpha(T_1p,T_1p)\left(1-\frac{ac}{b}\right)\hat{v}_\lambda(p,T_2p) = 0,$$

which implies $\hat{v}_\lambda(p,T_2p) = 0$, that is, $p = T_2p$. Therefore, $p = T_1p = T_2p$. Hence, it follows from Cases 5 and 6 that $p = T_1p = T_2p = T_3p$. In all the cases above, $p = T_1p = T_2p = T_3p$.

Now, using Cases 1a–1c above, we have

$$p = T_1p = T_6p, \quad p = T_2p = T_4p, \quad p = T_3p = T_5p.$$

Therefore, we have $p = T_1p = T_2p = T_3p = T_4p = T_5p = T_6p$ or $p \in Fix(T_i)$ for each $i = 1, 2, \cdots, 6$, which shows that p is a common fixed point of the mappings $T_1, T_2, T_3, T_4, T_5,$ *and* T_6.

To prove the uniqueness of the common fixed point p, suppose that another common fixed point x^* ($p \neq x^*$) of $T_1, T_2, T_3, T_4, T_5,$ *and* T_6 exists, namely

$$x^* = T_1x^* = T_2x^* = T_3x^* = T_4x^* = T_5x^* = T_6x^*.$$

Since $\alpha(x^*, p) \geq 1$, we have the following cases:
Case 1a*. $\alpha(x^*, y^*) \geq 1,\ x^* \neq y^* \implies \alpha(x^*, y^*)\hat{\nu}_\lambda(T_1x^*, T_2y^*) > 0.$
Case 1b*. $\alpha(x^*, y^*) \geq 1,\ x^* \neq y^* \implies \alpha(x^*, y^*)\hat{\nu}_\lambda(T_1x^*, T_3y^*) > 0.$
Case 1c*. $\alpha(x^*, y^*) \geq 1,\ x^* \neq y^* \implies \alpha(x^*, y^*)\hat{\nu}_\lambda(T_2x^*, T_3y^*) > 0.$

We can see that Case 1a* follows from Case 1a above and hence, $x^* = y^*$. Again, Case 1b* follows from Case 1b above and hence, $x^* = y^*$. Finally, Case 1c* follows from Case 1c above and therefore, $x^* = y^*$. Therefore, we have $x^* = y^*$. Hence, the common fixed-point p is a unique common fixed point T_i for each $i = 1, \cdots, 6$. We are now done with the proof. □

Remark 4. *Theorem 1 is a generalization of results in Karapinar et al. [26], Theorem 2.8 in Gholidahneh et al. [21], and Theorem 3.7 in Okeke et al. [34].*
Our result have made the following progress over the classical results in the following ways:

(a) This study extends classical metric and b-metric spaces by introducing modular extended b-metric spaces, which allow function-controlled distances.
(b) It defines the α-ν̂-A-B-C-Meir-Keeler-type contractions, a function-dependent contraction that generalizes Banach, Kannan, and Meir-Keeler contractions.
(c) Unlike traditional single-mapping results, this work establishes common fixed-point results for six self-mappings, significantly broadening the applicability of fixed-point theorems.
(d) The proposed contraction conditions use function-dependent inequalities instead of fixed contraction constants, making them adaptable to nonlinear and dynamic systems.
(e) This work bridges classical and modern fixed-point results by incorporating modular functions, b-metric spaces, and multi-mapping interactions, leading to a more generalized framework.

Now, we establish the following example to solidify Theorem 1.

Example 3. *Let* $X = (\mathbb{R} \setminus \{0\}) \cup \{\infty\}$ *with the modular extended b-metric defined by*

$$\frac{\ln(2)}{2}\hat{\nu}_\lambda(x, y) : = \frac{1}{1+\lambda} \max_{x,y \in X_{\hat{\nu}}} \{\|x - y\|\},$$

which is complete in $X_{\hat{\nu}}$ *for all* $\lambda > 0$*. Define the* $\hat{\nu}$*-weakly commuting mappings* $T_1, T_2, T_3, T_4, T_5, T_6 : (\mathbb{R} \setminus \{0\}) \cup \{\infty\} \to (\mathbb{R} \setminus \{0\}) \cup \{\infty\}$ *as follows:*

$$T_1x = \log_{64} x^6, \quad T_2x = \log_{32} x^5, \quad T_3x = \log_{16} x^4,$$

$$T_4x = \log_8 x^3, \quad T_5x = \log_4 x^2, \quad T_6x = \log_2 x,$$

for all $x \in (\mathbb{R} \setminus \{0\}) \cup \{\infty\}$ *and* $\lambda > 0$*, for each* $i = 1, 2, \cdots, 6$ *and also for* $x, y \in (\mathbb{R} \setminus \{0\}) \cup \{\infty\}$*,* $\alpha(x, T_ix) \geq 1 \implies \alpha(T_ix, T_i^2x) \geq 1$*,* $\alpha(x, y) \geq 1$ *and* $\alpha(y, T_iy) \geq 1 \implies \alpha(x, T_iy) \geq 1$*. Then the mappings* $T_1, T_2, T_3, T_4, T_5,$ *and* T_6 *satisfy the inequalities* (3.1)–(3.3) *of Theorem 1.*

In fact, let $T_i : X_{\hat{\nu}} \longrightarrow X_{\hat{\nu}}$ *be six orbitally continuous α-ν̂-A-B-C-Meir-Keeler-type contraction mappings for* $i = 1, 2, \cdots, 6$*, and let* $\{T_2, T_4\}$*,*$\{T_3, T_5\}$ *and* $\{T_1, T_6\}$ *be weakly commuting pairs of* *self-mappings. Indeed, consider* $\{T_2, T_4\}$. *Now for all* $\lambda > 0$, *and* $x \in (\mathbb{R} \setminus \{0\})$, *we show that* $\hat{v}_\lambda(T_2T_4x, T_4T_2x) \leq \hat{v}_\lambda(T_4x, T_2x)$. *Then, by the definition of a* $\hat{v}$*-modular extended b-metric, we have*

$$\begin{aligned}
\frac{1}{1+\lambda}\max_{x\in X_{\hat{v}}}\{\|T_2T_4x - T_4T_2x\|\} &= \frac{1}{1+\lambda}\max_{x\in X_{\hat{v}}}\left\{\left\|\log_{32}(\log_8 x^3)^5 - \log_8(\log_{32} x^5)^3\right\|\right\} \\
&= \frac{1}{1+\lambda}\max_{x\in X_{\hat{v}}}\left\{\left\|5\log_{32}(\log_8 x^3) - 3\log_8(\log_{32} x^5)\right\|\right\} \\
&= \frac{1}{1+\lambda}\max_{x\in X_{\hat{v}}}\left\|\frac{\ln(\log_8 x^3)}{\ln(2)} - \frac{\ln(\log_{32} x^5)}{\ln(2)}\right\| \\
&= \frac{1}{\ln(2)}\frac{1}{1+\lambda}\max_{x\in X_{\hat{v}}}\left\|\ln(\log_8 x^3) - \ln(\log_{32} x^5)\right\| \\
&= \frac{1}{\ln(2)}\frac{1}{1+\lambda}\max_{x\in X_{\hat{v}}}\left\|\ln\left(\frac{\log_8 x^3}{\log_{32} x^5}\right)\right\| \\
&= 0.
\end{aligned}$$

Again,

$$\begin{aligned}
\hat{v}_\lambda(T_4x, T_2x) &= \frac{1}{1+\lambda}\max_{x\in X_{\hat{v}}}\{\|T_4x - T_2x\|\} \\
&= \frac{1}{1+\lambda}\max_{x\in X_{\hat{v}}}\{\left\|\log_8 x^3 - \log_{32} x^5\right\|\} \\
&= \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{x\in X_{\hat{v}}}\{\left\|\ln(x) - \ln(x)\right\|\} \\
&= 0.
\end{aligned}$$

Thus $\hat{v}_\lambda(T_2T_4x, T_4T_2x) \leq \hat{v}_\lambda(T_4x, T_2x)$, *showing that* $\{T_2, T_4\}$ *is weakly commuting pair of self-mappings, and* $\{T_3, T_5\}$ *and* $\{T_1, T_6\}$ *are weakly commuting pairs of self-mappings according to the above mentioned procedure. It is clear that* $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$, *so that there exists a function* $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$, $a \neq 0, a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$ *and* $x_0 \in X_{\hat{v}}$ *such that* $\alpha(x_0, T_ix_0) \geq 1$. *Therefore by definition, we have*

$$\begin{aligned}
\hat{v}_\lambda(T_1x, T_2y) &= \frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\{\left\|T_1x - T_2y\right\|\} \\
&= \frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\left\{\left\|\log_{64} x^6 - \log_{32} y^5\right\|\right\} \\
&= \frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\{\left\|6\log_{64} x - 5\log_{32} y\right\|\} \\
&= \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{x,y\in X_{\hat{v}}}\{\left\|\ln(x) - \ln(y)\right\|\} \\
&= \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{x,y\in X_{\hat{v}}}\left\{\ln\left\|\frac{x}{y}\right\|\right\}.
\end{aligned} \tag{3.53}$$

$$\hat{v}_\lambda(T_1x, T_3y) = \frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\{\left\|T_1x - T_3y\right\|\}$$

$$
\begin{aligned}
&= \frac{1}{1+\lambda} \max_{x,y \in X_{\hat{\nu}}} \left\{ \left\| \log_{64} x^6 - \log_{16} y^4 \right\| \right\} \\
&= \frac{1}{1+\lambda} \max_{x,y \in X_{\hat{\nu}}} \{ \| 6 \log_{64} x - 4 \log_{16} y \| \} \\
&= \frac{1}{1+\lambda} \frac{1}{\ln(2)} \max_{x,y \in X_{\hat{\nu}}} \{ \| \ln(x) - \ln(y) \| \} \\
&= \frac{1}{1+\lambda} \frac{1}{\ln(2)} \max_{x,y \in X_{\hat{\nu}}} \left\{ \ln \left\| \frac{x}{y} \right\| \right\}.
\end{aligned} \tag{3.54}
$$

$$
\begin{aligned}
\hat{\nu}_\lambda(T_2 x, T_3 y) &= \frac{1}{1+\lambda} \max_{x,y \in X_{\hat{\nu}}} \{ \| T_2 x - T_3 y \| \} \\
&= \frac{1}{1+\lambda} \max_{x,y \in X_{\hat{\nu}}} \left\{ \left\| \log_{32} x^5 - \log_{16} y^4 \right\| \right\} \\
&= \frac{1}{1+\lambda} \max_{x,y \in X_{\hat{\nu}}} \{ \| 5 \log_{32} x - 4 \log_{16} y \| \} \\
&= \frac{1}{1+\lambda} \frac{1}{\ln(2)} \max_{x,y \in X_{\hat{\nu}}} \{ \| \ln(x) - \ln(y) \| \} \\
&= \frac{1}{1+\lambda} \frac{1}{\ln(2)} \max_{x,y \in X_{\hat{\nu}}} \left\{ \ln \left\| \frac{x}{y} \right\| \right\}.
\end{aligned} \tag{3.55}
$$

Now,

$$
\begin{aligned}
\hat{\nu}_\lambda(T_6 x, T_4 y) &= \frac{1}{1+\lambda} \max_{x,y \in X_{\hat{\nu}}} \{ \| T_6 x - T_4 y \| \} \\
&= \frac{1}{1+\lambda} \max_{x,y \in X_{\hat{\nu}}} \left\{ \left\| \log_{2} x - \log_{8} y^3 \right\| \right\} \\
&= \frac{1}{1+\lambda} \max_{x,y \in X_{\hat{\nu}}} \{ \| \log_{2} x - 3 \log_{8} y \| \} \\
&= \frac{1}{1+\lambda} \frac{1}{\ln(2)} \max_{x,y \in X_{\hat{\nu}}} \{ \| \ln(x) - \ln(y) \| \} \\
&= \frac{1}{1+\lambda} \frac{1}{\ln(2)} \max_{x,y \in X_{\hat{\nu}}} \left\{ \ln \left\| \frac{x}{y} \right\| \right\}.
\end{aligned} \tag{3.56}
$$

$$
\begin{aligned}
\hat{\nu}_\lambda(T_1 x, T_6 y) &= \frac{1}{1+\lambda} \max_{x,y \in X_{\hat{\nu}}} \{ \| T_1 x - T_6 y \| \} \\
&= \frac{1}{1+\lambda} \max_{x,y \in X_{\hat{\nu}}} \left\{ \left\| \log_{64} x^6 - \log_{2} y \right\| \right\} \\
&= \frac{1}{1+\lambda} \max_{x,y \in X_{\hat{\nu}}} \{ \| 6 \log_{64} x - \log_{2} y \| \} \\
&= \frac{1}{1+\lambda} \frac{1}{\ln(2)} \max_{x,y \in X_{\hat{\nu}}} \{ \| \ln(x) - \ln(y) \| \} \\
&= \frac{1}{1+\lambda} \frac{1}{\ln(2)} \max_{x,y \in X_{\hat{\nu}}} \left\{ \ln \left\| \frac{x}{y} \right\| \right\}.
\end{aligned} \tag{3.57}
$$

$$\begin{aligned}\hat{v}_\lambda(T_3x, T_4y) &= \frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\{\|T_3x - T_4y\|\}\\ &= \frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\left\{\|\log_{16} x^4 - \log_8 y^3\|\right\}\\ &= \frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\{\|4\log_{16} x - 3\log_8 y\|\}\\ &= \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{x,y\in X_{\hat{v}}}\{\|\ln(x) - \ln(y)\|\}\\ &= \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{x,y\in X_{\hat{v}}}\left\{\ln\left\|\frac{x}{y}\right\|\right\}. \end{aligned} \tag{3.58}$$

$$\begin{aligned}\hat{v}_\lambda(T_2x, T_5y) &= \frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\{\|T_2x - T_5y\|\}\\ &= \frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\left\{\|\log_{32} x^5 - \log_4 y^2\|\right\}\\ &= \frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\{\|5\log_{32} x - 2\log_4 y\|\}\\ &= \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{x,y\in X_{\hat{v}}}\{\|\ln(x) - \ln(y)\|\}\\ &= \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{x,y\in X_{\hat{v}}}\left\{\ln\left\|\frac{x}{y}\right\|\right\}. \end{aligned} \tag{3.59}$$

Again,

$$\begin{aligned}\hat{v}_\lambda(T_5x, T_4y) &= \frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\{\|T_5x - T_4y\|\}\\ &= \frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\{\|\log_4 x^2 - \log_8 y^3\|\}\\ &= \frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\{\|2\log_4 x - 3\log_8 y\|\}\\ &= \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{x,y\in X_{\hat{v}}}\{\|\ln(x) - \ln(y)\|\}\\ &= \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{x,y\in X_{\hat{v}}}\left\{\ln\left\|\frac{x}{y}\right\|\right\}. \end{aligned} \tag{3.60}$$

$$\begin{aligned}\hat{v}_\lambda(T_1x, T_6^2x) &= \frac{1}{1+\lambda}\max_{x\in X_{\hat{v}}}\left\{\|T_1x - T_6^2x\|\right\}\\ &= \frac{1}{1+\lambda}\max_{x\in X_{\hat{v}}}\left\{\|\log_{64} x^6 - (\log_2 x)^2\|\right\}\\ &= \frac{1}{1+\lambda}\max_{x\in X_{\hat{v}}}\left\{\|6\log_{64} x - (\log_2 x)^2\|\right\}\end{aligned}$$

$$= \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{x\in X_{\hat{v}}}\left\{\left\|\ln(x) - \frac{\ln(x)^2}{\ln(2)}\right\|\right\}$$
$$\leq \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{x\in X_{\hat{v}}}\left\{\ln\|x\|\right\}. \tag{3.61}$$

$$\hat{v}_\lambda(T_3 y, T_4^2 y) = \frac{1}{1+\lambda}\max_{y\in X_{\hat{v}}}\left\{\left\|T_3 y - T_4^2 y\right\|\right\}$$
$$= \frac{1}{1+\lambda}\max_{y\in X_{\hat{v}}}\left\{\left\|\log_{16} y^4 - (\log_8 y^3)^2\right\|\right\}$$
$$= \frac{1}{1+\lambda}\max_{y\in X_{\hat{v}}}\left\{\left\|4\log_{16} y - (3\log_8 y)^2\right\|\right\}$$
$$= \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{y\in X_{\hat{v}}}\left\{\left\|\ln(y) - \frac{\ln(y)^2}{\ln(2)}\right\|\right\}$$
$$\leq \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{y\in X_{\hat{v}}}\left\{\ln\left\|y\right\|\right\}. \tag{3.62}$$

$$\hat{v}_\lambda(T_6 x, T_2 x) = \frac{1}{1+\lambda}\max_{x\in X_{\hat{v}}}\{\|T_6 x - T_2 x\|\}$$
$$= \frac{1}{1+\lambda}\max_{x\in X_{\hat{v}}}\left\{\left\|\log_2 x - \log_{32} x^5\right\|\right\}$$
$$= \frac{1}{1+\lambda}\max_{x\in X_{\hat{v}}}\left\{\left\|\log_2 x - 5\log_{32} x\right\|\right\}$$
$$= \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{x\in X_{\hat{v}}}\left\{\left\|\ln(x) - \ln(x)\right\|\right\}$$
$$= \frac{1}{1+\lambda}\frac{1}{\ln(2)}\max_{x\in X_{\hat{v}}}\left\{\|0\|\right\} = 0. \tag{3.63}$$

Therefore, we have $F_\lambda^A(T_1 x, T_2 y) = a\max_{x,y\in X_{\hat{v}}}\left\{\ln\left\|\frac{x}{y}\right\|, \ln\left\|\frac{x}{y}\right\|, \ln\left\|\frac{x}{y}\right\|, \ln\left\|\frac{x}{y}\right\|\right\}$, *where* $a = \dfrac{1}{2}$. *Given* $\frac{1}{2}\ln(\epsilon)$, *we choose* $\ln(\delta(\epsilon)) = \frac{1}{2}\ln(\epsilon)$. *If* $\epsilon \leq \exp\{\ln(F_\lambda^A(T_1 x, T_2 y))\} < \frac{1}{2}\ln(\epsilon) + \ln(\delta(\epsilon)) = \ln(\epsilon) > \epsilon$, *therefore, we have, as* $\epsilon > 0$,

$$\alpha(x,y)\hat{v}_\lambda(T_1 x, T_2 y) \leq a\frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\left\{\ln\left\|\frac{x}{y}\right\|\right\}$$
$$\leq a\exp\left\{\ln\left\{\frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\left\{\ln\left\|\frac{x}{y}\right\|, \ln\left\|\frac{x}{y}\right\|, \ln\left\|\frac{x}{y}\right\|, \ln\left\|\frac{x}{y}\right\|\right\}\right\}\right\}$$
$$= a\exp\ln(F_\lambda^A(T_1 x, T_2 y))$$
$$< \ln(\epsilon)$$
$$< \epsilon.$$

Again, $F_\lambda^B(T_1 x, T_3 y) = b\max_{x,y\in X_{\hat{v}}}\left\{\ln\left\|\frac{x}{y}\right\|, \ln\left\|\frac{x}{y}\right\|, \ln\|x\|, \ln\left\|y\right\|\right\}$, *where* $b = \dfrac{1}{2}$. *Given* $\frac{1}{2}\ln(\epsilon)$, *we choose* $\ln(\delta(\epsilon)) = \frac{1}{2}\ln(\epsilon)$. *If* $\epsilon \leq exp\ln(F_\lambda^B(T_1 x, T_3 y)) < \frac{1}{2}\ln(\epsilon) + \ln(\delta(\epsilon)) = \ln(\epsilon) < \epsilon$, *therefore, we* *have* $\epsilon > 0$ *and*

$$\begin{aligned}\alpha(x,y)\hat{v}_\lambda(T_1x,T_3y) &\le b\frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\left\{\ln\left\|\frac{x}{y}\right\|\right\}\\ &\le bexp\left\{\ln\left\{\frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\left\{\ln\left\|\frac{x}{y}\right\|, \ln\left\|\frac{x}{y}\right\|, \ln\|x\|, \ln\|y\|\right\}\right\}\right\}\\ &= bexp\ln(F_\lambda^B(T_1x,T_3y))\\ &< \ln(\epsilon)\\ &< \epsilon.\end{aligned}$$

Lastly, $F_\lambda^C(T_2x,T_3y) = c\max_{x,y\in X_{\hat{v}}}\left\{\ln\left\|\frac{x}{y}\right\|, \ln\left\|\frac{x}{y}\right\|, \ln\left\|\frac{x}{y}\right\|\right\}$*, where* $c = \dfrac{1}{2}$*. Given* $\frac{1}{2}\ln(\epsilon)$*, we choose* $\ln(\delta(\epsilon)) = \frac{1}{2}\ln(\epsilon)$*. If* $\epsilon \le exp\ln(F_\lambda^C(T_2x,T_3y)) < \frac{1}{2}\ln(\epsilon) + \ln(\delta(\epsilon)) = \ln(\epsilon) < \epsilon$*, Therefore, we have*

$$\begin{aligned}\alpha(x,y)\hat{v}_\lambda(T_2x,T_3y) &\le c\frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\left\{\ln\left\|\frac{x}{y}\right\|\right\}\\ &\le c\exp\left\{\ln\left\{\frac{1}{1+\lambda}\max_{x,y\in X_{\hat{v}}}\left\{\ln\left\|\frac{x}{y}\right\|, \ln\left\|\frac{x}{y}\right\|, \ln\left\|\frac{x}{y}\right\|\right\}\right\}\right\}\\ &= cexp\ln(F_\lambda^C(T_2x,T_3y))\\ &< \ln(\epsilon)\\ &< \epsilon.\end{aligned}$$

Thus we have

$$\alpha(x,y)\hat{v}_\lambda(T_1x,T_2y) < \mathfrak{A}(\epsilon) \implies \epsilon \le \mathfrak{A}^{-1}(F_\lambda^A(T_1x,T_2y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.64}$$
$$\alpha(x,y)\hat{v}_\lambda(T_1x,T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \le \mathfrak{A}^{-1}(F_\lambda^B(T_1x,T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.65}$$
$$\alpha(x,y)\hat{v}_\lambda(T_2x,T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \le \mathfrak{A}^{-1}(F_\lambda^C(T_2x,T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.66}$$

where

$$F_\lambda^A(T_1x,T_2y) := a\max\{\hat{v}_\lambda(T_6x,T_4y), \hat{v}_\lambda(T_1x,T_6y), \hat{v}_\lambda(T_3x,T_4y), \hat{v}_\lambda(T_2x,T_5y)\}; \tag{3.67}$$
$$F_\lambda^B(T_1x,T_3y) := b\max\{\hat{v}_\lambda(T_5x,T_4y), \hat{v}_\lambda(T_2x,T_5y), \hat{v}_\lambda(T_1x,T_6^2x), \hat{v}_\lambda(T_3y,T_4^2y)\}; \tag{3.68}$$
$$F_\lambda^C(T_2x,T_3y) := c\max\{\hat{v}_\lambda(T_6x,T_2x), \hat{v}_\lambda(T_2x,T_5y), \hat{v}_\lambda(T_3x,T_4y), \hat{v}_\lambda(T_1x,T_6y)\}. \tag{3.69}$$

Since T_i *is orbitally continuous, all the conditions of Theorem 1 are fulfilled.*

Corollary 1. *Let* $X_{\hat{v}}$ *be a* $\hat{v}$*-regular and* $\hat{v}$*-complete modular extended b-metric space. Consider six mappings* $T_i : X_{\hat{v}} \to X_{\hat{v}}$ *for* $i = 1, 2, \ldots, 6$*, which are orbitally continuous and satisfy a specific type of contraction known as the* α*-*$\hat{v}$*-A-B-C-Meir-Keeler condition. The pairs* $\{T_2, T_4\}$*,* $\{T_3, T_5\}$*, and* $\{T_1, T_6\}$ *are weakly commuting self-mappings, with the following inclusions holding:*

$$T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}}), \quad T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}}), \quad T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}}).$$

Additionally, a function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$ *exists with the parameters* $a \neq 0$*,* $a < 1$*,* $b \neq 0$*, and the condition* $\frac{ac}{b} < 1$*. Let* $x_0 \in X_{\hat{v}}$ *be given such that*

$$\alpha(x_0, x_1) \geq 1, \quad \epsilon, \delta > 0.$$

For each mapping T_i *(where* $i = 1, \ldots, 6$*), we find that* T_i *remains triangular* α*-orbital admissible mapping for all* $\lambda > 0$*. The following conditions are satisfied for some positive integer* $m \geq 1$*:*

$$\alpha(x, y)\hat{v}_\lambda(T_1^m x, T_2^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^A(T_1^m x, T_2^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.70}$$

$$\alpha(x, y)\hat{v}_\lambda(T_1^m x, T_3^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^B(T_1^m x, T_3^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.71}$$

$$\alpha(x, y)\hat{v}_\lambda(T_2^m x, T_3^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^C(T_2^m x, T_3^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.72}$$

where

$$F_\lambda^A(T_1^m x, T_2^m y) := a \max\Big\{\hat{v}_\lambda(T_6^m x, T_4^m y), \hat{v}_\lambda(T_1^m x, T_6^m y), \hat{v}_\lambda(T_3^m x, T_4^m y), \hat{v}_\lambda(T_2^m x, T_5^m y)\Big\}; \tag{3.73}$$

$$F_\lambda^B(T_1^m x, T_3^m y) := b \max\Big\{\hat{v}_\lambda(T_5^m x, T_4^m y), \hat{v}_\lambda(T_2^m x, T_5^m y), \hat{v}_\lambda(T_1^m x, T_6^{2m} x), \hat{v}_\lambda(T_3^m y, T_4^{2m} y)\Big\}; \tag{3.74}$$

$$F_\lambda^C(T_2^m x, T_3^m y) := c \max\Big\{\hat{v}_\lambda(T_6^m x, T_2^m x), \hat{v}_\lambda(T_2^m x, T_5^m y), \hat{v}_\lambda(T_3^m x, T_4^m y), \hat{v}_\lambda(T_1^m x, T_6^m y)\Big\}. \tag{3.75}$$

Let the sequences $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{\xi_n\}_{n\in\mathbb{N}}$ *be in* $X_{\hat{v}}$ *so that for* x_n *in* $X_{\hat{v}}$*, we choose* x_{n+1} *such that* $\xi_n = T_1 x_n = T_6 x_{n+1}$*; again, for* x_{n+1} *in* $X_{\hat{v}}$*, we choose* x_{n+2} *such that* $\xi_{n+1} = T_2 x_{n+1} = T_5 x_{n+2}$ *and, for a point* x_{n+2} *in* $X_{\hat{v}}$*, we choose* x_{n+3} *such that* $\xi_{n+2} = T_3 x_{n+2} = T_4 x_{n+3}$ *for* $n = 0, 1, 2, \cdots$*. Then* T_i *has a fixed point in* $X_{\hat{v}}$*. Moreover, if* $\alpha(x^*, y^*) \geq 1$ *for all* $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$*, then* T_i *has common unique fixed-point in* $\bigcap_{i=1} Fix(T_i)$ *for* $i = 1, \cdots, 6$ *for some positive integer,* $m \geq 1$*.*

Proof. According to Theorem 1, for a certain positive integer $m \geq 1$, we have the following equalities: $p = T_1^m p = T_6^m p$, $p = T_2^m p = T_4^m p$, and $p = T_3^m p = T_5^m p$. This implies that p can be expressed as:

$$p = T_1^m p = T_2^m p = T_3^m p = T_4^m p = T_5^m p = T_6^m p,$$

indicating that p lies in the fixed-point set $\mathrm{Fix}(T_i)$ for each $i = 1, 2, \ldots, 6$ and the positive integer $m \geq 1$. Consequently, p serves as a fixed point for each mapping $T_1^m, T_2^m, T_3^m, T_4^m, T_5^m, T_6^m$. The uniqueness of this point can be derived similarly to the argument presented in Theorem 1. Thus, it follows that the mappings T_i possess a unique common fixed point located in the intersection $\bigcap_{i=1} \mathrm{Fix}(T_i)$ for $i = 1, \ldots, 6$ and for some positive integer $m \geq 1$. □

Corollary 2. *Let* $X_{\hat{v}}$ *be a* $\hat{v}$*-regular and* $\hat{v}$*-complete modular extended b-metric space. Consider six orbitally continuous mappings* $T_i : X_{\hat{v}} \to X_{\hat{v}}$ *(*$i = 1, 2, \ldots, 6$*) that satisfy the* α*-*$\hat{v}$*-A-B-C-Meir-Keeler-type contraction condition. Additionally, assume that the pairs* $\{T_2, T_4\}$*,* $\{T_3, T_5\}$*, and* $\{T_1, T_6\}$ *are weakly commuting, with the following inclusions holding:*

$$T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}}), \quad T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}}), \quad T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}}).$$

Assume that a function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$ *exists, along with the constants* $a \neq 0$*,* $a < 1$*,* $b \neq 0$*, and* $\frac{ac}{b} < 1$*, as well as a point* $x_0 \in X_{\hat{v}}$*, such that:*

$$\alpha(x_0, x_1) \geq 1, \quad \epsilon, \delta > 0.$$

Furthermore, for all $\lambda > 0$, each T_i is assumed to be a triangular α-orbital admissible mapping and satisfies the following conditions:

$$\alpha(x,y)\hat{v}_\lambda(T_1x,T_2y) < \mathfrak{A}(\epsilon) \implies \epsilon \le \mathfrak{A}^{-1}(F_\lambda^A(T_1x,T_2y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \quad (3.76)$$

$$\alpha(x,y)\hat{v}_\lambda(T_1x,T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \le \mathfrak{A}^{-1}(F_\lambda^B(T_1x,T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \quad (3.77)$$

$$\alpha(x,y)\hat{v}_\lambda(T_2x,T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \le \mathfrak{A}^{-1}(F_\lambda^C(T_2x,T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \quad (3.78)$$

where

$$F_\lambda^A(T_1x,T_2y) := a\max\left\{\max\{\hat{v}_\lambda(T_6x,T_4y),\hat{v}_\lambda(T_1x,T_6y)\},\min\{\hat{v}_\lambda(T_3x,T_4y),\hat{v}_\lambda(T_2x,T_5y)\}\right\}; \quad (3.79)$$

$$F_\lambda^B(T_1x,T_3y) := b\max\left\{\max\{\hat{v}_\lambda(T_5x,T_4y),\hat{v}_\lambda(T_2x,T_5y)\},\min\{\hat{v}_\lambda(T_1x,T_6^2x),\hat{v}_\lambda(T_3y,T_4^2y)\}\right\}; \quad (3.80)$$

$$F_\lambda^C(T_2x,T_3y) := c\max\left\{\max\{\hat{v}_\lambda(T_6x,T_2x),\hat{v}_\lambda(T_2x,T_5y)\},\min\{\hat{v}_\lambda(T_3x,T_4y),\hat{v}_\lambda(T_1x,T_6y)\}\right\}. \quad (3.81)$$

Let the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ be in $X_{\hat{v}}$ so that for x_n be defined in $X_{\hat{v}}$, we choose x_{n+1} such that $\xi_n = T_1x_n = T_6x_{n+1}$; again for x_{n+1} in $X_{\hat{v}}$, we choose x_{n+2} such that $\xi_{n+1} = T_2x_{n+1} = T_5x_{n+2}$ and, for a point x_{n+2} in $X_{\hat{v}}$, we choose x_{n+3} such that $\xi_{n+2} = T_3x_{n+2} = T_4x_{n+3}$ for $n = 0,1,2,\cdots$. Then T_i has a fixed point in $X_{\hat{v}}$. Moreover, if $\alpha(x^*,y^*) \ge 1$ for all $x^*,y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has a unique common fixed point in $\bigcap_{i=1} Fix(T_i)$ for each $i = 1,\cdots,6$.

Proof. If we take $X_{\hat{v}}$ to be empty, then there is nothing to prove. Henceforth, we assume that $X_{\hat{v}} \neq \emptyset$. Then a function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0,+\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{v}}$ exists such that $\alpha(x_0,x_1) \ge 1$, $\epsilon,\delta > 0$, and, for each $i = 1,\cdots,6$, T_i remains a triangular α-orbital admissible mappings for every $\lambda > 0$. Suppose that the mappings, T_i for $i = 1,\cdots,6$ satisfy theinequalities (3.76)–(3.78). Since x_0,x_1, and x_2 are points in $X_{\hat{v}}$ and $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$, we can find a point x_1 in $X_{\hat{v}}$ such that $\xi_0 = T_1x_0 = T_6x_1$. For $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, we can find a point $x_2 \in X_{\hat{v}}$ such that $\xi_1 = T_2x_1 = T_5x_2$, and for $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, we can find a point x_3 in $X_{\hat{v}}$ such that $\xi_2 = T_3x_2 = T_4x_3$. Now for all $\lambda > 0$, in general, one can find sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ residing in the space $X_{\hat{v}}$ that fulfill the relationships defined in Eq (3.7). If there is an integer $n_0 \in \mathbb{N}$ such that $\xi_{n_0} = \xi_{n_0+1}$, it follows that $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, and $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$ hold. In fact, if $m \in \mathbb{N}$ exists such that $\xi_{m+2} = \xi_{m+3}$, then $T_1u = T_6u$, where $u = x_{m+3}$. Therefore, the pair $\{T_1,T_6\}$ has a coincidence point $u \in X_{\hat{v}}$. If $\xi_m = \xi_{m+1}$, then $T_2u = T_4u$, where $u = x_{m+1}$. Therefore, the pair $\{T_2,T_4\}$ has a coincidence point $u \in X_{\hat{v}}$. If $\xi_{m+1} = \xi_{m+3}$, then $T_3u = T_5u$, where $u = x_{m+2}$. Thus, the pair $\{T_3,T_5\}$ has a coincidence point $u \in X_{\hat{v}}$. Again, if there is an $n_0 \in \mathbb{N}$ such that $\xi_{n_0} = \xi_{n_0+1} = \xi_{n_0+2}$, then $\xi_n = \xi_{n_0}$ for any $n \ge n_0$. This implies that $\{\xi_n\}$ is a modular $\hat{v}$ Cauchy sequence in $X_{\hat{v}}$. Actually, if $\eta \in \mathbb{N}$ exists such that (1) $\xi_\eta = \xi_{\eta+1} = \xi_{\eta+2}$, (2) $\xi_\eta \neq \xi_{\eta+1} = \xi_{\eta+2}$, (3) $\xi_\eta \neq \xi_{\eta+2} = \xi_{\eta+1}$, and (4) $\xi_\eta \neq \xi_{\eta+1} \neq \xi_{\eta+2}$ hold. In fact, Case (1) is easy, and Case (3) is similar to Case (2); therefore, from the inequalities (3.76)–(3.78), we get the result by setting $x = \xi_{\eta+2}$ and $y = \xi_{\eta+3}$. Following the proof of Theorem 1, the conclusion is now evident. □

Remark 5. *Corollary 2 is a generalization of [34, Corollaries 3.16 and 3.18].*

Corollary 3. *Let $X_{\hat{v}}$ be a modular extended b-metric space that is both $\hat{v}$-regular and $\hat{v}$-complete. Consider six mappings $T_i : X_{\hat{v}} \to X_{\hat{v}}$ for $i = 1, 2, \ldots, 6$ that exhibit orbital continuity and adhere to the α-$\hat{v}$-A-B-C-Meir-Keeler-type contraction conditions. The pairs $\{T_2, T_4\}$, $\{T_3, T_5\}$, and $\{T_1, T_6\}$ are considered weakly commuting self-mappings, satisfying the following inclusions:*

$$T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}}), \quad T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}}), \quad T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}}).$$

A function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$ exists such that $a \neq 0$, $a < 1$, $b \neq 0$, and $\frac{ac}{b} < 1$. Additionally, let $x_0 \in X_{\hat{v}}$ satisfy $\alpha(x_0, x_1) \geq 1$ for some $\epsilon, \delta > 0$. For each mapping T_i with $i = 1, \ldots, 6$, it is required that T_i qualifies as triangular α-orbital admissible mapping for all $\lambda > 0$, and the following conditions hold for some positive integer $m \geq 1$:

$$\alpha(x,y)\hat{v}_\lambda(T_1^m x, T_2^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^A(T_1^m x, T_2^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.82}$$

$$\alpha(x,y)\hat{v}_\lambda(T_1^m x, T_3^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^B(T_1^m x, T_3^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.83}$$

$$\alpha(x,y)\hat{v}_\lambda(T_2^m x, T_3^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^C(T_2^m x, T_3^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.84}$$

where

$$F_\lambda^A(T_1^m x, T_2^m y) : = a \max\Big\{\max\{\hat{v}_\lambda(T_6^m x, T_4^m y), \hat{v}_\lambda(T_1^m x, T_6^m y)\}, \tag{3.85}$$

$$\min\{\hat{v}_\lambda(T_3^m x, T_4^m y), \hat{v}_\lambda(T_2^m x, T_5^m y)\}\Big\};$$

$$F_\lambda^B(T_1^m x, T_3^m y) : = b \max\Big\{\max\{\hat{v}_\lambda(T_5^m x, T_4^m y), \hat{v}_\lambda(T_2^m x, T_5^m y)\}, \tag{3.86}$$

$$\min\{\hat{v}_\lambda(T_1^m x, T_6^{2m} x), \hat{v}_\lambda(T_3^m y, T_4^{2m} y)\}\Big\};$$

$$F_\lambda^C(T_2^m x, T_3^m y) : = c \max\Big\{\max\{\hat{v}_\lambda(T_6^m x, T_2^m x), \hat{v}_\lambda(T_2^m x, T_5^m y)\}, \tag{3.87}$$

$$\min\{\hat{v}_\lambda(T_3^m x, T_4^m y), \hat{v}_\lambda(T_1^m x, T_6^m y)\}\Big\}.$$

Let the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ be defined in $X_{\hat{v}}$ so that for x_n in $X_{\hat{v}}$, we choose x_{n+1} such that $\xi_n = T_1 x_n = T_6 x_{n+1}$; again for x_{n+1} in $X_{\hat{v}}$, we choose x_{n+2} such that $\xi_{n+1} = T_2 x_{n+1} = T_5 x_{n+2}$ and, for a point x_{n+2} in $X_{\hat{v}}$, we choose x_{n+3} such that $\xi_{n+2} = T_3 x_{n+2} = T_4 x_{n+3}$ for $n = 0, 1, 2, \cdots$. Then T_i has a fixed point in $X_{\hat{v}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$ for some positive integer, where $m \geq 1$.

Proof. According to Corollary 2, for a certain positive integer $m \geq 1$, we have the equalities $p = T_1^m p = T_6^m p$, $p = T_2^m p = T_4^m p$, and $p = T_3^m p = T_5^m p$. Consequently, it follows that

$$p = T_1^m p = T_2^m p = T_3^m p = T_4^m p = T_5^m p = T_6^m p,$$

indicating that p is a fixed point for each mapping T_i^m where $i = 1, 2, \ldots, 6$ for the specified positive integer $m \geq 1$. This establishes that p is, in fact, a fixed point of T_1^m as well as $T_2^m, T_3^m, T_4^m, T_5^m$, and T_6^m individually. Hence, the uniqueness of this fixed point can be derived as shown in Theorem 1 above. Therefore, the mappings T_i possess a unique common fixed point within the intersection $\bigcap_{i=1} \mathrm{Fix}(T_i)$ for $i = 1, \ldots, 6$ for some positive integer $m \geq 1$. □

Remark 6. *Corollary 3 is a generalization of [34, Corollary 3.19].*

Corollary 4. *Consider the space $X_{\hat{v}}$, which is characterized as a $\hat{v}$-regular and $\hat{v}$-complete modular extended b-metric space. Within this framework, let $T_i : X_{\hat{v}} \to X_{\hat{v}}$ for $i = 1, 2, \ldots, 6$ denote six mappings that are orbitally continuous and adhere to the α-$\hat{v}$-A-B-C-Meir-Keeler-type contraction conditions. The pairs $\{T_2, T_4\}$, $\{T_3, T_5\}$, and $\{T_1, T_6\}$ are weakly commuting self-mappings, and the following inclusions are satisfied:*

$$T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}}), \quad T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}}), \quad T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}}).$$

A function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$ exists with parameters such that $a \neq 0$, $a < 1$, $b \neq 0$, and $\frac{ac}{b} < 1$. Additionally, let $x_0 \in X_{\hat{v}}$ satisfy $\alpha(x_0, x_1) \geq 1$ for some $\epsilon, \delta > 0$. Each mapping T_i for $i = 1, \ldots, 6$ is required to be triangular α-orbital admissible mapping for all $\lambda > 0$, fulfilling the following specific conditions:

$$\alpha(x, y)\hat{v}_\lambda(T_1x, T_2y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^A_\lambda(T_1x, T_2y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.88}$$
$$\alpha(x, y)\hat{v}_\lambda(T_1x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^B_\lambda(T_1x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.89}$$
$$\alpha(x, y)\hat{v}_\lambda(T_2x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^C_\lambda(T_2x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.90}$$

where

$$F^A_\lambda(T_1x, T_2y) := a \max\left\{\max\{\hat{v}_\lambda(T_6x, T_4y), \hat{v}_\lambda(T_1x, T_6y)\} + \min\{\hat{v}_\lambda(T_3x, T_4y), \hat{v}_\lambda(T_2x, T_5y)\}\right\}; \tag{3.91}$$

$$F^B_\lambda(T_1x, T_3y) := b \max\left\{\max\{\hat{v}_\lambda(T_5x, T_4y), \hat{v}_\lambda(T_2x, T_5y)\} + \min\{\hat{v}_\lambda(T_1x, T_6^2x), \hat{v}_\lambda(T_3y, T_4^2y)\}\right\}; \tag{3.92}$$

$$F^C_\lambda(T_2x, T_3y) := c \max\left\{\max\{\hat{v}_\lambda(T_6x, T_2x), \hat{v}_\lambda(T_2x, T_5y)\} + \min\{\hat{v}_\lambda(T_3x, T_4y), \hat{v}_\lambda(T_1x, T_6y)\}\right\}. \tag{3.93}$$

Let the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ be in $X_{\hat{v}}$ so that for x_n in $X_{\hat{v}}$, we choose x_{n+1} such that $\xi_n = T_1x_n = T_6x_{n+1}$; again, for x_{n+1} in $X_{\hat{v}}$, we choose x_{n+2} such that $\xi_{n+1} = T_2x_{n+1} = T_5x_{n+2}$, and for a point x_{n+2} in $X_{\hat{v}}$, we choose x_{n+3} such that $\xi_{n+2} = T_3x_{n+2} = T_4x_{n+3}$ for $n = 0, 1, 2, \cdots$. Then T_i has a fixed point in $X_{\hat{v}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has a unique common fixed point in $\bigcap_{i=1} Fix(T_i)$ for each $i = 1, \cdots, 6$.

Proof. Suppose that $X_{\hat{v}}$ is empty; then there is nothing to prove. We now assume that $X_{\hat{v}} \neq \emptyset$. Then a function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{v}}$ exist such that $\alpha(x_0, x_1) \geq 1$, $\epsilon, \delta > 0$ and, for each $i = 1, \cdots, 6$, T_i remains a triangular α-orbital admissible mappings for every $\lambda > 0$. Suppose that the mappings, T_i for $i = 1, \cdots, 6$ satisfy the inequalities (3.88)–(3.90). Since x_0, x_1 and x_2 are points in $X_{\hat{v}}$ and $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$, we can find a point x_1 in $X_{\hat{v}}$ such that $\xi_0 = T_1x_0 = T_6x_1$. For $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, we can find a point $x_2 \in X_{\hat{v}}$ such that $\xi_1 = T_2x_1 = T_5x_2$; for $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, we can find a point x_3 in $X_{\hat{v}}$ such that $\xi_2 = T_3x_2 = T_4x_3$. Now for all $\lambda > 0$, in general, sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ exist within $X_{\hat{v}}$ that satisfy the conditions outlined in Eq (3.7). If a natural number n_0 exists for which $\xi_{n_0} = \xi_{n_0+1}$, then it can be inferred that $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, and $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$ hold. In fact, $m \in \mathbb{N}$ exists such that $\xi_{m+2} = \xi_{m+3}$, then $T_1u = T_6u$, where $u = x_{m+3}$. Therefore, the pair $\{T_1, T_6\}$ has a coincidence point

$u \in X_{\hat{v}}$. If $\xi_m = \xi_{m+1}$, then $T_2u = T_4u$, where $u = x_{m+1}$. Therefore, the pair $\{T_2, T_4\}$ has a coincidence point $u \in X_{\hat{v}}$. If $\xi_{m+1} = \xi_{m+3}$, then $T_3u = T_5u$, where $u = x_{m+2}$. Thus, the pair $\{T_3, T_5\}$ has a coincidence point $u \in X_{\hat{v}}$. Again, if there is $n_0 \in \mathbb{N}$ such that $\xi_{n_0} = \xi_{n_0+1} = \xi_{n_0+2}$, then $\xi_n = \xi_{n_0}$ for any $n \geq n_0$. This implies that $\{\xi_n\}$ is a modular $\hat{v}$ Cauchy sequence in $X_{\hat{v}}$. Actually, $\eta \in \mathbb{N}$ exists such that (1) $\xi_\eta = \xi_{\eta+1} = \xi_{\eta+2}$, (2) $\xi_\eta \neq \xi_{\eta+1} = \xi_{\eta+2}$, (3) $\xi_\eta \neq \xi_{\eta+2} = \xi_{\eta+1}$, and (4) $\xi_\eta \neq \xi_{\eta+1} \neq \xi_{\eta+2}$ hold. In fact, Case (1) is easy, and Case (3) is similar to Case (2). Therefore, from the inequalities (3.88)–(3.90), we get the result by setting $x = \xi_{\eta+2}$ and $y = \xi_{\eta+3}$. By Theorem 1, T_i has a fixed point in $X_{\hat{v}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$. □

Remark 7. *Corollary 4 is a generalization of [34, Corollary 3.14].*

Corollary 5. *Let $X_{\hat{v}}$ be a $\hat{v}$-regular $\hat{v}$-complete modular extended b-metric space and let $T_i : X_{\hat{v}} \longrightarrow X_{\hat{v}}$ be six orbitally continuous mappings satisfying the α-$\hat{v}$-A-B-C-Meir-Keeler-type contraction for $i = 1, 2, \cdots, 6$, the pairs $\{T_2, T_4\}$, $\{T_3, T_5\}$, and $\{T_1, T_6\}$ are considered weakly commuting self-mappings that satisfy the condition that $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$. Moreover, a function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{v}}$ exist such that $\alpha(x_0, x_1) \geq 1$, $\epsilon, \delta > 0$, and, for each $i = 1, \cdots, 6$. For all $\lambda > 0$, T_i is classified as a triangular α-orbital admissible mapping, provided that the following conditions are met for some positive integer $m \geq 1$:*

$$\alpha(x,y)\hat{v}_\lambda(T_1^m x, T_2^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^A(T_1^m x, T_2^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.94}$$

$$\alpha(x,y)\hat{v}_\lambda(T_1^m x, T_3^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^B(T_1^m x, T_3^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.95}$$

$$\alpha(x,y)\hat{v}_\lambda(T_2^m x, T_3^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^C(T_2^m x, T_3^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.96}$$

where

$$F_\lambda^A(T_1^m x, T_2^m y) := a \max\Big\{ \max\{\hat{v}_\lambda(T_6^m x, T_4^m y), \hat{v}_\lambda(T_1^m x, T_6^m y)\} + \min\{\hat{v}_\lambda(T_3^m x, T_4^m y), \hat{v}_\lambda(T_2^m x, T_5^m y)\} \Big\}; \tag{3.97}$$

$$F_\lambda^B(T_1^m x, T_3^m y) := b \max\Big\{ \max\{\hat{v}_\lambda(T_5^m x, T_4^m y), \hat{v}_\lambda(T_2^m x, T_5^m y)\} + \min\{\hat{v}_\lambda(T_1^m x, T_6^{2m} x), \hat{v}_\lambda(T_3^m y, T_4^{2m} y)\} \Big\}; \tag{3.98}$$

$$F_\lambda^C(T_2^m x, T_3^m y) := c \max\Big\{ \max\{\hat{v}_\lambda(T_6^m x, T_2^m x), \hat{v}_\lambda(T_2^m x, T_5^m y)\} + \min\{\hat{v}_\lambda(T_3^m x, T_4^m y), \hat{v}_\lambda(T_1^m x, T_6^m y)\} \Big\}. \tag{3.99}$$

Let the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ be defined in $X_{\hat{v}}$ so that for x_n in $X_{\hat{v}}$, we choose x_{n+1} such that $\xi_n = T_1x_n = T_6x_{n+1}$; again, for x_{n+1} in $X_{\hat{v}}$, we choose x_{n+2} such that $\xi_{n+1} = T_2x_{n+1} = T_5x_{n+2}$ and, for a point x_{n+2} in $X_{\hat{v}}$, we choose x_{n+3} such that $\xi_{n+2} = T_3x_{n+2} = T_4x_{n+3}$ for $n = 0, 1, 2, \cdots$. Then T_i has a fixed point in $X_{\hat{v}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$ for some positive integer, $m \geq 1$.

Proof. By Corollary 4, $p = T_1^m p = T_6^m p$, $p = T_2^m p = T_4^m p$, and $p = T_3^m p = T_5^m p$. Thus, $p = T_1^m p = T_2^m p = T_3^m p = T_4^m p = T_5^m p = T_6^m p$ or $p \in Fix(T_i^m)$ for $i = 1, 2, \cdots, 6$ and some positive integer $m \geq 1$, showing that p is a fixed point of T_1^m and also a fixed point of $T_2^m, T_3^m, T_4^m, T_5^m,$ *and* T_6^m respectively. Therefore, the uniqueness follows as in Theorem 1 above. Hence, T_i has a common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$ and some positive integer, $m \geq 1$. □

Corollary 6. *Consider $X_{\hat{v}}$ to be a $\hat{v}$-regular and $\hat{v}$-complete modular extended b-metric space. Let $T_i : X_{\hat{v}} \to X_{\hat{v}}$ represent six orbitally continuous mappings that adhere to the α-$\hat{v}$-A-B-C-Meir-Keeler-type contraction for $i = 1, 2, \ldots, 6$. The pairs $\{T_2, T_4\}$, $\{T_3, T_5\}$, and $\{T_1, T_6\}$ are identified as weakly commuting self-mappings, satisfying the inclusions:*

$$T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}}), \quad T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}}), \quad T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}}).$$

Additionally, a function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$ exist with the parameters $a \neq 0$, $a < 1$, $b \neq 0$, and $\frac{ac}{b} < 1$. Let $x_0 \in X_{\hat{v}}$ be such that $\alpha(x_0, x_1) \geq 1$ and $\epsilon, \delta > 0$. For each $i = 1, \ldots, 6$, it follows that T_i qualifies as triangular α-orbital admissible mapping for every $\lambda > 0$, with the following specific conditions being satisfied:

$$\alpha(x,y)\hat{v}_\lambda(T_1x, T_2y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^A(T_1x, T_2y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.100}$$

$$\alpha(x,y)\hat{v}_\lambda(T_1x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^B(T_1x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.101}$$

$$\alpha(x,y)\hat{v}_\lambda(T_2x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^C(T_2x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.102}$$

where

$$F_\lambda^A(T_1x, T_2y) := a \max\left\{\hat{v}_\lambda(T_6x, T_4y), \hat{v}_\lambda(T_1x, T_6y)\right\}; \tag{3.103}$$

$$F_\lambda^B(T_1x, T_3y) := b \max\left\{\hat{v}_\lambda(T_5x, T_4y), \hat{v}_\lambda(T_2x, T_5y)\right\}; \tag{3.104}$$

$$F_\lambda^C(T_2x, T_3y) := c \max\left\{\hat{v}_\lambda(T_6x, T_2x), \hat{v}_\lambda(T_2x, T_5y)\right\}. \tag{3.105}$$

Let the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ be defined in $X_{\hat{v}}$ as follows. For a given x_n in $X_{\hat{v}}$, select x_{n+1} such that $\xi_n = T_1x_n = T_6x_{n+1}$. Next, for x_{n+1} in $X_{\hat{v}}$, we choose x_{n+2} so that $\xi_{n+1} = T_2x_{n+1} = T_5x_{n+2}$. Furthermore, for the point x_{n+2} in $X_{\hat{v}}$, we determine x_{n+3} such that $\xi_{n+2} = T_3x_{n+2} = T_4x_{n+3}$, for $n = 0, 1, 2, \ldots$. Then T_i has a fixed point in $X_{\hat{v}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has a unique common fixed point in $\bigcap_{i=1} Fix(T_i)$ for each $i = 1, \cdots, 6$.

Proof. If we take $X_{\hat{v}}$ to be empty, then there is nothing to prove. Henceforth, we assume that $X_{\hat{v}} \neq \emptyset$. Then a function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{v}}$ exist such that $\alpha(x_0, x_1) \geq 1$, $\epsilon, \delta > 0$ and, for each $i = 1, \cdots, 6$, T_i remains a triangular α-orbital admissible mapping for every $\lambda > 0$. Suppose that the mappings, T_i for $i = 1, \cdots, 6$ satisfies the inequalities (3.100)–(3.102). Since x_0, x_1, and x_2 are points in $X_{\hat{v}}$ and $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$, we can find a point x_1 in $X_{\hat{v}}$ such that $\xi_0 = T_1x_0 = T_6x_1$. For $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, we can find a point $x_2 \in X_{\hat{v}}$ such that $\xi_1 = T_2x_1 = T_5x_2$; for $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, we can find a point x_3 in $X_{\hat{v}}$ such that $\xi_2 = T_3x_2 = T_4x_3$. Now for all $\lambda > 0$, in general, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ in $X_{\hat{v}}$ such that Eq (3.7) holds. For any

$n_0 \in \mathbb{N}$ such that $\xi_{n_0} = \xi_{n_0+1}$, and $T_3(X_{\hat{\nu}}) \subseteq T_4(X_{\hat{\nu}})$, $T_2(X_{\hat{\nu}}) \subseteq T_5(X_{\hat{\nu}})$, and $T_1(X_{\hat{\nu}}) \subseteq T_6(X_{\hat{\nu}})$ hold, if, $m \in \mathbb{N}$ such that $\xi_{m+2} = \xi_{m+3}$, then $T_1u = T_6u$, where $u = x_{m+3}$. Therefore, the pair $\{T_1, T_6\}$ has a coincidence point $u \in X_{\hat{\nu}}$. If $\xi_m = \xi_{m+1}$, then $T_2u = T_4u$, where $u = x_{m+1}$. Therefore, the pair $\{T_2, T_4\}$ has a coincidence point $u \in X_{\hat{\nu}}$. If $\xi_{m+1} = \xi_{m+3}$, then $T_3u = T_5u$, where $u = x_{m+2}$. Thus, the pair $\{T_3, T_5\}$ has a coincidence point $u \in X_{\hat{\nu}}$. Again, if there is an $n_0 \in \mathbb{N}$ such that $\xi_{n_0} = \xi_{n_0+1} = \xi_{n_0+2}$, then $\xi_n = \xi_{n_0}$ for any $n \geq n_0$. This implies that $\{\xi_n\}$ is a modular $\hat{\nu}$ Cauchy sequence in $X_{\hat{\nu}}$. Actually, if $\eta \in \mathbb{N}$ exists such that (1) $\xi_\eta = \xi_{\eta+1} = \xi_{\eta+2}$, (2) $\xi_\eta \neq \xi_{\eta+1} = \xi_{\eta+2}$, (3) $\xi_\eta \neq \xi_{\eta+2} = \xi_{\eta+1}$, and (4) $\xi_\eta \neq \xi_{\eta+1} \neq \xi_{\eta+2}$ hold. In fact, Case (1) is easy, and Case (3) is similar to Case (2); therefore, from inequalities (3.100)–(3.102), we get the result by setting $x = \xi_{\eta+2}$ and $y = \xi_{\eta+3}$. By Theorem 1, T_i has a fixed point in $X_{\hat{\nu}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has a common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$. □

Remark 8. *Corollary 6 is a generalization of [34, Corollary 3.12] and the results in Karapinar et al. [26].*

Corollary 7. *Let $X_{\hat{\nu}}$ be a $\hat{\nu}$-regular $\hat{\nu}$-complete modular extended b-metric space and let $T_i : X_{\hat{\nu}} \longrightarrow X_{\hat{\nu}}$ be six orbitally continuous mappings satisfying the α-$\hat{\nu}$-A-B-C-Meir-Keeler-type contraction for $i = 1, 2, \cdots, 6$, where $\{T_2, T_4\}$, $\{T_3, T_5\}$, and $\{T_1, T_6\}$ are weakly commuting pairs of self-mappings such that $T_3(X_{\hat{\nu}}) \subseteq T_4(X_{\hat{\nu}})$, $T_2(X_{\hat{\nu}}) \subseteq T_5(X_{\hat{\nu}})$, $T_1(X_{\hat{\nu}}) \subseteq T_6(X_{\hat{\nu}})$. A function $\alpha : X_{\hat{\nu}} \times X_{\hat{\nu}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{\nu}}$ exist such that $\alpha(x_0, x_1) \geq 1$, $\epsilon, \delta > 0$ and, for each $i = 1, \cdots, 6$, the mapping T_i is classified as a triangular α-orbital admissible function for every $\lambda > 0$, provided that certain conditions are satisfied for a positive integer $m \geq 1$.*

$$\alpha(x,y)\hat{\nu}_\lambda(T_1^m x, T_2^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^A(T_1^m x, T_2^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.106}$$

$$\alpha(x,y)\hat{\nu}_\lambda(T_1^m x, T_3^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^B(T_1^m x, T_3^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.107}$$

$$\alpha(x,y)\hat{\nu}_\lambda(T_2^m x, T_3^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^C(T_2^m x, T_3^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.108}$$

where

$$F_\lambda^A(T_1^m x, T_2^m y) := a \max\left\{ \hat{\nu}_\lambda(T_6^m x, T_4^m y), \hat{\nu}_\lambda(T_1^m x, T_6^m y) \right\}; \tag{3.109}$$

$$F_\lambda^B(T_1^m x, T_3^m y) := b \max\left\{ \hat{\nu}_\lambda(T_5^m x, T_4^m y), \hat{\nu}_\lambda(T_2^m x, T_5^m y) \right\}; \tag{3.110}$$

$$F_\lambda^C(T_2^m x, T_3^m y) := c \max\left\{ \hat{\nu}_\lambda(T_6^m x, T_2^m x), \hat{\nu}_\lambda(T_2^m x, T_5^m y) \right\}. \tag{3.111}$$

Let the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ be in $X_{\hat{\nu}}$ so that for x_n in $X_{\hat{\nu}}$, we choose x_{n+1} such that $\xi_n = T_1x_n = T_6x_{n+1}$; again, for x_{n+1} in $X_{\hat{\nu}}$, we choose x_{n+2} such that $\xi_{n+1} = T_2x_{n+1} = T_5x_{n+2}$ and, for a point x_{n+2} in $X_{\hat{\nu}}$, we choose x_{n+3} such that $\xi_{n+2} = T_3x_{n+2} = T_4x_{n+3}$ for $n = 0, 1, 2, \cdots$. Then T_i has a fixed point in $X_{\hat{\nu}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$ for some positive integer, $m \geq 1$.

Proof. By Corollary 6, we get $p = T_1^m p = T_6^m p$, $p = T_2^m p = T_4^m p$, and $p = T_3^m p = T_5^m p$. Thus, $p = T_1^m p = T_2^m p = T_3^m p = T_4^m p = T_5^m p = T_6^m p$, or $p \in Fix(T_i^m)$ for $i = 1, 2, \cdots, 6$ and some positive integer, $m \geq 1$, showing that p is a fixed point of T_1^m and also a fixed point of $T_2^m, T_3^m, T_4^m, T_5^m, T_6^m$ respectively. Therefore, the uniqueness follows as in Theorem 1 above. Hence, T_i has a common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$ and some positive integer $m \geq 1$. □

Remark 9. *Corollary 7 is a generalization of [34, Corollary 3.13].*

Corollary 8. *Let $X_{\hat{\nu}}$ be a $\hat{\nu}$-regular $\hat{\nu}$-complete modular extended b-metric space and let $T_i : X_{\hat{\nu}} \longrightarrow X_{\hat{\nu}}$ be six orbitally continuous mappings satisfying the α-$\hat{\nu}$-A-B-C-Meir-Keeler-type contraction for $i = 1, 2, \cdots, 6$, where $\{T_2, T_4\}$, $\{T_3, T_5\}$ and $\{T_1, T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\hat{\nu}}) \subseteq T_4(X_{\hat{\nu}})$, $T_2(X_{\hat{\nu}}) \subseteq T_5(X_{\hat{\nu}})$, $T_1(X_{\hat{\nu}}) \subseteq T_6(X_{\hat{\nu}})$. A function $\alpha : X_{\hat{\nu}} \times X_{\hat{\nu}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{\nu}}$ exist such that $\alpha(x_0, x_1) \geq 1$, $\epsilon, \delta > 0$ and, for each $i = 1, \cdots, 6$, T_i remains a triangular α-orbital admissible mapping for all $\lambda > 0$ satisfying the following conditions:*

$$\alpha(x, y)\hat{\nu}_\lambda(T_1x, T_2y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^A_\lambda(T_1x, T_2y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.112}$$

$$\alpha(x, y)\hat{\nu}_\lambda(T_1x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^B_\lambda(T_1x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.113}$$

$$\alpha(x, y)\hat{\nu}_\lambda(T_2x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^C_\lambda(T_2x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.114}$$

where

$$F^A_\lambda(T_1x, T_2y) := a\hat{\nu}_\lambda(T_6x, T_4y); \tag{3.115}$$

$$F^B_\lambda(T_1x, T_3y) := b\hat{\nu}_\lambda(T_5x, T_4y); \tag{3.116}$$

$$F^C_\lambda(T_2x, T_3y) := c\hat{\nu}_\lambda(T_6x, T_2x). \tag{3.117}$$

Let the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ be in $X_{\hat{\nu}}$ so that for x_n in $X_{\hat{\nu}}$, we choose x_{n+1} such that $\xi_n = T_1x_n = T_6x_{n+1}$; again, for x_{n+1} in $X_{\hat{\nu}}$, we choose x_{n+2} such that $\xi_{n+1} = T_2x_{n+1} = T_5x_{n+2}$ and, for a point x_{n+2} in $X_{\hat{\nu}}$, we choose x_{n+3} such that $\xi_{n+2} = T_3x_{n+2} = T_4x_{n+3}$ for $n = 0, 1, 2, \cdots$. Then T_i has a fixed point in $X_{\hat{\nu}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has a unique common fixed point in $\bigcap_{i=1} Fix(T_i)$ for each $i = 1, \cdots, 6$.

Proof. If we take $X_{\hat{\nu}}$ to be empty, then there is nothing to prove. Now, we assume that $X_{\hat{\nu}} \neq \emptyset$. Then a function $\alpha : X_{\hat{\nu}} \times X_{\hat{\nu}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{\nu}}$ exist such that $\alpha(x_0, x_1) \geq 1$, $\epsilon, \delta > 0$ and, for each $i = 1, \cdots, 6$, T_i remains a triangular α-orbital admissible mappings for every $\lambda > 0$. Suppose that the mappings, T_i for $i = 1, \cdots, 6$ satisfy inequalities the (3.112)–(3.114). Since x_0, x_1, and x_2 are points in $X_{\hat{\nu}}$ and $T_1(X_{\hat{\nu}}) \subseteq T_6(X_{\hat{\nu}})$, we can find a point x_1 in $X_{\hat{\nu}}$ such that $\xi_0 = T_1x_0 = T_6x_1$. For $T_2(X_{\hat{\nu}}) \subseteq T_5(X_{\hat{\nu}})$, we can find a point $x_2 \in X_{\hat{\nu}}$ such that $\xi_1 = T_2x_1 = T_5x_2$; for $T_3(X_{\hat{\nu}}) \subseteq T_4(X_{\hat{\nu}})$, we can find a point x_3 in $X_{\hat{\nu}}$ such that $\xi_2 = T_3x_2 = T_4x_3$. Now for all $\lambda > 0$, in general, the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ are in $X_{\hat{\nu}}$ such that Eq (3.7) hold. If $n_0 \in \mathbb{N}$ such that $\xi_{n_0} = \xi_{n_0+1}$, then $T_3(X_{\hat{\nu}}) \subseteq T_4(X_{\hat{\nu}})$, $T_2(X_{\hat{\nu}}) \subseteq T_5(X_{\hat{\nu}})$, and $T_1(X_{\hat{\nu}}) \subseteq T_6(X_{\hat{\nu}})$ hold. In fact, if there is an $m \in \mathbb{N}$ such that $\xi_{m+2} = \xi_{m+3}$, then $T_1u = T_6u$, where $u = x_{m+3}$. Therefore, the pair $\{T_1, T_6\}$ has a coincidence point $u \in X_{\hat{\nu}}$. If $\xi_m = \xi_{m+1}$, then $T_2u = T_4u$, where $u = x_{m+1}$. Therefore, the pair $\{T_2, T_4\}$ has a coincidence point $u \in X_{\hat{\nu}}$. If $\xi_{m+1} = \xi_{m+3}$, then $T_3u = T_5u$, where $u = x_{m+2}$. Thus, the pair $\{T_3, T_5\}$ has a coincidence point $u \in X_{\hat{\nu}}$. Again, if there is an $n_0 \in \mathbb{N}$ such that $\xi_{n_0} = \xi_{n_0+1} = \xi_{n_0+2}$, then $\xi_n = \xi_{n_0}$ for any $n \geq n_0$. This implies that $\{\xi_n\}$ is a modular $\hat{\nu}$ Cauchy sequence in $X_{\hat{\nu}}$. Actually, if $\eta \in \mathbb{N}$ exists such that (1) $\xi_\eta = \xi_{\eta+1} = \xi_{\eta+2}$, (2) $\xi_\eta \neq \xi_{\eta+1} = \xi_{\eta+2}$, (3) $\xi_\eta \neq \xi_{\eta+2} = \xi_{\eta+1}$, and (4) $\xi_\eta \neq \xi_{\eta+1} \neq \xi_{\eta+2}$ hold. In fact, Case (1) is easy, and Case (3) is similar to Case (2); therefore, from the inequalities (3.112)–(3.114), we get the result by setting $x = \xi_{\eta+2}$ and $y = \xi_{\eta+3}$. By Theorem 1, T_i has a fixed point in $X_{\hat{\nu}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has a common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$. □

Remark 10. *Corollary 8 is a generalization of [20, Theorem 2.8] and results in Karapinar et al. [26].*

Corollary 9. *Let $X_{\hat{v}}$ be a $\hat{v}$-regular $\hat{v}$-complete modular extended b-metric space and let $T_i : X_{\hat{v}} \longrightarrow X_{\hat{v}}$ be six orbitally continuous mappings satisfying the α-$\hat{v}$-A-B-C-Meir-Keeler-type contraction for $i = 1, 2, \cdots, 6$, and let $\{T_2, T_4\}$, $\{T_3, T_5\}$, and $\{T_1, T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$. A function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{v}}$ exist such that $\alpha(x_0, x_1) \geq 1$, $\epsilon, \delta > 0$, and, for each $i = 1, \cdots, 6$, T_i remain a triangular α-orbital admissible mapping for all $\lambda > 0$ satisfying the following conditions for some positive integer $m \geq 1$:*

$$\alpha(x,y)\hat{v}_\lambda(T_1^m x, T_2^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^A(T_1^m x, T_2^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.118}$$

$$\alpha(x,y)\hat{v}_\lambda(T_1^m x, T_3^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^B(T_1^m x, T_3^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.119}$$

$$\alpha(x,y)\hat{v}_\lambda(T_2^m x, T_3^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^C(T_2^m x, T_3^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.120}$$

where,

$$F_\lambda^A(T_1^m x, T_2^m y) := a\hat{v}_\lambda(T_6^m x, T_4^m y); \tag{3.121}$$

$$F_\lambda^B(T_1^m x, T_3^m y) := b\hat{v}_\lambda(T_5^m x, T_4^m y); \tag{3.122}$$

$$F_\lambda^C(T_2^m x, T_3^m y) := c\hat{v}_\lambda(T_6^m x, T_2^m x). \tag{3.123}$$

Let the two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ be in $X_{\hat{v}}$ so that for x_n in $X_{\hat{v}}$, we choose x_{n+1} such that $\xi_n = T_1 x_n = T_6 x_{n+1}$; again, for x_{n+1} in $X_{\hat{v}}$, we choose x_{n+2} such that $\xi_{n+1} = T_2 x_{n+1} = T_5 x_{n+2}$ and, for a point x_{n+2} in $X_{\hat{v}}$, we choose x_{n+3} such that $\xi_{n+2} = T_3 x_{n+2} = T_4 x_{n+3}$ for $n = 0, 1, 2, \cdots$. Then T_i has a fixed point in $X_{\hat{v}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$ for some positive integer, $m \geq 1$.

Proof. By Corollary 8, $p = T_1^m p = T_6^m p$, $p = T_2^m p = T_4^m p$ and $p = T_3^m p = T_5^m p$. Thus, $p = T_1^m p = T_2^m p = T_3^m p = T_4^m p = T_5^m p = T_6^m p$ or $p \in Fix(T_i^m)$ for $i = 1, 2, \cdots, 6$ and some positive integer, $m \geq 1$; showing that p is a fixed point of T_1^m and also a fixed point of $T_2^m, T_3^m, T_4^m, T_5^m,$ *and* T_6^m respectively. Therefore, the uniqueness follows as in Theorem 1 above. Hence, T_i has common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$ and some positive integer, $m \geq 1$. □

Corollary 10. *Let $X_{\hat{v}}$ be a $\hat{v}$-regular $\hat{v}$-complete modular extended b-metric space and $T_i : X_{\hat{v}} \longrightarrow X_{\hat{v}}$ be six orbitally continuous mappings satisfying the α-$\hat{v}$-A-B-C-Meir-Keeler-type contraction for $i = 1, 2, \cdots, 6$, and let $\{T_2, T_4\}$, $\{T_3, T_5\}$ and $\{T_1, T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$. A function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{v}}$ exist such that $\alpha(x_0, x_1) \geq 1$, $\epsilon, \delta > 0$ and, for each $i = 1, \cdots, 6$, T_i remains a triangular α-orbital admissible mappings for all $\lambda > 0$ for which the following hold:*

$$\alpha(x,y)\hat{v}_\lambda(T_1 x, T_2 y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^A(T_1 x, T_2 y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.124}$$

$$\alpha(x,y)\hat{v}_\lambda(T_1 x, T_3 y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^B(T_1 x, T_3 y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.125}$$

$$\alpha(x,y)\hat{v}_\lambda(T_2 x, T_3 y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^C(T_2 x, T_3 y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.126}$$

where

$$F_\lambda^A(T_1 x, T_2 y) := a\hat{v}_\lambda(T_6 x, T_4 y); \tag{3.127}$$

$$F_\lambda^B(T_1 x, T_3 y) := b\hat{v}_\lambda(T_5 x, T_4 y); \tag{3.128}$$

$$F_\lambda^C(T_2 x, T_3 y) := c\hat{v}_\lambda(T_2 x, T_5 y). \tag{3.129}$$

Let the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ be in $X_{\hat{v}}$ so that for x_n in $X_{\hat{v}}$, we choose x_{n+1} such that $\xi_n = T_1x_n = T_6x_{n+1}$; again. for x_{n+1} in $X_{\hat{v}}$, we choose x_{n+2} such that $\xi_{n+1} = T_2x_{n+1} = T_5x_{n+2}$ and, for a point x_{n+2} in $X_{\hat{v}}$, we choose x_{n+3} such that $\xi_{n+2} = T_3x_{n+2} = T_4x_{n+3}$ for $n = 0, 1, 2, \cdots$. Then T_i has a fixed point in $X_{\hat{v}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has a unique common fixed point in $\bigcap_{i=1} Fix(T_i)$ for each $i = 1, \cdots, 6$.

Proof. Suppose that $X_{\hat{v}}$ is empty; then there is nothing to prove. We now assume that $X_{\hat{v}} \neq \emptyset$. Then a function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{v}}$ exist such that $\alpha(x_0, x_1) \geq 1$, $\epsilon, \delta > 0$ and, for each $i = 1, \cdots, 6$, T_i remains a triangular α-orbital admissible mappings for every $\lambda > 0$. Suppose that the mappings T_i for $i = 1, \cdots, 6$ satisfy the inequalities (3.124)–(3.126). Since x_0, x_1, and x_2 are points in $X_{\hat{v}}$ and $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$, we can find a point x_1 in $X_{\hat{v}}$ such that $\xi_0 = T_1x_0 = T_6x_1$. For $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, we can find a point $x_2 \in X_{\hat{v}}$ such that $\xi_1 = T_2x_1 = T_5x_2$, and for $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, we can find a point x_3 in $X_{\hat{v}}$ such that $\xi_2 = T_3x_2 = T_4x_3$. Now for all $\lambda > 0$, induce on n so that the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ are in $X_{\hat{v}}$ such that Eq (3.7) hold. For any $n_0 \in \mathbb{N}$ such that $\xi_{n_0} = \xi_{n_0+1}$, then $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$ holds. In fact, if $m \in \mathbb{N}$ exists such that $\xi_{m+2} = \xi_{m+3}$, then $T_1u = T_6u$, where $u = x_{m+3}$. Therefore, the pair $\{T_1, T_6\}$ has a coincidence point $u \in X_{\hat{v}}$. If $\xi_m = \xi_{m+1}$, then $T_2u = T_4u$, where $u = x_{m+1}$. Therefore, the pair $\{T_2, T_4\}$ has a coincidence point $u \in X_{\hat{v}}$. If $\xi_{m+1} = \xi_{m+3}$, then $T_3u = T_5u$, where $u = x_{m+2}$. Thus, the pair $\{T_3, T_5\}$ has a coincidence point $u \in X_{\hat{v}}$. Again, if there is an $n_0 \in \mathbb{N}$ such that $\xi_{n_0} = \xi_{n_0+1} = \xi_{n_0+2}$, then $\xi_n = \xi_{n_0}$ for any $n \geq n_0$. This implies that $\{\xi_n\}$ is a modular $\hat{v}$ Cauchy sequence in $X_{\hat{v}}$. Actually, if $\eta \in \mathbb{N}$ exists such that (1) $\xi_\eta = \xi_{\eta+1} = \xi_{\eta+2}$, (2) $\xi_\eta \neq \xi_{\eta+1} = \xi_{\eta+2}$, (3) $\xi_\eta \neq \xi_{\eta+2} = \xi_{\eta+1}$, and (4) $\xi_\eta \neq \xi_{\eta+1} \neq \xi_{\eta+2}$ hold. In fact, Case (1) is easy, and Case (3) is similar to Case (2); therefore, from the inequalities (3.124)–(3.126), we get the result by setting $x = \xi_{\eta+2}$ and $y = \xi_{\eta+3}$. By Theorem 1, T_i has a fixed point in $X_{\hat{v}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has a common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$. □

Remark 11. *(1) If $T = T_1 = \cdots = T_6$, then the inequalities* (3.124)–(3.126) *and Eqs* (3.127)–(3.129) *of Corollary 10 coincides, which is a modification of [20, Theorem 2.8].*
(2) $T_1 = T_2 = T_3 = T$ and $T_4 = T_5 = T_6 = I$, $c = 0$, and $a+b = 1$, then the inequalities (3.124)–(3.126) *and Eqs* (3.127)–(3.129) *of Corollary 10 coincide with [20, Theorem 2.8].*

Corollary 11. *Let $X_{\hat{v}}$ be a $\hat{v}$-regular $\hat{v}$-complete modular extended b-metric space and let $T_i : X_{\hat{v}} \longrightarrow X_{\hat{v}}$ be six orbitally continuous mappings satisfying the α-$\hat{v}$-A-B-C-Meir-Keeler-type contraction for $i = 1, 2, \cdots, 6$, and let $\{T_2, T_4\}$, $\{T_3, T_5\}$, and $\{T_1, T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$. A function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{v}}$ exist such that $\alpha(x_0, x_1) \geq 1$, $\epsilon, \delta > 0$, and, for each $i = 1, \cdots, 6$, T_i is triangular α-orbital admissible mapping for all $\lambda > 0$ and for some positive integer $m \geq 1$, for which:*

$$\alpha(x,y)\hat{v}_\lambda(T_1^m x, T_2^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^A(T_1^m x, T_2^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.130}$$

$$\alpha(x,y)\hat{v}_\lambda(T_1^m x, T_3^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^B(T_1^m x, T_3^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.131}$$

$$\alpha(x,y)\hat{v}_\lambda(T_2^m x, T_3^m y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^C(T_2^m x, T_3^m y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.132}$$

where

$$F_\lambda^A(T_1^m x, T_2^m y) := a\hat{v}_\lambda(T_6^m x, T_4^m y); \tag{3.133}$$

$$F_\lambda^B(T_1^m x, T_3^m y) := b\hat{\nu}_\lambda(T_5^m x, T_4^m y); \tag{3.134}$$
$$F_\lambda^C(T_2^m x, T_3^m y) := c\hat{\nu}_\lambda(T_2^m x, T_5^m y). \tag{3.135}$$

Let the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ be in $X_{\hat{\nu}}$ so that for x_n in $X_{\hat{\nu}}$, we choose x_{n+1} such that $\xi_n = T_1x_n = T_6x_{n+1}$; again, for x_{n+1} in $X_{\hat{\nu}}$, we choose x_{n+2} such that $\xi_{n+1} = T_2x_{n+1} = T_5x_{n+2}$ and, for a point x_{n+2} in $X_{\hat{\nu}}$, we choose x_{n+3} such that $\xi_{n+2} = T_3x_{n+2} = T_4x_{n+3}$ for $n = 0, 1, 2, \cdots$. Then T_i has a fixed point in $X_{\hat{\nu}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$ for some positive integer, $m \geq 1$.

Proof. By Corollary 10, $p = T_1^m p = T_6^m p$, $p = T_2^m p = T_4^m p$, and $p = T_3^m p = T_5^m p$. Thus, $p = T_1^m p = T_2^m p = T_3^m p = T_4^m p = T_5^m p = T_6^m p$, or $p \in Fix(T_i^m)$ for $i = 1, 2, \cdots, 6$ and some positive integer $m \geq 1$, showing that p is a fixed point of T_1^m and also a fixed point of $T_2^m, T_3^m, T_4^m, T_5^m$, *and* T_6^m respectively. Therefore, the uniqueness follows as in Theorem 1 above. Hence, T_i has a common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$ and some positive integer, $m \geq 1$. □

Remark 12. *Corollary 11 is a generalization of [20, Theorem 2.8].*

Corollary 12. *Let $X_{\hat{\nu}}$ be a $\hat{\nu}$-regular $\hat{\nu}$-complete modular extended b-metric space and let $T_i : X_{\hat{\nu}} \longrightarrow X_{\hat{\nu}}$ be six orbitally continuous mappings satisfying the α-$\hat{\nu}$-A-B-C-Meir-Keeler-type contraction for $i = 1, 2, \cdots, 6$, and let $\{T_2, T_4\}$, $\{T_3, T_5\}$, and $\{T_1, T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\hat{\nu}}) \subseteq T_4(X_{\hat{\nu}})$, $T_2(X_{\hat{\nu}}) \subseteq T_5(X_{\hat{\nu}})$, $T_1(X_{\hat{\nu}}) \subseteq T_6(X_{\hat{\nu}})$. A function $\alpha : X_{\hat{\nu}} \times X_{\hat{\nu}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{\nu}}$ exist such that $\alpha(x_0, x_1) \geq 1$, $\epsilon, \delta > 0$, and, for each $i = 1, \cdots, 6$, T_i remains a triangular α-orbital admissible mapping for all $\lambda > 0$ satisfying the following conditions:*

$$\alpha(x, y)\hat{\nu}_\lambda(T_1x, T_2y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^A(T_1x, T_2y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.136}$$
$$\alpha(x, y)\hat{\nu}_\lambda(T_1x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^B(T_1x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.137}$$
$$\alpha(x, y)\hat{\nu}_\lambda(T_2x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^C(T_2x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.138}$$

where,

$$F_\lambda^A(T_1x, T_2y) := a\max\left\{\hat{\nu}_\lambda(T_6x, T_4y), \frac{1}{3}\{\hat{\nu}_\lambda(T_1x, T_6y) + \hat{\nu}_\lambda(T_3x, T_4y) + \hat{\nu}_\lambda(T_2x, T_5y)\}\right\}; \tag{3.139}$$
$$F_\lambda^B(T_1x, T_3y) := b\max\left\{\hat{\nu}_\lambda(T_5x, T_4y), \frac{1}{3}\{\hat{\nu}_\lambda(T_2x, T_5y) + \hat{\nu}_\lambda(T_1x, T_6^2x) + \hat{\nu}_\lambda(T_3y, T_4^2y)\}\right\}; \tag{3.140}$$
$$F_\lambda^C(T_2x, T_3y) := c\max\left\{\hat{\nu}_\lambda(T_6x, T_2x), \frac{1}{3}\{\hat{\nu}_\lambda(T_2x, T_5y) + \hat{\nu}_\lambda(T_3x, T_4y) + \hat{\nu}_\lambda(T_1x, T_6y)\}\right\}. \tag{3.141}$$

Let the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ be in $X_{\hat{\nu}}$ so that for x_n in $X_{\hat{\nu}}$, we choose x_{n+1} such that $\xi_n = T_1x_n = T_6x_{n+1}$; again, for x_{n+1} in $X_{\hat{\nu}}$, we choose x_{n+2} such that $\xi_{n+1} = T_2x_{n+1} = T_5x_{n+2}$ and, for a point x_{n+2} in $X_{\hat{\nu}}$, we choose x_{n+3} such that $\xi_{n+2} = T_3x_{n+2} = T_4x_{n+3}$ for $n = 0, 1, 2, \cdots$. Then T_i has a fixed point in $X_{\hat{\nu}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has a unique common fixed point in $\bigcap_{i=1} Fix(T_i)$ for each $i = 1, \cdots, 6$.

Proof. Upon appeal to Theorem 1, we can see that T_i has a common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$. □

Corollary 13. *Let $X_{\hat{v}}$ be a $\hat{v}$-regular $\hat{v}$-complete modular extended b-metric space and let $T_i : X_{\hat{v}} \longrightarrow X_{\hat{v}}$ be six orbitally continuous mappings satisfying the α-$\hat{v}$-A-B-C-Meir-Keeler-type contraction for $i = 1, 2, \cdots, 6$, and let $\{T_2, T_4\}$, $\{T_3, T_5\}$, and $\{T_1, T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$. A function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{v}}$ exist such that $\alpha(x_0, x_1) \geq 1$, $\epsilon, \delta > 0$ and, for each $i = 1, \cdots, 6$, T_i remains a triangular α-orbital admissible mapping for all $\lambda > 0$ satisfying the following conditions:*

$$\alpha(x,y)\hat{v}_\lambda(T_1x, T_2y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^A(T_1x, T_2y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.142}$$
$$\alpha(x,y)\hat{v}_\lambda(T_1x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^B(T_1x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.143}$$
$$\alpha(x,y)\hat{v}_\lambda(T_2x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^C(T_2x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.144}$$

where

$$F_\lambda^A(T_1x, T_2y) := a^2 \max\left\{ \frac{\max\{\hat{v}_\lambda(T_6x, T_4y), \hat{v}_\lambda(T_1x, T_6y)\}}{a + \min\{\hat{v}_\lambda(T_3x, T_4y), \hat{v}_\lambda(T_2x, T_5y)\}}, \frac{\min\{\hat{v}_\lambda(T_3x, T_4y), \hat{v}_\lambda(T_2x, T_5y)\}}{a + \min\{\hat{v}_\lambda(T_3x, T_4y), \hat{v}_\lambda(T_2x, T_5y)\}} \right\}; \tag{3.145}$$

$$F_\lambda^B(T_1x, T_3y) := b^2 \max\left\{ \frac{\max\{\hat{v}_\lambda(T_5x, T_4y), \hat{v}_\lambda(T_2x, T_5y)\}}{b + \min\{\hat{v}_\lambda(T_1x, T_6^2x), \hat{v}_\lambda(T_3y, T_4^2y)\}}, \frac{\min\{\hat{v}_\lambda(T_1x, T_6^2x), \hat{v}_\lambda(T_3y, T_4^2y)\}}{b + \min\{\hat{v}_\lambda(T_1x, T_6^2x), \hat{v}_\lambda(T_3y, T_4^2y)\}} \right\}; \tag{3.146}$$

$$F_\lambda^C(T_2x, T_3y) := c^2 \max\left\{ \frac{\max\{\hat{v}_\lambda(T_6x, T_2x), \hat{v}_\lambda(T_2x, T_5y)\}}{c + \min\{\hat{v}_\lambda(T_3x, T_4y), \hat{v}_\lambda(T_1x, T_6y)\}}, \frac{\min\{\hat{v}_\lambda(T_3x, T_4y), \hat{v}_\lambda(T_1x, T_6y)\}}{c + \min\{\hat{v}_\lambda(T_3x, T_4y), \hat{v}_\lambda(T_1x, T_6y)\}} \right\}. \tag{3.147}$$

Let the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ be in $X_{\hat{v}}$ so that for x_n in $X_{\hat{v}}$, we choose x_{n+1} such that $\xi_n = T_1x_n = T_6x_{n+1}$; again, for x_{n+1} in $X_{\hat{v}}$, we choose x_{n+2} such that $\xi_{n+1} = T_2x_{n+1} = T_5x_{n+2}$ and, for a point x_{n+2} in $X_{\hat{v}}$, we choose x_{n+3} such that $\xi_{n+2} = T_3x_{n+2} = T_4x_{n+3}$ for $n = 0, 1, 2, \cdots$. Then T_i has a fixed point in $X_{\hat{v}}$. Moreover, if $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has a unique common fixed point in $\bigcap_{i=1} Fix(T_i)$ for each $i = 1, \cdots, 6$.

Proof. Thanks to Corollary 2, we can see that T_i has a common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1, \cdots, 6$. □

Corollary 14. *Let $X_{\hat{v}}$ be a $\hat{v}$-regular $\hat{v}$-complete modular extended b-metric space and let $T_i : X_{\hat{v}} \longrightarrow X_{\hat{v}}$ be six orbitally continuous mappings satisfying the α-$\hat{v}$-A-B-C-Meir-Keeler-type contraction for $i = 1, 2, \cdots, 6$, and let $\{T_2, T_4\}$, $\{T_3, T_5\}$, and $\{T_1, T_6\}$ be weakly commuting pairs of self-mappings such that $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$. A function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0, +\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{v}}$ exist such that $\alpha(x_0, x_1) \geq 1$, $\epsilon, \delta > 0$ and, for each $i = 1, \cdots, 6$, T_i remains a triangular α-orbital admissible mapping for all $\lambda > 0$ satisfying the following conditions:*

$$\alpha(x,y)\hat{v}_\lambda(T_1x, T_2y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^A(T_1x, T_2y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.148}$$
$$\alpha(x,y)\hat{v}_\lambda(T_1x, T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F_\lambda^B(T_1x, T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)); \tag{3.149}$$

$$\alpha(x,y)\hat{v}_\lambda(T_2x,T_3y) < \mathfrak{A}(\epsilon) \implies \epsilon \leq \mathfrak{A}^{-1}(F^C_\lambda(T_2x,T_3y)) < \mathfrak{A}(\epsilon) + \mathfrak{A}(\delta(\epsilon)), \tag{3.150}$$

where

$$F^A_\lambda(T_1x,T_2y) := a\max\Big\{\hat{v}_\lambda(T_6x,T_4y), \hat{v}_\lambda(T_1x,T_6y), \hat{v}_\lambda(T_3x,T_4y), \hat{v}_\lambda(T_2x,T_5y), \frac{\hat{v}_\lambda(T_3x,T_4y)+\hat{v}_\lambda(T_2x,T_5y)}{2}\Big\}; \tag{3.151}$$

$$F^B_\lambda(T_1x,T_3y) := b\max\Big\{\hat{v}_\lambda(T_5x,T_4y), \hat{v}_\lambda(T_2x,T_5y), \hat{v}_\lambda(T_1x,T_6^2x), \hat{v}_\lambda(T_3y,T_4^2y), \frac{\hat{v}_\lambda(T_5x,T_4y)+\hat{v}_\lambda(T_2x,T_5y)}{2}\Big\}; \tag{3.152}$$

$$F^C_\lambda(T_2x,T_3y) := c\max\Big\{\hat{v}_\lambda(T_6x,T_2x), \hat{v}_\lambda(T_2x,T_5y), \hat{v}_\lambda(T_3x,T_4y), \hat{v}_\lambda(T_1x,T_6y), \frac{\hat{v}_\lambda(T_3x,T_4y)+\hat{v}_\lambda(T_1x,T_6y)}{2}\Big\}. \tag{3.153}$$

Let the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ be in $X_{\hat{v}}$ so that for x_n in $X_{\hat{v}}$, we choose x_{n+1} such that $\xi_n = T_1x_n = T_6x_{n+1}$; again, for x_{n+1} in $X_{\hat{v}}$, we choose x_{n+2} such that $\xi_{n+1} = T_2x_{n+1} = T_5x_{n+2}$ and, for a point x_{n+2} in $X_{\hat{v}}$, we choose x_{n+3} such that $\xi_{n+2} = T_3x_{n+2} = T_4x_{n+3}$ for $n = 0,1,2,\cdots$. Then T_i has a fixed point in $X_{\hat{v}}$. Moreover, if $\alpha(x^*,y^*) \geq 1$ for all $x^*,y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has a unique common fixed point in $\bigcap_{i=1} Fix(T_i)$ for each $i = 1,\cdots,6$.

Proof. If we take $X_{\hat{v}}$ to be empty, then there is nothing to prove. Now, we assume that $X_{\hat{v}} \neq \emptyset$. Then a function $\alpha : X_{\hat{v}} \times X_{\hat{v}} \to [0,+\infty)$, $a \neq 0$, $a < 1$, $b \neq 0$, $\frac{ac}{b} < 1$, and $x_0 \in X_{\hat{v}}$ exist such that $\alpha(x_0,x_1) \geq 1$, $\epsilon,\delta > 0$ and, for each $i = 1,\cdots,6$, T_i remains a triangular α-orbital admissible mappings for every $\lambda > 0$. Suppose that the mappings, T_i for $i = 1,\cdots,6$ satisfy inequalities the (3.148)–(3.150). Since x_0, x_1, and x_2 are points in $X_{\hat{v}}$ and $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$, we can find a point x_1 in $X_{\hat{v}}$ such that $\xi_0 = T_1x_0 = T_6x_1$. For $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, we can find a point $x_2 \in X_{\hat{v}}$ such that $\xi_1 = T_2x_1 = T_5x_2$; for $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, we can find a point x_3 in $X_{\hat{v}}$ such that $\xi_2 = T_3x_2 = T_4x_3$. Now for all $\lambda > 0$, inductively, sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ in $X_{\hat{v}}$ exist such that Eq (3.7) hold. For any integer, $n_0 \in \mathbb{N}$ such that $\xi_{n_0} = \xi_{n_0+1}$, then $T_3(X_{\hat{v}}) \subseteq T_4(X_{\hat{v}})$, $T_2(X_{\hat{v}}) \subseteq T_5(X_{\hat{v}})$, and $T_1(X_{\hat{v}}) \subseteq T_6(X_{\hat{v}})$ hold. In fact, if $m \in \mathbb{N}$ exists such that $\xi_{m+2} = \xi_{m+3}$, then $T_1u = T_6u$, where $u = x_{m+3}$. Therefore, the pair $\{T_1,T_6\}$ has a coincidence point $u \in X_{\hat{v}}$. If $\xi_m = \xi_{m+1}$, then $T_2u = T_4u$, where $u = x_{m+1}$. Therefore, the pair $\{T_2,T_4\}$ has a coincidence point $u \in X_{\hat{v}}$. If $\xi_{m+1} = \xi_{m+3}$, then $T_3u = T_5u$, where $u = x_{m+2}$. Thus, the pair $\{T_3,T_5\}$ has a coincidence point $u \in X_{\hat{v}}$. Again, if there is an $n_0 \in \mathbb{N}$ such that $\xi_{n_0} = \xi_{n_0+1} = \xi_{n_0+2}$, then $\xi_n = \xi_{n_0}$ for any $n \geq n_0$. This implies that $\{\xi_n\}$ is a modular $\hat{v}$ Cauchy sequence in $X_{\hat{v}}$. Actually, if $\eta \in \mathbb{N}$ exists such that (1) $\xi_\eta = \xi_{\eta+1} = \xi_{\eta+2}$, (2) $\xi_\eta \neq \xi_{\eta+1} = \xi_{\eta+2}$, (3) $\xi_\eta \neq \xi_{\eta+2} = \xi_{\eta+1}$, and (4) $\xi_\eta \neq \xi_{\eta+1} \neq \xi_{\eta+2}$ hold. In fact, Case (1) is easy, and Case (3) is similar to Case (2); therefore, from the inequalities (3.148)–(3.150), we get the result by setting $x = \xi_{\eta+2}$ and $y = \xi_{\eta+3}$. By Theorem 1, T_i has a fixed point in $X_{\hat{v}}$. Moreover, if $\alpha(x^*,y^*) \geq 1$ for all $x^*,y^* \in \bigcap_{i=1} Fix(T_i)$, then T_i has common unique fixed point in $\bigcap_{i=1} Fix(T_i)$ for $i = 1,\cdots,6$. □

Example 4. *Suppose that $X_{\hat{v}} = (\mathbb{R} \setminus \{0\}) \cup \{\infty\}$ with the modular extended b-metric as defined in Example 3, which is complete in $X_{\hat{v}}$ for all $\lambda > 0$. Define the $\hat{v}$-weakly commuting mappings*

$T_1, T_2, T_3, T_4, T_5, T_6 : (\mathbb{R} \setminus \{0\}) \cup \{\infty\} \rightarrow (\mathbb{R} \setminus \{0\}) \cup \{\infty\}$ *as follows:* $T_1 x = \log_{64} x^6$, $T_2 x = \log_{32} x^5$, $T_3 x = \log_{16} x^4$, $T_4 x = \log_8 x^3$, $T_5 x = \log_4 x^2$, $T_6 x = \log_2 x$ *for all* $x \in (\mathbb{R} \setminus \{0\}) \cup \{\infty\}$ *and* $\lambda > 0$, *for each* $i = 1, 2, \cdots, 6$, *and also for* $x, y \in (\mathbb{R} \setminus \{0\}) \cup \{\infty\}$, $\alpha(x, T_i x) \geq 1 \implies \alpha(T_i x, T_i^2 x) \geq 1$, *and* $\alpha(x, y) \geq 1$ *and* $\alpha(y, T_i y) \geq 1 \implies \alpha(x, T_i y) \geq 1$. *Then the mappings* T_1, T_2, T_3, T_4, T_5, *and* T_6 *satisfy the inequalities* (3.151)–(3.153) *of Corollary 14. In fact, let* $T_i : X_{\hat{v}} \longrightarrow X_{\hat{v}}$ *be six orbitally continuous* α*-*$\hat{v}$*-A-B-C-Meir-Keeler-type contraction mappings for* $i = 1, 2, \cdots, 6$, *and let* $\{T_2, T_4\}$, $\{T_3, T_5\}$, *and* $\{T_1, T_6\}$ *be weakly commuting pairs of self-mappings. Actually, it suffices to show that the inequalities* (3.1)–(3.3) *coincide with inequalities* (3.148)–(3.150). *Now observe that from Example 3,* $\hat{v}_\lambda(T_3 x, T_4 y) = \dfrac{1}{1+\lambda} \dfrac{1}{\ln(2)} \max\limits_{x,y \in X_{\hat{v}}} \left\{ \ln \left\| \dfrac{x}{y} \right\| \right\}$ *and* $\hat{v}_\lambda(T_2 x, T_5 y) = \dfrac{1}{1+\lambda} \dfrac{1}{\ln(2)} \max\limits_{x,y \in X_{\hat{v}}} \left\{ \ln \left\| \dfrac{x}{y} \right\| \right\}$, *and from inequality* (3.151), *we can see that* $\dfrac{\hat{v}_\lambda(T_3 x, T_4 y) + \hat{v}_\lambda(T_2 x, T_5 y)}{2} = \dfrac{1}{1+\lambda} \dfrac{1}{\ln(2)} \max\limits_{x,y \in X_{\hat{v}}} \left\{ \ln \left\| \dfrac{x}{y} \right\| \right\}$, *which is either* $\hat{v}_\lambda(T_3 x, T_4 y)$ *or* $\hat{v}_\lambda(T_2 x, T_5 y)$. *Similarly, the inequalities* (3.152) *and* (3.153) *hold. It is now evident that the inequalities* (3.1)–(3.3) *and* (3.148)–(3.150) *coincides.*

4. Conclusions

In this paper, we introduced the concept of α-$\hat{v}$-A-B-C-Meir-Keeler-type nonlinear contractions in modular extended b-metric spaces. We established common unique fixed-point theorems that unify and extend existing results in modular b-metric and modular extended b-metric spaces. These findings provide a broader perspective on contraction mappings and deepen the understanding of fixed-point theory in modular settings. The results presented here build on and generalize classical fixed-point theorems, demonstrating the richness of modular extended b-metric spaces.

Author Contributions

Daniel Francis: Conceptualization, methodology, validation, formal analysis, investigation, writing-original draft preparation, writing-review and editing, visualization; Godwin Amechi Okeke: Methodology, validation, resources, writing-original draft preparation, Aviv Gibali: Formal analysis, investigation; writing-original draft preparation, and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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