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Research article

Pseudo-ordering and δ^1-level mappings: A study in fuzzy interval convex analysis

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Abstract: This work utilized the concepts of fuzzy interval analysis and convexity to explore some novel refinements of classical counterparts. The main goal was to look into a type of strong convexity that connected the ideas of pseudo-ordering, δ^1-level mappings, and the control function $\hbar_\circ$. This type of mapping is called a fuzzy number-valued $\hbar_\circ$-super-quadratic mapping. An interesting fact is that all the function classes extracted from this class were new and novel and quite useful in the optimization and approximation theory. We assessed this class of functions pertaining to essential properties, examples, and various integral inequalities such as Jensen's, reverse Jensen's, Jensen-Mercer, Hermite-Hadamard and Fejer's like inequalities in the classical, and fractional framework. Furthermore, we delivered the accuracy of our findings through graphical and tabular approaches, particularly a novel application for means.

Keywords: fuzzy convex mappings; Super-quadratic mappings; Jensen's inequality; Jensen-Mercer's inequality; Hermite-Hadamard's inequality; Fejer's inequality
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1. Introduction

Inequalities are essential and instrumental in dealing with several complex mathematical quantities that appeared in diverse domains of physical sciences. They have been investigated from multiple aspects, including expanding the applicable domain and eliminating the limitations of already proved results and utilizing various approaches from functional analysis, generalized calculus, and convex analysis. Originally, they were pivotal to acquiring bounds for different mappings, integrals, the uniqueness and stability of solutions, error analysis of quadrature algorithms, information theory, etc. Based on these factors, this theory has grown exponentially through convex analysis. Additionally, the impact of concavity is exemplary in inequalities due to plenty of factors, particularly the fact that several inequalities can be derived from the premise of convexity. This suggests that one of the main motives for studying classical inequalities is to characterize convexity and its generalizations. One of the notable classes of convexity depending upon quadratic support is known to us as strong convexity, which generalizes the classical concepts and is highly applied to conclude the novel refinements of already proven results. This class has inspired the development of several new mapping classes in the literature. For comprehensive details on generalizations of strongly convex, consult [1–6]. In 2007, Abramovich et al. [7] demonstrated another concept of a super-quadratic mapping incorporated with translations of itself and a support line. It is defined as:

A mapping $\Psi : [0, \infty) \to \mathbb{R}$ is considered to be a super-quadratic for $\mu \geq 0$ if there exist a constant $C(\mu) \in \mathbb{R}$, such that

$$\Psi(\mu_1) \geq \Psi(\mu) + C(\mu)(\mu_1 - \mu) + \Psi(|\mu - \mu_1|), \quad \mu_1 \geq 0.$$

It can also be interpreted as:

Definition 1.1 ([7]). *A mapping $\Psi : [0, \infty) \to \mathbb{R}$ is called super-quadratic if, and only if,*

$$\Psi((1-\varphi)\mu + \varphi y) \leq (1-\varphi)\Psi(\mu) + \varphi\Psi(y) - \varphi\Psi((1-\varphi)|\mu - y|) - (1-\varphi)\Psi(\varphi|\mu - y|), \tag{1.1}$$

holds $\forall\, \mu, y \geq 0$ and $0 \leq \varphi \leq 1$.

Here, we provide some instrumental results to discuss super-quadratic mappings.

Lemma 1.1 ([7]). *If $\Psi : [0, \infty) \to \mathbb{R}$ is a super-quadratic mapping, then*

(1) $\Psi(0) \leq 0$,
(2) $C(\mu) = \Psi'(\mu)$, *when $\Psi(\mu)$ is differentiable with $\Psi(0) = \Psi'(0) = 0$, for all $\mu \geq 0$,*
(3) for all $\mu \geq 0$, if $\Psi(\mu) \geq 0$, then Ψ is convex and $\Psi(0) = \Psi'(0) = 0$.

Kian and his coauthors [8, 9] came up with the idea of operator super-quadratic mappings and Jensen's kinds of inequalities that are related to them. Oguntuase and Persson [10] discussed Hardy-like inequalities utilizing the notion of super-quadratic mappings. Study these additional papers for more comprehensive research on super-quadraticity, [11–15].

Varosanec [16] proposed the unified class of convexity through control mapping and provided new insight to conduct research in the following field. Throughout the investigation, let $\hbar_\circ : (0, 1) \to \mathbb{R}$ be a mapping such that $\hbar_\circ \geq 0$.

Definition 1.2 ([16]). *A mapping $\Psi : [s_3, s_4] \to \mathbb{R}$ is considered to be a $\hbar_\circ$-convex, if*

$$\Psi((1-\varphi)\mu + \varphi y) \leq \hbar_\circ(\varphi)\Psi(\mu) + \hbar_\circ(1-\varphi)\Psi(y).$$

Inspired by the idea presented in [16], Alomari and Chesneau [17] developed a general class of super-quadratic mappings and investigated some of their essential properties, defined as:

Definition 1.3 ([17]). *Any mapping $\Psi : [s_3, s_4] \to \mathbb{R}$ is regraded as $\hbar_\circ$-super-quadratic, If*

$$\Psi((1-\varphi)\mu + \varphi y) \leq \hbar_\circ(\varphi)[\Psi(y) - \Psi((1-\varphi)|y - \mu|)] + \hbar_\circ(1-\varphi)[\Psi(\mu) - \Psi(\varphi|\mu - y|)].$$

Lemma 1.2 ([17]). *Suppose* $\Psi : [s_3, s_4] \to \mathbb{R}$ *is a* $\hbar_\circ$*-super-quadratic mapping, then*

(1) $\Psi(0) \leq 0$,
(2) for all $\mu \geq 0$, *if* $\Psi(\mu) \geq 0$, *then* Ψ *is* $\hbar_\circ$*-convex such that* $\Psi(0) = \Psi'(0) = 0$.

The Jensen's inequality for this class of mappings is given as

Theorem 1.1 ([17]). *Suppose* $\Psi : [s_3, s_4] \to \mathbb{R}$ *is a* $\hbar_\circ$*-super-quadratic mapping, then*

$$\Psi\left(\frac{1}{C_\vartheta}\sum_{\nu=1}^{\vartheta}\varphi_\nu\mu_\nu\right) \leq \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi(\mu_\nu) - \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=1}^{\vartheta}\varphi_\nu\mu_\nu\right|\right).$$

Also, they proved the Jensen-Mercer inequality for $\hbar_\circ$-super-quadratic mappings.

Theorem 1.2 ([17]). *Let* $\Psi : [s_3, s_4] \to \mathbb{R}$ *be a* $\hbar_\circ$*-super-quadratic mapping, then*

$$\begin{aligned}\Psi\left(s_3 + s_4 - \frac{1}{C_\vartheta}\sum_{\nu=1}^{\vartheta}\varphi_\nu\mu_\nu\right) &\leq \Psi(s_3) + \Psi(s_4) - \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi(\mu_\nu)\\ -\sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)[\Psi(\mu_\nu - s_3) + \Psi(s_4 - \mu_\nu)] &\quad - \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=1}^{\vartheta}\varphi_\nu\mu_\nu\right|\right).\end{aligned}$$

Set-valued analysis and its subdomains are cornerstones in mathematical sciences to generalize the previously obtained results. In this regard, Moore [18] applied the set-valued mappings to establish bounded solutions of differential equations. Recently, researchers have focused on decision-making, multi-objective optimization, numerical analysis, mathematical modeling, and advanced nonlinear analysis through interval-valued mapping. Probabilistic and interval-valued techniques are utilized to extract the results from data having randomness. However, these approaches are not applicable to quantities that possess vagueness. To deal with such problems, Zadeh [19] proposed the idea of a fuzzy set based on generalized indicator mapping and also presented the idea of a fuzzy convex set. This theory emerged as a potential theory in the last few decades. The contribution of these concepts in optimization, decision-making, inequalities, differential equations, mathematical modeling, approximation methodologies, dynamic systems, and computer science is unprecedented. Note that this theory is not statistical in nature but sets new trends in possibility theory. Dubois and Prade [20] researched preliminary terminologies related to area and tangent problems and offered new insights to carry new developments. Nanda and Kar [21] explored diverse groups of fuzzy convex mappings and reported their essential characterization.

As we move ahead, let us go over certain previously laid-out concepts and implications of fuzzy interval analysis. Assume that K_c symbolizes the space of all closed and bounded intervals in $\mathbb{R}$, while K_c^+ represents the space of positive intervals. The interval ${}_1\chi$ is defined as:

$$[{}_1\chi] = [{}_1\underline{\chi}, {}_1\overline{\chi}] = \{\mu : {}_1\underline{\chi} \leq \mu \leq {}_1\overline{\chi}, \mu \in \mathbb{R}\}.$$

Given $\chi, \phi \in K_c$ and $\delta^1 \in \mathbb{R}$, Minkowski's operations are given as:

$$\delta^1.\chi := \begin{cases} [\delta^1\chi_*, \delta^1\chi^*] & \text{if } \delta^1 \geq 0, \\ [\delta^1\chi^*, \delta^1\chi_*] & \text{if } \delta^1 < 0. \end{cases}$$

Then the Minkowski addition $\chi + \phi$ and $\chi \times \phi$ for $\chi, \phi \in K_c$ are defined by

$$[\phi_*, \phi^*] + [\chi_*, \chi^*] := [\phi_* + \chi_*, \quad \phi^* + \chi^*],$$

and

$$[\phi_*, \phi^*] \times [\chi_*, \chi^*] := [\min\{\phi_*\chi_*, \phi^*\chi_*, \phi_*\chi^*, \phi^*\chi^*\}, \ \max\{\phi_*\chi_*, \phi^*\chi_*, \phi_*\chi^*, \phi^*\chi^*\}].$$

Definition 1.4 ([22]). *For any compact intervals $A = [s_{3*}, s_3^*]$, $B = [s_{4*}, s_4^*]$ and $C = [c_*, c^*]$, the generalized Hukuhara difference (gH-difference) is explored as:*

$$[s_{3*}, s_3^*] \ominus_g [s_{4*}, s_4^*] = [c_*, c^*] \Leftrightarrow \begin{cases} (i) & \begin{cases} s_{3*} = s_{4*} + c_* \\ s_3^* = s_4^* + c^*, \end{cases} \\ \\ (ii) & \begin{cases} s_{4*} = s_{3*} - c_* \\ s_4^* = s_3^* - c^*. \end{cases} \end{cases}$$

Also, the gH-difference can be illustrated as:

$$[s_{3*}, s_3^*] \ominus_g [s_{4*}, s_4^*] = \left[\min\{s_{3*} - s_{4*}, \ s_3^* - s_4^*\}, \ \max\{s_{3*} - s_{4*}, \ s_3^* - s_4^*\}\right].$$

Also for $A \in K_c$, the length of interval is computed by $l(A) = s_3^* - s_{3*}$. Then, for all $A, B \in K_c$, we have

$$A \ominus_g B = \begin{cases} [s_{3*} - s_{4*}, \ s_3^* - s_4^*], & l(A) \geq l(B), \\ [s_3^* - s_4^*, \ s_{3*} - s_{4*}], & l(A) \leq l(B). \end{cases}$$

Definition 1.5 ([23]). *The "$\leq_\rho$" relation over K_c is provided as:*

$$[\phi_*, \phi^*] \leq_\rho [\chi_*, \chi^*],$$

if and only if,

$$\phi_* \leq \chi_*, \ \phi^* \leq \chi^*,$$

for all $[\phi_*, \phi^*], [\chi_*, \chi^*] \in K_c$ is a pseudo-order or left-right (LR) ordering relation.

Theorem 1.3 ([23, 24]). *Every fuzzy set and $\delta^1 \in (0, 1]$, the representations of δ^1-level set of ${}^1\pi$ are examined in the following order: ${}^1\pi_{\delta^1} = \{\mu \in \mathbb{R} : {}^1\pi(\mu) \geq \delta^1\}$ and $supp({}^1\pi) = cl\{\mu \in \mathbb{R} : {}^1\pi(\mu) > 0\}$. The fuzzy sets in $\mathbb{R}$ are represented by Θ_{δ^1} and ${}^1\pi \in \Theta_{\delta^1}$. A ${}^1\pi$ is considered to be a fuzzy number (interval) if it is normal, fuzzy convex, semi-continuous, and has compact support. The space of all real fuzzy numbers are specified by Υ_{δ^1}.*

Let ${}^1\pi \in \Upsilon_{\delta^1}$ be a fuzzy interval if, and only if, δ^1-levels $[{}^1\pi]^{\delta^1}$ is a compact convex set of $\mathbb{R}$. Now, we deliver the representation of fuzzy number:

$$[{}^1\pi]^{\delta^1} = [{}^1\pi_*(\delta^1), {}^1\pi^*(\delta^1)],$$

where

$${}^1\pi_*(\delta^1) := \inf\{\mu \in \mathbb{R} : {}^1\pi(\mu) \geq \delta^1\}, \ {}^1\pi^*(\delta^1) := \sup\{\mu \in \mathbb{R} : {}^1\pi(\mu) \geq \delta^1\}.$$

Thus, a fuzzy-interval can be investigated and characterized by a parameterized triplet. For more details, see [26].

$$\left\{{}^1\pi_*(\delta^1), {}^1\pi^*(\delta^1); \delta^1 \in [0, 1]\right\}.$$

These two endpoint mappings ${}^1\pi_*(\delta^1)$ and ${}^1\pi^*(\delta^1)$ play a vital role in exploring the fuzzy numbers.

Proposition 1.1 ([25]). *If $V,\chi \in \Upsilon_{\delta^1}$, then the relation "$\preceq$" explored on Υ_{δ^1} by*

$V \preceq \chi$ if, and only if, $[V]^{\delta^1} \leq_\rho [\chi]^{\delta^1}$, for all $\delta^1 \in [0,1]$, this relation is known as a partial order relation.

For $V,\chi \in \Upsilon_{\delta^1}$ and $c \in \mathbb{R}$, the scalar product $c \cdot \chi$, sum with constant, the sum $V \oplus \chi$, and product $V \odot \chi$ are defined by:

$$\begin{aligned}
[c \cdot V]^{\delta^1} &= c \cdot [V]^{\delta^1}, \\
[c \oplus V]^{\delta^1} &= c + [V]^{\delta^1}. \\
[V \oplus \chi]^{\delta^1} &= [V]^{\delta^1} + [\chi]^{\delta^1}, \\
[V \odot \chi]^{\delta^1} &= [V]^{\delta^1} \times [\chi]^{\delta^1}.
\end{aligned}$$

The level wise difference of the fuzzy number is stated as follows:

Definition 1.6 ([27]). *Let V, χ be the two fuzzy numbers. Then the level-wise difference is defined as*

$$[V \ominus \chi]^{\delta^1} = [V_*(\delta^1) - \chi^*(\delta^1),\ V^*(\delta^1) - \chi_*(\delta^1)].$$

To overcome the limitations of Hukuhara difference, the following difference is defined as follows.

Definition 1.7 ([22]). *Let V, χ be the two fuzzy numbers. Then the generalized Hukuhara difference (gH-difference) of $V \ominus_g \chi$ is a fuzzy number ξ such that*

$$V \ominus_g \chi = \xi \Leftrightarrow \begin{cases} (i) & V = \chi \oplus \xi \\ (ii) & \chi = V \oplus (-1)\xi. \end{cases}$$

Also the gH-difference based on δ^1 can be illustrated as:

$$[V \ominus_g \chi]^{\delta^1} = [\min\{V_*(\delta^1) - \chi_*(\delta^1),\ V^*(\delta^1) - \chi^*(\delta^1)\}, \max\{V_*(\delta^1) - \chi_*(\delta^1),\ V^*(\delta^1) - \chi^*(\delta^1)\}].$$

Also for $V \in \Upsilon_{\delta^1}$, the length of the fuzzy interval is given by $l(V(\delta^1)) = V^*(\delta^1) - V_*(\delta^1)$. Then, for all $V,\chi \in \Upsilon_{\delta^1}$, we have

$$V \ominus_g \chi = \begin{cases} [V_*(\delta^1) - \chi_*(\delta^1),\ V^*(\delta^1) - \chi^*(\delta^1)], & l(V(\delta)) \geq l(\chi(\delta)), \\ [V^*(\delta^1) - \chi^*(\delta^1),\ V_*(\delta^1) - \chi_*(\delta^1)], & l(V(\delta)) \leq l(\chi(\delta)). \end{cases} \tag{1.2}$$

Note that a function $\Psi : [s_3, s_4] \subseteq \mathbb{R} \rightarrow \Upsilon_{\delta^1}$ is said to be l-increasing, if length function $len\left([\Psi(\mu)]^{\delta^1}\right) = \Psi^*(\mu, \delta^1) - \Psi_*(\mu, \delta^1)$ is increasing with respect μ for all $\delta^1 \in [0,1]$. Mathematically, for any $\mu_1, \mu_2 \in [s_3, s_4]$ and $\mu_1 \leq \mu_2$. Then $len\left([\Psi(\mu_2)]^{\delta^1}\right) \geq len\left([\Psi(\mu_1)]^{\delta^1}\right)$, $\forall \delta^1 \in [0,\ 1]$. For more details, see [28].

Proposition 1.2 ([28]). *Let $\Psi : T = (s_3, s_4) \subseteq \mathbb{R} \rightarrow \Upsilon_{\delta^1}$ be a Fuzzy number valued ($\mathtt{F.N.V}$) mapping. If $\Psi(\mu + h) \ominus_g \Psi(\mu)$ exists for some h such that $\mu + h \in T$, then one of the following conditions hold:*

$$\textit{Case (i)} \begin{cases} len\left([\Psi(\mu+h)]^{\delta^1}\right) \geq len\left([\Psi(\mu)]^{\delta^1}\right), & \forall \delta^1 \in [0,\ 1] \\ \Psi_*(\mu+h, \delta^1) - \Psi_*(\mu, \delta^1), & \textit{is a monotonic increasing with respect to } \delta^1 \\ \Psi^*(\mu+h, \delta^1) - \Psi^*(\mu, \delta^1), & \textit{is a monotonic decreasing with respect to } \delta^1. \end{cases}$$

$$\textit{Case (ii)}\begin{cases} len\left([\Psi(\mu+h)]^{\delta^1}\right) \leq len\left([\Psi(\mu)]^{\delta^1}\right), & \forall \delta^1 \in [0,\ 1] \\ \Psi_*(\mu+h,\delta^1) - \Psi_*(\mu,\delta^1), & \textit{is a monotonic decreasing with respect to } \delta^1 \\ \Psi^*(\mu+h,\delta^1) - \Psi^*(\mu,\delta^1), & \textit{is a monotonic increasing with respect to } \delta^1. \end{cases}$$

Remark 1.1. *From Proposition 2.4, the $\Psi(\mu + h) \ominus_g \Psi(\mu)$ can be written by the definition of the fuzzy interval as:*

$$\textit{Case (i)}\begin{cases} len\left([\Psi(\mu)]^{\delta^1}\right), & \textit{is a monotonic increasing with respect to } \mu,\ \forall \delta^1 \in [0,\ 1] \\ \Psi(\mu+h) \ominus_g \Psi(\mu) = [\Psi_*(\mu+h,\delta^1) - \Psi_*(\mu,\delta^1),\ \Psi^*(\mu+h,\delta^1) - \Psi^*(\mu,\delta^1)]. \end{cases}$$

$$\textit{Case (ii)}\begin{cases} len\left([\Psi(\mu)]^{\delta^1}\right), & \textit{is a monotonic decreasing with respect to } \mu,\ \forall \delta^1 \in [0,\ 1] \\ \Psi(\mu+h) \ominus_g \Psi(\mu) = [\Psi^*(\mu+h,\delta^1) - \Psi^*(\mu,\delta^1),\ \Psi_*(\mu+h,\delta^1) - \Psi_*(\mu,\delta^1)]. \end{cases}$$

Definition 1.8 ([25]). *If $\Psi : [s_3, s_4] \subset \mathbb{R} \to \Upsilon_{\delta^1}$ is an* F.N.V *mapping. For each $\delta^1 \in [0, 1]$, whose δ^1-cuts highlight the bundle of* I.V.F*, such that $\Psi_{\delta^1} : [s_3, s_4] \subset \mathbb{R} \to K_c$ is described as $\Psi_{\delta^1}(\mu) = [\Psi_*(\mu, \delta^1), \Psi^*(\mu, \delta^1)]$, $\mu \in [s_3, s_4]$. Every $\delta^1 \in [0, 1]$, the left and right real valued mappings $\Psi_*(\mu, \delta^1), \Psi^*(\mu, \delta^1) : [s_3, s_4] \to \mathbb{R}$ are sometimes referred to as Ψ end points.*

Definition 1.9 ([29]). *Let $\Psi : [s_3, s_4] \subset \mathbb{R} \to \Upsilon_{\delta^1}$ be an* F.N.V *mapping. Then, fuzzy integral of Ψ over $[s_3, s_4]$ is projected as $(FR) \int_{s_3}^{s_4} \Psi(\mu)\mathrm{d}\mu$,*

$$\begin{aligned} \left[(FR)\int_{s_3}^{s_4} \Psi(\mu)\mathrm{d}\mu\right]^{\delta^1} &= (FR)\int_{s_3}^{s_4} \Psi_{\delta^1}(\mu)\mathrm{d}\mu \\ &= \left\{\int_{s_3}^{s_4} \Psi(\mu,\delta^1)\mathrm{d}\mu : \Psi(\mu,\delta^1) \in \mathcal{R}_{([s_3,s_4],\delta^1)}\right\}, \end{aligned}$$

for all $\delta^1 \in [0, 1]$, where $\mathcal{R}_{([s_3,s_4],\delta^1)}$ describes the space of integrable mappings.

Theorem 1.4 ([26]). *If $\Psi : [s_3, s_4] \subset \mathbb{R} \to \Upsilon_{\delta^1}$ is an* F.N.V *mapping. For each $\delta^1 \in [0, 1]$, whose δ^1-cuts highlight the bundle of* I.V.F*, such that $\Psi_{\delta^1} : [s_3, s_4] \subset \mathbb{R} \to K_c$ is described as $\Psi_{\delta^1}(\mu) = [\Psi_*(\mu, \delta^1), \Psi^*(\mu, \delta^1)]$, $\mu \in [s_3, s_4]$. Then, Ψ is fuzzy Riemann integrable (FR-integrable) over $[s_3, s_4]$,$\Leftrightarrow$ $\Psi_*(\mu, \delta^1), \Psi^*(\mu, \delta^1) \in \mathcal{R}_{([s_3,s_4],\delta^1)}$, then*

$$\begin{aligned} \left[(FR)\int_{s_3}^{s_4} \Psi(\mu)\mathrm{d}\mu\right]^{\delta^1} &= \left[(R)\int_{s_3}^{s_4} \Psi_*(\mu,\delta^1), (R)\int_{s_3}^{s_4} \Psi^*(\mu,\delta^1)\right] \\ &= (FR)\int_{s_3}^{s_4} \Psi_{\delta^1}(\mu)\mathrm{d}\mu, \end{aligned}$$

for all $\delta^1 \in [0, 1]$, where FR represents interval Riemann integration of $\Psi_{\delta^1}(\mu)$. For all $\delta^1 \in [0, 1]$, $F\mathcal{R}_{([s_3,s_4],\delta^1)}$ specifies the class of all FR-integrable F.N.V *mappings over $[s_3, s_4]$.*

Definition 1.10 ([21]). *A mapping $\Psi : [s_3, s_4] \to \Upsilon_{\delta^1}$ is termed as an* F.N.V *LR-convex mapping on $[s_3, s_4]$ if*

$$\Psi((1-\varphi)\mu + \varphi\mathcal{Y}) \preceq (1-\varphi)\Psi(\mu)\tilde{+}\varphi\Psi(\mathcal{Y}), \tag{1.3}$$

for all $\mu, \mathcal{Y} \in [s_3, s_4], \varrho_\circ \in [0, 1]$, where $\Psi(\mu) \succcurlyeq \tilde{0}$ for all $\mu \in [s_3, s_4]$.

Definition 1.11 ([26]). *Let $\tau > 0$ and $Ł(s_3, s_4, \Upsilon_{\delta^1})$ be the space of all Lebesgue measurable* F.N.V *mapping on $[s_3,\ s_4]$. Then, the fuzzy left and right RL-fractional integral operator of $\Psi \in Ł(s_3, s_4, \Upsilon_{\delta^1})$ are defined as:*

$$J^{\tau}_{a^+}\Psi(s_3) = \frac{1}{\Gamma(\tau)} \int_{s_3}^{s_4} (\mu - s_3)^{\tau-1}\Psi(\mu)\mathrm{d}\mu, \quad \mu \geq a$$

and

$$J^{\tau}_{s_4^-}\Psi(s_4) = \frac{1}{\Gamma(\tau)} \int_{s_3}^{s_4} (s_4 - \mu)^{\tau-1}\Psi(\mu)\mathrm{d}\mu, \quad \mu \leq s_4.$$

Furthermore, the left and right RL-fractional operator based on left and right endpoint mappings can be defined, that is,

$$\begin{aligned}
\left[J^{\tau}_{a^+}\Psi(s_3)\right]^{\delta^1} &= \frac{1}{\Gamma(\tau)} \int_{s_3}^{s_4} (\mu - s_3)^{\tau-1}\Psi_{\delta^1}(\varrho_\circ)\mathrm{d}\varrho_\circ \\
&= \frac{1}{\Gamma(\tau)} \int_{s_3}^{s_4} (\mu - s_3)^{\tau-1}\Psi_{\delta^1}\left[\Psi_*(\varrho_\circ, \delta^1),\ \Psi^*(\varrho_\circ, \delta^1)\right]\mathrm{d}\varrho_\circ,
\end{aligned}$$

where

$$J^{\tau}_{a^+}\Psi_*(a, \delta^1) = \frac{1}{\Gamma(\tau)} \int_{s_3}^{s_4} (\mu - s_3)^{\tau-1}\Psi_*(\varrho_\circ, \delta^1)\mathrm{d}\varrho_\circ,$$

and

$$J^{\tau}_{a^+}\Psi^*(a, \delta^1) = \frac{1}{\Gamma(\tau)} \int_{s_3}^{s_4} (\mu - s_3)^{\tau-1}\Psi^*(\varrho_\circ, \delta^1)\mathrm{d}\varrho_\circ.$$

By similar argument, we can define the right operator.

The authors [30] employed interval-valued unified approximate convexity to examine new refinements of inequalities. Nwaeze et al. [31] proposed the class of interval-valued ϑ-polynomial convex mappings and reported several interesting inequalities. Abdeljawad et al. [32] utilized the p mean to develop the idea of interval-valued p convexity and presented some corresponding general inequalities of Hermite-Hadamard type. Shi and his colleagues [33] studied the totally ordered unified convexity in the perspective of integral inequalities. Through interval-valued log-convexity and cr-harmonic convexity, Liu et al. [34, 35] found the trapezium and Jensen's-like inequalities.

Budak et al. [36] implemented the interval-valued RL-fractional operators and convexity to derive the trapezoidal inequalities. Vivas-Cortez [37] introduced the totally ordered τ-convex mappings and analyzed several Jensen's, Schur's, and fractional Hadamard's and kinds of inequalities. Cheng et al. [38] looked at new kinds of Hadamard-like inequalities using fuzzy-valued mapping and fractional quantum calculus. Bin-Mohsin et al. [39] bridged the harmonic coordinated convexity and fractional operators relying on Raina's special mapping to establish new 2-dimensional inequalities. For comprehensive details, consult [40–44].

Recently, Fahad [45] proposed some novel bounds of classical inequalities pertaining to center-radius ordered geometric-arithmetic convexity and some interesting applications to information theory.

Authors [46] explored the unified class of stochastic convex processes relying on quasi-weighted mean and cr ordering relation to conclude new forms of inequalities. For the first time, Khan and Butt established the new counterparts of classical inequalities depending upon partially and totally ordered super-quadraticity, respectively, in [47,48].

Costa et al. [49] implemented the fuzzy-valued mappings to acquire some boundaries in a one-point quadrature scheme. Zhang and his coauthors [50] focused on set-valued Jensen-like inequalities along with some interesting applications. Khan et al. [51] examined fractional analogues of fuzzy interval-valued integral inequalities. In [52], authors discussed fuzzy valued Hadamard-like inequalities associated with log convexity. Abbaszadeh and Eshaghi employed the fuzzy valued r convex mappings to establish the trapezium type inequalities. Bin-Mohsin [53] introduced the idea of fuzzy bi-convex mappings and derived the various inequalities. For comprehensive details, see [54–58].

The above literature is evidence that theories of inequalities are interlinked with convexity. Several classes of convexity have been introduced to reduce the limitations of classical convexity or to acquire better estimations of existing results. Researchers have applied several techniques to produce better estimations of mathematical quantities. The principle motivation is to explore super-quadratic mappings in fuzzy environments through a unified approach. First, we will propose a new class of fuzzy number-valued super-quadratic mappings based on left and right ordering relations. Further, we will give a detailed description of newly developed concepts along with their potential cases. Most importantly, we will derive classical inequalities like the trapezoidal inequality, the weighted form for symmetric mappings, and Jensen's and its related inequalities. Later on, some fractional inequalities will be presented and graphed. To increase the reliability and accuracy, some visuals and related numerical data will be provided. Finally, we will present applications based on our primary findings. This is the first study regarding super-quadraticity via fuzzy calculus.

2. Primary findings

This part contains the results related to newly proposed notion of fuzzy super-quadratic mappings.

2.1. *Fuzzy-number-valued $\hbar_\circ$-super-quadratic functions*

First, we investigate the fuzzy-valued $\hbar_\circ$-super-quadratic mapping.

Definition 2.1. *Suppose $\hbar_\circ \geq 0$. Let $\Psi : [s_3, s_4] \subseteq [0, \infty) \to \Upsilon_{\delta^1}$ be an* F.N.V *mapping such that $\Psi_{\delta^1}(\gamma) = [\Psi_*(\delta^1, \gamma), \Psi^*(\delta^1, \gamma)]$, and $len\left([\Psi(\gamma)]^{\delta^1}\right) = \Psi^*(\gamma, \delta^1) - \Psi_*(\gamma, \delta^1)$ is increasing with respect γ for all $\delta^1 \in [0, 1]$. Then Ψ is considered to be an* F.N.V *$\hbar_\circ$-super-quadratic mapping if*

$$\Psi((1-\varphi)\gamma + \varphi y) \preceq \hbar_\circ(\varphi)[\Psi(y) \ominus_g \Psi((1-\varphi)|\gamma - y|)] \oplus \hbar_\circ(1-\varphi)[\Psi(\gamma) \ominus_g \Psi(\varphi|\gamma - y|)],$$

holds $\forall \gamma, y \in [s_3, s_4]$ such that $\gamma < y$ and $|y - \gamma| < \gamma$ where $\varphi \in [0, 1]$.

Now, we enlist some potential deductions of Definitions 2.1.

- Inserting $\hbar_\circ(\varphi) = \varphi$, we recapture the class of F.N.V super-quadratic mappings.
- Inserting $\hbar_\circ(\varphi) = \varphi^s$, we recapture the class of F.N.V-s-super-quadratic mappings:

$$\Psi((1-\varphi)\gamma + \varphi y) \preceq \varphi^s[\Psi(y) \ominus_g \Psi((1-\varphi)|\gamma - y|)] \oplus (1-\varphi)^s[\Psi(\gamma) \ominus_g \Psi(\varphi|\gamma - y|)].$$

- Inserting $\hbar_\circ(\varphi)=\varphi^{-s}$, we recapture the class of $\mathtt{F.N.V}$-s Godunova super-quadratic mappings:

$$\Psi((1-\varphi)\gamma+\varphi y)\leq\varphi^{-s}[\Psi(y)\ominus_g\Psi((1-\varphi)|\gamma-y|)]\oplus(1-\varphi)^{-s}[\Psi(\gamma)\ominus_g\Psi(\varphi|\gamma-y|)].$$

- Inserting $\hbar_\circ(\varphi)=\varphi(1-\varphi)$, we recapture the class of $\mathtt{F.N.V}$-tgs super-quadratic mappings:

$$\Psi((1-\varphi)\gamma+\varphi y)\leq\varphi(1-\varphi)[\Psi(y)\ominus_g\Psi((1-\varphi)|\gamma-y|)]\oplus\varphi(1-\varphi)[\Psi(\gamma)\ominus_g\Psi(\varphi|\gamma-y|)].$$

- Inserting $\hbar_\circ(\varphi)=1$, we recapture the class of $\mathtt{F.N.V}$-P super-quadratic mappings:

$$\Psi((1-\varphi)\gamma+\varphi y)\leq[\Psi(y)\ominus_g\Psi((1-\varphi)|\gamma-y|)]\oplus[\Psi(\gamma)\ominus_g\Psi(\varphi|\gamma-y|)].$$

- Inserting $\hbar_\circ(\varphi)=\exp(\varphi)-1$, we recapture the class of $\mathtt{F.N.V}$-exponential super-quadratic mappings:

$$\begin{aligned}&\Psi((1-\varphi)\gamma+\varphi y)\\&\leq[\exp(\varphi)-1][\Psi(y)\ominus_g\Psi((1-\varphi)|\gamma-y|)]\oplus[\exp(1-\varphi)-1][\Psi(\gamma)\ominus_g\Psi(\varphi|\gamma-y|)].\end{aligned}$$

- Selecting $\Psi_*(\gamma,\delta^1)=\Psi_*(\gamma,\delta^1)$ and $\delta^1=1$, we acquire the notion of $\hbar_\circ$-super-quadratic mapping defined in [17].

The spaces of $\hbar_\circ$-super-quadratic mappings and ($\mathtt{F.N.V}$)–$\hbar_\circ$ l-increasing super-quadratic mappings defined over $[s_3,s_4]$ are represented by $SSQF([s_3,s_4],\hbar_\circ)$ and $SSQFNF([s_3,s_4],\hbar_\circ)$ respectively.

Proposition 2.1. *Let $\Psi,g:[s_3,s_4]\to\Upsilon_{\delta^1}$ be two $\mathtt{F.N.V}$ mappings. If $\Psi,g\in SSQFNF([s_3,s_4],\hbar_\circ)$, then*

- $\Psi+g\in SSQFNF([s_3,s_4],\hbar_\circ)$.
- $c\Psi\in SSQFNF([s_3,s_4],\hbar_\circ),\quad c\geq0$.

Proof. The proof is obvious. □

Proposition 2.2. *If $\Psi\in SSQFNF([s_3,s_4],{\hbar_\circ}^1)$ and ${\hbar_\circ}^1(\varphi)\leq\hbar_\circ^2(\varphi)$, then $\Psi\in SSQFNF([s_3,s_4],{\hbar_\circ}^2)$.*

Now we prove the criteria to investigate the class of $\mathtt{F.N.V}$–$\hbar_\circ$ super-quadratic mappings.

Proposition 2.3. *If $\Psi\in SSQFNF([s_3,s_4],{\hbar_\circ}^1)$ and ${\hbar_\circ}^1(\varphi)\leq\hbar_\circ^2(\varphi)$, then $\Psi\in SSQFNF([s_3,s_4],{\hbar_\circ}^2)$.*

Now we prove the criteria to investigate class of $\mathtt{F.N.V}$–$\hbar_\circ$ super-quadratic mappings.

Proposition 2.4. *Let $\Psi:[s_3,s_4]\subseteq[0,\infty)\to\Upsilon_{\delta^1}$ be an $\mathtt{F.N.V}$ mapping. For any $\gamma,y\in[s_3,\ s_4]$ such that $\gamma<y$ and satisfying the condition that $|y-\gamma|<\gamma$. Then, $\Psi\in SSQFNF([s_3,s_4],\hbar_\circ)$ if, and only if, $\Psi_*(\gamma,\delta^1),\Psi^*(\gamma,\delta^1)\in SSQF([s_3,s_4],\hbar_\circ)$ and $len\left([\Psi(\gamma)]^{\delta^1}\right)=\Psi^*(\gamma,\delta^1)-\Psi_*(\gamma,\delta^1)$ is increasing with respect γ for all $\delta^1\in[0,1]$.*

Proof. Let $\Psi_*,\Psi^*\in SSQF([s_3,s_4],\hbar_\circ)$ and $\forall\gamma,y\in[s_3,s_4]$ such that $\gamma<y$ and satisfying the condition $|y-\gamma|<\gamma$. Then:

$$\begin{aligned}\Psi_*((1-\varphi)y+\varphi\gamma,\delta^1)\leq{}&\hbar_\circ(1-\varphi)\left[\Psi_*(y,\delta^1)-\Psi_*(\varphi|y-\gamma|,\delta^1)\right]\\&+\hbar_\circ(\varphi)\left[\Psi_*(\gamma,\delta^1)-\Psi_*((1-\varphi)|y-\gamma|,\delta^1)\right],\end{aligned}\tag{2.1}$$

and

$$\Psi^*((1-\varphi)y+\varphi\gamma,\delta^1) \le \hbar_\circ(1-\varphi)\left[\Psi^*(y,\delta^1)-\Psi^*(\varphi|y-\gamma|,\delta^1)\right] + \hbar_\circ(\varphi)\left[\Psi^*(\gamma,\delta^1)-\Psi^*((1-\varphi)|y-\gamma|,\delta^1)\right]. \tag{2.2}$$

Combining (2.1) and (2.2) by definition of pseudo ordering relation, we have

$$\begin{aligned}
&\left[\Psi_*((1-\varphi)y+\varphi\gamma,\delta^1),\ \Psi^*((1-\varphi)y+\varphi\gamma,\delta^1)\right] \\
&\le \left[\hbar_\circ(1-\varphi)\left[\Psi_*(y,\delta^1)-\Psi_*(\varphi|y-\gamma|,\delta^1)\right]+\hbar_\circ(\varphi)\left[\Psi_*(\gamma,\delta^1)-\Psi_*((1-\varphi)|y-\gamma|,\delta^1)\right],\right. \\
&\quad \left.\hbar_\circ(1-\varphi)\left[\Psi^*(y,\delta^1)-\Psi^*(\varphi|y-\gamma|,\delta^1)\right]+\hbar_\circ(\varphi)\left[\Psi^*(\gamma,\delta^1)-\Psi^*((1-\varphi)|y-\gamma|,\delta^1)\right]\right].
\end{aligned}$$

Then by Case (i) of Remark (1.1), we have

$$\Psi((1-\varphi)y+\varphi\gamma) \le \hbar_\circ(1-\varphi)\left[\Psi(y)\ominus_g\Psi(\varphi|y-\gamma|)\right]\oplus\hbar_\circ(\varphi)\left[\Psi(\gamma)\ominus_g\Psi((1-\varphi)|y-\gamma|)\right].$$

This completes the proof of first part. For the converse part, consider $\Psi \in SSQFNF([s_3,s_4],\hbar_\circ)$, then

$$\Psi((1-\varphi)y+\varphi\gamma) \le \hbar_\circ(1-\varphi)\left[\Psi(y)\ominus_g\Psi(\varphi|y-\gamma|)\right]\oplus\hbar_\circ(\varphi)\left[\Psi(\gamma)\ominus_g\Psi((1-\varphi)|y-\gamma|)\right].$$

The above inequality can be written as

$$\begin{aligned}
&\left[\Psi_*((1-\varphi)y+\varphi\gamma,\delta^1),\ \Psi^*((1-\varphi)y+\varphi\gamma,\delta^1)\right] \\
&\le \left[\min\left\{\hbar_\circ(1-\varphi)\left[\Psi_*(y,\delta^1)-\Psi_*(\varphi|y-\gamma|,\delta^1)\right]+\hbar_\circ(\varphi)\left[\Psi_*(\gamma,\delta^1)-\Psi_*((1-\varphi)|y-\gamma|,\delta^1)\right],\right.\right. \\
&\quad \left.\hbar_\circ(1-\varphi)\left[\Psi^*(y,\delta^1)-\Psi^*(\varphi|y-\gamma|,\delta^1)\right]+\hbar_\circ(\varphi)\left[\Psi^*(\gamma,\delta^1)-\Psi^*((1-\varphi)|y-\gamma|,\delta^1)\right]\right\}, \\
&\quad \max\left\{\hbar_\circ(1-\varphi)\left[\Psi_*(y,\delta^1)-\Psi_*(\varphi|y-\gamma|,\delta^1)\right]+\hbar_\circ(\varphi)\left[\Psi_*(\gamma,\delta^1)-\Psi_*((1-\varphi)|y-\gamma|,\delta^1)\right],\right. \\
&\quad \left.\left.\hbar_\circ(1-\varphi)\left[\Psi^*(y,\delta^1)-\Psi^*(\varphi|y-\gamma|,\delta^1)\right]+\hbar_\circ(\varphi)\left[\Psi^*(\gamma,\delta^1)-\Psi^*((1-\varphi)|y-\gamma|,\delta^1)\right]\right\}\right].
\end{aligned}$$

From Definition 2.1, it is clear that $len\left([\Psi(\gamma)]^{\delta^1}\right)$ is increasing. Using Case (i) of Remark 1.1, we have

$$\begin{aligned}
&\left[\Psi_*((1-\varphi)y+\varphi\gamma,\delta^1),\ \Psi^*((1-\varphi)y+\varphi\gamma,\delta^1)\right] \\
&\le \left[\hbar_\circ(1-\varphi)\left[\Psi_*(y,\delta^1)-\Psi_*(\varphi|y-\gamma|,\delta^1)\right]+\hbar_\circ(\varphi)\left[\Psi_*(\gamma,\delta^1)-\Psi_*((1-\varphi)|y-\gamma|,\delta^1)\right],\right. \\
&\quad \left.\hbar_\circ(1-\varphi)\left[\Psi^*(y,\delta^1)-\Psi^*(\varphi|y-\gamma|,\delta^1)\right]+\hbar_\circ(\varphi)\left[\Psi^*(\gamma,\delta^1)-\Psi^*((1-\varphi)|y-\gamma|,\delta^1)\right]\right].
\end{aligned} \tag{2.3}$$

From (2.3), we can write as

$$\Psi_*((1-\varphi)y+\varphi\gamma,\delta^1) \le \hbar_\circ(1-\varphi)\left[\Psi_*(y,\delta^1)-\Psi_*(\varphi|y-\gamma|,\delta^1)\right] + \hbar_\circ(\varphi)\left[\Psi_*(\gamma,\delta^1)-\Psi_*((1-\varphi)|y-\gamma|,\delta^1)\right], \tag{2.4}$$

and

$$\Psi^*((1-\varphi)y+\varphi\gamma,\delta^1) \le \hbar_\circ(1-\varphi)\left[\Psi^*(y,\delta^1)-\Psi^*(\varphi|y-\gamma|,\delta^1)\right] + \hbar_\circ(\varphi)\left[\Psi^*(\gamma,\delta^1)-\Psi^*((1-\varphi)|y-\gamma|,\delta^1)\right]. \tag{2.5}$$

It is evident from (2.4) and (2.5) that both $\Psi_*, \Psi^* \in SSQF([s_3,s_4],\hbar_\circ)$. Hence, the result is proved. □

It is noteworthy to mention that Proposition 2.4 provides the necessary and sufficient condition for $\mathtt{F.N.V}$–$\hbar_\circ$ super-quadratic mapping. It is noteworthy to mention that Proposition 2.4 provides the necessary and sufficient condition for the $\mathtt{F.N.V}$–$\hbar_\circ$ super-quadratic mapping.

Example 2.1. *Let us consider* $\mathtt{F.N.V.}$ $\Psi : [s_3,\ s_4] = [0, 2] \to \mathbb{R}_\circ$*, which is defined as follows*

$$\Psi_\mu(\mu_1) = \begin{cases} \frac{\mu_1}{3\mu^3}, & \mu_1 \in [0, 3\mu^3] \\ \frac{6\mu^3-\mu_1}{3\mu^3}, & \mu_1 \in (3\mu^3, 6\mu^3]. \end{cases}$$

Then, for $\delta^1 \in [0, 1]$*, we have*

$$\Psi_{\delta^1} = \left[3\delta^1\mu^3,\ (6 - 3\delta^1)\mu^3\right].$$

Notice that both endpoint mappings $\Psi(\mu, \delta^1) = 3\delta^1\mu^3$ *and* $\Psi(\mu, \delta^1) = (6-3\delta^1)\mu^3$ *are* $\hbar_\circ$*-super-quadratic mappings, respectively. So,* $\Psi \in SSQFNF([s_3, s_4], \hbar_\circ)$*. Also len* $\left([\Psi(\mu)]^{\delta^1}\right) = (6 - 6\delta^1)\mu^3$ *is increasing with respect to* μ *for all* $\delta^1 \in [0, 1]$.

Now, we prove an alternative definition of this class of convexity for ϑ different points of $[s_3, s_4]$ known as Jensen's inequality. This inequality is useful for the development of further integral inequalities.

Theorem 2.1. *Let* $\hbar_\circ : (0, 1] :\to [0, \infty)$ *be a nonnegative super-multiplicative mapping. If* $\Psi \in SSQFNF([s_3, s_4], \hbar_\circ)$*, then*

$$\Psi\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right) \leq \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi(\mu_\nu)\ominus_g\sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right|\right), \tag{2.6}$$

for $\mu_\nu \in [s_3, s_4], \varphi_\nu \in [0, 1]$ *such that* $C_\vartheta = \sum_{\nu=1}^{\vartheta}\varphi_\nu$.

Proof. If $\Psi \in SSQFNF([s_3, s_4], \hbar_\circ)$, then it can be written as:

$$\begin{aligned}
&\left[\Psi_*\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu), \delta^1\right), \Psi^*\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu), \delta^1\right)\right] \\
&\leq \left[\sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi_*(\mu_\nu, \delta^1) - \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi_*\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right|, \delta^1\right),\right. \\
&\quad \left.\sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi^*(\mu_\nu, \delta^1) - \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi^*\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right|, \delta^1\right)\right].
\end{aligned}$$

By pseudo order relation, one can resolve the above inequality as:

$$\Psi_*\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu), \delta^1\right) \leq \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi_*(\mu_\nu, \delta^1) - \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi_*\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right|, \delta^1\right), \tag{2.7}$$

and

$$\Psi^*\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu), \delta^1\right) \leq \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi^*(\mu_\nu, \delta^1) - \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi^*\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right|, \delta^1\right). \tag{2.8}$$

We employ induction technique to prove both inequalities (2.7) and (2.8). Fixing $\vartheta = 2$ and $\frac{\varphi_1}{C_2} = \alpha$ and $\frac{\varphi_2}{C_2} = 1 - \alpha$ in (2.7), we acquire the definition of $\hbar_\circ$-super-quadratic mapping. To proceed further, we assume that (2.7) holds true for $\vartheta = \nu - 1$, then

$$\Psi_*\left(\sum_{\nu=1}^{\vartheta-1}\frac{\varphi_\nu}{C_{\vartheta-1}}(\mu_\nu),\delta^1\right) \leq \sum_{\nu=1}^{\vartheta-1}\hbar_\circ\left(\frac{\varphi_\nu}{C_{\vartheta-1}}\right)\Psi_*(\mu_\nu,\delta^1) - \sum_{\nu=1}^{\vartheta-1}\hbar_\circ\left(\frac{\varphi_\nu}{C_{\vartheta-1}}\right)\Psi_*\left(\left|\mu_\nu - \sum_{\nu=1}^{\vartheta-1}\frac{\varphi_\nu}{C_{\vartheta-1}}(\mu_\nu)\right|,\delta^1\right). \quad (2.9)$$

Next, we prove the validity of (2.6).

$$\begin{aligned}
&\Psi_*\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu),\delta^1\right) = \Psi_*\left(\frac{\varphi_n x_\vartheta}{C_\vartheta} + \frac{C_{\vartheta-1}}{C_\vartheta}\sum_{\nu=1}^{\vartheta-1}\frac{\varphi_\nu}{C_{\vartheta-1}}(\mu_\nu),\delta^1\right)\\
&\leq \hbar_\circ\left(\frac{\varphi_\vartheta}{C_\vartheta}\right)\Psi_*(\mu_\vartheta,\delta^1) + \hbar_\circ\left(\frac{C_{\vartheta-1}}{C_\vartheta}\right)\Psi_*\left(\sum_{\nu=1}^{\vartheta-1}\left(\frac{\varphi_\nu\mu_\nu}{C_{\vartheta-1}}\right),\delta^1\right)\\
&-\hbar_\circ\left(\frac{\varphi_\vartheta}{C_\vartheta}\right)\Psi_*\left(\frac{C_{\vartheta-1}}{C_\vartheta}\left|\mu_\vartheta - \sum_{\nu=1}^{\vartheta-1}\left(\frac{\varphi_\nu\mu_\nu}{C_{\vartheta-1}}\right)\right|,\delta^1\right) - \hbar_\circ\left(\frac{C_{\vartheta-1}}{C_\vartheta}\right)\Psi_*\left(\frac{\varphi_\vartheta}{C_\vartheta}\left|\mu_\vartheta - \sum_{\nu=1}^{\vartheta-1}\left(\frac{\varphi_\nu\mu_\nu}{C_{\vartheta-1}}\right)\right|,\delta^1\right). \quad (2.10)
\end{aligned}$$

Using (2.9) in (2.10) and the super-multiplicative property of $\hbar_\circ$, we recapture

$$\begin{aligned}
&\Psi_*\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu),\delta^1\right) \leq \hbar_\circ\left(\frac{\varphi_\vartheta}{C_\vartheta}\right)\Psi_*(\mu_\vartheta,\delta^1) + \hbar_\circ\left(\frac{C_{\vartheta-1}}{C_\vartheta}\right)\left[\sum_{\nu=1}^{\vartheta-1}\hbar_\circ\left(\frac{\varphi_\nu}{C_{\vartheta-1}}\right)\Psi_*(\mu_\nu,\delta^1)\right.\\
&\left.-\sum_{\nu=1}^{\vartheta-1}\hbar_\circ\left(\frac{\varphi_\nu}{C_{\vartheta-1}}\right)\Psi_*\left(\left|\mu_\nu - \sum_{\nu=1}^{\vartheta-1}\frac{\varphi_\nu}{C_{\vartheta-1}}(\mu_\nu)\right|,\delta^1\right)\right] - \hbar_\circ\left(\frac{\varphi_\vartheta}{C_\vartheta}\right)\Psi_*\left(\frac{C_{\vartheta-1}}{C_\vartheta}\left|\mu_\vartheta - \sum_{\nu=1}^{\vartheta-1}\left(\frac{\varphi_\nu\mu_\nu}{C_{\vartheta-1}}\right)\right|,\delta^1\right)\\
&-\hbar_\circ\left(\frac{C_{\vartheta-1}}{C_\vartheta}\right)\Psi_*\left(\frac{\varphi_\vartheta}{C_\vartheta}\left|\mu_\vartheta - \sum_{\nu=1}^{\vartheta-1}\left(\frac{\varphi_\nu\mu_\nu}{C_{\vartheta-1}}\right)\right|,\delta^1\right).
\end{aligned}$$

Thus, we have

$$\Psi_*\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu\mu_\nu,\delta^1\right) \leq \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi_*(\mu_\nu,\delta^1) - \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi_*\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu\mu_\nu\right|,\delta^1\right), \quad (2.11)$$

By similar proceedings, we have

$$\Psi^*\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu),\delta^1\right) \leq \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi^*(\mu_\nu,\delta^1) - \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi^*\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu\mu_\nu\right|,\delta^1\right). \quad (2.12)$$

Comparing inequalities (2.11) and (2.12) through Pseudo ordering relation, we have

$$\begin{aligned}
&\left[\Psi_*\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu\mu_\nu,\delta^1\right),\ \Psi^*\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu),\delta^1\right)\right]\\
&\preceq \left[\sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi_*(\mu_\nu,\delta^1) - \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi_*\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu\mu_\nu\right|,\delta^1\right),\right.
\end{aligned}$$

$$\sum_{\nu=1}^{\vartheta} \hbar_{\circ}\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi^*(\mu_\nu, \delta^1) - \sum_{\nu=1}^{\vartheta} \hbar_{\circ}\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi^*\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu\mu_\nu\right|, \delta^1\right)\Bigg].$$

Finally, it can be transformed as

$$\Psi\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right) \leq \sum_{\nu=1}^{\vartheta} \hbar_{\circ}\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi(\mu_\nu)\ominus_g \sum_{\nu=1}^{\vartheta} \hbar_{\circ}\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right|\right).$$

Hence, the result is accomplished. □

We deliver some corollaries of Theorem 2.1.

- Setting $\sum_{\nu=1}^{\vartheta}\varphi_\nu = C_\vartheta = 1$ in Theorem 2.1, we attain the generalized Jensen's inequality:

$$\Psi\left(\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right) \leq \sum_{\nu=1}^{\vartheta} \hbar_{\circ}(\varphi_\nu)\Psi(\mu_\nu)\ominus_g \sum_{\nu=1}^{\vartheta} \hbar_{\circ}(\varphi_\nu)\Psi\left(\left|\mu_\nu - \sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right|\right).$$

- To attain Jensen's inequality for the fuzzy interval-valued super-quadratic mapping, we set $\hbar_{\circ}(\varphi) = \varphi$ in Theorem 2.1.

$$\Psi\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right) \leq \sum_{\nu=1}^{\vartheta} \frac{\varphi_\nu}{C_\vartheta}\Psi(\mu_\nu)\ominus_g \sum_{\nu=1}^{\vartheta} \frac{\varphi_\nu}{C_\vartheta}\Psi\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right|\right).$$

- To attain Jensen's inequality for the fuzzy interval-valued s-super-quadratic mapping, we set $\hbar_{\circ}(\varphi) = \varphi^s$ in Theorem 2.1.

$$\Psi\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right) \leq \sum_{\nu=1}^{\vartheta} \left(\frac{\varphi_\nu}{C_\vartheta}\right)^s f(\mu_\nu)\ominus_g \sum_{\nu=1}^{\vartheta} \left(\frac{\varphi_\nu}{C_\vartheta}\right)^s f\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right|\right).$$

- To attain Jensen's inequality for the fuzzy interval-valued s Godunova-Levin super-quadratic mapping, we set $\hbar_{\circ}(\varphi) = \varphi^{-s}$ in Theorem 2.1.

$$\Psi\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right) \leq \sum_{\nu=1}^{\vartheta} \left(\frac{\varphi_\nu}{C_\vartheta}\right)^{-s} \Psi(\mu_\nu)\ominus_g \sum_{\nu=1}^{\vartheta} \left(\frac{\varphi_\nu}{C_\vartheta}\right)^{-s} \Psi\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right|\right).$$

- To attain Jensen's inequality for the fuzzy interval-valued P-super-quadratic mapping, we set $\hbar_{\circ}(\varphi) = 1$ in Theorem 2.1.

$$\Psi\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right) \leq \sum_{\nu=1}^{\vartheta} \left(\frac{\varphi_\nu}{C_\vartheta}\right)^{-s} \Psi(\mu_\nu)\ominus_g \sum_{\nu=1}^{\vartheta} \left(\frac{\varphi_\nu}{C_\vartheta}\right)^{-s} \Psi\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right|\right).$$

- To attain Jensen's inequality for the fuzzy interval-valued exponential super-quadratic mapping, we set $\hbar_{\circ}(\varphi) = \exp(\varphi) - 1$ in Theorem 2.1.

$$\Psi\left(\frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right) \leq \sum_{\nu=1}^{\vartheta} \left[\exp\left(\frac{\varphi_\nu}{C_\vartheta}\right) - 1\right]\Psi(\mu_\nu)\ominus_g \sum_{\nu=1}^{\vartheta} \left[\exp\left(\frac{\varphi_\nu}{C_\vartheta}\right) - 1\right]\Psi\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=0}^{\vartheta}\varphi_\nu(\mu_\nu)\right|\right).$$

- By taking $\Psi_*(\mu, \delta^1) = \Psi_*(\mu, \delta^1)$ and δ^1 in Theorem 2.1, we get Theorem 1.1.

Next, the result is the Schur inequality for the fuzzy interval-valued super-quadratic mappings.

Theorem 2.2. *If $\Psi \in SSQFNF([s_3, s_4], \hbar_\circ)$ and $\mu, y, \mu_3 \in [s_3, s_4]$ with $\mu < y < \mu_3$ such that $y-\mu, \mu_3 - y$ and $\mu_3 - \mu \in [0, 1]$, then*

$$\hbar_\circ(\mu_3 - \mu)\Psi(y) \leq \hbar_\circ(\mu_3 - y)[\Psi(\mu) \ominus_g \Psi(y - \mu)] \oplus \hbar_\circ(y - \mu)[\Psi(\mu_3) \ominus_g \Psi(\mu_3 - y)].$$

Proof. Assume that $\mu, y, \mu_3 \in I$ with $\mu < y < \mu_3$ such that $y - \mu, \mu_3 - y$ and $\mu_3 - \mu \in [0, 1]$. Since $\Psi \in SSQFNF([s_3, s_4], \hbar_\circ)$, and $\hbar_\circ$ is a super-multiplicative mapping, then

$$\begin{aligned}
\Psi_*(y, \delta^1) &= \Psi_*\left(\frac{\mu_3 - y}{\mu_3 - \mu}\mu + \frac{y - \mu}{\mu_3 - \mu}y, \delta^1\right) \\
&\leq \hbar_\circ\left(\frac{\mu_3 - y}{\mu_3 - \mu}\right)\left[\Psi_*(\mu, \delta^1) - \Psi_*\left(\frac{y - \mu}{\mu_3 - \mu}|\mu_3 - \mu|, \delta^1\right)\right] \\
&\quad + \hbar_\circ\left(\frac{y - \mu}{\mu_3 - \mu}\right)\left[\Psi_*(\mu_3, \delta^1) - \Psi_*\left(\frac{\mu_3 - y}{\mu_3 - \mu}|\mu_3 - \mu|, \delta^1\right)\right].
\end{aligned}$$

Multiplying both sides of the aforementioned inequality by $\hbar_\circ(\mu_3 - \mu)$ and utilizing the supermultiplicative property, we recapture

$$\begin{aligned}
\hbar_\circ(\mu_3 - \mu)\Psi_*(y, \delta^1) &\leq \hbar_\circ(\mu_3 - y)[\Psi_*(\mu, \delta^1) - \Psi_*(y - \mu, \delta^1)] \\
&\quad + \hbar_\circ(y - \mu)[\Psi_*(\mu_3, \delta^1) - \Psi_*(\mu_3 - y, \delta^1)]. \qquad (2.13)
\end{aligned}$$

Likewise, we can prove that

$$\begin{aligned}
\hbar_\circ(\mu_3 - \mu)\Psi^*(y, \delta^1) &\leq \hbar_\circ(\mu_3 - y)[\Psi^*(\mu, \delta^1) - \Psi_*(y - \mu, \delta^1)] \\
&\quad + \hbar_\circ(y - \mu)[\Psi^*(\mu_3, \delta^1) - \Psi^*(\mu_3 - y, \delta^1)]. \qquad (2.14)
\end{aligned}$$

From (2.13) and (2.14), we get the desired containment. □

Remark 2.1. *For different substitution of $\hbar_\circ(\varphi) = \varphi, \varphi^s, \varphi^{-s}, \varphi(1 - \varphi)$, we get a blend of new counterparts for different classes of super-quadraticity. By taking $\Psi_*(\mu, \delta^1) = \Psi_*(\mu, \delta^1)$ and $\delta^1 = 1$ in Theorem 2.2, we get the reverse Jensen's inequality, and that is proved in [17].*

Through Theorem 2.2, we construct reverse Jensen's inequality leveraging the fuzzy number valued super-quadraticity.

Theorem 2.3. *For $\varphi_\nu \geq 0$ and $(v, V) \subseteq I$. Let $\hbar_\circ : (0, 1] \to [0, \infty)$ be nonnegative super-multiplicative mapping and $\Psi \in SSQFNF([s_3, s_4], \hbar_\circ)$. Then,*

$$\begin{aligned}
\sum_{\nu=1}^{\vartheta} \hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi(\mu_\nu) &\leq \sum_{\nu=1}^{\vartheta} \hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\left[\hbar_\circ\left(\frac{V - \mu_\nu}{V - v}\right)\Psi(v) + \hbar_\circ\left(\frac{\mu_\nu - v}{V - v}\right)\Psi(V)\right] \\
&\quad \ominus_g \hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\left[\hbar_\circ\left(\frac{V - \mu_\nu}{V - v}\right)\Psi(\mu_\nu - v) + \hbar_\circ\left(\frac{\mu_\nu - v}{V - v}\right)\Psi(V - \mu_\nu)\right].
\end{aligned}$$

Proof. Substitute $\mu_\nu = \nu, y = \mu_\nu$ and $\mu_3 = V$ in Theorem 2.2 and multiply both sides by $\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)$. Finally, we apply the sum up to ϑ to acquire the desired estimate. □

Now, we prove the Jensen-Mercer inequality pertaining to the fuzzy interval-valued super-quaraticity.

Theorem 2.4. *Let $\Psi \in SSQFNF([s_3, s_4], \hbar_\circ)$ and $\mu_\nu \in (s_3, s_4)$ and $\varphi_\nu \geq 0$, then*

$$\Psi\left(s_3 + s_4 - \frac{1}{C_\vartheta}\sum_{\nu=1}^{\vartheta}\varphi_\nu\mu_\nu\right) \leq \Psi(s_3)\oplus\Psi(s_4)\ominus_g \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi(\mu_\nu)\ominus_g \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)[\Psi(\mu_\nu - s_3)\oplus\Psi(s_4 - \mu_\nu)]$$
$$\ominus_g \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=1}^{\vartheta}\varphi_\nu\mu_\nu\right|\right).$$

Proof. Let $\Psi_*, \Psi^* \in SSQF([s_3, s_4], \hbar_\circ)$, then from Theorem 1.2, we get the following inequalities.

$$\begin{aligned}\Psi_*\left(s_3 + s_4 - \frac{1}{C_\vartheta}\sum_{\nu=1}^{\vartheta}\varphi_\nu\mu_\nu, \delta^1\right) &\leq \Psi_*(s_3, \delta^1) + \Psi_*(s_4, \delta^1) - \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi_*(\mu_\nu, \delta^1)\\ &- \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)[\Psi_*(\mu_\nu - s_3, \delta^1) + \Psi_*(s_4 - \mu_\nu, \delta^1)]\\ &- \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi_*\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=1}^{\vartheta}\varphi_\nu\mu_\nu\right|, \delta^1\right),\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\Psi^*\left(s_3 + s_4 - \frac{1}{C_\vartheta}\sum_{\nu=1}^{\vartheta}\varphi_\nu\mu_\nu, \delta^1\right) &\leq \Psi^*(s_3, \delta^1) + \Psi^*(s_4, \delta^1) - \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi^*(\mu_\nu, \delta^1)\\ &- \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)[\Psi^*(\mu_\nu - s_3, \delta^1) + \Psi^*(s_4 - \mu_\nu, \delta^1)]\\ &- \sum_{\nu=1}^{\vartheta}\hbar_\circ\left(\frac{\varphi_\nu}{C_\vartheta}\right)\Psi^*\left(\left|\mu_\nu - \frac{1}{C_\vartheta}\sum_{\nu=1}^{\vartheta}\varphi_\nu\mu_\nu\right|, \delta^1\right).\end{aligned} \tag{2.16}$$

Bridging (2.15) and (2.16) through Pseudo ordering, we acquire the fuzzy-valued Jensen-Mercer inequality. □

Remark 2.2. *For different substitution of $\hbar_\circ(\varphi) = \varphi, \varphi^s, \varphi^{-s}, \varphi(1-\varphi)$, we get fuzzy number-valued Jensen-Mercer inequalities for different classes of super-quadraticity. By taking $\Psi_*(\mu, \delta^1) = \Psi_*(\mu, \delta^1)$ and $\delta^1 = 1$ in Theorem 2.2, we get Jensen-Mercer inequality, and that is proved in [17].*

Theorem 2.5. *If $\Psi \in SSQFNF([s_3, s_4], \hbar_\circ)$, then*

$$\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi\left(\frac{s_3 + s_4}{2}\right)\oplus\frac{1}{(s_4 - s_3)}\int_{s_3}^{s_4}\Psi\left(\left|\mu - \frac{s_3 + s_4}{2}\right|\right)\mathrm{d}\mu \leq \frac{1}{s_4 - s_3}\int_{s_3}^{s_4}\Psi(\mu)\mathrm{d}\mu$$

$$\leq \frac{\Psi(s_3)\oplus\Psi(s_4)}{2}\cdot\frac{1}{(s_4-s_3)}\int_{s_3}^{s_4}\left(\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)+\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\right)\mathrm{d}\mu$$
$$\ominus_g \frac{1}{s_4-s_3}\int_{s_3}^{s_4}\left[\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi(\mu-s_3)\oplus\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi(s_4-\mu)\right]\mathrm{d}\mu.$$

Proof. Since $\Psi:[s_3,s_4]\to\mathbb{R}_I^+$ is a fuzzy interval-valued $\hbar_\circ$-super-quadratic mapping, we first consider $\varphi=\frac{1}{2}$, then we get

$$\Psi_*\left(\frac{\mu+y}{2},\delta^1\right)\leq\hbar_\circ\left(\frac{1}{2}\right)[\Psi_*(\mu,\delta^1)+\Psi_*(y,\delta^1)]-2\hbar_\circ\left(\frac{1}{2}\right)\Psi_*\left(\frac{1}{2}|y-\mu|,\delta^1\right).$$

The above inequality can be transformed as

$$\Psi_*\left(\frac{s_3+s_4}{2},\delta^1\right)\leq\hbar_\circ\left(\frac{1}{2}\right)\Psi_*\left(\varphi s_4+(1-\varphi)s_3,\delta^1\right)+\hbar_\circ\left(\frac{1}{2}\right)\Psi_*\left(\varphi s_3+(1-\varphi)s_4,\delta^1\right)$$
$$-2\hbar_\circ\left(\frac{1}{2}\right)\Psi_*\left(\frac{s_4-s_3}{2}|1-2\varphi|,\delta^1\right). \tag{2.17}$$

Integrating with respect to φ on $[0,1]$, we get

$$\Psi_*\left(\frac{s_3+s_4}{2},\delta^1\right)\leq\hbar_\circ\left(\frac{1}{2}\right)\int_0^1\Psi_*(\varphi s_4+(1-\varphi)s_3,\delta^1)\mathrm{d}\varphi+\hbar_\circ\left(\frac{1}{2}\right)\int_0^1\Psi_*(\varphi s_3+(1-\varphi)s_4,\delta^1)\mathrm{d}\varphi$$
$$-2\hbar_\circ\left(\frac{1}{2}\right)\int_0^1\Psi_*\left(\frac{s_4-s_3}{2}\,|1-2\varphi|\,,\delta^1\right)\mathrm{d}\varphi\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi_*\left(\frac{s_3+s_4}{2},\delta^1\right)$$
$$+\frac{1}{(s_4-s_3)}\int_{s_3}^{s_4}\Psi_*\left(\left|\mu-\frac{s_3+s_4}{2}\right|,\delta^1\right)\mathrm{d}\mu$$
$$\leq\frac{1}{s_4+s_3}\int_{s_3}^{s_4}\Psi_*(\mu,\delta^1)\mathrm{d}\mu.$$

By similar arguments, we have

$$\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi^*\left(\frac{s_3+s_4}{2},\delta^1\right)+\frac{1}{(s_4-s_3)}\int_{s_3}^{s_4}\Psi^*\left(\left|\mu-\frac{s_3+s_4}{2}\right|,\delta^1\right)\mathrm{d}\mu\leq\frac{1}{s_4+s_3}\int_{s_3}^{s_4}\Psi^*(\mu,\delta^1)\mathrm{d}\mu.$$

This implies that,

$$\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi\left(\frac{s_3+s_4}{2}\right)\oplus\frac{1}{(s_4-s_3)}\int_{s_3}^{s_4}\Psi\left(\left|\mu-\frac{s_3+s_4}{2}\right|\right)\mathrm{d}\mu\prec\frac{1}{s_4+s_3}\int_{s_3}^{s_4}\Psi(\mu)\mathrm{d}\mu. \tag{2.18}$$

Since $\Psi\in SSQFNF([s_3,s_4],\hbar_\circ)$, then

$$\Psi\left((1-\varphi)s_4+\varphi s_3\right)\leq\hbar_\circ(1-\varphi)\Psi(s_4)\oplus\hbar_\circ(\varphi)\Psi(s_3)$$
$$\ominus_g\hbar_\circ(\varphi)\Psi\left((1-\varphi)|s_4-s_3|\right)\ominus_g\hbar_\circ(1-\varphi)\Psi\left(\varphi|s_4-s_3|\right), \tag{2.19}$$

and

$$\Psi(\varphi s_4+(1-\varphi)s_3)\leq\hbar_\circ(\varphi)\Psi(s_4)\oplus\hbar_\circ(1-\varphi)\Psi(s_3)$$

$$\ominus_g \hbar_\circ(\varphi)\Psi((1-\varphi)|s_4-s_3|)\ominus_g \hbar_\circ(1-\varphi)\Psi(\varphi|s_4-s_3|). \tag{2.20}$$

Adding (2.19) and (2.20), we have

$$\begin{aligned}&\Psi\left((1-\varphi)s_4+\varphi s_3\right)\oplus\Psi(\varphi s_4+(1-\varphi)s_3)\\ &\leq [\hbar_\circ(\varphi)+\hbar_\circ(1-\varphi)][\Psi(a)\oplus\Psi(s_4)]\ominus_g 2\hbar_\circ(\varphi)\Psi((1-\varphi)|s_4-s_3|)\ominus_g 2\hbar_\circ(1-\varphi)\Psi(\varphi|s_4-s_3|).\end{aligned}$$

This can be transformed as

$$\begin{aligned}&\Psi_*(\varphi s_4+(1-\varphi)s_3,\delta^1)+\Psi(\varphi s_4+(1-\varphi)s_3,\delta^1)\\ &\leq [\hbar_\circ(\varphi)+\hbar_\circ(1-\varphi)][\Psi_*(s_4,\delta^1)+\Psi_*(s_3,\delta^1)]-2\hbar_\circ(\varphi)\Psi_*((1-\varphi)|s_4-s_3|,\delta^1)\\ &\quad -2\hbar_\circ(1-\varphi)\Psi_*(\varphi|s_4-s_3|,\delta^1),\end{aligned} \tag{2.21}$$

Integrating (2.21) with respect to φ on $[0,1]$,

$$\begin{aligned}\frac{1}{s_4-s_3}\int_{s_3}^{s_4}\Psi_*(\mu,\delta^1)\mathrm{d}\mu &\leq \frac{\Psi^*(s_3,\delta^1)+\Psi^*(s_4,\delta^1)}{2}\cdot\frac{1}{s_4-s_3}\int_{s_3}^{s_4}\left(\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3},\delta^1\right)+\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right),\delta^1\right)\mathrm{d}\mu\\ &-\frac{1}{s_4-s_3}\int_{s_3}^{s_4}\left[\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi_*(\mu-s_3,\delta^1)+\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi_*(s_4-\mu,\delta^1)\right]\mathrm{d}\mu.\end{aligned} \tag{2.22}$$

Through a similar strategy, we acquire

$$\begin{aligned}\frac{1}{s_4-s_3}\int_{s_3}^{s_4}\Psi^*(\mu)\mathrm{d}\mu &\leq \frac{\Psi^*(s_3)+\Psi^*(s_4)}{2}\cdot\frac{1}{s_4-s_3}\int_{s_3}^{s_4}\left(\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)+\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\right)\mathrm{d}\mu\\ &-\frac{1}{s_4-s_3}\int_{s_3}^{s_4}\left[\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi^*(\mu-s_3)+\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi^*(s_4-\mu)\right]\mathrm{d}\mu.\end{aligned} \tag{2.23}$$

Combining inequalities (2.25) and (2.26), we recapture the desired result. So, the result is accomplished. □

Now, we present some deductions of Theorem 2.5.

- Taking $\hbar_\circ(\varphi)=\varphi$, we recapture

$$\begin{aligned}&\Psi\left(\frac{s_3+s_4}{2}\right)\oplus\frac{1}{(s_4-s_3)}\int_{s_3}^{s_4}\Psi\left(\left|\mu-\frac{s_3+s_4}{2}\right|\right)\mathrm{d}\mu\leq\frac{1}{s_4-s_3}\int_{s_3}^{s_4}\Psi(\mu)\mathrm{d}\mu\\ &\leq\frac{\Psi(s_3)\oplus\Psi(s_4)}{2}\ominus_g\frac{1}{s_4-s_3}\int_{s_3}^{s_4}\left[\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi(\mu-s_3)\oplus\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi(s_4-\mu)\right]\mathrm{d}\mu.\end{aligned}$$

- Taking $\hbar_\circ(\varphi)=\varphi^s$, we recapture

$$\begin{aligned}&2^{s-1}\Psi\left(\frac{s_3+s_4}{2}\right)\oplus\frac{1}{(s_4-s_3)}\int_{s_3}^{s_4}\Psi\left(\left|\mu-\frac{s_3+s_4}{2}\right|\right)\mathrm{d}\mu\leq\frac{1}{s_4-s_3}\int_{s_3}^{s_4}\Psi(\mu)\mathrm{d}\mu\\ &\leq\frac{\Psi(s_3)\oplus\Psi(s_4)}{s+1}\ominus_g\frac{1}{s_4-s_3}\int_{s_3}^{s_4}\left[\left(\frac{s_4-\mu}{s_4-s_3}\right)^s f(\mu-s_3)\oplus\left(\frac{\mu-s_3}{s_4-s_3}\right)^s f(s_4-\mu)\right]\mathrm{d}\mu.\end{aligned}$$

- Taking $\hbar_\circ(\varphi) = \varphi^{-s}$, we recapture

$$2^{s+1}\Psi\left(\frac{s_3+s_4}{2}\right)\oplus\frac{1}{(s_4-s_3)}\int_{s_3}^{s_4}\Psi\left(\left|\mu-\frac{s_3+s_4}{2}\right|\right)\mathrm{d}\mu \leq \frac{1}{s_4-s_3}\int_{s_3}^{s_4}\Psi(\mu)\mathrm{d}\mu$$
$$\leq \frac{\Psi(s_3)\oplus\Psi(s_4)}{1-s}\ominus_g\frac{1}{s_4-s_3}\int_{s_3}^{s_4}\left[\left(\frac{s_4-\mu}{s_4-s_3}\right)^{-s}\Psi(\mu-s_3)\oplus\left(\frac{\mu-s_3}{s_4-s_3}\right)^{-s}\Psi(s_4-\mu)\right]\mathrm{d}\mu.$$

- Taking $\hbar_\circ(\varphi) = 1$, we recapture

$$2^{-1}\Psi\left(\frac{s_3+s_4}{2}\right)\oplus\frac{1}{(s_4-s_3)}\int_{s_3}^{s_4}\Psi\left(\left|\mu-\frac{s_3+s_4}{2}\right|\right)\mathrm{d}\mu \leq \frac{1}{s_4+s_3}\int_{s_3}^{s_4}\Psi(\mu)\mathrm{d}\mu$$
$$\leq [\Psi(s_3)\oplus\Psi(s_4)]\ominus_g\frac{1}{s_4-s_3}\int_{s_3}^{s_4}[\Psi(\mu-s_3)\oplus\Psi(s_4-\mu)]\,\mathrm{d}\mu.$$

Remark 2.3. *We can a get a blend of new Hermite-Hadamard's type inequalities for different values of $\hbar_\circ$ and by taking $\Psi_*(\mu,\delta^1) = \Psi_*(\mu,\delta^1)$ and $\delta^1 = 1$ in Theorem 2.5, we get the classical Hermite-Hadamard inequality for super-quadratic mappings, which is derived in [15].*

Example 2.2. *Let $\Psi : [s_3,\ s_4] = [0,2] \to \Upsilon_{\delta^1}$ be a fuzzy valued super-quadratic mapping, which is defined as*

$$\Psi_\mu(\mu_1) = \begin{cases} \frac{\mu_1}{3\mu^3}, & \mu_1 \in [0, 3\mu^3] \\ \frac{6\mu^3-\mu_1}{3\mu^3}, & \mu_1 \in (3\mu^3, 6\mu^3]. \end{cases}$$

and its level cuts are $\Psi_{\delta^1} = \left[3\delta^1\mu^3,\ (6-3\delta^1)\mu^3\right]$. It fulfils the condition of Theorem 2.5, then

$$\textit{Left Term} = \left[\frac{15{s_4}^3}{64},\ \frac{1}{32}\left(30-\frac{15}{2}\right){s_4}^3\right],$$
$$\textit{Middle Term} = \left[\frac{3{s_4}^3}{8},\ \frac{1}{4}\left(6-\frac{3}{2}\right){s_4}^3\right],$$
$$\textit{Right Term} = \left[\frac{6{s_4}^3}{10},\ \frac{2}{5}\left(6-\frac{3}{2}\right){s_4}^3\right].$$

To visualize the above formulations, we fix δ^1 and vary s_4.

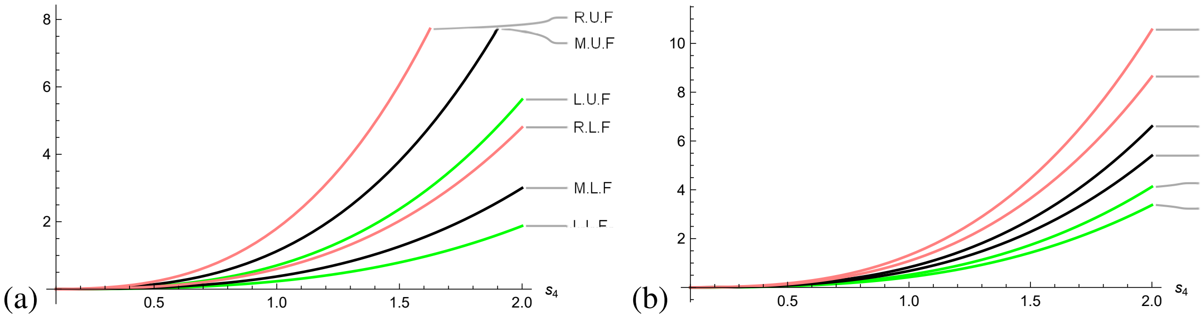

Figure 1. Visual analysis of Theorem 2.5.

Note that L.L.F, L.U.F, M.L.F, M.U.F, R.L.F, and R.U.F are specifying the endpoint mappings of left, middle, and right terms of Theorem 2.5.

Table 1. Numerical validation of Example 2.2.

(δ^1, s_4)	$L_*(\delta^1, s_4)$	$L^*(\delta^1, s_4)$	$M_*(\delta^1, s_4)$	$M^*(\delta^1, s_4)$	$R_*(\delta^1, s_4)$	$R^*(\delta^1, s_4)$
(0.3, 1)	0.140625	0.796875	0.2250	1.2750	0.3600	2.0400
(0.4, 1.3)	0.411938	1.64775	0.6591	2.6364	1.05456	4.21824
(0.5, 1.5)	0.791016	2.37305	1.26563	3.79688	2.025	6.0750
(0.8, 1.8)	0.1870	3.2805	3.4992	5.2488	5.59872	8.39808

Theorem 2.6. *If* $\Psi \in SSQFNF([s_3, s_4], \hbar_\circ)$ *and* $g : [s_3, s_4] \to \mathbb{R}$ *is a symmetric mapping, then*

$$\begin{aligned}
&\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi\left(\frac{s_3+s_4}{2}\right)\int_{s_3}^{s_4} g(\mu)\mathrm{d}\mu \oplus \int_{s_3}^{s_4} \Psi\left(\left|\mu - \frac{s_3+s_4}{2}\right|\right) g(\mu)\mathrm{d}\mu \\
&\leq \int_{s_3}^{s_4} \Psi(\mu, \delta^1) g(\mu)\mathrm{d}\mu \\
&\leq \frac{[\Psi(s_4)\oplus\Psi(s_3)]}{2}\int_{s_3}^{s_4}\left[\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right) + \hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\right] g(\mu)\mathrm{d}\mu \\
&\ominus_g \int_{s_3}^{s_4} \hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi(s_4-\mu)g(\mu)\mathrm{d}\mu \ominus_g \int_{s_3}^{s_4} \hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi(\mu-s_3)g(\mu)\mathrm{d}\mu.
\end{aligned}$$

Proof. Since $\Psi : [s_3, s_4] \to \mathbb{R}_I^+$ is a fuzzy interval-valued $\hbar_\circ$-super-quadratic mapping, then by multiplying (2.17) by $g(\varphi s_4 + (1-\varphi)a)$ and integrating with respect to φ on $[0, 1]$, we get

$$\begin{aligned}
\Psi_*\left(\frac{s_3+s_4}{2}, \delta^1\right)\int_0^1 g(\varphi s_4 + (1-\varphi)a)\mathrm{d}\varphi &\leq \hbar_\circ\left(\frac{1}{2}\right)\int_0^1 \Psi_*(\varphi s_4 + (1-\varphi)s_3, \delta^1) g(\varphi s_4 + (1-\varphi)a)\mathrm{d}\varphi \\
&+ \hbar_\circ\left(\frac{1}{2}\right)\int_0^1 \Psi_*(\varphi s_3 + (1-\varphi)s_4, \delta^1) g(\varphi s_4 + (1-\varphi)a)\mathrm{d}\varphi \\
&- 2\hbar_\circ\left(\frac{1}{2}\right)\int_0^1 \Psi_*\left(\frac{s_4-s_3}{2}|1-2\varphi|, \delta^1\right) g(\varphi s_4 + (1-\varphi)a)\mathrm{d}\varphi.
\end{aligned}$$

Since g is a symmetric mapping about $\frac{s_3+s_4}{2}$, then $g(\varphi s_4 + (1-\varphi)s_3) = g((1-\varphi)s_4 + \varphi s_3)$. Using this fact in the above inequality, we get

$$\begin{aligned}
&\Psi_*\left(\frac{s_3+s_4}{2}, \delta^1\right)\int_0^1 g(\varphi s_4 + (1-\varphi)a)\mathrm{d}\varphi \\
&\leq \hbar_\circ\left(\frac{1}{2}\right)\int_0^1 \Psi_*(\varphi s_4 + (1-\varphi)s_3, \delta^1) g(\varphi s_4 + (1-\varphi)a)\mathrm{d}\varphi \\
&+ \hbar_\circ\left(\frac{1}{2}\right)\int_0^1 \Psi_*(\varphi s_3 + (1-\varphi)s_4, \delta^1) g(\varphi s_3 + (1-\varphi)s_4)\mathrm{d}\varphi \\
&- 2\hbar_\circ\left(\frac{1}{2}\right)\int_0^a \Psi_{\circ *}\left(\frac{s_4-s_3}{2}|1-2\varphi|, \delta^1\right) g(\varphi s_4 + (1-\varphi)a)\mathrm{d}\varphi.
\end{aligned}$$

This implies that

$$\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi_{\circ*}\left(\frac{s_3+s_4}{2},\delta^1\right)\int_{s_3}^{s_4}g(\mu)\mathrm{d}\mu+\int_{s_3}^{s_4}\Psi_{\circ*}\left(\left|\mu-\frac{s_3+s_4}{2}\right|\right)g(\mu)\mathrm{d}\mu\leq\int_{s_3}^{s_4}\Psi_{\circ*}(\mu)g(\mu)\mathrm{d}\mu.$$

Similarly, we have

$$\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi^*\left(\frac{s_3+s_4}{2},\delta^1\right)\int_{s_3}^{s_4}g(\mu)\mathrm{d}\mu+\int_{s_3}^{s_4}\Psi^*\left(\left|\mu-\frac{s_3+s_4}{2}\right|,\delta^1\right)g(\mu)\mathrm{d}\mu\leq\int_{s_3}^{s_4}\Psi^*(\mu,\delta^1)g(\mu)\mathrm{d}\mu.$$

Combining the last two inequalities in the Pseudo ordering relation, we have

$$\begin{aligned}&\left[\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi_{\circ*}\left(\frac{s_3+s_4}{2},\delta^1\right)\int_{s_3}^{s_4}g(\mu)\mathrm{d}\mu+\int_{s_3}^{s_4}\Psi_{\circ*}\left(\left|\mu-\frac{s_3+s_4}{2}\right|\right)g(\mu)\mathrm{d}\mu\right.\\&\left.\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi^*\left(\frac{s_3+s_4}{2},\delta^1\right)\int_{s_3}^{s_4}g(\mu)\mathrm{d}\mu+\int_{s_3}^{s_4}\Psi^*\left(\left|\mu-\frac{s_3+s_4}{2}\right|,\delta^1\right)g(\mu)\mathrm{d}\mu\right]\\&\preceq\left[\int_{s_3}^{s_4}\Psi_*(\mu,\delta^1)g(\mu)\mathrm{d}\mu,\ \int_{s_3}^{s_4}\Psi^*(\mu,\delta^1)g(\mu)\mathrm{d}\mu\right].\end{aligned}$$

Finally, we can write

$$\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi\left(\frac{s_3+s_4}{2}\right)\int_{s_3}^{s_4}g(\mu)\mathrm{d}\mu\oplus\int_{s_3}^{s_4}\Psi\left(\left|\mu-\frac{s_3+s_4}{2}\right|\right)g(\mu)\mathrm{d}\mu\preceq\int_{s_3}^{s_4}\Psi(\mu)g(\mu)\mathrm{d}\mu.\tag{2.24}$$

Multiplying (2.21) by $g(\varphi s_4+(1-\varphi)s_3,\delta^1)$ and integrating with respect to φ on $[0,1]$, we get

$$\begin{aligned}&\int_0^1\Psi_*(\varphi s_3+(1-\varphi)s_4,\delta^1)g(\varphi s_4+(1-\varphi)s_3)\mathrm{d}\varphi+\int_0^1\Psi_*(\varphi s_4+(1-\varphi)s_3,\delta^1)g(\varphi s_4+(1-\varphi)s_3)\mathrm{d}\varphi\\&\leq[\Psi_*(s_4,\delta^1)+\Psi_*(s_3,\delta^1)]\int_0^1[\hbar_\circ(\varphi)+\hbar_\circ(1-\varphi)]g(\varphi s_4+(1-\varphi)s_3)\mathrm{d}\varphi\\&-2\int_0^1\hbar_\circ(\varphi)\Psi_*((1-\varphi)|s_4-s_3|,\delta^1)g(\varphi s_4+(1-\varphi)s_3)\mathrm{d}\varphi\\&-2\int_0^1\hbar_\circ(1-\varphi)\Psi_*(\varphi|s_4-s_3|,\delta^1)g(\varphi s_4+(1-\varphi)s_3)\mathrm{d}\varphi.\end{aligned}$$

We can write

$$\begin{aligned}\int_{s_3}^{s_4}\Psi_*(\mu,\delta^1)g(\mu)\mathrm{d}\varphi\leq&\frac{[\Psi_*(s_4,\delta^1)+\Psi_*(s_3,\delta^1)]}{2}\int_{s_3}^{s_4}\left[\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)+\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\right]g(\mu)\mathrm{d}\mu\\-\int_{s_3}^{s_4}\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)&\Psi_*(s_4-\mu,\delta^1)g(\mu)\mathrm{d}\mu-\int_{s_3}^{s_4}\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi_*(\mu-s_3,\delta^1)g(\mu)\mathrm{d}\mu.\end{aligned}\tag{2.25}$$

Also,

$$\int_{s_3}^{s_4}\Psi^*(\mu,\delta^1)g(\mu)\mathrm{d}\varphi\leq\frac{[\Psi^*(s_4,\delta^1)+\Psi^*(s_3,\delta^1)]}{2}\int_{s_3}^{s_4}\left[\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)+\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\right]g(\mu)\mathrm{d}\mu$$

$$-\int_{s_3}^{s_4} \hbar_\circ\left(\frac{\mu - s_3}{s_4 - s_3}\right)\Psi^*(s_4-\mu,\delta^1)g(\mu)\mathrm{d}\mu - \int_{s_3}^{s_4} \hbar_\circ\left(\frac{s_4-\mu}{s_4 - s_3}\right)\Psi^*(\mu - s_3,\delta^1)g(\mu)\mathrm{d}\mu. \tag{2.26}$$

Combining (2.25) and (2.26), we achieve the required inequality. Hence, the result is completed. □

Remark 2.4. *By selecting $\hbar_\circ(\varphi) = \varphi, \varphi^s, \varphi^{-s}, \varphi(1-\varphi)$ in Theorem 2.6, we get* $\mathtt{F.N.V}$ *Hermite-Hadamard-Fejer's inequalities for different classes of super-quadraticity. If we take $g(\mu) = 1$ in Theorem 2.6, we obtain the Hermite-Hadamard's inequality. Also by taking $\Psi_*(\mu,\delta^1) = \Psi_*(\mu,\delta^1)$ and $\delta^1 = 1$ in Theorem 2.6, we obtain the Hermite-Hadamard-Fejer inequality.*

Example 2.3. *Let $\Psi : [s_3, s_4] = [0, 2] \to \Upsilon_{\delta^1}$ be fuzzy valued super-quadratic mapping, which is defined as*

$$\Psi_\mu(\mu_1) = \begin{cases} \frac{\mu_1}{3\mu^3}, & \mu_1 \in [0, 3\mu^3] \\ \frac{6\mu^3-\mu_1}{3\mu^3}, & \mu_1 \in (3\mu^3, 6\mu^3]. \end{cases}$$

and its level cuts are $\Psi_{\delta^1} = \left[3\delta^1\mu^3,\ (6-3\delta^1)\mu^3\right]$. Also $g : [0,2] \to \mathbb{R}$ is a symmetric integrable mapping and is defined as $g(\mu) = \begin{cases} \mu, & \mu \in [0,1] \\ 2-\mu, & \mu \in (1,2]. \end{cases}$ Both mappings fulfill the condition of Theorem 2.6, then

$$\textit{Left Term} = \left[\frac{3}{16}{s_4}^3\left(2-(2-s_4)^2\right)\delta^1,\ \frac{1}{16}{s_4}^3\left(2-(2-s_4)^2\right)(6-3\delta^1)\right],$$

$$\textit{Middle Term} = \left[\frac{3}{10}\left({s_4}^4(5-2s_4)-3\right)\delta^1 + \frac{3\delta^1}{5},\ \frac{1}{10}\left({s_4}^4(5-2s_4)-3\right)(6-3\delta^1) + \frac{1}{5}(6-3\delta^1)\right],$$

$$\textit{Right Term} = \left[\frac{3}{4}{s_4}^3\left(2-(2-s_4)^2\right)\delta^1 - \frac{\left(-3{s_4}^6+12{s_4}^5-20{s_4}^3+30{s_4}^2-24s_4+8\right)(3\delta^1)}{60s_4},\right.$$
$$\left.\frac{1}{4}{s_4}^3\left(2-(2-s_4)^2\right)(6-3\delta^1) - \frac{\left(-3{s_4}^6+12{s_4}^5-20{s_4}^3+30{s_4}^2-24s_4+8\right)(6-3\delta^1)}{60s_4}\right].$$

To visualize the above formulations, we fix δ^1 and vary τ.

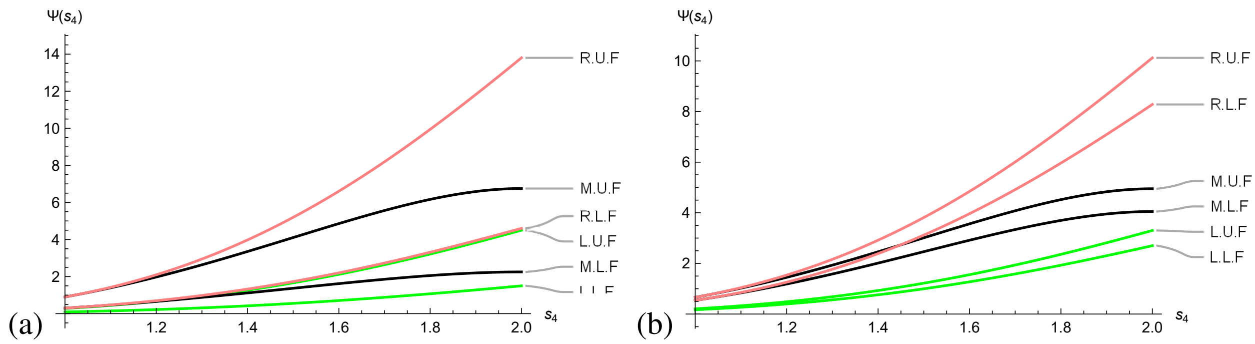

Figure 2. Visual analysis of Theorem 3.1.

Note that L.L.F, L.U.F, M.L.F, M.U.F, R.L.F, and R.U.F are specifying the endpoint mappings of left, middle, and right terms of Theorem 2.6.

Table 2. Numerical validation of Example 2.3.

(δ^1, s_4)	$L_*(\delta^1, s_4)$	$L^*(\delta^1, s_4)$	$M_*(\delta^1, s_4)$	$M^*(\delta^1, s_4)$	$R_*(\delta^1, s_4)$	$R^*(\delta^1, s_4)$
(0.3, 1)	0.05625	0.31875	0.1800	1.0200	0.1800	1.0200
(0.4, 1.3)	0.24881	0.995241	0.702557	2.81023	0.785476	3.1419
(0.5, 1.5)	0.553711	1.66113	1.36875	4.10625	1.73229	5.19688
(0.8, 1.8)	1.71461	2.57191	3.28719	4.93079	5.30129	7.95193

3. Fractional Hadamard and Fejer inequalities

This section contains fractional trapezoidal-like inequalities incorporated with $\mathtt{F.N.V}$-$\hbar_\circ$ super-quadratic mappings.

Lemma 3.1. *If* $\Psi \in SSQFNF([s_3, s_4], \hbar_\circ)$*, then*

$$\Psi\left(\frac{s_3 + s_4}{2}\right) \leq \hbar_\circ\left(\frac{1}{2}\right)\Psi(\mu)\oplus\hbar_\circ\left(\frac{1}{2}\right)\Psi(s_3 + s_4 - \mu)\ominus_g 2\hbar_\circ\left(\frac{1}{2}\right)\Psi\left(\left|\frac{s_3 + s_4}{2} - \mu\right|\right).$$

Proof. Let $\Psi : [s_3, s_4] \to \mathbb{R}_I^+$ be $\mathtt{F.N.V}$ $\hbar_\circ$-super-quadraticity, and we have

$$\Psi((1 - \varphi)y + \varphi\mu) \leq \hbar_\circ(1 - \varphi)\Psi(y)\oplus\hbar_\circ(\varphi)\Psi(\mu)\ominus_g\hbar_\circ(\varphi)\Psi((1 - \varphi)|y - \mu|)\ominus_g\hbar_\circ(1 - \varphi)\Psi(\varphi|y - \mu|).$$

We can break the above inequality as

$$\begin{aligned}\Psi_*((1 - \varphi)y + \varphi\mu, \delta^1) &< \hbar_\circ(1 - \varphi)\Psi_*(y, \delta^1) + \hbar_\circ(\varphi)\Psi_*(\mu, \delta^1)\\ &\quad - \hbar_\circ(\varphi)\Psi_*(|y - (tx_2 + (1 - \varphi)\mu|), \delta^1) - \hbar_\circ(1 - \varphi)\Psi_*(|\mu - ((1 - \varphi)\mu + \varphi y)|, \delta^1),\end{aligned}$$

and

$$\begin{aligned}\Psi^*((1 - \varphi)y + \varphi\mu, \delta^1) &< \hbar_\circ(1 - \varphi)\Psi^*(y, \delta^1) + \hbar_\circ(\varphi)\Psi^*(\mu, \delta^1)\\ &\quad - \hbar_\circ(\varphi)\Psi^*(|y - (tx_2 + (1 - \varphi)\mu|), \delta^1) - \hbar_\circ(1 - \varphi)\Psi_*(|\mu - ((1 - \varphi)\mu + \varphi y)|, \delta^1).\end{aligned}$$

Furthermore, we can write:

$$\begin{aligned}\Psi_*\left(\frac{s_3 + s_4}{2}, \delta^1\right) &\leq \hbar_\circ\left(\frac{1}{2}\right)\Psi_*(\mu, \delta^1) + \hbar_\circ\left(\frac{1}{2}\right)\Psi_*(s_3 + s_4 - \mu, \delta^1)\\ &\quad - \hbar_\circ\left(\frac{1}{2}\right)\Psi_*\left(\left|\mu - \left(\frac{s_3 + s_4}{2}\right)\right|, \delta^1\right) - \hbar_\circ\left(\frac{1}{2}\right)\Psi_*\left(\left|s_3 + s_4 - \mu - \left(\frac{s_3 + s_4}{2}\right)\right|, \delta^1\right)\\ &\leq \hbar_\circ\left(\frac{1}{2}\right)\Psi_*(\mu, \delta^1) + \hbar_\circ\left(\frac{1}{2}\right)\Psi_*(s_3 + s_4 - \mu, \delta^1) - 2\hbar_\circ\left(\frac{1}{2}\right)\Psi_*\left(\left|\left(\frac{s_3 + s_4}{2} - \mu\right)\right|, \delta^1\right).\end{aligned} \tag{3.1}$$

Moreover, we have

$$\begin{aligned}\Psi^*\left(\frac{s_3 + s_4}{2}, \delta^1\right) &\leq \hbar_\circ\left(\frac{1}{2}\right)\Psi^*(\mu, \delta^1) + \hbar_\circ\left(\frac{1}{2}\right)\Psi^*(s_3 + s_4 - \mu, \delta^1)\\ &\quad - 2\hbar_\circ\left(\frac{1}{2}\right)\Psi^*\left(\left|\left(\frac{s_3 + s_4}{2} - \mu\right)\right|, \delta^1\right).\end{aligned} \tag{3.2}$$

Comparing (3.1) and (3.2), we achieve the final result. □

Lemma 3.2. *If* $\Psi \in SSQFNF([s_3, s_4], \hbar_\circ)$*, then*

$$\Psi(\mu)\oplus\Psi(s_3+s_4-\mu) \leq \Psi(s_3)\oplus\Psi(s_4)\ominus_g 2\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi(s_4-\mu)\ominus_g 2\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi(\mu-s_3).$$

Proof. Assume that $\Psi : [s_3, s_4] \to \mathbb{R}_I^+$ is a $\mathtt{F.N.V}$-$\hbar_\circ$ super-quadratic mapping on $[s_3, s_4]$, then

$$\begin{aligned}\Psi_*((1-\varphi)s_4+\varphi s_3, \delta^1) \leq \hbar_\circ(1-\varphi)\Psi_*(s_4, \delta^1) + \hbar_\circ(\varphi)\Psi_*(s_3, \delta^1)&\\ - \hbar_\circ(\varphi)\Psi_*((1-\varphi)|s_3 - (\varphi s_3 + (1-\varphi)s_4|, \delta^1)&\\ - \hbar_\circ(1-\varphi)\Psi_*(\varphi|s_4 - (1-\varphi)s_4 + \varphi s_3|, \delta^1).&\end{aligned}$$

Substitute $\mu = ((1-\varphi)s_4 + \varphi s_3)$, then

$$\begin{aligned}\Psi_*(\mu, \delta^1) \leq \hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi_*(s_4, \delta^1) + \mu\left(\frac{\leftthreetimes_\circ - s_3}{s_4-s_3}\right)\Psi_*(s_3, \delta^1)&\\ - \hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi_*(\mu-s_3, \delta^1) - \hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi_*(s_4-\mu, \delta^1).&\end{aligned} \tag{3.3}$$

Replacing μ by $s_3 + s_4 - \mu$ in (3.3), we get

$$\begin{aligned}\Psi_*(s_3+s_4-\mu, \delta^1) \leq \hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi_*(s_4, \delta^1) + \hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi_*(s_3, \delta^1)&\\ - \hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi_*(\mu-s_3, \delta^1) - \hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi_*(s_4-\mu, \delta^1).&\end{aligned} \tag{3.4}$$

Adding (3.3) and (3.4), we have

$$\begin{aligned}\Psi_*(\mu, \delta^1) + \Psi_*(s_3+s_4-\mu, \delta^1) \leq \Psi_*(s_3, \delta^1) + \Psi_*(s_4, \delta^1) - 2\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}, \delta^1\right)\Psi_*(s_4-\mu, \delta^1)&\\ - 2\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi_*(\mu-s_3, \delta^1).&\end{aligned} \tag{3.5}$$

Similarly,

$$\begin{aligned}\Psi^*(\mu) + \Psi^*(s_3+s_4-\mu, \delta^1) \leq \Psi^*(s_3, \delta^1) + \Psi^*(s_4, \delta^1) - 2\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi^*(s_4-\mu, \delta^1)&\\ - 2\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi^*(\mu-s_3, \delta^1).&\end{aligned} \tag{3.6}$$

Comparison of (3.5) and (3.6) through Pseudo ordering relation, we get our final outcome. □

Theorem 3.1. *If* $\Psi \in SSQFNF([s_3, s_4], \hbar_\circ)$*, then*

$$\begin{aligned}&\frac{1}{\hbar_\circ\left(\frac{1}{2}\right)}\Psi\left(\frac{s_3+s_4}{2}\right)\oplus\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\Psi\left(\left|\frac{s_3+s_4}{2}-\mu\right|\right)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu\\ &\leq \frac{\Gamma(1+\tau)}{(s_4-s_3)^\tau}\left(J_{s_3+}^\tau\Psi(s_4)\oplus J_{s_4-}^\tau\Psi(s_3)\right)\\ &\leq \Psi(s_3)\oplus\Psi(s_4)\ominus_g\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left(\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi(s_4-\mu)\oplus\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi(\mu-s_3)\right) \qquad (3.7)\\ &\quad\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu. \qquad (3.8)\end{aligned}$$

Proof. Since Ψ is an `F.N.V` super-quadratic mapping, then

$$\Psi_*\left(\frac{s_3+s_4}{2},\delta^1\right)=\Psi_*\left(\frac{s_3+s_4}{2},\delta^1\right)\frac{\tau}{2(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu$$
$$=\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{\frac{s_3+s_4}{2}}\Psi_*\left(\frac{s_3+s_4}{2},\delta^1\right)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu.$$

Through Lemma 3.1, we can interpret

$$\Psi_*\left(\frac{s_3+s_4}{2},\delta^1\right)\le\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{\frac{s_3+s_4}{2}}\left[\hbar_\circ\left(\frac{1}{2}\right)\Psi_*(\mu,\delta^1)+\hbar_\circ\left(\frac{1}{2}\right)\Psi_*(s_3+s_4-\mu,\delta^1)\right.$$
$$\left.-2\hbar_\circ\left(\frac{1}{2}\right)\Psi_*\left(\left|\frac{s_3+s_4}{2}-\mu\right|,\delta^1\right)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\right]\mathrm{d}\mu. \tag{3.9}$$

From (3.9), we have

$$\frac{1}{\hbar_\circ\left(\frac{1}{2}\right)}\Psi_*\left(\frac{s_3+s_4}{2},\delta^1\right)\le\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{\frac{s_3+s_4}{2}}\left[\Psi_*(\mu,\delta^1)+\Psi_*(s_3+s_4-\mu,\delta^1)\right.$$
$$\left.-2\Psi_*\left(\left|\frac{s_3+s_4}{2}-\mu\right|,\delta^1\right)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\right]\mathrm{d}\mu$$
$$=\frac{\tau}{(s_4-s_3)^\tau}\left[\int_{s_3}^{\frac{s_3+s_4}{2}}\Psi_*(\mu,\delta^1)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu\right.$$
$$+\int_{\frac{s_3+s_4}{2}}^{s_4}\Psi_*(\mu,\delta^1)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu$$
$$-\int_{s_3}^{\frac{s_3+s_4}{2}}\Psi_*\left(\left|\frac{s_3+s_4}{2}-\mu\right|,\delta^1\right)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu$$
$$\left.-\int_{\frac{s_3+s_4}{2}}^{s_4}\Psi_*\left(\left|\frac{s_3+s_4}{2}-\mu\right|,\delta^1\right)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu\right]$$
$$=\frac{\tau}{(s_4-s_3)^\tau}\left[\int_{s_3}^{s_4}\Psi_*(\mu,\delta^1)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu\right.$$
$$\left.-\int_{s_3}^{s_4}\Psi_*\left(\left|\frac{s_3+s_4}{2}-\mu\right|,\delta^1\right)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu\right]$$
$$=\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3+}\Psi_*(s_4,\delta^1)+J^\tau_{s_4-}\Psi_*(s_3,\delta^1)\right]$$
$$-\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\Psi_*\left(\left|\frac{s_3+s_4}{2}-\mu\right|,\delta^1\right)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu. \tag{3.10}$$

Also,

$$\frac{1}{\hbar_\circ\left(\frac{1}{2}\right)}\Psi^*\left(\frac{s_3+s_4}{2},\delta^1\right)\le\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3+}\Psi^*(s_4,\delta^1)+J^\tau_{s_4-}\Psi^*(s_3,\delta^1)\right]$$
$$-\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\Psi^*\left(\left|\frac{s_3+s_4}{2}-\mu\right|,\delta^1\right)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu. \tag{3.11}$$

Combining (3.10) and (3.11) via pseudo order relation, we acquire the first inequality of (3.7). From Lemma 3.2, we can write

$$\begin{aligned}
&\frac{\Gamma(1+\tau)}{(s_4-s_3)^\tau}\left(J^\tau_{s_3+}\Psi_*(s_4,\delta^1)+J^\tau_{s_4-}\Psi_*(s_3,\delta^1)\right)\\
&=\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{\frac{\tau_1+s_1}{2}}[\Psi_*(\mu,\delta^1)+\Psi_*(s_3+s_4-\mu,\delta^1)]\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu\\
&\leq\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{\frac{s_3+s_4}{2}}\left(\Psi_*(s_3,\delta^1)+\Psi_*(s_4,\delta^1)-2\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi_*(s_4-\mu,\delta^1)\right.\\
&\left.-2\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi_*(\mu-s_3,\delta^1)\right)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu\\
&=\Psi_*(s_3)+\Psi_*(s_4)-\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left(\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi_*(s_4-\mu)\right.\\
&\left.+\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi_*(\mu-s_3)\right)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu. \qquad (3.12)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{\Gamma(1+\tau)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3+}\Psi^*(s_4,\delta^1)+J^\tau_{s_4-}\Psi^*(s_3,\delta^1)\right]\leq\Psi^*(s_3,\delta^1)+\Psi^*(s_4,\delta^1)\\
&-\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left(\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi^*(s_4-\mu,\delta^1)\right.\\
&\left.+\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi^*(\mu-s_3,\delta^1)\right)\left((s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right)\mathrm{d}\mu. \qquad (3.13)
\end{aligned}$$

Comparing (3.12) and (3.13) through Pseudo ordering relation, we achieve our desired result. □

Remark 3.1. *For $\tau = 1$, the Theorem 3.1 transformed into Theorem 2.5. By selecting $\hbar_\circ(\varphi) = \varphi, \varphi^s, \varphi^{-s}, \varphi(1-\varphi)$ in Theorem 3.1, we get various fractional* F.N.V *Hermite-Hadamard's inequalities for different classes of super-quadraticity. Also by taking $\Psi_*(\mu,\delta^1) = \Psi_*(\mu,\delta^1)$ and $\delta^1 = 1$ in Theorem 3.1, we obtain the fractional Hermite-Hadamard's inequality for $\hbar_\circ$–super-quadratic mappings, which is given in [59].*

Example 3.1. *Let $\Psi : [s_3,\ s_4] = [0,2] \to \Upsilon_{\delta^1}$ be a fuzzy valued super-quadratic mapping which is defined as*

$$\Psi_\mu(\mu_1)=\begin{cases}\frac{\mu_1}{3\mu^3}, & \mu_1\in[0,3\mu^3]\\ \frac{6\mu^3-\mu_1}{3\mu^3}, & \mu_1\in(3\mu^3,6\mu^3].\end{cases}$$

and its level cuts are $\Psi_{\delta^1} = \left[3\delta^1\mu^3,\ (6-3\delta^1)\mu^3\right]$. It fulfills the condition of Theorem 3.1, then

$$\textit{Left Term}=\left[\frac{\left(2\left(2^\tau\tau^3+5\ 2^\tau\tau-3\ 2^{\tau+1}+12\right)\right)(3\tau\delta^1)}{2^\tau(\tau(\tau+1)(\tau+2)(\tau+3))},\right.$$

$$+6\delta^1 \frac{\left(2\left(2^\tau\tau^3+5\,2^\tau\tau-3\,2^{\tau+1}+12\right)\right)(\tau(6-3\delta^1))}{2^\tau(\tau(\tau+1)(\tau+2)(\tau+3))}+(12-6\delta^1)\Bigg],$$

$$\textit{Middle Term}=\left[\frac{(3\tau\delta^1)\left(2^{\tau+3}\left(\frac{1}{\tau+3}+\frac{6\Gamma(\tau)}{\Gamma(\tau+4)}\right)\right)}{2^\tau},\frac{(\tau(6-3\delta^1))\left(2^{\tau+3}\left(\frac{1}{\tau+3}+\frac{6\Gamma(\tau)}{\Gamma(\tau+4)}\right)\right)}{2^\tau}\right],$$

$$\textit{Right Term}=\Bigg[24\delta^1-\frac{\left(2^{\tau+5}(\tau(\tau+3)+8)\right)(3\tau\delta^1)}{2^{\tau+1}((\tau+1)(\tau+2)(\tau+3)(\tau+4))},$$

$$(48-24\delta^1)-\frac{\left(2^{\tau+5}(\tau(\tau+3)+8)\right)(\tau(6-3\delta^1))}{2^{\tau+1}((\tau+1)(\tau+2)(\tau+3)(\tau+4))}\Bigg].$$

To visualize the above formulations, we fix δ^1 and vary τ.

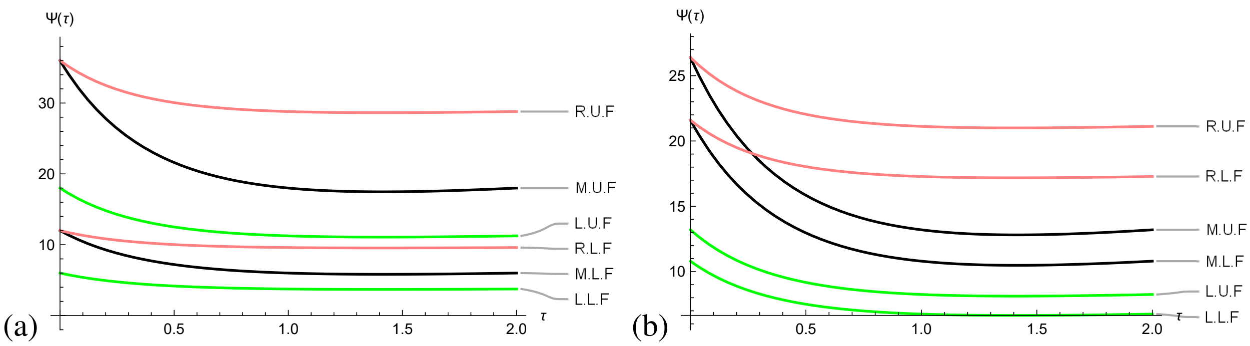

Figure 3. Graphical Validation of Theorem 3.1.

Note that L.L.F, L.U.F, M.L.F, M.U.F, R.L.F, and R.U.F are specifying the endpoint mappings of left, middle, and right terms of Theorem 3.1.

Table 3. Numerical validation of Example 3.1.

(δ^1, s_4)	$L_*(\delta^1, s_4)$	$L^*(\delta^1, s_4)$	$M_*(\delta^1, s_4)$	$M^*(\delta^1, s_4)$	$R_*(\delta^1, s_4)$	$R^*(\delta^1, s_4)$
$(0.3, 0.5)$	2.50084	14.1714	4.32	24.48	6.01143	34.0648
$(0.4, 1)$	3.0000	12.0000	4.8000	19.2000	5.7600	30.7200
$(0.5, 1.5)$	3.69468	11.084	5.82857	17.4857	9.54805	28.6442
$(0.8, 2)$	6.0000	9.0000	9.6000	14.4000	15.3600	23.0400

Now, we prove the weighted Hermite-Hadamard's inequality for symmetric mappings.

Theorem 3.2. *If $\Psi \in SSQFNF([s_3, s_4], \hbar_\circ)$ and $g : [s_3, s_4] \to \mathbb{R}$ is a nonnegative integrable symmetric mapping about $\frac{s_3+s_4}{2}$, then*

$$\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi\left(\frac{s_3+s_4}{2}\right)\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}g(s_4)+J^\tau_{s_4^-}g(s_3)\right]$$

$$\oplus\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right]\Psi\left(\left|\mu-\frac{s_3+s_4}{2}\right|\right)g(\mu)\mathrm{d}\mu$$

$$\leq \frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}\Psi g(s_4)\oplus J^\tau_{s_4^-}\Psi g(s_3)\right]$$

$$\leq (\Psi(s_3)\oplus\Psi(s_4))\cdot\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right]\left[\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)+\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\right]g(\mu)\mathrm{d}\mu$$

$$\ominus_g\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}]\left[\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi(\mu-s_3)\oplus\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi(s_4-\mu)\right]g(\mu)\mathrm{d}\mu.$$

Proof. Since $\Psi:[s_3,s_4]\to\mathbb{R}^+_I$ is a fuzzy interval-valued $\hbar_\circ$-super-quadratic mapping, then by multiplying (2.17) by $\varphi^{\tau-1}g(\varphi s_4+(1-\varphi)a)$ and applying integration with respect to φ on $[0,1]$, we get

$$\frac{1}{\hbar_\circ\left(\frac{1}{2}\right)}\Psi_*\left(\frac{s_3+s_4}{2},\delta^1\right)\int_0^1\varphi^{\tau-1}g(\varphi s_4+(1-\varphi)a)\mathrm{d}\varphi$$
$$\leq\int_0^1\varphi^{\tau-1}\Psi_*(\varphi s_4+(1-\varphi)s_3,\delta^1)g(\varphi s_4+(1-\varphi)a)\mathrm{d}\varphi$$
$$+\int_0^1\varphi^{\tau-1}\Psi_*(\varphi s_3+(1-\varphi)s_4,\delta^1)g(\varphi s_4+(1-\varphi)a)\mathrm{d}\varphi$$
$$-2\int_0^1\varphi^{\tau-1}\Psi_*\left(\frac{s_4-s_3}{2}\left|1-2\varphi\right|,\delta^1\right)g(\varphi s_4+(1-\varphi)a)\mathrm{d}\varphi.$$

Since g is symmetric mapping about $\frac{s_3+s_4}{2}$, then $g(\varphi s_4+(1-\varphi)s_3)=g((1-\varphi)s_4+\varphi s_3)$. Using this fact in the above inequality, we get

$$\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi_*\left(\frac{s_3+s_4}{2},\delta^1\right)\int_0^1\varphi^{\tau-1}[g(\varphi s_4+(1-\varphi)a)+g(\varphi s_3+(1-\varphi)s_4)]\mathrm{d}\varphi$$
$$\leq\int_0^1\varphi^{\tau-1}\Psi_*(\varphi s_4+(1-\varphi)s_3,\delta^1)g(\varphi s_4+(1-\varphi)a)\mathrm{d}\varphi$$
$$+\int_0^1\varphi^{\tau-1}\Psi_*(\varphi s_3+(1-\varphi)s_4,\delta^1)g(\varphi s_3+(1-\varphi)s_4)\mathrm{d}\varphi$$
$$-\int_0^1\varphi^{\tau-1}\Psi_*\left(\frac{s_4-s_3}{2}\left|1-2\varphi\right|,\delta^1\right)[g(\varphi s_4+(1-\varphi)a)+g(\varphi a+(1-\varphi)s_4)]\mathrm{d}\varphi.$$

After some simple computations, we have the following inequality

$$\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi_*\left(\frac{s_3+s_4}{2},\delta^1\right)\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}g(s_4)+J^\tau_{s_4^-}g(s_3)\right]$$
$$\leq\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}\Psi_*(s_4,\delta^1)g(s_4)+J^\tau_{s_4^-}\Psi_*(s_3,\delta^1)g(s_3)\right]$$
$$-\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right]\Psi_*\left(\left|\mu-\frac{s_3+s_4}{2}\right|,\delta^1\right)g(\mu)\mathrm{d}\mu. \tag{3.14}$$

By following a similar procedure, we get

$$\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi^*\left(\frac{s_3+s_4}{2},\delta^1\right)\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}g(s_4)+J^\tau_{s_4^-}g(s_3)\right]$$

$$\leq \frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}\Psi^*(s_4,\delta^1)g(s_4)+J^\tau_{s_4^-}\Psi^*(s_3,\delta^1)g(s_3)\right]$$
$$-\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right]\Psi^*\left(\left|\mu-\frac{s_3+s_4}{2}\right|,\delta^1\right)g(\mu)\mathrm{d}\mu. \tag{3.15}$$

Implementing the pseudo ordering relation on (3.14) and (3.15) results in the following relation

$$\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi\left(\frac{s_3+s_4}{2}\right)\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}g(s_4)+J^\tau_{s_4^-}g(s_3)\right]$$
$$\oplus\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right]\Psi\left(\left|\mu-\frac{s_3+s_4}{2}\right|\right)g(\mu)\mathrm{d}\mu$$
$$\leq \frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}\Psi g(s_4)\oplus J^\tau_{s_4^-}\Psi g(s_3)\right]. \tag{3.16}$$

Now, we establish our second inequality. Multiplying (2.21) by $\varphi^{\tau-1}g(\varphi s_4+(1-\varphi)s_3)$ and integrating with respect to φ on $[0,1]$, we get

$$\int_0^1\varphi^{\tau-1}\Psi_*(\varphi s_3+(1-\varphi)s_4,\delta^1)g(\varphi s_4+(1-\varphi)s_3)\mathrm{d}\varphi$$
$$+\int_0^1\varphi^{\tau-1}\Psi_*(\varphi s_4+(1-\varphi)s_3,\delta^1)g(\varphi s_4+(1-\varphi)s_3)\mathrm{d}\varphi$$
$$\leq[\Psi_*(s_4,\delta^1)+\Psi_*(s_3,\delta^1)]\int_0^1\varphi^{\tau-1}[\hbar_\circ(\varphi)+\hbar_\circ(1-\varphi)]g(\varphi s_4+(1-\varphi)s_3)\mathrm{d}\varphi$$
$$-2\int_0^1\varphi^{\tau-1}\hbar_\circ(\varphi)\Psi_*((1-\varphi)|s_4-s_3|,\delta^1)g(\varphi s_4+(1-\varphi)s_3)\mathrm{d}\varphi$$
$$-2\int_0^1\varphi^{\tau-1}\hbar_\circ(1-\varphi)\Psi_*(\varphi|s_4-s_3|,\delta^1)g(\varphi s_4+(1-\varphi)s_3)\mathrm{d}\varphi.$$

After performing some computations, we get

$$\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}\Psi_*(s_4,\delta^1)g(s_4)+J^\tau_{s_4^-}\Psi_*(s_3,\delta^1)g(s_3)\right]$$
$$\leq(\Psi_*(s_3,\delta^1)+\Psi_*(s_4,\delta^1))\cdot\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}$$
$$\times\left[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right]\left[\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)+\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\right]g(\mu)\mathrm{d}\mu$$
$$-\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}]$$
$$\times\left[\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi_*(\mu-s_3,\delta^1)+\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi_*(s_4-\mu,\delta^1)\right]g(\mu)\mathrm{d}\mu. \tag{3.17}$$

Similarly, we have

$$\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}\Psi^*(s_4,\delta^1)g(s_4)+J^\tau_{s_4^-}\Psi^*(s_3,\delta^1)g(s_3)\right]$$

$$
\begin{aligned}
&\leq (\Psi^*(s_3,\delta^1)+\Psi^*(s_4,\delta^1))\cdot\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\\
&\times\left[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right]\left[\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)+\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\right]g(\mu)\mathrm{d}\mu\\
&-\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}]\\
&\times\left[\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi^*(\mu-s_3,\delta^1)+\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi^*(s_4-\mu,\delta^1)\right]g(\mu)\mathrm{d}\mu.
\end{aligned}
\tag{3.18}
$$

Inequalities (3.17) and (3.18) produce the following relation

$$
\begin{aligned}
&\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}\Psi g(s_4)\oplus J^\tau_{s_4^-}\Psi g(s_3)\right]\\
&\leq (\Psi(s_3)\oplus\Psi(s_4))\cdot\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right]\left[\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)+\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\right]g(\mu)\mathrm{d}\mu\\
&\ominus_g\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}]\left[\hbar_\circ\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi(\mu-s_3)\oplus\hbar_\circ\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi(s_4-\mu)\right]g(\mu)\mathrm{d}\mu.
\end{aligned}
\tag{3.19}
$$

Finally, bridging inequalities (3.16) and (3.19), we achieve the Hermite-Hadmard-Fejer inequality. □

Now we discuss some special scenarios of Theorem 3.2.

- By setting $\hbar_\circ(\varphi)=\varphi$ in Theorem 3.2, we have

$$
\begin{aligned}
&\Psi\left(\frac{s_3+s_4}{2}\right)\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}g(s_4)+J^\tau_{s_4^-}g(s_3)\right]\\
&\quad\oplus\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right]\Psi\left(\left|\mu-\frac{s_3+s_4}{2}\right|\right)g(\mu)\mathrm{d}\mu\\
&\leq\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}\Psi g(s_4)\oplus J^\tau_{s_4^-}\Psi g(s_3)\right]\\
&\leq (\Psi(s_3)\oplus\Psi(s_4))\cdot\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right]g(\mu)\mathrm{d}\mu\\
&\quad\ominus_g\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}]\left[\left(\frac{s_4-\mu}{s_4-s_3}\right)\Psi(\mu-s_3)\oplus\left(\frac{\mu-s_3}{s_4-s_3}\right)\Psi(s_4-\mu)\right]g(\mu)\mathrm{d}\mu.
\end{aligned}
$$

- By setting $\hbar_\circ(\varphi)=\varphi^s$ in Theorem 3.2, we have

$$
\begin{aligned}
&2^{s-1}\Psi\left(\frac{s_3+s_4}{2}\right)\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}g(s_4)+J^\tau_{s_4^-}g(s_3)\right]\\
&\quad\oplus\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right]\Psi\left(\left|\mu-\frac{s_3+s_4}{2}\right|\right)g(\mu)\mathrm{d}\mu\\
&\leq\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}\Psi g(s_4)\oplus J^\tau_{s_4^-}\Psi g(s_3)\right]\\
&\leq (\Psi(s_3)\oplus\Psi(s_4))\cdot\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right]\left[\left(\frac{s_4-\mu}{s_4-s_3}\right)^s+\left(\frac{\mu-s_3}{s_4-s_3}\right)^s\right]g(\mu)\mathrm{d}\mu
\end{aligned}
$$

$$\ominus_g \frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}]\left[\left(\frac{s_4-\mu}{s_4-s_3}\right)^s\Psi(\mu-s_3)\oplus\left(\frac{\mu-s_3}{s_4-s_3}\right)^s\Psi(s_4-\mu)\right]g(\mu)\mathrm{d}\mu.$$

- By setting $\hbar_\circ(\varphi)=1$ in Theorem 3.2, we have

$$\begin{aligned}
&\frac{1}{2\hbar_\circ\left(\frac{1}{2}\right)}\Psi\left(\frac{s_3+s_4}{2}\right)\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}g(s_4)+J^\tau_{s_4^-}g(s_3)\right]\\
&\oplus\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right]\Psi\left(\left|\mu-\frac{s_3+s_4}{2}\right|\right)g(\mu)\mathrm{d}\mu\\
&\le\frac{\Gamma(\tau+1)}{(s_4-s_3)^\tau}\left[J^\tau_{s_3^+}\Psi g(s_4)\oplus J^\tau_{s_4^-}\Psi g(s_3)\right]\\
&\le 2(\Psi(s_3)\oplus\Psi(s_4))\cdot\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}\left[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}\right]g(\mu)\mathrm{d}\mu\\
&\ominus_g\frac{\tau}{(s_4-s_3)^\tau}\int_{s_3}^{s_4}[(s_4-\mu)^{\tau-1}+(\mu-s_3)^{\tau-1}]\left[\Psi(\mu-s_3)\oplus\Psi(s_4-\mu)\right]g(\mu)\mathrm{d}\mu.
\end{aligned}$$

Remark 3.2. *For $\tau=1$, the Theorem 3.2 transformed into Theorem 2.6. By selecting $\hbar_\circ(\varphi)=\varphi^{-s},\varphi(1-\varphi),\exp(\varphi)-1$ in Theorem 3.1, we get various fractional* F.N.V *Hermite-Hadamard-Fejer's inequalities for different classes of super-quadraticity. Also by taking $\Psi_*(\mu,\delta^1)=\Psi_*(\mu,\delta^1)$ and $\delta^1=1$ in Theorem 3.2, we obtain the fractional Hermite-Hadamard-Fejer's inequality for $\hbar_\circ$–super-quadratic mappings.*

Example 3.2. *Let $\Psi:[s_3,\ s_4]=[0,2]\to\Upsilon_{\delta^1}$ be a fuzzy valued super-quadratic mapping, which is defined as*

$$\Psi_\mu(\mu_1)=\begin{cases}\frac{\mu_1}{3\mu^3}, & \mu_1\in[0,3\mu^3]\\ \frac{6\mu^3-\mu_1}{3\mu^3}, & \mu_1\in(3\mu^3,6\mu^3].\end{cases}$$

and its level cuts are $\Psi_{\delta^1}=\left[3\delta^1\mu^3,\ (6-3\delta^1)\mu^3\right]$. Also $g:[0,2]\to\mathbb{R}$ is a symmetric integrable mapping and is defined as $g(\mu)=(\mu-1)^2$. Both mappings fulfill the condition of Theorem 3.2, then

$$\begin{aligned}
\textit{Left Term}=&\left[\frac{(2\,(2^\tau(\tau-1)(\tau(\tau+1)(\tau(\tau+5)+26)+120)+240))\,(3\delta^1)}{2^\tau((\tau+1)(\tau+2)(\tau+3)(\tau+4)(\tau+5))},\right.\\
&\left.\frac{(2\,(2^\tau(\tau-1)(\tau(\tau+1)(\tau(\tau+5)+26)+120)+240))\,(6-3\delta^1)}{2^\tau((\tau+1)(\tau+2)(\tau+3)(\tau+4)(\tau+5))}\right],\\
\textit{Middle Term}=&\left[\frac{\left(2^{\tau+3}(\tau(\tau(\tau(\tau+3)+10)-10)+24)\right)(3\delta^1)}{2^\tau((\tau+1)(\tau+2)(\tau+3)(\tau+4))},\ \frac{\left(2^{\tau+3}(\tau(\tau(\tau(\tau+3)+10)-10)+24)\right)(6-3\delta^1)}{2^\tau((\tau+1)(\tau+2)(\tau+3)(\tau+4))}\right],\\
\textit{Right Term}=&\left[\frac{\left(2^{\tau+1}\left(\tau^6+9\tau^5+55\tau^4+75\tau^3+304\tau^2-444\tau+720\right)\right)(21\delta^1)}{2^\tau(\tau(\tau+1)(\tau+2)(\tau+3)(\tau+4)(\tau+5)(\tau+6))},\right.\\
&\left.\frac{\left(2^{\tau+1}\left(\tau^6+9\tau^5+55\tau^4+75\tau^3+304\tau^2-444\tau+720\right)\right)(48-21\delta^1)}{2^\tau(\tau(\tau+1)(\tau+2)(\tau+3)(\tau+4)(\tau+5)(\tau+6))}\right].
\end{aligned}$$

To visualize the above formulations, we fix δ^1 and vary τ.

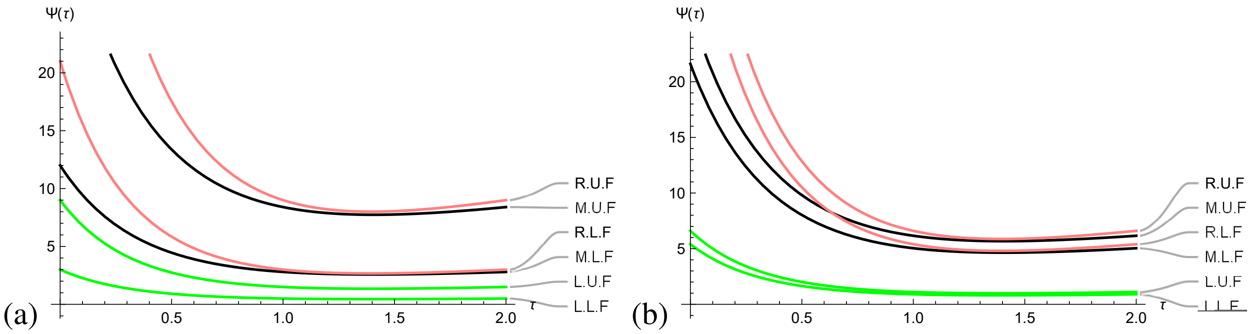

Figure 4. Graphical validation of Theorem 3.2.

Note that L.L.F, L.U.F, M.L.F, M.U.F, R.L.F, and R.U.F are specifying the endpoint mappings of left, middle, and right terms of Theorem 3.2.

Table 4. Numerical validation of Example 3.2.

(δ^1, s_4)	$L_*(\delta^1, s_4)$	$L^*(\delta^1, s_4)$	$M_*(\delta^1, s_4)$	$M^*(\delta^1, s_4)$	$R_*(\delta^1, s_4)$	$R^*(\delta^1, s_4)$
(0.3, 0.5)	0.548152	3.1062	2.67429	15.1543	7.00699	46.3796
(0.4, 1)	0.4000	1.6000	2.2400	8.9600	2.4000	9.6000
(0.5, 1.5)	0.451568	1.3547	2.58701	7.76104	2.68392	8.05175
(0.8, 2)	0.8000	1.2000	4.4800	6.7200	4.8000	7.2000

4. Applications

Now, we give some applications of our proposed results. First, we recall the binary means of positive real numbers.

(1) The arithmetic mean:

$$A(s_3, s_4) = \frac{s_3 + s_4}{2},$$

(2) The generalized log-mean:

$$L_r(s_3, s_4) = \left[\frac{{s_4}^{r+1} - {s_3}^{r+1}}{(r+1)(s_4 - s_3)}\right]^{\frac{1}{r}}; \quad r \in \mathfrak{R} \setminus \{-1, 0\}.$$

Proposition 4.1. *For* $s_3, s_4 \geq 0$*, then*

$$\left[3\delta^1, (6-3\delta^1)\right]\left[A^3(s_3, s_4) + \frac{(s_4 - s_3)^4}{2^5}\right] \leq \left[3\delta^1, (6-3\delta^1)\right] L_3^3(s_3, s_4)$$
$$\leq \left[3\delta^1, (6-3\delta^1)\right] A({s_3}^3, {s_4}^3) \ominus_g \left[3\delta^1, (6-3\delta^1)\right] \frac{2(s_4 - s_3)^3}{20}.$$

Proof. This result is acquired by applying $\Psi(\mu) = [3\delta^1\mu^3, (6-3\delta^1)\mu^3]$ on Theorem 2.5. □

Note that Proposition 4.1 provides the bounds for generalized logarithmic mean. Next, we give the refinements of the triangular inequality.

Proposition 4.2. *Let* $\{\mu_\nu\} \in [s_3, s_4]$ *be an increasing positive sequence. Then from Theorem 2.2, we have*

$$\left|\sum_{\nu=1}^{\vartheta}\mu_\nu\right| \leq \vartheta^2\sum_{\nu=1}^{\vartheta}|\mu_\nu| - \vartheta^2\sum_{\nu=1}^{\vartheta}\left|\mu_\nu - \frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mu_j\right|,$$

and

$$\|\sum_{\nu=1}^{\vartheta}\mu_\nu\| \leq \vartheta^2\sum_{\nu=1}^{\vartheta}\|\mu_\nu\| - \vartheta^2\sum_{\nu=1}^{\vartheta}\|\mu_\nu - \frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mu_j\|.$$

Proof. Since $\Psi(\mu) = |\mu|$ and $\Psi(\mu) = \|\mu\|$ are $\hbar_\circ$ super-quadratic mappings, by applying these mappings on Theorem 2.2 by taking $\hbar_\circ(\mu) = \frac{1}{\mu}$, $w_\nu = 1$, $\Psi_*(\mu, \delta^1) = \Psi^*(\mu, \delta^1)$, and $\delta^1 = 1$, we acquire our desired inequalities. □

Proposition 4.3. *Let* $\{\mu_\nu\} \in [s_3, s_4]$ *be an increasing positive sequence. Then, from Theorem 2.2, we have*

$$\left|\sum_{\nu=1}^{\vartheta}\mu_\nu\right|^r \leq \sum_{\nu=1}^{\vartheta}|\mu_\nu|^r - \sum_{\nu=1}^{\vartheta}\left|\mu_\nu - \frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mu_j\right|^r.$$

Particularly, we have

$$|\mu_1 + \mu_2|^r \leq |\mu_1|^r + |\mu_2|^r - 2^{1-r}|\mu_1 - \mu_2|^r.$$

Proof. Since $\Psi(\mu) = |\mu|^r$ where $r \geq 1$ is $\hbar_\circ$ super-quadratic mapping, by applying $\Psi(\mu)$ on Theorem 2.2, and taking $\hbar_\circ(\mu) = \mu$, $w_\nu = 1$, $\Psi_*(\mu, \delta^1) = \Psi^*(\mu, \delta^1)$, and $\delta^1 = 1$, we acquire our desired inequalities. □

Proposition 4.4. *Let* $\{\mu_\nu\} \in [s_3, s_4]$ *be an increasing positive sequence. Then, from Theorem 2.4, we have*

$$\begin{aligned}\sum_{\nu=1}^{\vartheta}|\mu_\nu|^r \leq{}& \vartheta[|\mu_1|^r + |\mu_\vartheta|^r] - \vartheta\left|\frac{\mu_1 + \mu_\vartheta - \vartheta^{-1}\sum_{\nu=1}^{\vartheta}\mu_j}{\vartheta}\right|^r \\ &- \sum_{\nu=1}^{\vartheta}[|\mu_\nu - \mu_1|^r + |\mu_\vartheta - \mu_\nu|^r] - \sum_{\nu=1}^{\vartheta}\left|\mu_\nu - \frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mu_j\right|^r,\end{aligned}$$

and

$$\begin{aligned}\sum_{\nu=1}^{\vartheta}\|\mu_\nu\|^r \leq{}& \vartheta[\|\mu_1\|^r + \|\mu_\vartheta\|^r] - \vartheta\left\|\frac{\mu_1 + \mu_\vartheta - \vartheta^{-1}\sum_{\nu=1}^{\vartheta}\mu_j}{\vartheta}\right\|^r \\ &- \sum_{\nu=1}^{\vartheta}[\|\mu_\nu - \mu_1\|^r + \|\mu_\vartheta - \mu_\nu\|^r] - \sum_{\nu=1}^{\vartheta}\left\|\mu_\nu - \frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mu_j\right\|^r.\end{aligned}$$

Proof. Since $\Psi(\mu) = |\mu|^r$ and $\Psi(\mu) = \|\mu\|^r$ for $r \geq 1$ are $\hbar_\circ$ super-quadratic mappings, by applying these mappings on Theorem 2.4, and taking $\hbar_\circ(\varphi) = \varphi$, $w_\nu = 1$, $\Psi_*(\mu, \delta^1) = \Psi^*(\mu, \delta^1)$, and $\delta^1 = 1$, we acquire our desired inequalities. □

5. Conclusions

The theory of inequalities is the main source used to investigate the various mapping classes. We've talked about the idea of a fuzzy number-valued super-quadratic mapping that works with the LR partially ordered ranking relation, δ^1-levels mappings, and a nonnegative mapping $\hbar_\circ$. This class is novel and new in literature; it reduces to several mapping classes of super-quadraticity, like fuzzy-valued super-quadratic mappings, fuzzy-valued s super-quadratics, fuzzy-valued Godunova s super-quadratic mappings, and many more. Also, results obtained from this class of mappings refined classical inequalities. We have developed several Jensen's and Hadamard's-like inequalities pertaining to this class of mappings. This study is significant due to various aspects because the first-time idea of super-quadratic mapping in a fuzzy environment is investigated. The proposed definition is novel due to its unified nature and strengthening the properties of the class of fuzzy numbered valued mappings. Also, this is the first study exploring the fuzzy numbered valued Hermite-Hadamard-Fejer type inequalities for strong convexity. The obtained results provides the better approximation as compared to existing results. In the future, we will try to address these inequalities by leveraging the concepts of generalized fractional operators, quantum, and symmetric quantum calculus to analyze the bounds for error inequalities like the Ostrowski inequality, Simpson's inequality, Bullen's and Boole's inequality, etc. By utilizing this class of mappings, Hausdorff-Pompeiu distance, and generalized differentiability of mappings based on Hukuhara differences as well as adopting a similar technique, one can introduce more general function classes and their applicable aspects in different domains. We will also talk about fuzzy-valued inequalities for totally ordered fuzzy-valued super-quadratic mappings and how they can be used in optimization. We hope the strategy, techniques, and ideas developed in our study will create new sights for research.

Author contributions

Muhammad Zakria Javed: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing-original draft, Writing-review and editing, Visualization; Muhammad Uzair Awan: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing-review and editing, Visualization, Supervision; Loredana Ciurdariu: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing-review and editing, Visualization; Omar Mutab Alsalami: Conceptualization, Software, Validation, Formal analysis, Investigation, Writing-review and editing, Visualization. All authors have read and agreed for the publication of this manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. K. Nikodem, On strongly convex functions and related classes of functions, In: *Handbook of functional equations: functional inequalities*, 2014, 365–405. https://doi.org/10.1007/978-1-4939-1246-9_16
2. A. W. Roberts, D. E. Varberg, *Convex functions*, New York: Academic Press, 1973.
3. K. Nikodem, Z. Pales, Characterizations of inner product spaces by strongly convex functions, *Banach J. Math. Anal.*, **5** (2011), 83–87. https://doi.org/10.15352/bjma/1313362982
4. N. Merentes, K. Nikodem, Remarks on strongly convex functions, *Aequat. Math.*, **80** (2010), 193–199. https://doi.org/10.1007/s00010-010-0043-0
5. M. A. Noor, K. I. Noor, Some characterizations of strongly preinvex functions, *J. Math. Anal.*, **316** (2006), 697–706. https://doi.org/10.1016/j.jmaa.2005.05.014
6. S. Z. Ullah, M. A. Khan, Z. A. Khan, Y. M. Chu, Coordinate strongly s-convex functions and related results, *J. Math. Inequal.*, **14** (2020), 829–843. http://doi.org/10.7153/jmi-2020-14-53
7. S. Abramovich, G. Jameson, G. Sinnamon, Refining Jensen's inequality, *Bull. Math. Soc. Sci. Math. Roumanie*, **47** (2004), 3–14.
8. M. Kian, Operator Jensen inequality for superquadratic functions, *Linear Algebra Appl.*, **456** (214), 82–87. https://doi.org/10.1016/j.laa.2012.12.011
9. M. Kian, S. S. Dragomir, Inequalities involving superquadratic functions and operators, *Mediterr. J. Math.*, **11** (2014), 1205–1214. https://doi.org/10.1007/s00009-013-0357-y
10. J. A. Oguntuase, L. E. Persson, Refinement of Hardy's inequalities via superquadratic and subquadratic functions, *J. Math. Anal. Appl.*, **339** (2008), 1305–1312. https://doi.org/10.1016/j.jmaa.2007.08.007
11. M. Krnic, H. R. Moradi, M. Sababheh, On superquadratic and logarithmically superquadratic functions, *Mediterr. J. Math.*, **20** (2023), 311. https://doi.org/10.1007/s00009-023-02514-y
12. S. Abramovich, S. Banic, M. Matic, J. Pecaric, Jensen-Steffensen's and related inequalities for superquadratic functions, *Math. Inequal. Appl.*, **11** (2008), 23.
13. D. Khan, S. I. Butt, Y. Seol, Analysis of (P, m)-superquadratic function and related fractional integral inequalities with applications, *J. Inequal. Appl.*, **2024** (2024), 137. https://doi.org/10.1186/s13660-024-03218-x

14. M. Niezgoda, An extension of Levin-Steckin's theorem to uniformly convex and superquadratic functions, *Aequat. Math.*, **94** (2020), 303–321. https://doi.org/10.1007/s00010-019-00675-4

15. S. Banic, J. Pecaric, S. Varosanec, Superquadratic functions and refinements of some classical inequalities, *J. Korean Math. Soc.*, **45** (2008), 513–525. https://doi.org/10.4134/JKMS.2008.45.2.513

16. S. Varosanec, On $\hbar_\circ$-convexity, *J. Math. Anal. Appl.*, **326** (2007), 303–311. https://doi.org/10.1016/j.jmaa.2006.02.086

17. M. W. Alomari, C. Chesneau, On $\hbar_\circ$-superquadratic functions, *Afr. Mat.*, **33** (2022), 41. https://doi.org/10.1007/s13370-022-00984-z

18. R. E. Moore, *Interval analysis*, Englewood Cliffs: Prentice-Hall, 1966.

19. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

20. D. Dubois, H. Prade, Towards fuzzy differential calculus part 1: Integration of fuzzy mappings, *Fuzzy Sets Syst.*, **8** (1982), 1–17. https://doi.org/10.1016/0165-0114(82)90025-2

21. S. Nanda, K. Kar, Convex fuzzy mappings, *Fuzzy Sets Syst.*, **48** (1992), 129–132. https://doi.org/10.1016/0165-0114(92)90256-4

22. L. Stefanini, A generalization of Hukuhara difference and division for interval and fuzzy arithmetic, *Fuzzy Sets Syst.*, **161** (2010), 1564–1584. https://doi.org/10.1016/j.fss.2009.06.009

23. D. Zhang, C. Guo, D. Chen, G. Wang, Jensen's inequalities for set-valued and fuzzy set-valued functions, *Fuzzy Sets Syst.*, **404** (2021), 178–204. https://doi.org/10.1016/j.fss.2020.06.003

24. U. W. Kulisch, W. L. Miranker, *Computer arithmetic in theory and practice*, New York: Academic press, 2014.

25. B. Bede, S. G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, *Fuzzy Sets Syst.*, **151** (2005), 581–599. https://doi.org/10.1016/j.fss.2004.08.001

26. M. B. Khan, H. G. Zaini, J. E. Macias-Diaz, S. Treanta, M. S. Soliman, Some fuzzy Riemann-Liouville fractional integral inequalities for Preinvex fuzzy interval-valued Functions, *Symmetry*, **14** (2022), 313. https://doi.org/10.3390/sym14020313

27. U. M. Pirzada, D. C. Vakaskar, Existence of Hukuhara differentiability of fuzzy-valued functions, preprint paper, 2017. https://doi.org/10.48550/arXiv.1609.04748 T

28. Y. Chalco-Cano, R. Rodriguez-Lopez, M. D. Jimenez-Gamero, Characterizations of generalized differentiable fuzzy functions, *Fuzzy Sets Syst.*, **95** (2016), 37–56. https://doi.org/10.1016/j.fss.2015.09.005

29. O. Kaleva, Fuzzy differential equations, *Fuzzy Sets Syst.*, **24** (1987), 301–317. https://doi.org/10.1016/0165-0114(87)90029-7

30. D. Zhao, M. A. Ali, A. Kashuri, H. Budak, M. Z. Sarikaya, Hermite-Hadamard-type inequalities for the interval-valued approximately $\hbar_\circ$-convex functions via generalized fractional integrals, *J. Inequal. Appl.*, **2020** (2020), 222. https://doi.org/10.1186/s13660-020-02488-5

31. E. R. Nwaeze, M. A. Khan, Y. M. Chu, Fractional inclusions of the Hermite-Hadamard type for m-polynomial convex interval-valued functions, *Adv. Differ. Equ.*, **2020** (2020), 507. https://doi.org/10.1186/s13662-020-02977-3

32. T. Abdeljawad, S. Rashid, H. Khan, Y. M. Chu, On new fractional integral inequalities for p-convexity within interval-valued functions, *Adv. Differ. Equ.*, **2020** (2020), 330. https://doi.org/10.1186/s13662-020-02782-y

33. F. Shi, G. Ye, W. Liu, D. Zhao, cr-h-convexity and some inequalities for CR-$\hbar_\circ$-convex function, *Filomat*, 2022, 1–17.

34. W. Liu, F. Shi, G. Ye, D. Zhao, Some inequalities for cr-log-$\hbar_\circ$-convex functions, *J. Inequal. Appl.*, **2022** (2022), 160. https://doi.org/10.1186/s13660-022-02900-2

35. W. Liu, F. Shi, G. Ye, D. Zhao, The properties of harmonically cr-h-convex function and its applications, *Mathematics*, **10** (2022), 2089. https://doi.org/10.3390/math10122089

36. H. Budak, T. Tunc, M. Sarikaya, Fractional Hermite-Hadamard-type inequalities for interval-valued functions, *Amer. Math. Soc.*, **148** (2020), 705–718. https://doi.org/10.1090/proc/14741

37. M. Vivas-Cortez, S. Ramzan, M. U. Awan, M. Z. Javed, A. G. Khan, M. A. Noor, IV-CR-γ-convex functions and their application in fractional Hermite-Hadamard inequalities, *Symmetry*, **15** (2023), 1405. https://doi.org/10.3390/sym15071405

38. H. Cheng, D. Zhao, M. Z. Sarikaya, Hermite-Hadamard type inequalities for $\hbar_\circ$-convex function via fuzzy interval-valued fractional q-integral, *Fractals*, **32** (2024), 2450042. https://doi.org/10.1142/S0218348X24500427

39. B. B. Mohsin, M. U. Awan, M. Javed, H. Budak, A. G. Khan, M. A. Noor, Inclusions involving interval-valued harmonically co-ordinated convex functions and Raina's fractional double integrals, *J. Math.*, **2022** (2022), 5815993. https://doi.org/10.1155/2022/5815993

40. W. Afzal, E. Y. Prosviryakov, S. M. El-Deeb, Y. Almalki, Some new estimates of Hermite-Hadamard, Ostrowski and Jensen-type inclusions for h-convex stochastic process via interval-valued functions, *Symmetry*, **15** (2023), 831. https://doi.org/10.3390/sym15040831

41. W. Afzal, A. Alb Lupas, K. Shabbir, Hermite-Hadamard and Jensen-type inequalities for harmonical (h_1, h_2)-Godunova-Levin interval-valued functions, *Mathematics*, **10** (2022), 2970. https://doi.org/10.3390/math10162970

42. B. Bin-Mohsin, M. Z. Javed, M. U. Awan, A. Kashuri, On some new AB-fractional inclusion relations, *Fractal Fract.*, **7** (2023), 725. https://doi.org/10.3390/fractalfract7100725

43. D. Zhao, G. Ye, W. Liu, D. F. Torres, Some inequalities for interval-valued functions on time scale, *Soft Comput.*, **23** (2019), 6005–6015. https://doi.org/10.1007/s00500-018-3538-6

44. B. Bin-Mohsin, M. Z. Javed, M. U. Awan, B. Meftah, A. Kashuri, Fractional reverse inequalities involving generic interval-valued convex functions and applications, *Fractal Fract.*, **8** (2024), 587. https://doi.org/10.3390/fractalfract8100587

45. A. Fahad, Y. Wang, Z. Ali, R. Hussain, S. Furuichi, Exploring properties and inequalities for geometrically arithmetically-cr-convex functions with Cr-order relative entropy, *Inform. Sci.*, **662** (2024), 120219. https://doi.org/10.1016/j.ins.2024.120219

46. M. Z. Javed, M. U. Awan, L. Ciurdariu, S. S. Dragomir, Y. Almalki, On extended class of totally ordered interval-valued convex stochastic processes and applications, *Fractal Fract.*, **8** (2024), 577. https://doi.org/10.3390/fractalfract8100577
47. S. I. Butt, D. Khan, Superquadratic function and its applications in information theory via interval calculus, *Chaos Solit. Fract.*, **190** (2025), 115748. https://doi.org/10.1016/j.chaos.2024.115748
48. D. Khan, S. I. Butt, Superquadraticity and its fractional perspective via center-radius cr-order relation, *Chaos Solit. Fract.*, **182** (2024), 114821. https://doi.org/10.1016/j.chaos.2024.114821
49. T. M. Costa, A. Flores-Franulic, Y. Chalco-Cano, I. Aguirre-Cipe, Ostrowski-type inequalities for fuzzy-valued functions and its applications in quadrature theory, *Inform. Sci.*, **529** (2020), 101–115. https://doi.org/10.1016/j.ins.2020.04.037
50. D. Zhang, C. Guo, D. Chen, G. Wang, Jensen's inequalities for set-valued and fuzzy set-valued functions, *Fuzzy Sets Syst.*, **404** (2021), 178–204. https://doi.org/10.1016/j.fss.2020.06.003
51. M. B. Khan, M. A. Noor, P. O. Mohammed, J. L. Guirao, K. I. Noor, Some integral inequalities for generalized convex fuzzy-interval-valued functions via fuzzy Riemann integrals, *Int. J. Comput. Intell. Syst.*, **14** (2021), 158. https://doi.org/10.1007/s44196-021-00009-w
52. M. B. Khan, H. M. Srivastava, P. O. Mohammed, J. E. Macias-Diaz, Y. S. Hamed, Some new versions of integral inequalities for log-preinvex fuzzy-interval-valued functions through fuzzy order relation, *Alex. Eng. J.*, **61** (2022), 7089–7101. https://doi.org/10.1016/j.aej.2021.12.052
53. S. Abbaszadeh, M. Eshaghi, A Hadamard-type inequality for fuzzy integrals based on r-convex functions, *Soft Comput.*, **20** (2016), 3117–3124. https://doi.org/10.1007/s00500-015-1934-8
54. B. Bin-Mohsin, S. Rafique, C. Cesarano, M. Z. Javed, M. U. Awan, A. Kashuri, et al., Some general fractional integral inequalities involving LR-bi-convex fuzzy interval-valued functions, *Fractal Fract.*, **6** (10), 565. https://doi.org/10.3390/fractalfract6100565
55. D. H. Hong, Berwald and Favard type inequalities for fuzzy integrals, *Int. J. Uncertain. Fuzz. Knowl. Based Syst.*, **24** (2016), 47. https://doi.org/10.1142/S0218488516500033
56. H. Agahi, M. A. Yaghoobi, A Minkowski type inequality for fuzzy integrals, *J. Uncertain Syst.*, **4** (2010), 187–194.
57. Y. Wang, M. Z. Javed, M. U. Awan, B. Bin-Mohsin, B. Meftah, S. Treanta, Symmetric quantum calculus in interval valued frame work: operators and applications, *AIMS Math.*, **9** (2024), 27664–27686. https://doi.org/10.3934/math.20241343
58. B. Daraby, A review on some fuzzy integral inequalities, *Sahand Commun. Math. Anal.*, **18** (2021), 153–185. https://doi.org/10.22130/scma.2022.555219.1125
59. S. I. Butt, D. Khan, Integral inequalities of h-superquadratic functions and their fractional perspective with applications, *Math. Meth. Appl. Sci.*, **48** (2025), 1952–1981. https://doi.org/10.1002/mma.10418

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