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Research article

Explicit evaluations of subfamilies of the hypergeometric function ${}_3F_2(1)$ along with specific fractional integrals

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Abstract: The present study explores the application of hypergeometric functions in evaluating fractional integrals, providing a comprehensive framework to bridge fractional calculus and special functions. As a generalization of classical integrals, fractional integrals have gained prominence due to their wide applicability in modeling anomalous diffusion, viscoelastic systems, and other non-local phenomena. Hypergeometric functions, renowned for their rich analytical properties and ability to represent solutions to differential equations, offer an elegant and versatile tool for solving fractional integrals. In this paper, we evaluate a new class of fractional integrals, presenting results that contribute significantly to the study of generalized hypergeometric functions, particularly ${}_3F_2(1)$. The results reveal previously unexplored connections within these functions, providing new insights and extending their applicability. Furthermore, evaluating these fractional integrals holds promise for advancing the theoretical understanding and practical applications of fractional differential equations.

Keywords: fractional integrals; fractional derivatives; Gauss hypergeometric functions; generalized hypergeometric functions; numerical series; power series; decomposition into partial fractions; differential equations
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1. Introduction

Historically, hypergeometric functions emerged during the 18th century in the work of mathematicians such as Euler and Gauss, who developed theories around hypergeometric series and associated integrals. These functions were first considered as generalizations of geometric series and solutions of higher-order linear differential equations. Today, hypergeometric functions are at the center of an entire field of research, that of special functions [1]. Moreover, it is now well known that most special functions can be expressed in terms of hypergeometric functions, and these functions can be represented using either series or integrals. This dual representation makes it an excellent tool for evaluating series and integrals or solving differential equations. Hypergeometric functions have a wide range of applications. From our literature review, we found that these functions are utilized to express solutions in various fields, including probability and statistics [2–4], combinatorics and number theory [5–7], random walks [8–10], random graphs [11], quantum mechanics (see [12], p. 89, 96, 127 and [13], p. 235, 290, 333), conformal mapping [14], and fractional hypergeometric differential equations [15, 16], among many other problems.

Hypergeometric functions usually have no explicit expression and are only represented by power series or integrals, which makes their evaluation time-consuming. Research on hypergeometric functions is divided into two main branches: continuous and discrete. The continuous branch focuses on the analytical study of these functions, treating their arguments as continuous variables (see [17–24]). In contrast, the discrete branch examines these functions by substituting their arguments with integers or specific values. Research in the discrete branch has surged in the past decade, driven by advancements in powerful computer algebra software and the wide range of problems that can be solved using hypergeometric functions. This study belongs to the discrete branch and aims to provide new results for different families of the generalized hypergeometric function ${}_3F_2(1)$. The literature includes several explicit forms of ${}_3F_2(a_1, a_2, a_3; b_1; b_2; 1)$, for particular choices of the parameters $(a_1, a_2, a_3, b_1, b_2)$. Thus, in [25], the authors used the Gamma function to derive explicit forms for ${}_3F_2(a, b, c; 1+a-b, 1+a-c; 1)$ and ${}_3F_2(a, b, c; 1+a-b, a+2b-c-1; 1)$. Moreover, in [26], the authors used specialized software and managed to generate, without any mathematical proof, about thirty explicit formulas of ${}_3F_2(a_1, a_2, a_3; b_1; b_2; 1)$, for particular choices of the parameters $(a_1, a_2, a_3, b_1, b_2)$. Furthermore, in [27], the authors exhibited the explicit expressions of ${}_3F_2(-2n, a, 1+d; 2a+1, d; 2)$ and ${}_3F_2(-2n-1, a, 1+d; 2a+1, d; 2)$ for $n \in \mathbb{N}$. Besides, in [28], the authors succeeded in determining the explicit expressions of the following sequences:

$$\begin{array}{ll}
{}_3F_2(\frac{1}{6}, \frac{5}{6}, \frac{1}{2}+n; 1, \frac{3}{2}+n; 1), & {}_3F_2(\frac{1}{6}, \frac{5}{6}, -\frac{1}{2}-n; 1, \frac{1}{2}-n; 1),\\
{}_3F_2(\frac{1}{6}, \frac{5}{6}, \frac{1}{3}+n; 1, \frac{4}{3}+n; 1), & {}_3F_2(\frac{1}{6}, \frac{5}{6}, -\frac{1}{3}-n; 1, \frac{2}{3}-n; 1),\\
{}_3F_2(\frac{1}{6}, \frac{5}{6}, \frac{2}{3}+n; 1, \frac{5}{3}+n; 1), & {}_3F_2(\frac{1}{6}, \frac{5}{6}, -\frac{2}{3}-n; 1, \frac{1}{3}-n; 1),\\
{}_3F_2(\frac{1}{6}, \frac{5}{6}, \frac{1}{4}+n; 1, \frac{5}{4}+n; 1), & {}_3F_2(\frac{1}{6}, \frac{5}{6}, -\frac{1}{4}-n; 1, \frac{3}{4}+n; 1),\\
{}_3F_2(\frac{1}{6}, \frac{5}{6}, \frac{3}{4}+n; 1, \frac{7}{4}+n; 1), & {}_3F_2(\frac{1}{6}, \frac{5}{6}, -\frac{1}{4}-n; 1, \frac{1}{4}-n; 1),
\end{array}$$

for all $n \in \mathbb{N}$. Finally, in [29], the authors provided the explicit forms of the sets ${}_3F_2(2x, 2x+\frac{1}{2}, x; \frac{1}{2}, 1+x; 1)$ and ${}_3F_2(2x, 2x-\frac{1}{2}, x; \frac{3}{2}, 1+x; 1)$, for all $x \in (-\infty, \frac{1}{4})$.

The main objective of our study is to find explicit forms for the sets:

$$\begin{array}{rcl}
K(\alpha) & = & {}_3F_2(1-\alpha, 1, \alpha+1; \alpha+1, \alpha+2; 1),\ \alpha \in (\frac{1}{2}, +\infty),\\
G(\alpha) & = & {}_3F_2(1-\alpha, 1, 2+\alpha; 1+\alpha, 3+\alpha; 1),\ \alpha \in (\frac{1}{2}, +\infty),\\
H(p) & = & {}_3F_2(\frac{1}{2}-p, 1, 1+2p; \frac{3}{2}+p, 2p+2; 1),\ p \in \mathbb{N}^*.
\end{array} \tag{1.1}$$

These sets emerge naturally from an exact evaluation of certain classes of fractional integrals, as we show in Section 4.

2. Preliminaries

In order to understand the notation used above and throughout, we present in this section some basic notations, definitions, and intermediate results, which will be useful to justify certain passages in the proofs of this manuscript. Let us now present a set of symbols and notations.

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of non-negative integers, integers, real numbers, and complex numbers, respectively,
- $\mathbb{X}^*$ denotes any set $\mathbb{X} \setminus \{0\}$,
- $\mathbb{Z}_0^-$ denotes set $-\mathbb{N} = \{\ldots, -n, \ldots, -1, 0\}$,
- $B(0,1) = \{z \in \mathbb{C} \mid |z| < 1\}$,
- $\overline{B}(0,1) = \{z \in \mathbb{C} \mid |z| \leq 1\}$,
- $\mathcal{R}(z)$ denotes the real part of z.

■ Now we present some basic notations, definitions, and intermediate results related to the so-called the Gauss hypergeometric function.

Definition 2.1. [30] The Euler gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad \forall\, z \in \mathbb{C} \,|\, \mathcal{R}(z) > 0. \tag{2.1}$$

Using integration by parts, one sees that

$$\Gamma(z+1) = z\Gamma(z), \quad \forall\, \mathcal{R}(z) > 0. \tag{2.2}$$

The extension of the Euler gamma function to the half-plane $\mathcal{R}(z) \leq 0$ is given by

$$\Gamma(z) = \frac{\Gamma(z+k)}{(z)_k}, \quad (\mathcal{R}(z) > -k; \;\; k \in \mathbb{N}^*; z \notin \mathbb{Z}_0^-),$$

where $(z)_k$ is the Pochhammer symbol defined for all $z \in \mathbb{C}$ and $k \in \mathbb{N}^*$ by

$$(z)_0 = 1 \;\text{ and }\; (z)_k = z(z+1)\cdots(z+k-1), \quad \forall\, k \in \mathbb{N}. \tag{2.3}$$

Relations (2.2) and (2.3) give

$$\Gamma(k+1) = (1)_k = k!, \quad \forall\, k \in \mathbb{N}. \tag{2.4}$$

Definition 2.2. [30] The Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ is defined in the unit disk as the sum of the hypergeometric series as follows:

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{2.5}$$

$$(a, b \in \mathbb{C}; c \in \mathbb{C} \setminus \mathbb{Z}_0^- \;;\; z \in \overline{B} \;;\; \mathcal{R}(c-b-a) > 0).$$

Furthermore, if $0 < \mathcal{R}(b) < \mathcal{R}(c)$ and $|\arg(1-z)| < \pi$, then ${}_2F_1(a,b;c;z)$ is given by the following Euler integral representation:

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\, dx. \tag{2.6}$$

If $z = 1$ with $\mathcal{R}(c - b - a) > 0$, the Gauss hypergeometric function has the following property:

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \tag{2.7}$$

A natural extension of ${}_2F_1$ to ${}_3F_2$ is defined by

$$ {}_3F_2(a, b, c; d, e; z) = \sum_{k=0}^{+\infty} \frac{(a)_k(b)_k(c)_k}{(d)_k(e)_k} \frac{z^k}{k!},$$

$$\forall\, (z \in \overline{B} \text{ and } \mathcal{R}(d + e - a - b - c) > 0).$$

In [31] Theorem 38, Rainville proves a general integral representation for ${}_{p+k}F_{q+k}$, but here we state the following three special cases,

Case 1: Let $p = 2$, $q = k = 1$, and choose $a = 1 - \alpha$, $b = 1$, $c = \alpha + 1$, $d = \alpha + 1$, $e = \alpha + 2$, and $z = 1$. Then, ${}_3F_2$ is given by the following integral representation:

$$ {}_3F_2(1 - \alpha, 1, \alpha + 1; \alpha + 1, \alpha + 2; 1) = (\alpha + 1) \int_0^1 x^{\alpha}\, {}_2F_1(1 - \alpha, 1; \alpha + 1; x)dx. \tag{2.8}$$

Case 2: Let $p = 2$, $q = k = 1$, and choose $a = 1 - \alpha$, $b = 1$, $c = \alpha + 2$, $d = \alpha + 1$, $e = \alpha + 3$, and $z = 1$. Then, ${}_3F_2$ is given by the following integral representation:

$$ {}_3F_2(1 - \alpha, 1, \alpha + 2; \alpha + 1, \alpha + 3; 1) = (\alpha + 2) \int_0^1 x^{\alpha+1}\, {}_2F_1(1 - \alpha, 1; \alpha + 1; x)dx. \tag{2.9}$$

Case 3: Let $p = 2$, $q = k = 1$, and choose $a = 1 - \alpha$, $b = 1$, $c = 2\alpha$, $d = \alpha + 1$, $e = 2\alpha + 1$, and $z = 1$. Then, ${}_3F_2$ is given by the following integral representation:

$$ {}_3F_2(1 - \alpha, 1, 2\alpha; \alpha + 1, 2\alpha + 1; 1) = 2\alpha \int_0^1 x^{2\alpha-1}\, {}_2F_1(1 - \alpha, 1; \alpha + 1; x)dx. \tag{2.10}$$

■ The following is the definition of the Riemann-Liouville fractional integral $I^{\alpha} f$ of order α.

Definition 2.3. [30] Let $\Omega = [\tau, \eta]$. The Riemann-Liouville fractional integral $I^{\alpha} f$ of order $\alpha \in \mathbb{C}$ $(\mathcal{R}(\alpha) > 0)$ is defined by

$$ I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{\tau}^{t} (t - s)^{\alpha-1} f(s)ds,\ \forall\, t > \tau \text{ and } \mathcal{R}(\alpha) > 0. \tag{2.11}$$

2.1. Useful results

In this section, we establish some results which will play an important role herein. We believe that some of these results may be new.

■ To justify the interchangeability between the integral and the sum, or to rewrite certain integrals, we give the following two lemmas that we will refer to several times in our work.

Lemma 2.1. Let $a, b, c \in \mathbb{C}$, $c \notin \mathbb{Z}_0^-$, and $\mathcal{R}(c - a - b) > 0$. Then, the series

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} z^k,$$

is normally convergent on the interval $[-1, 1]$.

Proof of Lemma 2.1. Since $a, b, c \in \mathbb{C}$, $c \notin \mathbb{Z}_0^-$, and $\mathcal{R}(c-a-b) > 0$, the Gauss hypergeometric function

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{2.12}$$

is defined for any complex number $z \in \overline{B}(0, 1)$. Moreover, the series (2.12) is absolutely convergent for all $z = 1$. Therefore, the series $\sum u_k$ is convergent, where u_k is defined by

$$u_k = \left| \frac{(a)_k (b)_k}{(c)_k} \right|, \ \forall \ k \in \mathbb{N}.$$

Furthermore, it is clear that for all $z \in [-1, 1]$, $k \in \mathbb{N}$, we have

$$\left| \frac{(a)_k (b)_k}{(c)_k} z^k \right| \leq u_k. \tag{2.13}$$

The convergence of the series $\sum u_k$ together with the relation (2.13) leads to the normal (therefore uniform) convergence of the series $\sum \frac{(a)_k (b)_k}{(c)_k} z^k$ on the interval $[-1, 1]$. The proof is complete. □

Lemma 2.2. Let $\Omega = [\tau, \eta]$ $(-\infty < \tau < \eta < \infty)$ be a finite interval on the real axis $\mathbb{R}$, and $\alpha > 1/2$. Then,

$$\int_{\tau}^{\eta} \sum_{k=0}^{+\infty} \frac{(1-\alpha)_k}{(\alpha+1)_k} \left(\frac{t-\tau}{\eta-\tau} \right)^k dt = \sum_{k=0}^{+\infty} \frac{(1-\alpha)_k}{(\alpha+1)_k} \int_{\tau}^{\eta} \left(\frac{t-\tau}{\eta-\tau} \right)^k dt. \tag{2.14}$$

Proof of Lemma 2.2. If we take $a = 1 - \alpha$, $b = 1$, and $c = \alpha + 1$, then $c \notin \mathbb{Z}_0^-$ and $\mathcal{R}(c - a - b) = 2\alpha - 1 > 0$, since $\alpha > 1/2$. Then by Lemma 2.1, the series $\sum \frac{(a)_k (b)_k}{(c)_k} z^k$ converges normally on the interval $[0, 1] \subset [-1, 1]$. Consequently, for all $\alpha > 1/2$, we have

$$\int_0^1 \sum_{k=0}^{+\infty} \frac{(1-\alpha)_k}{(\alpha+1)_k} z^k \, dz = \sum_{k=0}^{+\infty} \frac{(1-\alpha)_k}{(\alpha+1)_k} \int_0^1 z^k \, dz. \tag{2.15}$$

By using the change of variable with $z = \frac{t-\tau}{\eta-\tau}$, for relation (2.15) we have

$$\frac{1}{(\eta-\tau)} \int_{\tau}^{\eta} \sum_{k=0}^{+\infty} \frac{(1-\alpha)_k}{(\alpha+1)_k} \left(\frac{t-\tau}{\eta-\tau} \right)^k dt = \sum_{k=0}^{+\infty} \frac{(1-\alpha)_k}{(\alpha+1)_k} \frac{1}{(\eta-\tau)} \int_{\tau}^{\eta} \left(\frac{t-\tau}{\eta-\tau} \right)^k dt. \tag{2.16}$$

Thus, we have obtained (2.14). The proof is complete. □

■ We now establish some results, which we believe are new. These results give the limit of some series.

■ The following lemma gives the sum of the series $S_{\alpha,0}$ defined by the left-hand side of relation (2.17).

Lemma 2.3. For all $\alpha > 1/2$, we have

$$S_{\alpha,0} = \sum_{k=0}^{+\infty} \frac{(1-\alpha)_k}{(\alpha+1)_k\,(k+\alpha+1)} = \frac{1}{2\alpha}. \tag{2.17}$$

Proof of Lemma 2.3. For all $k \in \mathbb{N}$, let u_k be the general term of the series $S_{\alpha,0}$ given by the relation (2.17). If we take $a = 1-\alpha$, $b = 1$, and $c = \alpha+1$, then $c \notin \mathbb{Z}_0^-$ and $\mathcal{R}(c-a-b) = 2\alpha-1 > 0$, since $\alpha > 1/2$. Then, by Lemma 2.1 the series $\sum \frac{(1-\alpha)_k}{(\alpha+1)_k}$ is absolutely convergent, and since $0 < u_k \leq \frac{(1-\alpha)_k}{(\alpha+1)_k}$ for all $k \in \mathbb{N}$, we then deduce that the series $S_{\alpha,0}$ is absolutely convergent. Thus, multiplying and dividing the term u_k by $(0-\alpha)$, we obtain

$$\begin{aligned}
S_{\alpha,0} &= \sum_{k=0}^{+\infty} \frac{(0-\alpha)(1-\alpha)_k}{(0-\alpha)(\alpha+1)_k\,(k+\alpha+1)} \\
&= \sum_{k=0}^{+\infty} \frac{(-\alpha)_{k+1}}{-\alpha\,(\alpha+1)_{k+1}} \\
&= -\frac{1}{\alpha}\left[\sum_{k=1}^{+\infty} \frac{(-\alpha)_k}{(\alpha+1)_k}\right] \\
&= -\frac{1}{\alpha}\left[\sum_{k=0}^{+\infty} \frac{(-\alpha)_k}{(\alpha+1)_k} - 1\right] \\
&= \frac{1}{\alpha}\left[1 - {}_2F_1(-\alpha, 1; \alpha+1; 1)\right].
\end{aligned} \tag{2.18}$$

From the property of ${}_2F_1$, given by (2.7) we have

$${}_2F_1(-\alpha, 1; \alpha+1; 1) = \frac{1}{2}, \tag{2.19}$$

and so substituting (2.19) into (2.18), we obtain (2.17). This completes the proof. □

■ The following lemma gives the sum of the series $S_{\alpha,1}$ defined by the left-hand side of relation (2.20).

Lemma 2.4. For all $\alpha > 1/2$, we have

$$S_{\alpha,1} = \sum_{k=0}^{+\infty} \frac{(1-\alpha)_k}{(\alpha+1)_k\,(\alpha+2+k)} = \frac{1}{\alpha+1} + \frac{1}{2\alpha} - \frac{2}{2\alpha+1}. \tag{2.20}$$

Proof of Lemma 2.4. The proof is similar to the proof of Lemma 2.3. If we take $a = 1-\alpha$, $b = 1$, and $c = \alpha+1$, then $c \notin \mathbb{Z}_0^-$ and $\mathcal{R}(c-a-b) = 2\alpha-1 > 0$, since $\alpha > 1/2$. We deduce that the series $S_{\alpha,1}$ is absolutely convergent. Thus, multiplying and dividing the term u_k by the same quantity $(k+\alpha+1)$, we obtain

$$\begin{aligned}
u_k &= \frac{(1-\alpha)_k\,(k+\alpha+1)}{(\alpha+1)_k\,(\alpha+1+k)\,(\alpha+1+k+1)} \\
&= \frac{(1-\alpha)_k\,(1-\alpha+k+2\alpha)}{(\alpha+1)_{k+2}}
\end{aligned}$$

$$= \frac{(1-\alpha)_{k+1}}{(\alpha+1)_{k+2}} + 2\alpha\frac{(1-\alpha)_k}{(\alpha+1)_{k+2}}. \tag{2.21}$$

Applying the identity $(a)_{k+1} = a(1+a)_k$ to $\alpha+1$, we obtain the following two relations:

$$\begin{aligned} (\alpha+1)_{k+2} &= (\alpha+1)(\alpha+2)_{k+1}, \\ (\alpha+1)_{k+2} &= (\alpha+1)(\alpha+2)(\alpha+3)_k, \end{aligned} \tag{2.22}$$

and using the above relations and the fact that $(a)_0 = 1$, the series $S_{\alpha,1}$ can be rewritten as follows:

$$\begin{aligned} S_{\alpha,1} &= \frac{1}{\alpha+1}\sum_{k=0}^{+\infty}\frac{(1-\alpha)_{k+1}}{(\alpha+2)_{k+1}} + \frac{2\alpha}{(\alpha+1)(\alpha+2)}\sum_{k=0}^{+\infty}\frac{(1-\alpha)_k}{(\alpha+3)_k} \\ &= \frac{1}{\alpha+1}\sum_{k=1}^{+\infty}\frac{(1-\alpha)_k}{(\alpha+2)_k} + \frac{2\alpha}{(\alpha+1)(\alpha+2)}\sum_{k=0}^{+\infty}\frac{(1-\alpha)_k}{(\alpha+3)_k} \\ &= \frac{1}{\alpha+1}\left[\sum_{k=0}^{+\infty}\frac{(1-\alpha)_k}{(\alpha+2)_k} - 1\right] + \frac{2\alpha}{(\alpha+1)(\alpha+2)}\sum_{k=0}^{+\infty}\frac{(1-\alpha)_k}{(\alpha+3)_k} \\ &= \frac{1}{\alpha+1}\Big[{}_2F_1(1-\alpha,1;\alpha+2;1) - 1\Big] + \frac{2\alpha}{(\alpha+1)(\alpha+2)}\,{}_2F_1(1-\alpha,1;\alpha+3;1). \end{aligned}$$

From the property of ${}_2F_1$ given by (2.7), we have

$${}_2F_1(1-\alpha,1;\alpha+2;1) = \frac{\alpha+1}{2\alpha}, \tag{2.23}$$

and

$${}_2F_1(1-\alpha,1;\alpha+3;1) = \frac{\alpha+2}{2\alpha+1}. \tag{2.24}$$

Thus, we obtain

$$\begin{aligned} S_{\alpha,1} &= \frac{1}{\alpha+1}\Big[\frac{\alpha+1}{2\alpha} - 1\Big] + \frac{2\alpha}{(\alpha+1)(\alpha+2)}\,\frac{\alpha+2}{2\alpha+1} \\ &= \frac{1}{\alpha+1} + \frac{1}{2\alpha} - \frac{2}{(2\alpha+1)}. \end{aligned} \tag{2.25}$$

The proof is complete. □

■ The following lemma gives the limit of the series $S_{\alpha,2}$ defined by the left-hand side of relation (2.26).

Lemma 2.5. For all $\alpha > 1/2$, we have

$$S_{\alpha,2} = \sum_{k=0}^{+\infty}\frac{(1-\alpha)_k}{(\alpha+1)_k\,(k+2\alpha)} = \frac{1}{2\alpha}\,{}_3F_2(1-\alpha,1,2\alpha;\alpha+1,2\alpha+1;1). \tag{2.26}$$

Proof of Lemma 2.5. If $a = 1-\alpha$, $b = 1$, and $c = \alpha+1$, then $c \notin \mathbb{Z}_0^-$, and $\mathcal{R}(c-a-b) = 2\alpha-1 > 0$, since $\alpha > 1/2$. Then, expressing the rightside of (2.10) as a series and changing the order of integration and summation which is justified by Lemma 2.1 (due to the uniform convergence of the series) gives

$$2\alpha\int_0^1 x^{2\alpha-1}\,{}_2F_1(1-\alpha,1;\alpha+1;x)dx = 2\alpha\sum_{k=0}^{+\infty}\frac{(1-\alpha)_k(1)_k}{(\alpha+1)_k}\int_0^1\frac{x^{2\alpha-1}x^k}{k!}dx$$

$$= 2\alpha \sum_{k=0}^{+\infty} \frac{(1-\alpha)_k}{(\alpha+1)_k} \frac{1}{(k+2\alpha)}.$$

Thus, we obtained (2.26). The proof is complete. □

■ The following lemma gives the sum of the series $S_{\alpha,3}$ defined by the left-hand side of relation (2.27).

Corollary 2.1. *For all* $\alpha > 1$, *we have*

$$S_{\alpha,3} = \sum_{k=0}^{+\infty} \frac{(2-\alpha)_k}{(\alpha+1)_k\,(2\alpha+k)} = \frac{\alpha}{(1-\alpha)(2\alpha-1)} + \left[\frac{1-3\alpha}{2\alpha(1-\alpha)}\right] {}_3F_2(1-\alpha, 1, 2\alpha; \alpha+1, 2\alpha+1; 1). \tag{2.27}$$

Proof of Corollary 2.1. The proof is similar to the proof of Lemma 2.4, and so we just sketch the basic idea. If we take $a = 2-\alpha$, $b = 1$, and $c = \alpha+1$, then $c \notin \mathbb{Z}_0^-$ and $\mathcal{R}(c-a-b) = 2\alpha - 2 > 0$, since $\alpha > 1$. Then, by Lemma 2.1 we deduce that the series $S_{\alpha,3}$ is absolutely convergent. Thus, multiplying and dividing the term u_k by the same quantity $(1-\alpha)$, we obtain

$$\begin{aligned}
S_{\alpha,3} &= \sum_{k=0}^{+\infty} \frac{(2-\alpha)_k}{(\alpha+1)_k\,(2\alpha+k)} \\
&= \sum_{k=0}^{+\infty} \frac{(1-\alpha)(2-\alpha)_k}{(\alpha+1)_k\,(1-\alpha)\,(2\alpha+k)} \\
&= \sum_{k=0}^{+\infty} \frac{(1-\alpha)_{k+1}}{(1-\alpha)\,(\alpha+1)_k\,(2\alpha+k)} \\
&= \frac{1}{1-\alpha}\left[\sum_{k=0}^{+\infty} \frac{(1-\alpha)_k\,(k+1-\alpha)}{(\alpha+1)_k\,(2\alpha+k)}\right] \\
&= \frac{1}{1-\alpha}\left[\sum_{k=0}^{+\infty} \frac{(1-\alpha)_k\,(k+2\alpha+1-3\alpha)}{(\alpha+1)_k\,(2\alpha+k)}\right] \\
&= \frac{1}{1-\alpha}\left[\sum_{k=0}^{+\infty} \frac{(1-\alpha)_k}{(\alpha+1)_k} + (1-3\alpha)\sum_{k=0}^{+\infty} \frac{(1-\alpha)_k}{(\alpha+1)_k\,(2\alpha+k)}\right] \\
&= \frac{1}{1-\alpha}\left[{}_2F_1(1-\alpha, 1; \alpha+1; 1) + (1-3\alpha)\sum_{k=0}^{+\infty} \frac{(1-\alpha)_k}{(\alpha+1)_k\,(2\alpha+k)}\right].
\end{aligned}$$

Above we use the property of ${}_2F_1$ (2.7) and the result of Lemma 2.5 to obtain (2.27). The proof is complete. □

■ In the following corollary, we give the calculation of ${}_3F_2(-1, 1, 4; 3, 5; 1)$, which we believe may be new.

Corollary 2.1.

$${}_3F_2(-1, 1, 4; 3, 5; 1) = \frac{11}{15}. \tag{2.28}$$

Proof of Corollary 2.1. Note that when $\alpha = 2$, the numerical series introduced in Lemmas 2.4 and 2.5 coincide. Therefore, the right-hand terms of relations (2.20) and (2.26) are equal when $\alpha = 2$. Thus,

$$\begin{aligned} {}_3F_2(-1, 1, 4; 3, 5; 1) &= 4\left(\frac{1}{3} + \frac{1}{4} - \frac{2}{5}\right) \\ &= \frac{7}{3} - \frac{8}{5} = \frac{11}{15}. \end{aligned} \tag{2.29}$$

The proof is complete. □

3. Explicit form of two subfamilies of ${}_3F_2(1)$

This section explores specific subfamilies of ${}_3F_2(1)$ and is structured into two subsections. The first subsection presents the explicit forms of the subfamilies $\{{}_3F_2(1-\alpha, 1, \alpha+1; \alpha+1, \alpha+2; 1)\}$ and $\{{}_3F_2(1-\alpha, 1, \alpha+2; \alpha+1, \alpha+3; 1)\}$ for all $\alpha > \frac{1}{2}$, while the second subsection provides the explicit form of the subfamily $\{{}_3F_2(\frac{1}{2}-p, 1, 1+2p; \frac{3}{2}+p, 2p+2; 1)\}$ for all $p \in \mathbb{N}^*$.

3.1. Explicit form of ${}_3F_2(1-\alpha, 1, \alpha+1; \alpha+1, \alpha+2; 1)$

This part aims to find the explicit form of the function $K(\alpha) =: {}_3F_2(1-\alpha, 1, \alpha+1; \alpha+1, \alpha+2; 1)$ (Theorem 3.1). To do this, we write K in the form of a series of functions (Lemma 3.1), and then we use the result of Lemma 2.3 to prove our main result of this section, Theorem 3.1.

■ In the following lemma, we express the function $K(\alpha)$ as a series.

Lemma 3.1. For all $\alpha > 1/2$, we have

$${}_3F_2(1-\alpha, 1, \alpha+1; \alpha+1, \alpha+2; 1) = (\alpha+1)\sum_{k=0}^{+\infty} \frac{(1-\alpha)_k}{(\alpha+1)_k\,(k+\alpha+1)}. \tag{3.1}$$

Proof of Lemma 3.1. If $a = 1-\alpha$, $b = 1$, and $c = \alpha+1$, then $c \notin \mathbb{Z}_0^-$ and $\mathcal{R}(c-a-b) = 2\alpha - 1 > 0$, since $\alpha > 1/2$. Then, expressing the rightside of (2.8) as a series, changing the order of integration and summation, which is justified by Lemma 2.1, and applying the steps of the proof of Lemma 2.5, we can readily derive the proof of Lemma 3.1. □

■ Now we state and prove our main result of this section, Theorem 3.1.

Theorem 3.1. For all $\alpha > 1/2$, we have

$${}_3F_2(1-\alpha, 1, \alpha+1; \alpha+1, \alpha+2; 1) = \frac{\alpha+1}{2\alpha}. \tag{3.2}$$

Proof of Theorem 3.1. By identifying relations (2.17) and (3.1), we obtain

$${}_3F_2(1-\alpha, 1, \alpha+2; \alpha+1, \alpha+3; 1) = (\alpha+1)\left(\frac{1}{2\alpha}\right) = \frac{\alpha+1}{2\alpha}.$$

The proof is complete. □

3.2. *Explicit form of* ${}_3F_2(1-\alpha, 1, \alpha+2; \alpha+1, \alpha+3; 1)$

This part aims to find the explicit form of the function $G(\alpha) =: {}_3F_2(1-\alpha, 1, \alpha+2; \alpha+1, \alpha+3; 1)$ (Theorem 3.2). To do this, we write G in the form of a series of functions (Lemma 3.2), then we use the result of Lemma 2.4 to prove our main result of this section, Theorem 3.2. Our argument in this section is similar to the previous section.

■ In the following lemma, we express the function $G(\alpha)$ as a series.

Lemma 3.2. For all $\alpha > 1/2$, we have

$$ {}_3F_2(1-\alpha, 1, \alpha+2; \alpha+1, \alpha+3; 1) = (\alpha+2) \sum_{k=0}^{+\infty} \frac{(1-\alpha)_k}{(\alpha+1)_k\,(k+\alpha+2)}. \tag{3.3} $$

Proof of Lemma 3.2. If $a = 1-\alpha$, $b = 1$, and $c = \alpha+1$, then $c \notin \mathbb{Z}_0^-$ and $\mathcal{R}(c-a-b) = 2\alpha-1 > 0$, since $\alpha > 1/2$. Then, expressing the rightside of (2.9) as a series, changing the order of integration and summation, which is justified by Lemma 2.1, and applying the steps of the proof of Lemma 2.5, we can readily derive the proof of Lemma 3.2. □

■ Now we state and prove our main result of this section, Theorem 3.2.

Theorem 3.2. For all $\alpha > 1/2$, we have

$$ {}_3F_2(1-\alpha, 1, \alpha+2; \alpha+1, \alpha+3; 1) = \frac{1}{2} + \frac{1}{\alpha} + \frac{1}{\alpha+1} - \frac{3}{2\alpha+1}. \tag{3.4} $$

Proof of Theorem 3.2. By identifying relations (2.20) and (3.3), we obtain

$$ \begin{aligned} {}_3F_2(1-\alpha, 1, \alpha+2; \alpha+1, \alpha+3; 1) &= (\alpha+2)\left(\frac{1}{\alpha+1} + \frac{1}{2\alpha} - \frac{2}{2\alpha+1}\right) \\ &= \frac{1}{2} + \frac{1}{\alpha} + \frac{1}{\alpha+1} - \frac{3}{2\alpha+1}. \end{aligned} $$

The proof is complete. □

3.3. *Explicit form for some family of functions* ${}_3F_2(1-\alpha, 1, 2\alpha; \alpha+1, 2\alpha+1; 1)$

This section deals with the family of functions $F(\alpha) = {}_3F_2(1-\alpha, 1, 2\alpha; \alpha+1, 2\alpha+1; 1)$. It is easy to find a representation for it by a series of functions $\sum_{k=0}^{+\infty} f_k(\alpha)$ (see Lemma 2.5). However, it is not obvious to find an explicit one unless $\alpha \in \mathbb{N}^*$ or a rational number of the form $\alpha = \frac{2p+1}{2}$ and $p \in \mathbb{N}^*$. Note that when $\alpha = p \in \mathbb{N}^*$, then $\sum_{k=0}^{+\infty} f_k(\alpha)$ is equal to $\sum_{k=0}^{p-1} f_k(\alpha)$. Therefore, this case will be excluded from this study. In summary, this part aims to determine the explicit form of $F(\alpha)$ when $\alpha = \frac{2p+1}{2}$, which coincides with ${}_3F_2(\frac{1}{2}-p, 1, 1+2p; \frac{3}{2}+p, 2p+2; 1)$ and which we will designate by $H(p)$. To do this, we first calculate $H(1)$ in Section 3.3.1 to make the calculation of $H(p)$ easy to follow in Section 3.3.2.

■ The main result of this section is summarized in the following theorem. The proof of this theorem is postponed to the end of this section.

Theorem 3.3. For all $p \in \mathbb{N}^*$, we have

$$ {}_3F_2(\tfrac{1}{2}-p, 1, 1+2p; \tfrac{3}{2}+p, 2p+2; 1) = -(2p+1)K_p\left(2a_{2p+1}\ln(2) + 2a_{2p+1}(E_p+B_p) + \sum_{j=1}^{2p} a_jB_j\right), $$

where

$$ \begin{aligned} K_p &= (1-2p)(3-2p)(5-2p)\cdots(-1+2p)(1+2p) \\ a_{2p+1} &= -\frac{2^{2p}(2p)!\,(3p)!}{(6p+1)!\,(p)!} \\ E_p &= \sum_{k=1}^{2p}\frac{1}{2k} \\ B_j &= \sum_{k=0}^{j-1}\frac{1}{2k-2p+1}, \ j=1,\dots,2p \\ a_j &= \frac{2}{(-1)^j\,(2)^{2p}\,(j)!\,(2p-j)!\,(6p-2j+1)}, \ j=1,\dots,2p. \end{aligned} \tag{3.5} $$

3.3.1. Calculation of $H(p) := {}_3F_2(\frac{1}{2}-p, 1, 1+2p; \frac{3}{2}+p, 2p+2; 1)$, for $p=1$

This part aims to find the explicit form of the function $H(1)$. To do this, we write $H(1)$ in the form of a numerical series. Then, we determine the limit of this series.

■ In Lemma 3.3, we give the exact value of ${}_3F_2(-\dfrac{1}{2}, 1, 3; \dfrac{5}{2}, 4; 1)$.

Lemma 3.3.

$$ {}_3F_2(-\frac{1}{2}, 1, 3; \frac{5}{2}, 4; 1) = \frac{33}{35} - \frac{6}{35}\ln(2). \tag{3.6} $$

Proof of Lemma 3.3. For $p=1$ (i.e., $\alpha = \frac{3}{2}$), after simplifications, relation (2.26) gives

$$ -3\sum_{k=0}^{+\infty}\frac{1}{(2k-1)(2k+1)(2k+3)(k+3)} = \frac{1}{3}\,{}_3F_2(-\frac{1}{2}, 1, 3; \frac{5}{2}, 4; 1), \tag{3.7} $$

that is, ${}_3F_2(-\dfrac{1}{2}, 1, 3; \dfrac{5}{2}, 4; 1) = -9S$, where

$$ S = \sum_{k=0}^{+\infty}\frac{1}{(2k-1)(2k+1)(2k+3)(k+3)}. \tag{3.8} $$

△ The decomposition into partial fractions of the general term u_k, of the series $S = \sum u_k$, gives

$$ \begin{aligned} u_k &= \frac{1}{(2k-1)(2k+1)(2k+3)(k+3)} \\ &= \frac{1}{28(2k-1)} - \frac{1}{10(2k+1)} + \frac{1}{12(2k+3)} - \frac{1}{105(k+3)}. \end{aligned} \tag{3.9} $$

△ Let n be a fixed positive integer. Then, $(S_n)_{n\geq 0}$ and $(T_n)_{n\geq 1}$ are sequences defined by

$$ \begin{aligned} S_n &= \sum_{k=0}^{n}\frac{1}{(2k-1)(2k+1)(2k+3)(k+3)} \\ T_n &= \sum_{k=1}^{n}\frac{1}{2k-1}. \end{aligned} \tag{3.10} $$

△ We will simplify the partial sum S_n in order to find its limit when n tends to infinity. To do this, we will express the partial sums,

$$U_n = \sum_{k=0}^{n} \frac{1}{2k-1} \ ; \ V_n = \sum_{k=0}^{n} \frac{1}{2k+1} \ ; \ W_n = \sum_{k=0}^{n} \frac{1}{2k+3},$$

as a function of T_n. It is easy to check the following equalities:

$$\begin{aligned} U_n &= -1 + T_n \\ V_n &= T_n + \frac{1}{2n+1} \\ W_n &= -1 + T_n + \frac{1}{2n+1} + \frac{1}{2n+3}. \end{aligned} \tag{3.11}$$

△ We then deduce S_n as a function of T_n:

$$\begin{aligned} S_n &= \frac{U_n}{28} - \frac{V_n}{10} + \frac{W_n}{12} - \frac{1}{105} \sum_{k=0}^{n} \frac{1}{k+3} \\ &= -\frac{5}{42} + A_n + B_n, \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} A_n &= \frac{2}{105} T_n - \frac{1}{105} \sum_{k=0}^{n} \frac{1}{k+3} \\ B_n &= -\frac{1}{60(2n+1)} + \frac{1}{12(2n+3)}. \end{aligned} \tag{3.13}$$

△ Furthermore, we have

$$\begin{aligned} \sum_{k=0}^{n} \frac{1}{k+3} &= 2 \sum_{k=0}^{n} \frac{1}{2(k+3)} \\ &= 2 \sum_{k=3}^{n+3} \frac{1}{2k} \\ &= 2 \left(-\frac{1}{2} - \frac{1}{4} + \sum_{k=1}^{n} \frac{1}{2k} + \frac{1}{2(n+1)} + \frac{1}{2(n+2)} + \frac{1}{2(n+3)} \right) \\ &= 2 \left(-\frac{3}{4} + \sum_{k=1}^{n} \frac{1}{2k} + \frac{1}{2(n+1)} + \frac{1}{2(n+2)} + \frac{1}{2(n+3)} \right) \\ &= 2 \left(-\frac{3}{4} + \sum_{k=1}^{n} \frac{1}{2k} + C_n \right), \end{aligned} \tag{3.14}$$

where

$$C_n = \frac{1}{2(n+1)} + \frac{1}{2(n+2)} + \frac{1}{2(n+3)}.$$

Consequently,

$$\begin{aligned} A_n &= \frac{2}{105}T_n - \frac{2}{105}\left(-\frac{3}{4} + \sum_{k=1}^{n}\frac{1}{2k} + C_n\right) \\ &= \frac{2}{105}\left(\sum_{k=1}^{n}\frac{1}{2k-1} - \sum_{k=1}^{n}\frac{1}{2k} + \frac{3}{4} - C_n\right) \\ &= \frac{2}{105}\left(\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k} + \frac{3}{4} - C_n\right) \\ &= \frac{2}{105}\left(D_n + \frac{3}{4} - C_n\right), \end{aligned} \tag{3.15}$$

where

$$D_n = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}.$$

△ Based on relations (3.12) and (3.15), we obtain

$$\begin{aligned} S_n &= -\frac{5}{42} + \frac{2}{105}\left(D_n + \frac{3}{4} - C_n\right) + B_n \\ &= -\frac{11}{105} + \frac{2}{105}\left(D_n - C_n\right) + B_n. \end{aligned} \tag{3.16}$$

△ Furthermore, we know that for all real numbers $x \neq -1$ such that $|x| \leq 1$, we have

$$\lim_{n\to+\infty}\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k}x^k = \ln(1+x). \tag{3.17}$$

Consequently,

$$\lim_{n\to+\infty} D_n = \ln(2). \tag{3.18}$$

△ Moreover, it is easy to verify that

$$\begin{aligned} \lim_{n\to+\infty} B_n &= 0, \\ \lim_{n\to+\infty} C_n &= 0. \end{aligned} \tag{3.19}$$

△ Finally, we have

$$\lim_{n\to+\infty} S_n = -\frac{11}{105} + \frac{2}{105}\ln(2). \tag{3.20}$$

△ By grouping relations (3.7) and (3.20), we deduce that

$$\begin{aligned} {}_3F_2(-\frac{1}{2}, 1, 3; \frac{5}{2}, 4; 1) &= -9\left(-\frac{11}{105} + \frac{2}{105}\ln(2)\right) \\ &= \frac{33}{35} - \frac{6}{35}\ln(2). \end{aligned} \tag{3.21}$$

The proof is complete. □

3.3.2. Calculation of ${}_3F_2(\frac{1}{2}-p,1,1+2p;\frac{3}{2}+p,2p+2;1)$, for all $p\in\mathbb{N}^*$

This section explores specific subfamilies of ${}_3F_2(1)$ and is structured into two subsections. The first subsection presents the explicit forms of the subfamilies F(1) and F(2), while the second subsection provides the explicit form of the subfamily F(3). This part aims to find the explicit form of the function $H(p)$, $p\in\mathbb{N}^*$. To do this, we will follow the same approach as that used in Section 3.3.1 to calculate $H(1)$.

■ In Lemma 3.5, we give the explicit expression of $\dfrac{(1-\alpha)_k}{1+\alpha}$ when $\alpha=\frac{2p+1}{2}$.

Lemma 3.4. For all $p\in\mathbb{N}^*$, $k\in\mathbb{N}$, we have

$$\frac{(1-\frac{2p+1}{2})_k}{(1+\frac{2p+1}{2})_k}=\frac{(1-2p)(1-2p+1))\cdots(-1+2p)(1+2p)}{(2k+1-2p)(2k+3-2p)\cdots(2k-1+2p)(2k+1+2p)}. \tag{3.22}$$

Proof of Lemma 3.5. If $\alpha=\frac{2p+1}{2}$, then we have

$$\begin{aligned}(1-\alpha)_k &= (\tfrac{1}{2}-p)(\tfrac{3}{2}-p)(\tfrac{5}{2}-p)\cdots(k-\tfrac{3}{2}-p)(k-\tfrac{1}{2}-p),\\ (\alpha+1)_k &= (\tfrac{3}{2}+p)(\tfrac{5}{2}+p)\cdots(k-\tfrac{1}{2}+p)(k+\tfrac{1}{2}+p).\end{aligned} \tag{3.23}$$

■ Note that when $k\geq 2p+2$, there are $j=k-2p-1$ terms in common between $(1-\alpha)_k$ and $(\alpha+1)_k$. Indeed, we observe the first number in common if $k-\frac{1}{2}-p=\frac{3}{2}+p$, that is $k=2p+2$. Therefore, if $k=2p+3$, then there are two terms in common, and so on. Thus, if $k\geq 2p+2$, then $(1-\alpha)_k$ and $(1+\alpha)_k$ are written as follows:

$$\begin{aligned}(1-\alpha)_k &= (\tfrac{1}{2}-p)\cdots(\tfrac{1}{2}-p+2p)(\tfrac{3}{2}+p)\cdots(k-\tfrac{1}{2}-p),\\ (\alpha+1)_k &= (\tfrac{3}{2}+p)(\tfrac{5}{2}+p)\cdots(k-p-\tfrac{1}{2})(k-p+\tfrac{1}{2})\cdots(k+\tfrac{1}{2}+p).\end{aligned} \tag{3.24}$$

Therefore, their quotient can be simplified as follows:

$$\begin{aligned}\frac{(1-\alpha)_k}{(\alpha+1)_k} &= \frac{(\frac{1}{2}-p)(\frac{1}{2}+1-p)\cdots(\frac{1}{2}+2p-p)}{(k-p+\frac{1}{2})(k-p+1+\frac{1}{2})\cdots(k-p+2p+\frac{1}{2})}\\ &= \frac{(\frac{1}{2}-(p-0))(\frac{1}{2}-(p-1))\cdots(\frac{1}{2}-(p-2p))}{(k+\frac{1}{2}-(p-0))(k+\frac{1}{2}-(p-1))\cdots(k+\frac{1}{2}-(p-2p))}\\ &= \frac{2^{2p+1}(1-2(p-0))(1-2(p-1))\cdots(1-2(p-2p))}{2^{2p+1}(2k+1-2(p-0))(2k+1-2(p-1))\cdots(2k+1-2(p-2p))}\\ &= \frac{(1-2p)(1-2p+2))\cdots(-1+2p)(1+2p)}{(2k+1-2p)(2k+3-2p)\cdots(2k-1+2p)(2k+1+2p)}\\ &= \frac{(1-2p)(3-2p))\cdots(-1+2p)(1+2p)}{(2k+1-2p)(2k+3-2p)\cdots(2k-1+2p)(2k+1+2p)}.\end{aligned} \tag{3.25}$$

■ When $k\leq 2p+1$, we will show that the quotient $\frac{(1-\alpha)_k}{(\alpha+1)_k}$ can be reduced to the form given by the last line of relation (3.25).

△ When $k=0$, we have

$$\begin{aligned}\frac{(1-\alpha)_0}{(\alpha+1)_0} &= \frac{1}{1}\\ &= \frac{(1-2p)(3-2p)\cdots(-1+2p)(1+2p)}{(1-2p)(3-2p))\cdots(-1+2p)(1+2p)}\\ &= \frac{(1-2p)(3-2p)\cdots(-1+2p)(1+2p)}{(2\times 0+1-2p)(2\times 0+3-2p))\cdots(2\times 0-1+2p)(2\times 0+1+2p)}.\end{aligned} \tag{3.26}$$

△ When $k = 1$, we have

$$\begin{aligned}\frac{(1-\alpha)_1}{(\alpha+1)_1} &= \frac{1-2p}{(3+2p)}\\ &= \frac{(1-2p)\quad[(3-2p)\cdots(-1+2p)(1+2p)]}{[(3-2p)\cdots(-1+2p)(1+2p)]\quad(3+2p)}\\ &= \frac{(1-2p)(3-2p)\cdots(-1+2p)(1+2p)}{(2\times1+1-2p)(2\times1+3-2p)\cdots(2\times1-1+2p)(2\times1+1+2p)}.\end{aligned} \tag{3.27}$$

△ More generally, when k is an integer such that $0 \le k \le 2p+1$, we have

$$\begin{aligned}\frac{(1-\alpha)_k}{(\alpha+1)_k} &= \frac{(1-2p)(3-2p)\cdots(2k-3-2p)(2k-1-2p)}{(3+2p)(5+2p)\cdots(2k-1+2p)(2k+1+2p)}\\ &= \frac{(1-2p)(3-2p)\cdots(2k-1-2p)\quad[(2k+1-2p)\cdots(1+2p)]}{[(2k+1-2p)\cdots(1+2p)]\quad(3+2p)(5+2p)\cdots(2k+1+2p)}\\ &= \frac{(1-2p)(3-2p)\cdots(-1+2p)(1+2p)}{(2k+1-2p)\cdots(1+2p)(3+2p)(5+2p)\cdots(2k+1+2p)}.\end{aligned} \tag{3.28}$$

Thus, we have shown that $\frac{(1-\alpha)_k}{(\alpha+1)_k}$ is written in the form (3.22) for all $k \in \mathbb{N}$. The proof is complete. □

Remark 3.1. For $\alpha = \frac{2p+1}{2}$, we deduce from Lemma 3.5 that

$$\frac{(1-\alpha)_k}{(\alpha+1)_k(k+2p+1)} = K_p u_k, \tag{3.29}$$

where

$$\begin{aligned}K_p &= (1-2p)(3-2p)\cdots(-1+2p)(1+2p)\\ u_k &= \frac{1}{(2k+1-2p)(2k+3-2p)\cdots(2k+1+2p)(k+2p+1)}.\end{aligned} \tag{3.30}$$

■ From the above remark, the decomposition into partial fractions of u_k gives

$$\begin{aligned}u_k &= \frac{a_0}{2k+1-2p} + \frac{a_1}{2k+3-2p} + \cdots + \frac{a_{2p}}{2k+1+2p} + \frac{a_{2p+1}}{k+2p+1}\\ &= \frac{a_0}{2k+1-2(p-0)} + \frac{a_1}{2k+1-2(p-1)} + \cdots + \frac{a_{2p}}{2k+1-2(p-2p)} + \frac{a_{2p+1}}{k+2p+1}\\ &= \sum_{i=0}^{2p} \frac{a_i}{2k+1-2(p-i)} + \frac{a_{2p+1}}{k+2p+1}.\end{aligned} \tag{3.31}$$

For all $i = 0, \ldots, 2p$, to find a_i, simply multiply u_k by $(2k+1-2(p-i))$, Then, evaluate the resulting expression at $k = \frac{2(p-i)-1}{2}$. For now, let us calculate only the first three coefficients a_0, a_1, and a_2.

△ For a_0, we evaluate the resulting expression at $k = \frac{2p-1}{2}$ (i.e., $2k = 2p-1$), so we obtain

$$\begin{aligned}\frac{1}{a_0} &= (2p-1+1-2(p-1))\cdots(2p-1+1-2(p-2p))\frac{1}{2}(2p-1+4p+2)\\ &= (2\times1)(2\times2)\cdots(2\times2p)\tfrac{1}{2}(6p+1),\\ &= 2^{2p}\ (2p)!\ \frac{1}{2}(6p+1)\\ &= \frac{1}{2}\ (-1)^0\ (0!)\ 2^{2p}\ (2p-0)!\ (6p-2\times0+1).\end{aligned} \tag{3.32}$$

Consequently,

$$a_0 = \frac{2}{(-1)^0\ (0!)\ 2^{2p}\ (2p-0)!\ (6p-2\times 0+1)}. \tag{3.33}$$

△ For a_1, we evaluate the resulting expression at $k = \frac{2p-3}{2}$ (i.e., $2k = 2p-3$), so we obtain

$$\begin{aligned}
\frac{1}{a_1} &= (2p-3+1-2(p-0))\cdots(2p-3+1-2(p-2p))\frac{1}{2}(2p-3+4p+2)\\
&= (-2)(2\times 1)(2\times 3)\cdots(2\times(2p-1))\frac{1}{2}(6p-1)\\
&= (-1)^1\ 2^1\ 2^{2p-1}\ (2p-1)!\ \frac{1}{2}(6p-1)\\
&= \frac{1}{2}\ (-1)^1\ (1!)\ (2)^{2p}\ (2p-1)!\ (6p-2\times 1+1).
\end{aligned} \tag{3.34}$$

Consequently,

$$a_1 = \frac{2}{(-1)^1\ (1!)\ (2)^{2p}\ (2p-1)!\ (6p-2\times 1+1)}. \tag{3.35}$$

△ For a_2, we evaluate the resulting expression at $k = \frac{2p-5}{2}$ (i.e., $2k = 2p-5$), so we obtain

$$\begin{aligned}
\frac{1}{a_2} &= (2p-5+1-2(p-0))\cdots(2p-5+1-2(p-2p))\frac{1}{2}(2p-5+4p+2)\\
&= (-2^2)(-2^1)(2\times 1)(2\times 3)\cdots(2\times(2p-2))\frac{1}{2}(6p-3)\\
&= (-1)^2\ 2^2\ (2!)\ 2^{2p-2}\ (2p-2)!\ \frac{1}{2}(6p-3)\\
&= \frac{1}{2}\ (-1)^2\ (2!)\ (2)^{2p}\ (2p-2)!\ (6p-2\times 2+1).
\end{aligned} \tag{3.36}$$

Consequently,

$$a_2 = \frac{2}{(-1)^2\ (2)^{2p}\ (2p-2)!\ (6p-2\times 2+1)}. \tag{3.37}$$

■ The calculation of the first three coefficients a_0, a_1, a_2 allowed us to guess the general expression of a_i for all $i = 0, \ldots, 2p$, which is stated in the following Lemma 3.5.

Lemma 3.5. Let $p \in \mathbb{N}^*$. Then, for all $i = 0, \ldots, 2p$, we have

$$a_i = \frac{2}{(-1)^i\ (2)^{2p}\ (i)!\ (2p-i)!\ (6p-2i+1)}. \tag{3.38}$$

Proof of Lemma 3.5. We will confirm this expression by a direct calculation of a_i. To do this, we evaluate the resulting expression at $k = \frac{2p-(2i+1)}{2}$ (i.e., $2k = 2p - (2i + 1)$), so we obtain

$$\begin{aligned} \frac{1}{a_i} &= \Big(2p-(2i+1)+1-2(p-0)\Big)\Big(2p-(2i+1)+1-2(p-1)\Big)\cdots \\ &\cdots \Big(2p-(2i+1)+1-2(p-(i-1))\Big)\Big(2p-(2i+1)+1-2(p-(i+1))\Big)\cdots \\ &\cdots \Big(2p-(2i+1)+1-2(p-2p)\Big)\frac{1}{2}\Big(2p-(2i+1)+4p+2\Big) \\ &= (-2(i-0))(-2(i-1))\cdots(-2(i-(i-1)))(2\times 1)(2\times 2)\cdots \\ &\cdots (2(2p-i))\frac{1}{2}(6p-2i+1) \\ &= \frac{1}{2}\,(-2)^i\,(i)!\,2^{2p-i}\,(2p-i)!\,(6p-2i+1) \\ &= \frac{1}{2}\,(-1)^i\,(2)^i\,(i)!\,2^{2p-i}\,(2p-i)!\,(6p-2i+1) \\ &= \frac{1}{2}\,(-1)^i\,(2)^{2p}\,(i)!\,(2p-i)!\,(6p-2i+1). \end{aligned} \tag{3.39}$$

Thus, we find the expression of a_i stated in relation (3.38). The proof is then complete. □

■ All that remains is to determine the expression of a_{2p+1}. This will be the subject of Lemma 3.6.

Lemma 3.6. The last coefficient of the decomposition into partial fractions (3.31) is given by

$$a_{2p+1} = -\frac{2^{2p}\,(2p)!\,(3p)!}{(6p+1)!\,(p)!}. \tag{3.40}$$

Proof of Lemma 3.6. To find a_{2p+1}, multiply u_k by $(k + 2p + 1)$. Then, evaluate the resulting expression at $k = -2p - 1$ (i.e., $2k = -4p - 2$). We then obtain

$$\begin{aligned} \frac{1}{a_{2p+1}} &= (-4p-2-2p+1)(-4p-2-2p+3)\cdots(-4p-2+2p-1)(-4p-2+2p+1) \\ &= (-6p-1)(-6p+1)(-6p+3)\cdots(-2p-3)(-2p-1) \\ &= \frac{(-6p-1)(-6p)(-6p+1)(-6p+2)(-6p+3)\cdots(-2p-3)(-2p-2)(-2p-1)}{(-6p)(-6p+2)\cdots(-2p-4)(-2p-2)} \\ &= \frac{(-1)^{2p+1}(6p+1)(6p)(6p-1)(6p-2)(6p-3)\cdots(2p+3)(2p+2)(2p+1)}{(-1)^{2p}(6p)(6p-2)\cdots(2p+4)(2p+2)} \\ &= -\frac{(2p+1)(2p+2)\cdots(6p+1)}{2(p+1)2(p+2)\cdots 2(3p)} \\ &= -\frac{[1\times 2\times\cdots\times 2p]\,(2p+1)(2p+2)\cdots(6p+1)\;[1\times 2\times\cdots\times p]}{[1\times 2\times\cdots\times 2p]\,2^{3p-(p+1)+1}\,[1\times 2\times\cdots\times p]\,(p+1)(p+2)\cdots(3p)} \\ &= -\frac{(6p+1)!\,(p)!}{2^{2p}\,(2p)!\,(3p)!}. \end{aligned} \tag{3.41}$$

The proof is complete. □

■ The following remark shows the validation of formulas (3.38) and (3.40) when $p = 1$.

Remark 3.2. For $p = 1$, we will check the concordance of the coefficients $a_0, \ldots, a_3$ given by relations (3.40) and (3.38) with those presented in relation (3.9). When $p = 1$, relations (3.38) and (3.40) give

$$\begin{aligned} a_0 &= \frac{2}{(-1)^0\,(2)^2\,(0)!\,(2)!\,(7)} = \frac{1}{28} \\ a_1 &= \frac{2}{(-1)^1\,(2)^2\,(1)!\,(1)!\,(5)} = -\frac{1}{10} \\ a_2 &= \frac{2}{(-1)^2\,(2)^2\,(2)!\,(0)!\,(3)} = \frac{1}{12} \\ a_3 &= -\frac{2^2\,(2)!\,(3)!}{(7)!\,(1)!} = -\frac{1}{105}. \end{aligned} \tag{3.42}$$

Thus, we find the same coefficients of the decomposition into partial fractions of u_k given by (3.9).

■ Lemma 3.7 presents a relation between the coefficient a_{2p+1} and the coefficients $(a_i)_{0\leq i\leq 2p}$.

Lemma 3.7. For all $p \in \mathbb{N}^*$, we have

$$\sum_{i=0}^{2p} a_i = -2a_{2p+1}. \tag{3.43}$$

Proof of Lemma 3.7. We have shown in relation (3.31) that, for all $k > 0$, the term u_k is written as a partial fraction (3.31), where $(a_i)_{0\leq i\leq 2p}$ and a_{2p+1} are given by (3.38) and (3.40).

By reducing all the partial fractions to the same denominator and identifying the numerators, we then obtain $2p + 2$ equations with unknowns $a_0, \ldots, a_{2p}, a_{2p+1}$. It is easy to see that the equation that relates the coefficients of the monomial k^{2p+1} is written in the following form $\sum_{i=0}^{2p} k(2k)^{2p} a_i + (2k)^{2p+1} a_{2p+1} = 0$, or in the equivalent form

$$\sum_{i=0}^{2p} a_i = -2a_{2p+1}. \tag{3.44}$$

The proof is complete. □

■ In the following Lemma 3.8, we establish a supporting result that arises from the calculations performed in the previous results of this section. This result provides an explicit form of a finite sum.

Lemma 3.8. For all $p \in \mathbb{N}^*$, we have

$$\sum_{i=0}^{2p} \frac{(-1)^i}{(2)^{2p}\,(i)!\,(2p-i)!(6p-2i+1)} = \frac{2^{2p}\,(2p)!\,(3p)!}{(6p+1)!\,(p)!}. \tag{3.45}$$

Proof of Lemma 3.8. Based on formula (3.43), we have

$$\sum_{i=0}^{2p} a_i = -2a_{2p+1}. \tag{3.46}$$

By replacing in the previous equality the coefficients $(a_i)_{0\leq i\leq 2p+1}$ by their expressions presented in relations (3.38) and (3.40), we directly obtain relation (3.45). The proof is complete. □

■ In the following section, we will prove that the series $\sum u_k$ converges and we calculate its sum, where

u_k is defined by (3.30).

Convergence of the series $\sum u_k$

Let n be a fixed positive integer, and $(S_n)_{n\geq 0}$ and $(T_n)_{n\geq 1}$ be the sequences defined by

$$\begin{aligned} S_n &= \sum_{k=0}^{n} u_k, \\ T_n &= \sum_{k=0}^{n} \frac{1}{2k-2p+1}. \end{aligned} \tag{3.47}$$

■ We will write S_n as a function of T_n, prove that the sequence S_n converges, and determine its limit.

△ For all $j = 1, \dots, 2p$, we have

$$\begin{aligned} \sum_{k=0}^{n} \frac{1}{2k-2(p-j)+1} &= \sum_{k=0}^{n} \frac{1}{2(k+j)-2p+1} \\ &= \sum_{k=j}^{n+j} \frac{1}{2k-2p+1} \\ &= \sum_{k=0}^{n} \frac{1}{2k-2p+1} + \sum_{k=n+1}^{n+j} \frac{1}{2k-2p+1} - \sum_{k=0}^{j-1} \frac{1}{2k-2p+1} \\ &= T_n + A_{j,n} - B_j, \end{aligned} \tag{3.48}$$

where

$$\begin{aligned} A_{j,n} &= \sum_{k=n+1}^{n+j} \frac{1}{2k-2p+1}, \\ B_j &= \sum_{k=0}^{j-1} \frac{1}{2k-2p+1}. \end{aligned} \tag{3.49}$$

Note that for $j = 0$, we also have

$$\begin{aligned} \sum_{k=0}^{n} \frac{1}{2k-2(p-0)+1} &= \sum_{k=0}^{n} \frac{1}{2k-2p+1} \\ &= T_n + A_{0,n} - B_0, \end{aligned} \tag{3.50}$$

where

$$A_{0,n} = B_0 = 0. \tag{3.51}$$

△ Furthermore, we also have

$$\begin{aligned} \sum_{k=0}^{n} \frac{1}{k+2p+1} &= 2\sum_{k=0}^{n} \frac{1}{2(k+2p+1)} \\ &= 2\sum_{k=2p+1}^{n+2p+1} \frac{1}{2k} \\ &= 2\sum_{k=1}^{n} \frac{1}{2k} + 2\sum_{k=n+1}^{n+2p+1} \frac{1}{2k} - 2\sum_{k=1}^{2p} \frac{1}{2k} \\ &= 2\sum_{k=1}^{n} \frac{1}{2k} + 2D_n - 2E_p, \end{aligned} \tag{3.52}$$

where

$$\begin{aligned} D_n &= \sum_{k=n+1}^{n+2p+1} \frac{1}{2k}, \\ E_p &= \sum_{k=1}^{2p} \frac{1}{2k}. \end{aligned} \tag{3.53}$$

△ Moreover, we have

$$\begin{aligned} T_n &= \sum_{k=0}^{n} \frac{1}{2k-2p+1} \\ &= \sum_{k=0}^{p-1} \frac{1}{2k-2p+1} + \sum_{k=p}^{n+p} \frac{1}{2k-2p+1} \\ &= \sum_{k=0}^{p-1} \frac{1}{2k-2p+1} + \sum_{k=0}^{n} \frac{1}{2k+1} \\ &= B_p + \sum_{k=0}^{n} \frac{1}{2k+1}, \end{aligned} \tag{3.54}$$

where

$$B_p = \sum_{k=0}^{p-1} \frac{1}{2k-2p+1}. \tag{3.55}$$

■ After writing S_n as a function of T_n, we will inject the expression of T_n given by (3.54) into S_n to prove that S_n converges and deduce its limit.

△ Using relations (3.48), (3.54) (3.52), and (3.43), the partial sum S_n equals

$$\begin{aligned} S_n &= \sum_{j=0}^{2p} a_j \sum_{k=0}^{n} \frac{1}{2k-2(p-j)+1} + a_{2p+1} \sum_{k=0}^{n} \frac{1}{k+2p+1} \\ &= \sum_{j=0}^{2p} a_j (T_n + A_{j,n} - B_j) + a_{2p+1} \left(2 \sum_{k=1}^{n} \frac{1}{2k} + 2D_n - 2E_p \right) \\ &= T_n \sum_{j=0}^{2p} a_j + 2a_{2p+1} \sum_{k=1}^{n} \frac{1}{2k} + \sum_{j=1}^{2p} a_j (A_{j,n} - B_j) + 2a_{2p+1} \left(D_n - E_p \right) \\ &= -2a_{2p+1} T_n + 2a_{2p+1} \sum_{k=1}^{n} \frac{1}{2k} + \sum_{j=1}^{2p} a_j A_{j,n} - \sum_{j=1}^{2p} a_j B_j + 2a_{2p+1} D_n - 2a_{2p+1} E_p \\ &= -2a_{2p+1} \sum_{k=0}^{n} \frac{1}{2k+1} - 2a_{2p+1} B_p + 2a_{2p+1} \sum_{k=1}^{n} \frac{1}{2k} + \sum_{j=1}^{2p} a_j A_{j,n} - \sum_{j=1}^{2p} a_j B_j + 2a_{2p+1} D_n - 2a_{2p+1} E_p \\ &= -2a_{2p+1} \left(\sum_{k=0}^{n} \frac{1}{2k+1} - \sum_{k=1}^{n} \frac{1}{2k} \right) - 2a_{2p+1} E_p - \sum_{j=1}^{2p} a_j B_j - 2a_{2p+1} B_p + \sum_{j=1}^{2p} a_j A_{j,n} + 2a_{2p+1} D_n \\ &= -2a_{2p+1} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} - 2a_{2p+1} E_p - \sum_{j=1}^{2p} a_j B_j - 2a_{2p+1} B_p + \sum_{j=1}^{2p} a_j A_{j,n} + 2a_{2p+1} D_n. \end{aligned} \tag{3.56}$$

△ Since

$$\begin{aligned} \lim_{n\to+\infty}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k} &= \ln(2) \\ \lim_{n\to+\infty} A_{j,n} &= 0, \ \forall\, j=1,\ldots,2p \\ \lim_{n\to+\infty} D_n &= 0, \end{aligned} \tag{3.57}$$

we deduce that S_n converges, and that its limit verifies

$$\lim_{n\to+\infty} S_n = -2a_{2p+1}\ln(2) - 2a_{2p+1}(E_p + B_p) - \sum_{j=1}^{2p} a_j B_j. \tag{3.58}$$

■ The following Lemma 3.9 gives the sum of the series $\sum u_k$.

Lemma 3.9. For all $p \in \mathbb{N}^*$, we have

$$\sum_{k=0}^{+\infty} \frac{(\frac{1}{2}-p)_k}{(\frac{1}{2}+p)_k(k+2p+1)} = -K_p\left(2a_{2p+1}\ln(2) + 2a_{2p+1}(E_p+B_p) + \sum_{j=1}^{2p} a_j B_j\right), \tag{3.59}$$

where K_p, a_{2p+1}, E_p, B_j, and a_j are defined in (3.5).

Proof of Lemma 3.9. By grouping relations (3.29), (3.47), and (3.58), we obtain

$$\begin{aligned} \sum_{k=0}^{+\infty} \frac{(\frac{1}{2}-p)_k}{(\frac{1}{2}+p)_k(k+2p+1)} &= K_p \lim_{n\to+\infty} S_n \\ &= -K_p\left(2a_{2p+1}\ln(2) + 2a_{2p+1}(E_p+B_p) + \sum_{j=1}^{2p} a_j B_j\right). \end{aligned} \tag{3.60}$$

The proof is complete. □

■ The following remark gives the validation of formula (3.59) when $p = 1$.

Remark 3.3. For $p = 1$, we will check the concordance of the limit given by relation (3.20) with that presented in relation (3.58). When $p = 1$, we easily obtain $E_1 = \frac{3}{4}$, $B_1 = -1$, and $B_2 = 0$. Moreover, the values of $(a_i)_{0\le i\le 3}$ are given in (3.42). Thus, relation (3.58) gives

$$\begin{aligned} \lim_{n\to+\infty} S_n &= -2(-\frac{1}{105})\ln(2) - 2(-\frac{1}{105})(\frac{3}{4}-1) - [-\frac{1}{10}(-1) + \frac{1}{12}(0)] \\ &= \frac{2}{105}\ln(2) + \frac{1}{105}(\frac{3}{2}-2) - \frac{1}{10} \\ &= \frac{2}{105}\ln(2) - \frac{1}{210} - \frac{21}{210} \\ &= \frac{2}{105}\ln(2) - \frac{22}{210} \\ &= \frac{2}{105}\ln(2) - \frac{11}{105}. \end{aligned} \tag{3.61}$$

Thus, we find the same limit as that found in relation (3.20).

■ With the intermediate results from Section 3.3 now found, we can prove Theorem 3.3 stated at the beginning of this section.

Proof of Theorem 3.3. Based on relation (2.26) and taking $\alpha = \frac{2p+1}{2}$, we obtain

$$\begin{aligned}\sum_{k=0}^{+\infty} \frac{(\frac{1}{2}-p)_k}{(\frac{1}{2}+p)_k(k+2p+1)} &= \frac{{}_3F_2(1-\alpha, 1, 2\alpha; \alpha+1, 2\alpha+1; 1)}{2\alpha} \\ &= \frac{{}_3F_2(\frac{1}{2}-p, 1, 1+2p; \frac{3}{2}+p, 2p+2; 1)}{2p+1}.\end{aligned} \tag{3.62}$$

By identifying the right-hand sides of relations (3.59) and (3.62), we obtain

$$-K_p\left(2a_{2p+1}\ln(2) + 2a_{2p+1}(E_p+B_p) + \sum_{j=1}^{2p} a_jB_j\right) = \frac{{}_3F_2(\frac{1}{2}-p, 1, 1+2p; \frac{3}{2}+p, 2p+2; 1)}{2p+1},$$

that is,

$$-(2p+1)K_p\left(2a_{2p+1}\ln(2) + 2a_{2p+1}(E_p+B_p) + \sum_{j=1}^{2p} a_jB_j\right) = {}_3F_2(\tfrac{1}{2}-p, 1, 1+2p; \tfrac{3}{2}+p, 2p+2; 1).$$

The proof is complete. □

■ We now state and prove the following two results of Theorem 3.3 corresponding to the special cases $p = 2$ and $p = 3$.

△ For $p = 2$, Theorem 3.3 yields the first new special result.

Corollary 3.1.

$${}_3F_2(-\frac{3}{2}, 1, 5; \frac{7}{2}, 6; 1) = \frac{4045}{6006} + \frac{10}{1001}\ln(2). \tag{3.63}$$

Proof of corollary 3.1. Let $p = 2$. Then, from Theorem 3.3, we easily compute the following quantities:

$$\begin{aligned}&K_2 = 45,\ E_2 = \frac{25}{24},\ B_1 = -\frac{1}{3},\ B_2 = -\frac{4}{3},\ B_3 = -\frac{1}{3},\ B_4 = 0,\ a_1 = -\frac{1}{528},\\ &a_2 = \frac{1}{288},\ a_3 = -\frac{1}{336},\ a_4 = \frac{1}{960},\ a_5 = -\frac{1}{45045},\ \sum_{j=1}^{4} a_jB_j = -\frac{25}{8316}.\end{aligned}$$

Thus, to derive relation (3.63), we simply substitute the values above into Theorem 3.3. The proof is complete. □

△ Also, for $p = 3$, Theorem 3.3 yields the second new special result.

Corollary 3.2.

$${}_3F_2(-\frac{5}{2}, 1, 7; \frac{9}{2}, 8; 1) = \frac{221158}{415701} - \frac{70}{138567}\ln(2). \tag{3.64}$$

Proof of Corollary 3.2. The proof follows similar lines of argument to that of Corollary 3.1. Let $p = 3$. Then, from Theorem 3.3, we easily compute the following quantities:

$$B_1 = -\frac{1}{5},\ B_2 = -\frac{8}{15},\ B_3 = -\frac{23}{15},\ B_4 = -\frac{8}{15},\ B_5 = -\frac{3}{15},\ B_6 = 0,$$
$$a_1 = -\frac{1}{65280},\ a_2 = \frac{1}{23040},\ a_3 = -\frac{1}{14976},\ a_4 = \frac{1}{16896},\ a_5 = -\frac{1}{34560}, \tag{3.65}$$
$$a_6 = \frac{1}{161280},\ a_7 = -\frac{1}{43648605},\ K_3 = -1575,\ E_3 = \frac{49}{40},\ \sum_{j=1}^{6} a_j B_j = \frac{371}{6563700}.$$

Thus, to derive relation (3.64), we simply substitute the values above into Theorem 3.3. The proof is complete. □

4. Explicit evaluation of certain classes of fractional integrals

This section is devoted to giving our new evaluation of a certain class of fractional integrals whose values are written in terms of hypergeometric functions ${}_3F_2$ which we obtained in the previous sections.

Theorem 4.1. For all $\tau \le s \le t \le \eta$, and $\alpha > 1/2$, the following integral representations for the Gauss hypergeometric function hold true.

(1)

$$I_1^\alpha(t) := \int_\tau^t (t-s)^{\alpha-1}(\eta - s)^{\alpha-1}\, ds = \left[\frac{(\eta-\tau)^{\alpha-1}(t-\tau)^\alpha}{\alpha}\right] {}_2F_1(1-\alpha, 1; \alpha+1; g(t)), \tag{4.1}$$

where $g(t) := \dfrac{t-\tau}{\eta-\tau}$.

(2)

$$\int_\tau^\eta I_1^\alpha(t)dt = \left[\frac{(\eta-\tau)^{2\alpha}}{\alpha(\alpha+1)}\right] {}_3F_2(1-\alpha, 1, \alpha+1; \alpha+1, \alpha+2; 1) \tag{4.2}$$
$$= \frac{(\eta-\tau)^{2\alpha}}{2\alpha^2}. \tag{4.3}$$

Proof of Theorem 4.1.

(1) Let $s = \tau + x(t-\tau)$. By changing the integration variable from s to x, the integral $I_1^\alpha(t)$ becomes

$$\begin{aligned} I_1^\alpha(t) &= (t-\tau)\int_0^1 ((t-\tau) - x(t-\tau))^{\alpha-1}\, ((\eta-\tau) - x(t-\tau))^{\alpha-1}\, dx \\ &= (t-\tau)(t-\tau)^{\alpha-1}\int_0^1 (1-x)^{\alpha-1}\left((\eta-\tau) - \frac{(\eta-\tau)(t-\tau)}{(\eta-\tau)}x\right)^{\alpha-1} dx \\ &= (\eta-\tau)^{\alpha-1}(t-\tau)^\alpha \int_0^1 (1-x)^{\alpha-1}\left(1 - \frac{(t-\tau)}{(\eta-\tau)}x\right)^{\alpha-1} dx. \end{aligned}$$

Above, we have the Euler integral representation of ${}_2F_1(a, b; c; z)$ with $a = 1-\alpha$, $b = 1$, $c = \alpha+1$, and $z = g(t) = \frac{t-\tau}{\eta-\tau}$. Thus,

$$I_1^\alpha(t) = \frac{(\eta-\tau)^{\alpha-1}(t-\tau)^\alpha}{\alpha} {}_2F_1(1-\alpha, 1; \alpha+1; g(t)).$$

This completes the proof of (4.1).

(2) Now we are in a position to evaluate the integral (4.2), so denoting the left-hand side of (4.2) by λ_1, we have

$$\lambda_1 := \frac{(\eta-\tau)^{\alpha-1}}{\alpha}\int_\tau^\eta (t-\tau)^\alpha {}_2F_1(1-\alpha,1;\alpha+1;g(t))dt. \tag{4.4}$$

Now, expressing ${}_2F_1$ as a series and changing the order of integration and summation, which is justified by Lemma 2.2, we have

$$\begin{aligned}\lambda_1 &:= \left[\frac{(\eta-\tau)^{\alpha-1}}{\alpha}\right]\sum_{k=0}^{\infty}\frac{(1-\alpha)_k}{(\alpha+1)_k}\int_\tau^\eta (t-\tau)^\alpha\left(\frac{t-\tau}{\eta-\tau}\right)^k dt\\ &= \left[\frac{(\eta-\tau)^{\alpha-1}}{\alpha}\right]\sum_{k=0}^{\infty}\frac{(1-\alpha)_k}{(\alpha+1)_k}\left(\frac{1}{\eta-\tau}\right)^k\int_\tau^\eta (t-\tau)^{k+\alpha}\, dt\\ &= \left[\frac{(\eta-\tau)^{2\alpha}}{\alpha}\right]\sum_{k=0}^{\infty}\frac{(1-\alpha)_k}{(\alpha+1)_k}\frac{1}{(k+\alpha+1)}.\end{aligned}$$

Now, by Lemma 3.1, we obtain (4.2), that is

$$\lambda_1 := \left[\frac{(\eta-\tau)^{2\alpha}}{\alpha(\alpha+1)}\right]{}_3F_2(1-\alpha,1,\alpha+1;\alpha+1,\alpha+2;1),$$

and Theorem 3.1 gives (4.3). This completes the proof. □

Theorem 4.2. For all $\tau\le s\le t\le\eta$ and $\alpha>1/2$, the following integral representations for the Gauss hypergeometric function hold true.
(1)

$$\begin{aligned}I_2^\alpha(t) &:= \int_\tau^t (t-\tau)(t-s)^{\alpha-1}(\eta-s)^{\alpha-1}\, ds\\ &= \left[\frac{(\eta-\tau)^{\alpha-1}(t-\tau)^{\alpha+1}}{\alpha}\right]{}_2F_1(1-\alpha,1;\alpha+1;g(t)),\end{aligned} \tag{4.5}$$

where $g(t) := \frac{t-\tau}{\eta-\tau}$.
(2)

$$\int_\tau^\eta I_2^\alpha(t)dt = \left[\frac{(\eta-\tau)^{2\alpha+1}}{\alpha(\alpha+2)}\right]{}_3F_2(1-\alpha,1,\alpha+2;\alpha+1,\alpha+3;1) \tag{4.6}$$

$$= \frac{(\eta-\tau)^{2\alpha+1}}{\alpha}\left[\frac{1}{\alpha+1}+\frac{1}{2\alpha}-\frac{2}{2\alpha+1}\right]. \tag{4.7}$$

Proof of Theorem 4.2.
(1) The proof is similar to the proof of Theorem 4.1, and so we just sketch the basic idea. In exactly the same manner, the integral $I_2^\alpha(t)$ can be obtained.

(2) Now denoting the left-hand side of (4.6) by λ_2, we have

$$\lambda_2 := \frac{(\eta-\tau)^{\alpha-1}}{\alpha}\int_\tau^\eta (t-\tau)^{\alpha+1}{}_2F_1(1-\alpha,1;\alpha+1;g(t))dt. \tag{4.8}$$

Now, expressing ${}_2F_1$ as a series and changing the order of integration and summation, which is justified by Lemma 2.2, we have

$$\begin{aligned}\lambda_2 &:= \left[\frac{(\eta-\tau)^{\alpha-1}}{\alpha}\right]\sum_{k=0}^{\infty}\frac{(1-\alpha)_k}{(\alpha+1)_k}\int_\tau^\eta (t-\tau)^{\alpha+1}\left(\frac{t-\tau}{\eta-\tau}\right)^k dt\\ &= \left[\frac{(\eta-\tau)^{\alpha-1}}{\alpha}\right]\sum_{k=0}^{\infty}\frac{(1-\alpha)_k}{(\alpha+1)_k}\left(\frac{1}{\eta-\tau}\right)^k\int_\tau^\eta (t-\tau)^{k+\alpha+1}\,dt\\ &= \left[\frac{(\eta-\tau)^{2\alpha+1}}{\alpha}\right]\sum_{k=0}^{\infty}\frac{(1-\alpha)_k}{(\alpha+1)_k}\frac{1}{(k+\alpha+2)}.\end{aligned}$$

Now, by Lemma 3.2, we obtain (4.6), that is

$$\lambda_2 := \left[\frac{(\eta-\tau)^{2\alpha+1}}{\alpha(\alpha+2)}\right]{}_3F_2(1-\alpha,1,\alpha+2;\alpha+1,\alpha+3;1),$$

and Theorem 3.2 gives (4.7). This completes the proof. □

Theorem 4.3. For all $\tau\le s\le t\le\eta$, and $\alpha>1/2$, the following integral representations for the Gauss hypergeometric function hold true.
(1)

$$\begin{aligned}I_3^\alpha(t) &:= \int_\tau^t (t-\tau)^{\alpha-1}(t-s)^{\alpha-1}(\eta-s)^{\alpha-1}\,ds\\ &= \left[\frac{(\eta-\tau)^{\alpha-1}(t-\tau)^{2\alpha-1}}{\alpha}\right]{}_2F_1(1-\alpha,1,\alpha+1;g(t)),\end{aligned} \tag{4.9}$$

where $g(t) := \frac{t-\tau}{\eta-\tau}$.
(2)

$$\int_\tau^\eta I_3^\alpha(t)dt = \left[\frac{(\eta-\tau)^{3\alpha-1}}{2\alpha^2}\right]{}_3F_2(1-\alpha,1,2\alpha;\alpha+1,2\alpha+1;1). \tag{4.10}$$

Proof of Theorem 4.3.
(1) The proof is similar to the proof of Theorem 4.1, and so we just sketch the basic idea. In exactly the same manner, the integral $I_3^\alpha(t)$ can be obtained.
(2) Now, denoting the left-hand side of (4.10) by λ_3, we have

$$\lambda_3 := \frac{(\eta-\tau)^{\alpha-1}}{\alpha}\int_\tau^\eta (t-\tau)^{2\alpha-1}{}_2F_1(1-\alpha,1;\alpha+1;g(t))dt. \tag{4.11}$$

By expressing ${}_2F_1$ as a series and change the order of integration and summation, which is justified by justified by Lemma 2.2, we have

$$\lambda_3 = \left[\frac{(\eta-\tau)^{\alpha-1}}{\alpha}\right]\sum_{k=0}^{\infty}\frac{(1-\alpha)_k}{(\alpha+1)_k}\left(\frac{1}{\eta-\tau}\right)^k\int_\tau^\eta (t-\tau)^{k+2\alpha-1}\,dt$$

$$= \left[\frac{(\eta-\tau)^{3\alpha-1}}{\alpha}\right]\sum_{k=0}^{\infty}\frac{(1-\alpha)_k}{(\alpha+1)_k}\frac{1}{(k+2\alpha)}. \tag{4.12}$$

Thus, the result of Lemma 2.5 gives (4.10). This completes the proof. □

Remark 4.1. Note that when $\alpha = 2$, the results of Theorems 4.2 and 4.3 coincide. Therefore, the right-hand terms of relations (4.6) and (4.10) are equal when $\alpha = 2$, which can be justified by Lemma 2.1. Thus,

$$\int_\tau^\eta I_2^2(t)dt = \int_\tau^\eta I_3^2(t)dt = \frac{11(\eta-\tau)^5}{120}. \tag{4.13}$$

■ For the choices $\alpha = \frac{2p+1}{2}$ and $p \in \mathbb{N}^*$, we have the following new explicit evaluation of a certain class of integrals as a special case from our Theorem 4.3.

Theorem 4.4. For all $\tau \le s \le t \le \eta$ and $p \in \mathbb{N}^*$, the following integral holds true.
(1)

$$\int_\tau^\eta I_3^{\frac{2p+1}{2}}(t)dt = \left[\frac{2(\eta-\tau)^{\frac{6p+1}{2}}}{(2p+1)^2}\right] {}_3F_2(\tfrac{1}{2}-p, 1, 1+2p; \tfrac{3}{2}+p, 2p+2; 1) \tag{4.14}$$

$$= -K_p\left[\frac{2(\eta-\tau)^{\frac{6p+1}{2}}}{(2p+1)}\right]\left(2a_{2p+1}\ln(2) + 2a_{2p+1}(E_p+B_p) + \sum_{j=1}^{2p} a_j B_j\right), \tag{4.15}$$

where $I_3^{\frac{2p+1}{2}}$ is defined in (4.9) with $\alpha = \frac{2p+1}{2}$, and K_p, a_{2p+1}, E_p, B_j, and a_j are defined in (3.5).

Proof of Theorem 4.4. By letting $\alpha = \frac{2p+1}{2}$ and $p \in \mathbb{N}^*$ in the integral (4.10), we obtain (4.14) and (4.15) is obtained by replacing the explicit evaluation of the fucntion ${}_3F_2(\frac{1}{2}-p, 1, 1+2p; \frac{3}{2}+p, 2p+2; 1)$ given by Theorem 3.3. The proof is complete. □

■ The following result presents the integrals of $I_3^\alpha(t)$ for some α when $\alpha = 3/2$, $\alpha = 5/2$, and $\alpha = 7/2$.

Corollary 4.1. *For all $\tau \le s \le t \le \eta$, the following integrals hold true.*
(1)

$$\int_\tau^\eta I_3^{3/2}(t)dt = \left[\frac{2(\eta-\tau)^{\frac{7}{2}}}{3^2}\right]\left(\frac{33}{35} - \frac{6}{35}\ln(2)\right), \tag{4.16}$$

where $I_3^{3/2}$ is defined in (4.9) *with $\alpha = 3/2$.*
(2)

$$\int_\tau^\eta I_3^{5/2}(t)dt = \left[\frac{2(\eta-\tau)^{\frac{13}{2}}}{5^2}\right]\left(\frac{4045}{6006} + \frac{10}{1001}\ln(2)\right), \tag{4.17}$$

where $I_3^{5/2}$ is defined in (4.9) *with $\alpha = 5/2$.*
(3)

$$\int_\tau^\eta I_3^{7/2}(t)dt = \left[\frac{2(\eta-\tau)^{\frac{19}{2}}}{7^2}\right]\left(\frac{221158}{415701} - \frac{70}{138567}\ln(2)\right), \tag{4.18}$$

where $I_3^{7/2}$ is defined in (4.9) *with $\alpha = 7/2$.*

Proof of corollary 4.1.
(1) Let $p = 1$. Then, from Theorem 4.4 (4.14), we obtain

$$\int_{\tau}^{\eta} I_3^{3/2}(t)dt = \left[\frac{2(\eta-\tau)^{\frac{7}{2}}}{3^2}\right] {}_3F_2(-\frac{1}{2}, 1, 3; \frac{5}{2}, 4; 1). \tag{4.19}$$

From Lemma 3.3, we have

$${}_3F_2(-\frac{1}{2}, 1, 3; \frac{5}{2}, 4; 1) = \frac{33}{35} - \frac{6}{35}\ln(2), \tag{4.20}$$

and so substituting (4.20) into (4.19), we obtain (4.31). This completes the proof of (4.31).
(2) Similarly, let $p = 2$. Then, from Theorem 4.4 (4.14), we obtain

$$\int_{\tau}^{\eta} I_3^{5/2}(t)dt = \left[\frac{2(\eta-\tau)^{\frac{13}{2}}}{5^2}\right] {}_3F_2(-\frac{3}{2}, 1, 5; \frac{7}{2}, 6; 1). \tag{4.21}$$

From Corollary 3.1, we have

$${}_3F_2(-\frac{3}{2}, 1, 5; \frac{7}{2}, 6; 1) = \frac{4045}{6006} + \frac{10}{1001}\ln(2), \tag{4.22}$$

and so substituting (4.22) into (4.21), we obtain (4.32). This completes the proof of (4.32).
(3) Also, if we let $p = 2$, then from Theorem 4.4 (4.14), we obtain

$$\int_{\tau}^{\eta} I_3^{7/2}(t)dt = \left[\frac{2(\eta-\tau)^{\frac{19}{2}}}{7^2}\right] {}_3F_2(-\frac{5}{2}, 1, 7; \frac{9}{2}, 8; 1). \tag{4.23}$$

From Corollary 3.2, we have

$${}_3F_2(-\frac{5}{2}, 1, 7; \frac{9}{2}, 8; 1) = \frac{221158}{415701} - \frac{70}{138567}\ln(2), \tag{4.24}$$

and so substituting (4.24) into (4.23), we obtain (4.33). This completes the proof. □

Theorem 4.5. For all $\tau \le s \le t \le \eta$ and $\alpha > 1$, the following integral representations for the Gauss hypergeometric functions hold true.
(1)

$$\begin{aligned} I_4^{\alpha}(t) &:= \int_{\tau}^{t} (t-\tau)^{\alpha-1}(t-s)^{\alpha-1}(\eta-s)^{\alpha-2}\,ds \\ &= \left[\frac{(\eta-\tau)^{\alpha-2}(t-\tau)^{2\alpha-1}}{\alpha}\right] {}_2F_1(2-\alpha, 1; \alpha+1; g(t)), \end{aligned} \tag{4.25}$$

where $g(t) := \frac{t-\tau}{\eta-\tau}$.
(2)

$$\int_{\tau}^{\eta} I_4^{\alpha}(t)dt = \frac{(\eta-\tau)^{3\alpha-2}}{(1-\alpha)(2\alpha-1)} + \left[\frac{(1-3\alpha)(\eta-\tau)^{3\alpha-2}}{2\alpha^2(1-\alpha)}\right] {}_3F_2(1-\alpha, 1, 2\alpha; \alpha+1, 2\alpha+1; 1). \tag{4.26}$$

Proof of Theorem 4.5.
(1) The proof follows similar lines of argument to that of the above theorems and so we just sketch the basic idea. In exactly the same manner, we have

$$I_4^{\alpha}(t) = (\eta-\tau)^{\alpha-2}(t-\tau)^{2\alpha-1}\int_0^1 (1-x)^{\alpha-1}\left(1-\frac{(t-\tau)}{(\eta-\tau)}x\right)^{\alpha-2} dx.$$

Above, we have the Euler integral representation of ${}_2F_1(a,b;c;z)$ with $a = 2-\alpha$, $b = 1$, $c = \alpha+1$, and $z = g(t) = \frac{t-\tau}{\eta-\tau}$, thus

$$I_4^{\alpha}(t) = \frac{(\eta-\tau)^{\alpha-2}(t-\tau)^{2\alpha-1}}{\alpha}{}_2F_1(2-\alpha,1;\alpha+1;g(t)). \tag{4.27}$$

This completes the proof of (4.25).
(2) Now denoting the left-hand side of (4.26) by λ_4, we have

$$\lambda_4 := \frac{(\eta-\tau)^{\alpha-2}}{\alpha}\int_\tau^\eta (t-\tau)^{2\alpha-1}{}_2F_1(2-\alpha,1;\alpha+1;g(t))dt, \tag{4.28}$$

and by expressing ${}_2F_1$ as a series and change the order of integration and summation, which is justified by justified by Lemma 2.2, we have

$$\begin{aligned}\lambda_4 &= \left[\frac{(\eta-\tau)^{\alpha-2}}{\alpha}\right]\sum_{k=0}^{\infty}\frac{(2-\alpha)_k}{(\alpha+1)_k}\left(\frac{1}{\eta-\tau}\right)^k\int_\tau^\eta (t-\tau)^{k+2\alpha-1}\,dt\\ &= \left[\frac{(\eta-\tau)^{3\alpha-2}}{\alpha}\right]\sum_{k=0}^{\infty}\frac{(2-\alpha)_k}{(\alpha+1)_k}\frac{1}{(k+2\alpha)}.\end{aligned} \tag{4.29}$$

Thus, the result of Lemma 2.1 gives (4.26). This completes the proof. □

■ For the choices $\alpha = \frac{2p+1}{2}$ and $p \in \mathbb{N}^*$, we have also the following new explicit evaluation of a certain class of Integrals as special case from our Theorem 4.5.

Theorem 4.6. For all $\tau \le s \le t \le \eta$ and $p \in \mathbb{N}^*$, the following integral holds true.

$$\begin{aligned}\int_\tau^\eta I_4^{\frac{2p+1}{2}}(t)dt &= \frac{(\eta-\tau)^{\frac{6p-1}{2}}}{p(1-2p)} + \left[\frac{(6p+1)(\eta-\tau)^{\frac{6p-1}{2}}}{(2p-1)(2p+1)^2}\right]{}_3F_2(\tfrac{1}{2}-p,1,1+2p;\tfrac{3}{2}+p,2p+2;1)\\ &= \frac{(\eta-\tau)^{\frac{6p-1}{2}}}{p(1-2p)} - \left[\frac{K_p(6p+1)(\eta-\tau)^{\frac{6p-1}{2}}}{(2p-1)(2p+1)}\right]\Big(2a_{2p+1}\ln(2) + 2a_{2p+1}(E_p+B_p) + \textstyle\sum_{j=1}^{2p} a_jB_j\Big),\end{aligned} \tag{4.30}$$

where $I_4^{\frac{2p+1}{2}}$ is defined in (4.25) with $\alpha = \frac{2p+1}{2}$, and K_p, a_{2p+1}, E_p, B_j, and a_j are defined in (3.5)

Proof of Theorem 4.6. The proof is similar to the proof of Theorem 4.4, so we omit the proof for brevity. □

■ The following result presents the values of I_4^{α} for some α that are when $\alpha = 3/2$, $\alpha = 5/2$, and $\alpha = 7/2$.

Corollary 4.2. *For all $\tau \le s \le t \le \eta$, the following integrals hold true.*
(1)

$$\int_{\tau}^{\eta} I_4^{3/2}(t)dt = -(\eta-\tau)^{\frac{5}{2}} + \frac{7(\eta-\tau)^{\frac{5}{2}}}{9}\left(\frac{33}{35} - \frac{6}{35}\ln(2)\right), \tag{4.31}$$

where $I_3^{3/2}$ is defined in (4.25) *with $\alpha = 3/2$.*
(2)

$$\int_{\tau}^{\eta} I_4^{5/2}(t)dt = -\frac{(\eta-\tau)^{\frac{11}{2}}}{6} + \frac{12(\eta-\tau)^{\frac{11}{2}}}{75}\left(\frac{4045}{6006} + \frac{10}{1001}\ln(2)\right), \tag{4.32}$$

where $I_3^{5/2}$ is defined in (4.25) *with $\alpha = 5/2$.*
(3)

$$\int_{\tau}^{\eta} I_4^{7/2}(t)dt = -\frac{(\eta-\tau)^{\frac{17}{2}}}{15} + \frac{19(\eta-\tau)^{\frac{17}{2}}}{245}\left(\frac{221158}{415701} - \frac{70}{138567}\ln(2)\right), \tag{4.33}$$

where $I_3^{7/2}$ is defined in (4.25) *with $\alpha = 7/2$.*

Proof of Corollary 4.2. The proof is similar to the proof of Theorem 4.1, so we omit the proof for brevity. □

5. Conclusions and future work

In this study, we expressed four families of fractional integrals, denoted as $\mathcal{F}_1^{\alpha} = \{I_1^{\alpha}(\alpha), I_2^{\alpha}(\alpha)\ ;\ \alpha > \frac{1}{2}\}$, $\mathcal{F}_2^{\alpha} = \{I_3^{\alpha}(\alpha),\ ;\ \alpha > \frac{1}{2}\}$ and $\mathcal{F}_3^{\alpha} = \{I_4^{\alpha}(\alpha)\ ;\ \alpha > 1\}$, using the class of hypergeometric functions ${}_3F_2(1)$. The series representation of the hypergeometric functions allowed us to derive explicit forms for the integrals of the family $\mathcal{F}_1^{\alpha}$ for all $\alpha > \frac{1}{2}$, as well as for the integrals of the subfamily $\mathcal{F}_2^{\frac{2p+1}{2}}$ and $\mathcal{F}_3^{\frac{2p+1}{2}}$ for all $p \in \mathbb{N}^*$.

For the integrals of the family $\mathcal{F}_2^{\alpha}$ and $\mathcal{F}_3^{\alpha}$, we have only calculated the explicit forms of those of the subfamily $\mathcal{F}_2^{\frac{2p+1}{2}}$ and $\mathcal{F}_3^{\frac{2p+1}{2}}$. However, by examining other subfamilies of $\mathcal{F}_2^{\alpha}$ and $\mathcal{F}_3^{\alpha}$, we could derive more interesting formulas relating fractional integrals to the family of functions ${}_3F_2(1)$.

Author contributions

Saleh S. Almuthaybiri: Conceptualization, supervision, validation, investigation, writing original draft preparation, formal analysis, writing review and editing. Abdelhamid Zaidi: Investigation, validation, writing original draft preparation, methodology, formal analysis, writing review and editing, doing the revision, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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