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Research article

On the numerical solution of highly oscillatory Fredholm integral equations using a generalized quadrature method

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Abstract: In this paper, a numerical method is presented for solving Fredholm integral equations with highly oscillatory kernels. The proposed method combined piecewise collocation with a generalized quadrature rule in a uniform mesh. Due to the oscillatory nature of the kernels of integral equation, the discretized collocation equations required the evaluation of oscillatory integrals, which were computed using an efficient generalized quadrature rule. Convergence was analyzed in terms of both asymptotic and classical accuracy. The method's practical performance and reliability were showcased with two numerical examples.

Keywords: highly oscillatory; integral equation; asymptotic order; convergence
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1. Introduction

Numerous scientific and engineering problems, including those found in electric circuits and chemical kinetics, can be mathematically expressed as differential equations. These equations can subsequently be converted into equivalent integral equations of Volterra and Fredholm types. Additionally, within various fields such as physics, biology, and engineering, a significant number of phenomena–such as the propagation of stocked fish in newly established lakes, Volterra's model of population growth, the coexistence of biological species, as well as heat transfer and radiation–can also be characterized by integral or integrodifferential equations [1, 35]. There has been substantial interest in the numerical approach to solving integral equations, primarily because analytical solutions are often unavailable. A variety of effective methods have been employed to compute numerical solutions for these equations including the spectral method [11, 29], the collocation method [4], the iterated collocation method [4], the Runge-Kutta type method [6], the Galerkin type method [16, 46], and the multistep collocation method [8, 10, 13]. For a more comprehensive understanding of these techniques, readers are encouraged to refer to the monographs [1, 2, 35].

On the other hand, the mathematical modeling of oscillatory phenomena in various fields such as electrodynamics, quantum chemistry, nonlinear optics, fluid mechanics, plasma transport, computerized tomography, celestial mechanics, and the computation of Schrödinger spectra and Bose-Einstein condensates results in integral equations that encompass highly oscillatory integrals, commonly referred to as highly oscillatory integral equations (HOIEs). Preliminary investigations conducted by numerous researchers have indicated that classical methods to HOIEs are not effective, as the use of classical quadrature rules for calculating highly oscillatory integrals (HOIs) leads to a significant increase in computational costs as the frequency increases. Contrary to widespread belief, the efficient computation of HOIs is indeed feasible and notably, the accuracy of the approximation improves with increased oscillation of the integral. Researches have illustrated that HOI operators can be precisely approximated through various methods including the Filon-type method [20, 21], Levin-type method [15, 26, 27], steepest descent method [12, 17], exponential fitting (EF) quadrature rule [18, 19, 31] and Gaussian integration rule [32].

More recently, a stable and accurate algorithm based on reproducing kernel functions was introduced for the numerical evaluation of Fourier-type HOIs [45]. Additionally, a generalized bivariate Filon-Clenshaw-Curtis method was proposed for double HOIs on the square [14]. More recent methods on this topic can be found in [38, 41], among others. However, these methods each have implementation limitations. For instance, the Filon method's moments are themselves oscillatory integrals, and their explicit values are known only for specific, simple forms of the oscillator. Furthermore, complex-valued Gaussian quadrature rules require the amplitude function f to be analytic in an infinitely large region of the complex plane encompassing the integration interval. Transitioning from HOIs to HOIEs, integral equations with oscillatory integrands can be approximated using the quadrature rules mentioned.

In recent decades, several papers have explored the existence, uniqueness, and numerical solutions of HOIEs. For highly oscillatory Volterra integral equations (HOVIEs), the research of Brunner et al. [3, 7] demonstrated both the existence and uniqueness of solutions for Volterra integral and integrodifferential equations with kernels that exhibit high oscillatory behavior. In a separate study, Wang and Xiang [36] introduced a Filon-type method designed for addressing a Volterra integral equation of the first kind that features a highly oscillatory Bessel kernel. Additionally, Xiang and Brunner [40] proposed a Clenshaw-Curtis-Filon-type method aimed at solving Volterra integral equations involving oscillatory Bessel kernels. Moreover, the study presented in [25] concentrated on obtaining numerical solutions for a specific category of HOVIEs, employing collocation methods that are based on the EF technique. Furthermore, Conte et al. [9] introduced effective collocation techniques that employ the Filon-type method to address nonlinear Volterra integral equations with an oscillatory kernel. Other new numerical methods can be found in [43, 47]. For an elaborate discussion on the structure and numerical solutions of HOVIEs, please see Brunner's monograph [2], which provides an in-depth description of HOVIEs along with their numerical solutions.

In addition to HOVIEs, there are some papers which have studied the numerical solution of the highly oscillatory Fredholm integral equations (HOFIEs). The general form of HOFIEs is given by

$$u(t) = f(t) + \int_a^b K_\omega(t,\tau)u(\tau)\mathrm{d}\tau,\ t \in I := [a,b], \tag{1.1}$$

where $K_\omega(t,\tau)$ is the oscillatory kernel function. The computation of HOFIEs is addressed in the literature, specifically in references [5, 30, 34]. The authors focused on analyzing the spectra problem associated with oscillatory Fredholm integral operators. The asymptotic properties of Fredholm integral equations (FIEs) with an oscillatory kernel $k(t,\tau)e^{i\omega|t-\tau|}$ were examined in [34]. In [24], a collocation method using Clenshaw-Curtis points was employed to solve FIEs with oscillatory kernels, where the oscillatory integral component was computed with the efficient Filon method. The authors of [37] developed oscillatory function spaces to address the oscillatory components of FIEs with such kernels and applied the Galerkin method to achieve optimal convergence rates and stability. Moreover, Li et al. [30] introduced an enhanced Levin approach to effectively tackle Fredholm oscillatory integral equations. Some more recent methods to approximate HOFIEs can be found in [22, 23].

In this study, we consider a quadrature method for solving HOFIEs of the form

$$u(t) = f(t) + \int_a^b k(t,\tau)e^{i\omega(g(t)-g(\tau))}u(\tau)d\tau, \quad t \in I := [a,b], \tag{1.2}$$

where the term $\omega \gg 1$ indicates the oscillation parameter, and $u(t)$ refers to the unknown function that is to be determined. The functions $k(t,\tau)$, $f(t)$, and $g(t)$ are considered to be sufficiently smooth within the specified domains $D := \{(t,\tau) : \ a \le t, \tau \le b\}$ and I, respectively. Under these assumptions, the second kind FIE (1.2) has a unique solution [24]. In this paper, we are concerned with the HOFIEs in which $g'(t) \neq 0$ for $t \in I$.

Following the strategy recommended for tackling HOVIEs, as mentioned in [9], we utilize the traditional collocation method using predetermined collocation points for (1.2). Following this, to effectively discretize the integrals derived from the collocation equation, we use a two-point quadrature formula, commonly referred to as the generalized quadrature (GQ) method, as introduced in [44]. We further examine the error associated with both exact and discrete collocation methods by employing various auxiliary lemmas and theorems. Then, we confirm theoretical results by numerical examples. An important remark concerning the error analysis of the proposed methods for HOIEs is their dependence on the frequency, step size, or both factors. Some researchers analyzed the connection between the error estimates and the frequency, i.e., the influence of frequency ω on error [39, 42]. Furthermore, some studies have investigated the relationship between error and step size, focusing on how step size h influences error [25, 33]. Nevertheless, as far as we know, only a few studies have addressed the dependence of error on both ω and h [9, 44]. In this paper, our error analysis illustrates the combined impact of step size h and frequency ω on the error.

The rest of this paper is organized as follows. In Section 2, the collocation technique is utilized to tackle HOFIE (1.2) through the application of the GQ rule. Section 3 is dedicated to the convergence analysis. Numerical examples illustrating the performance are discussed in Section 4. Finally, Section 5 offers the concluding remarks of the paper.

2. Description of the proposed method

In this section, we focus on analyzing piecewise polynomial collocation methods for addressing the HOFIE specified in Eq (1.2). Since any finite interval I in Eq (1.2) can be converted to $[0,T]$ using a linear transformation, we will treat the interval of integration as $I = [0,T]$.

2.1. The collocation method

Considering the key prerequisites for the m-points collocation methods, we define the uniform distribution I_h over the interval $I = [0, T]$ in the following manner:

$$I_h := \Big\{t_n := nh,\ n = 0, 1, \ldots, N,\ h \geq 0,\ Nh = T\Big\},$$

where h denotes the diameter associated with the uniform mesh and for each $n = 0, 1, \ldots, N-1$, the subintervals σ_n are defined as $\sigma_n := (t_n, t_{n+1}]$.

Now, consider the following linear HOFIE:

$$u(t) = f(t) + \int_0^T k(t, \tau)\mathrm{e}^{\mathrm{i}\omega(g(t)-g(\tau))}u(\tau)\mathrm{d}\tau, \quad t \in I := [0, T]. \tag{2.1}$$

In keeping with the principles of classical collocation methods, our primary objective is to approximate the solution to Eq (2.1) by using a set of piecewise algebraic polynomials within a finite-dimensional space defined by the following structure:

$$S_{m-1}^{(-1)}(I_h) := \Big\{p(s) : p(s)\Big|_{\sigma_n} \in \pi_{m-1},\ 0 \leq n \leq N-1\Big\}, \tag{2.2}$$

where π_{m-1} stands for the set of all polynomials possessing degrees less than or equal to $m-1$ and the dimension of this space is Nm [1, 2].

In this position, the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for Eq (2.1) is defined by the following collocation equation:

$$u_h(t) = f(t) + \int_0^T k(t, \tau)\mathrm{e}^{\mathrm{i}\omega(g(t)-g(\tau))}u_h(\tau)\mathrm{d}\tau, \quad t \in X_h, \tag{2.3}$$

where

$$X_h := \{t = t_{n,j} := t_n + c_j h;\ j = 1, \ldots, m,\ 0 \leq n \leq N-1\}$$

denotes the set of collocation points with the collocation parameters c_j, which can be properly chosen as

$$0 \leq c_1 < c_2 < \cdots < c_m \leq 1.$$

By the assumptions $U_{n,i} := u_h(t_{n,i})$, the collocation solution u_h of Eq (2.1) in subintervals σ_n, $n = 0, \ldots, N-1$ can be expressed as follows:

$$u_h(t_n + sh) = \sum_{j=1}^{m} \ell_j(s)U_{n,j}, \quad s \in (0, 1], \tag{2.4}$$

where

$$\ell_j(s) = \prod_{\substack{k=1 \\ k\neq j}}^{m} \frac{s - c_k}{c_j - c_k}.$$

In the sequel, inserting $t = t_{n,j}$, the collocation equation (2.3) can be expressed in the form presented below:

$$u_h(t_{n,j}) = f(t_{n,j}) + \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} k(t_{n,j}, \tau)\mathrm{e}^{\mathrm{i}\omega\big(g(t_{n,j})-g(\tau)\big)}u_h(\tau)\mathrm{d}\tau. \tag{2.5}$$

In other words,

$$U_{n,j} = f(t_{n,j}) + \sum_{i=0}^{N-1} h e^{i\omega g(t_{n,j})} \int_0^1 k(t_{n,j}, t_i + sh) e^{-i\omega g(t_i+sh)} u_h(t_i + sh) ds. \tag{2.6}$$

In this position, to achieve a linear collocation system for every $n = 0, \ldots, N-1$, which is predominantly linked to the unknowns $U_{n,j}$, $j = 1, \ldots, m$, we insert the collocation polynomial (2.4) within the collocation equation (2.6). Consequently, the attained semi-discretized system is obtained as follows:

$$U_{n,j} = f(t_{n,j}) + \sum_{k=1}^{m} \sum_{i=0}^{N-1} h e^{i\omega g(t_{n,j})} \left(\int_0^1 k(t_{n,j}, t_i + sh) e^{-i\omega g(t_i+sh)} \ell_k(s) ds \right) U_{i,k}. \tag{2.7}$$

The aforementioned linear system gives Nm algebraic equations in hand, with Nm unknowns to be identified precisely.

2.2. The fully discrete collocation method

Due to the nature of HOIs, the approach outlined in the previous section may not always be reliable in practical situations. Therefore, it is necessary to evaluate HOIs efficiently. As mentioned earlier, several numerical techniques have been developed to approximate HOIs such as Filon-type methods, Levin-type methods, the steepest descent method, EF, and the GQ method.

Here, in order to deal with numerical evaluation of HOIs in collocation system (2.7), we use a GQ method similar to what is introduced in [44]. In this method, the authors proposed a two-point quadrature rule of the form

$$Q_1(f; a, b) := \int_a^b f(\tau) \cos(\omega q(\tau)) d\tau \approx w_1 f(a) + w_2 f(b), \tag{2.8}$$

where $q'(\tau) \neq 0$, $\forall \tau \in [a, b]$, the weights w_1 and w_2 are found by solving a linear system, and the error of the quadrature method is given (further details can be found in [44]). A similar scheme can be constructed for evaluating the HOIs of the form

$$Q_2(f; a, b) := \int_a^b f(\tau) \sin(\omega q(\tau)) d\tau.$$

Since

$$e^{i\omega q(\tau)} = \cos(\omega q(\tau)) + i \sin(\omega q(\tau)),$$

this technique can also be extended for numerical computation of HOIs including the term $e^{i\omega q(\tau)}$. More precisely,

$$Q(f; a, b) := \int_a^b f(\tau) e^{i\omega q(\tau)} d\tau \approx w_1 f(a) + w_2 f(b). \tag{2.9}$$

We note that the same error bound in [44] is expected for the two-point GQ method (2.9).

By using the quadrature formula (2.9) to approximate the integrals on the right side of (2.7) and ignoring the associated quadrature errors, we derive the fully discrete version

$$\hat{U}_{n,j} = f(t_{n,j}) + \sum_{k=1}^{m} \sum_{i=0}^{N-1} h e^{i\omega g(t_{n,j})} \Big(w_1^{(i)} k(t_{n,j}, t_i) \ell_k(0) + w_2^{(i)} k(t_{n,j}, t_i + h) \ell_k(1) \Big) \hat{U}_{i,k}, \tag{2.10}$$

with $\hat{U}_{n,j} := \hat{u}_h(t_{n,j})$, where

$$\hat{u}_h(t_n + sh) = \sum_{j=1}^{m} \ell_j(s)\hat{U}_{n,j}, \quad s \in (0, 1] \tag{2.11}$$

is the local representation of $\hat{u}_h$.

3. Convergence analysis

In this section, our aim is to explore the principal results of the paper, which are expressed in the form of two theorems that provide theoretical validation for the practicality of the piecewise collocation method.

Lemma 3.1. *Suppose that the real-valued smooth function $q(t)$ in (a, b) satisfies $\left|q^{(k)}(t) \geq 1\right|, \forall t \in (a, b)$. Then,*

$$\left|\int_a^b e^{i\omega q(t)} dt\right| \leq c(k)\omega^{-1/k}$$

holds for $k \geq 2$, or $k = 1$ and $q'(t)$ is monotonic, such that $c(k) = 5 \times 2^{k-1} - 2$.

Lemma 3.2. *Given the same assumptions regarding $q(t)$ as stated in Lemma 3.1, we achieve that*

$$\left|\int_a^b e^{i\omega q(t)}\phi(t) dt\right| \leq c(k)\omega^{-1/k}\left(|\phi(b)| + \int_a^b |\phi'(t)| dt\right).$$

Theorem 3.1. *Let $\phi(t) \in C^1$, $q(t)$ adhere to the conditions given in Lemma 3.2, and suppose there exists a point $t_0 \in [a, b]$ for which $\phi(t_0) = 0$. Then, we will have the following inequality:*

$$\left|\int_a^b e^{i\omega q(t)}\phi(t) dt\right| \leq 2c(k)\frac{\|\phi'(t)\|_\infty}{\omega^{1/k}}(b - a).$$

Furthermore, if $\phi(t) \in C^2$, $q(t) \in C^3$ with $k = 1$, and given that $\phi(a) = \phi(b) = 0$, it follows that

$$\left|\int_a^b e^{i\omega q(t)}\phi(t) dt\right| \leq \min\left\{C_1\frac{b - a}{\omega^2},\ C_2\frac{(b - a)^2}{\omega}\right\},$$

where $C_1 = 6\left\|\left(\frac{\phi(t)}{q'(t)}\right)''\right\|_\infty$ and $C_2 = 3\|\phi''(t)\|_\infty$.

Following the idea in [44], the expected error estimate for the proposed GQ method can be stated in the following theorem:

Theorem 3.2. *Suppose that $q(\tau)$ belongs to $C^2([a, b])$, $q'(\tau) \neq 0$, and that $q^{(j)}(\tau)$ $(j = 0, 1, 2)$ remains uniformly bounded for each ω. Under these conditions, the error of the GQ method* (2.9) *is expressed as*

$$\left|\int_a^b f(\tau)e^{i\omega q(\tau)} d\tau - (w_1 f(a) + w_2 f(b))\right| \leq \min\left\{C_1(b - a)^3, C_2\frac{b - a}{\omega^2}\right\},$$

in which the constants C_1 and C_2 are independent of ω.

Now, we turn our attention to the convergence property of the proposed method. To do so, by putting $t = t_{n,j}$ in the underlying HOFIE (2.1), we get

$$\begin{aligned} u(t_{n,j}) &= f(t_{n,j}) + \int_0^T k(t_{n,j},\tau)\mathrm{e}^{\mathrm{i}\omega\left(g(t_{n,j})-g(\tau)\right)}u(\tau)\mathrm{d}\tau \\ &= f(t_{n,j}) + \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} k(t_{n,j},\tau)\mathrm{e}^{\mathrm{i}\omega\left(g(t_{n,j})-g(\tau)\right)}u(\tau)\mathrm{d}\tau \\ &= f(t_{n,j}) + \sum_{i=0}^{N-1} h\Big(\int_0^1 k(t_{n,j},t_i+sh)\mathrm{e}^{\mathrm{i}\omega\left(g(t_{n,j})-g(t_i+sh)\right)}u(t_i+sh)\mathrm{d}s\Big). \end{aligned} \tag{3.1}$$

Subtracting the newly obtained equation from Eq (2.6), we will have

$$\epsilon_{n,j} = \sum_{i=0}^{N-1} h\mathrm{e}^{\mathrm{i}\omega g(t_{n,j})}\left(\int_0^1 k(t_{n,j},t_i+sh)\mathrm{e}^{-\mathrm{i}\omega g(t_i+sh)}(u(t_i+sh)-u_h(t_i+sh))\mathrm{d}s\right), \tag{3.2}$$

such that $\epsilon_{n,j} := u(t_{n,j}) - U_{n,j}$. Then, as $e_h = u - u_h$, Eq (3.2) can be written as follows:

$$\epsilon_{n,j} = \sum_{i=0}^{N-1} h\mathrm{e}^{\mathrm{i}\omega g(t_{n,j})}\left(\int_0^1 k(t_{n,j},t_i+sh)\mathrm{e}^{-\mathrm{i}\omega g(t_i+sh)}e_h(t_i+sh)\mathrm{d}s\right). \tag{3.3}$$

In this position, having $u \in C^{\nu}(I), (-1 \leq \nu \leq m)$, we can express the interpolation error according to Peano's theorem (as outlined in Corollary 1.8.2 of the monograph [1]) as

$$u(t_n + sh) = \sum_{j=1}^{m} \ell_j(s)u(t_{n,j}) + h^{\nu}R_{\nu,n}(s), \quad s \in (0,1]. \tag{3.4}$$

On the one hand, according to Eqs (2.4) and (3.4), the error e_h possesses a local representation in relation to the exact collocation solution as

$$\begin{aligned} e_h(t_i+sh) &= u(t_i+sh) - u_h(t_i+sh) \\ &= \sum_{k=1}^{m} \ell_k(s)u(t_{i,j}) + h^{\nu}R_{\nu,i}(s) - \sum_{k=1}^{m}\ell_k(s)U_{i,k} \\ &= \sum_{k=1}^{m} \ell_k(s)\epsilon_{i,k} + h^{\nu}R_{\nu,i}(s). \end{aligned} \tag{3.5}$$

Therefore, substituting (3.5) into (3.3) gives

$$\begin{aligned} \epsilon_{n,j} &= \sum_{i=0}^{N-1} h\mathrm{e}^{\mathrm{i}\omega g(t_{n,j})}\left(\int_0^1 k(t_{n,j},t_i+sh)\mathrm{e}^{-\mathrm{i}\omega g(t_i+sh)}\left(\sum_{k=1}^{m}\ell_k(s)\epsilon_{i,k} + h^{\nu}R_{\nu,i}(s)\right)\mathrm{d}s\right) \\ &= \sum_{i=0}^{N-1} h\mathrm{e}^{\mathrm{i}\omega g(t_{n,j})}\sum_{k=1}^{m}\left(\int_0^1 k(t_{n,j},t_i+sh)\ell_k(s)\mathrm{e}^{-\mathrm{i}\omega g(t_i+sh)}\mathrm{d}s\right)\epsilon_{i,k} \\ &\quad + \sum_{i=0}^{N-1} h^{\nu+1}\mathrm{e}^{\mathrm{i}\omega g(t_{n,j})}\left(\int_0^1 k(t_{n,j},t_i+sh)R_{\nu,i}(s)\mathrm{e}^{-\mathrm{i}\omega g(t_i+sh)}\mathrm{d}s\right), \quad n = 0,\ldots,N-1, \ j = 1,\ldots,m. \end{aligned} \tag{3.6}$$

Here, in order to achieve a compact representation of the method, the following matrices must be assumed:

$$\mathbf{A}_n^{(i)} := \left(\mathrm{e}^{\mathrm{i}\omega g(t_{n,j})} \int_0^1 k(t_{n,j}, t_i + sh)\ell_k(s)\mathrm{e}^{-\mathrm{i}\omega g(t_i+sh)}\mathrm{d}s \right)_{j,k=1,\ldots,m}, \tag{3.7}$$

$$\mathbf{B}_n^{(i)} := \left(\mathrm{e}^{\mathrm{i}\omega g(t_{n,j})} \int_0^1 k(t_{n,j}, t_i + sh)R_{\nu,i}(s)\mathrm{e}^{-\mathrm{i}\omega g(t_i+sh)}\mathrm{d}s \right)_{j,k=1,\ldots,m}, \tag{3.8}$$

and for $n = 0, \ldots, N-1$,

$$\mathbf{E}_n := (\epsilon_{n,1}, \ldots, \epsilon_{n,m})^T. \tag{3.9}$$

Thus, Eq (3.6) can be written as

$$\left[\mathbf{I}_m - h\mathbf{A}_n^{(i)}\right]\mathbf{E}_n = \sum_{i=0}^{N-1} h^{\nu+1}\mathbf{B}_n^{(i)}, \quad n = 0, \ldots, N-1, \tag{3.10}$$

where $\mathbf{I}_m$ denotes the $m \times m$ identity matrix.

Thanks to the continuity of the kernel function $k(t, s)$, we conclude that the elements of the matrices $\mathbf{A}_n^{(i)}$ are bounded. Subsequently, with respect to the Neumann lemma [1], whenever $h\|\mathbf{A}_n^{(i)}\| < 1$ for some matrix norm, then the matrix $\left(\mathbf{I}_m - h\mathbf{A}_n^{(i)}\right)$ has an inverse. This assertion is clearly valid when h is sufficiently small. In particular, for any mesh I_h characterized by a diameter h that lies within the range $(0, \bar{h})$, with $\bar{h}$ being appropriately small, it follows that each matrix $\left(\mathbf{I}_m - h\mathbf{A}_n^{(i)}\right)$ has an inverse that is uniformly bounded. Hence, for sufficiently small values of h, we can assume the existence of a constant D_0 such that

$$\left\|\mathbf{I}_m - h\mathbf{A}_n^{(i)}\right\|_1 \le D_0, \quad n = 0, \ldots, N-1.$$

Also, we can ensure that $\left\|\mathbf{A}_n^{(i)}\right\|_1 \le D_1$ for $i < n \le N-1$, according to the continuity of the kernel function of the integral equation, where D_1 is a constant.

On the other hand, given that $R_{\nu,i}(c_1) = \cdots = R_{\nu,i}(c_m) = 0$, we will have

$$\left| \int_0^1 k(t_{n,j}, t_i + sh)R_{\nu,i}(s)\mathrm{e}^{-\mathrm{i}\omega g(t_i+sh)}\mathrm{d}s \right| \le c\frac{M_\nu}{\omega h}, \tag{3.11}$$

thanks to Theorem 3.1, for $a = 0, b = 1$, where $M_\nu := \left\|u^{(\nu)}(t)\right\|_\infty$.

In the continuation of this analysis, it is essential to highlight that the notation C represents a constant that can vary in value across different inequalities, but does not rely on h and ω. In addition, if $\nu \ge 2$ and $c_1 = 0, c_m = 1$ which means that $R_{\nu,i}(0) = R_{\nu,i}(1) = 0$, then we have

$$\left| \int_0^1 k(t_{n,j}, t_i + sh)R_{\nu,i}(s)\mathrm{e}^{-\mathrm{i}\omega g(t_i+sh)}\mathrm{d}s \right| \le CM_\nu \min\left\{\frac{1}{\omega^2 h^2}, \frac{1}{\omega h}\right\}. \tag{3.12}$$

Now, according to (3.11) and (3.12), we have the following estimate:

$$\left\|\mathbf{B}_n^{(i)}\right\|_1 \le CM_\nu \begin{cases} \min\left\{\frac{1}{\omega^2 h^2}, \frac{1}{\omega h}\right\}, & \text{if } c_1 = 0, c_m = 1 \text{ and } \nu \ge 2, \\ \frac{1}{\omega h}, & \text{otherwise.} \end{cases} \tag{3.13}$$

Then, Eqs (3.10) and (3.13) give

$$\begin{aligned}\|\mathbf{E}_n\|_1 &\le \sum_{i=0}^{N-1} h^{\nu+1}\left\|\mathbf{B}_n^{(i)}\right\|_1\\ &\le CM_\nu\begin{cases}\dfrac{h^{\nu-1}}{\omega}\min\Big\{\dfrac{1}{\omega h},1\Big\}, & \text{if } c_1=0, c_m=1 \text{ and } \nu\ge 2,\\ \dfrac{h^{\nu-1}}{\omega}, & \text{otherwise.}\end{cases}\end{aligned}\tag{3.14}$$

On the other hand, we have

$$\begin{aligned}\left\|\mathbf{B}_n^{(i)}\right\|_1 &\le \left|\mathrm{e}^{\mathrm{i}\omega g(t_{n,j})}\int_0^1 k(t_{n,j},t_i+sh)R_{\nu,i}(s)\mathrm{e}^{-\mathrm{i}\omega g(t_i+sh)}\mathrm{d}s\right|\\ &\le \int_0^1\left|k(t_{n,j},t_i+sh)R_{\nu,i}(s)\right|\mathrm{d}s\\ &\le \bar{K}k_\nu M_\nu,\end{aligned}\tag{3.15}$$

where $\bar{K} := \max_{t\in I}\int_0^1 |k(t,s)|\mathrm{d}s$, $k_\nu := \max_{s\in[0,1]}\int_0^1 |k(s,z)|\mathrm{d}z$. Now, due to the (3.10) and (3.15), we get

$$\|\mathbf{E}_n\|_1 \le CM_\nu h^\nu.\tag{3.16}$$

Therefore, combining (3.14) and (3.16) results in

$$\max_{t\in X_h}|u(t)-u_h(t)| \le CM_\nu\begin{cases}\min\Big\{h^\nu,\dfrac{h^{\nu-1}}{\omega},\dfrac{h^{\nu-2}}{\omega^2}\Big\}, & \text{if } c_1=0, c_m=1 \text{ and } \nu\ge 2,\\ \min\Big\{h^\nu,\dfrac{h^{\nu-1}}{\omega}\Big\}, & \text{otherwise,}\end{cases}\tag{3.17}$$

where $M_\nu := \left\|u^{(\nu)}(t)\right\|_\infty$.

To conclude the above analysis, we summarize it in the following theorem:

Theorem 3.3. *Suppose that the following are assumed for the HOFIE* (2.1) *with* $1\le\nu\le m$*:*

(1) $f(t)\in C^\nu(I)$,
(2) $k(t,\tau)\in C^\nu(D)$.

Then, the error of the numerical method defined by (2.10) *and* (2.11) *is estimated by*

$$\max_{t\in X_h}|u(t)-u_h(t)| \le CM_\nu\begin{cases}\min\Big\{h^\nu,\dfrac{h^{\nu-1}}{\omega},\dfrac{h^{\nu-2}}{\omega^2}\Big\}, & \textit{for } c_1=0, c_m=1 \textit{ and } \nu\ge 2,\\ \min\Big\{h^\nu,\dfrac{h^{\nu-1}}{\omega}\Big\}, & \textit{otherwise,}\end{cases}$$

where C is a constant independent of h and ω *and* $M_\nu := \left\|u^{(\nu)}(t)\right\|_\infty$.

We are now in a position to establish the order of the discretized collocation solution $\hat{u}_h(t)$. By employing the triangle inequality, we are able to express it as follows:

$$|u(t) - \hat{u}_h(t)| \leq |u(t) - u_h(t)| + |u_h(t) - \hat{u}_h(t)|\,. \tag{3.18}$$

The global estimate for the exact collocation error, as previously stated, is given by Theorem 3.3. To evaluate the perturbation error $|u_h(t) - \hat{u}_h(t)|$ caused by the quadrature process, let $z_h(t) := u_h(t) - \hat{u}_h(t)$. Then, on σ_n,

$$z_h(t_n + sh) := u_h(t_n + sh) - \hat{u}_h(t_n + sh) = \sum_{j=1}^{m} \ell_j(s) Z_{n,j}, \tag{3.19}$$

where $Z_{n,j} := U_{n,j} - \hat{U}_{n,j}$. According to (2.7) and (2.10), we get

$$\begin{aligned} Z_{n,j} = \sum_{k=1}^{m}\sum_{i=0}^{N-1} h \mathrm{e}^{\mathrm{i}\omega g(t_{n,j})}\Big(\Big(\int_0^1 k(t_{n,j}, t_i + sh)\mathrm{e}^{-\mathrm{i}\omega g(t_i+sh)}\ell_k(s)\mathrm{d}s\Big) U_{i,k} \\ - \big(w_1^{(i)} k(t_{n,j}, t_i)\ell_k(0) + w_2^{(i)} k(t_{n,j}, t_i + h)\ell_k(1)\big)\hat{U}_{i,k}\Big). \end{aligned} \tag{3.20}$$

Define the operator $\mathcal{F} : C(I) \to C(I)$ by

$$(\mathcal{F}u)(t) := \int_0^T k(t, \tau)\mathrm{e}^{\mathrm{i}\omega(g(t)-g(\tau))} u(\tau)\mathrm{d}\tau,$$

and for $t = t_{n,j} = t_n + c_j h \in \sigma_n$, let

$$\begin{aligned} (Q_h^{(i)} u_h)(t) &:= \int_0^1 k(t, t_i + sh)\mathrm{e}^{\mathrm{i}\omega(g(t)-g(t_i+sh))} u_h(t_i + sh)\mathrm{d}s, \\ (\hat{Q}_h^{(i)} u_h)(t) &:= w_1^{(i)} k(t, t_i)\mathrm{e}^{\mathrm{i}\omega g(t)} u_h(t_i) + w_2^{(i)} k(t, t_{i+1})\mathrm{e}^{\mathrm{i}\omega g(t)} u_h(t_{i+1}), \end{aligned}$$

then, Eqs (2.7) and (2.10) can then be represented as the following operator equations, respectively:

$$\begin{aligned} u_h(t) &= f(t) + (\mathcal{F} u_h)(t), \quad t \in X_h, \\ \hat{u}_h(t) &= f(t) + (\mathcal{F}_h \hat{u}_h)(t), \quad t \in X_h. \end{aligned}$$

In other words,

$$U_{n,j} = f(t_{n,j}) + \sum_{i=0}^{N-1} h(Q_h^{(i)} u_h)(t_{n,j}), \quad n = 0, \ldots, N-1, \;\; j = 1, \ldots, m,$$

$$\hat{U}_{n,j} = f(t_{n,j}) + \sum_{i=0}^{N-1} h(\hat{Q}_h^{(i)} \hat{u}_h)(t_{n,j}), \quad n = 0, \ldots, N-1, \;\; j = 1, \ldots, m.$$

In this position, by taking the quadrature error as

$$E_h^{(i)}(t) = (Q_h^{(i)} \hat{u}_h)(t) - (\hat{Q}_h^{(i)} \hat{u}_h)(t),$$

Equation (3.20) can be written as

$$\begin{aligned}
Z_{n,j} &= \sum_{i=0}^{N-1} h\left((Q_h^{(i)}u_h)(t_{n,j}) - (\hat{Q}_h^{(i)}\hat{u}_h)(t_{n,j})\right) \\
&= \sum_{i=0}^{N-1} h\left((Q_h^{(i)}u_h)(t_{n,j}) + E_h^{(i)}(t_{n,j}) - (Q_h^{(i)}\hat{u}_h)(t_{n,j})\right) \\
&= \sum_{i=0}^{N-1} h\mathrm{e}^{\mathrm{i}\omega g(t_{n,j})} \sum_{k=1}^{m} \int_0^1 k(t_{n,j}, t_i + sh)\ell_k(s)\mathrm{e}^{\mathrm{i}\omega g(t_i+sh)}\mathrm{d}sZ_{i,k} + \sum_{i=0}^{N-1} hE_h^{(i)}(t_{n,j}).
\end{aligned}$$

Therefore, recalling the definition of $\mathbf{A}_n^{(i)}$, the above system can be written as

$$\left[\mathbf{I}_m - h\mathbf{A}_n^{(i)}\right]\mathbf{Z}_n = \sum_{i=0}^{N-1} hE_h^{(i)}(t_{n,j}).$$

Since

$$\left|E_h^{(i)}(t)\right| \leq C\min\left\{1, \frac{1}{\omega^2}\right\} \leq \frac{C}{\omega^2},$$

therefore,

$$\|\mathbf{Z}_n\|_1 \leq D_0 \sum_{i=0}^{N-1} h\left\|E_h^{(i)}(t_{n,j})\right\|_1 \leq \frac{C}{\omega^2}.$$

Consequently,

$$|u(t) - \hat{u}_h(t)| \leq C \begin{cases} \max\left\{M_\nu \min\left\{h^\nu, \dfrac{h^{\nu-1}}{\omega}, \dfrac{h^{\nu-2}}{\omega^2}\right\}, \dfrac{1}{\omega^2}\right\}, & \text{for } \nu \geq 2 \text{ and } c_1 = 0, c_m = 1, \\ \max\left\{M_\nu \min\left\{h^\nu, \dfrac{h^{\nu-1}}{\omega}\right\}, \dfrac{1}{\omega^2}\right\}, & \text{otherwise.} \end{cases} \tag{3.21}$$

Finally, the estimation derived from Eq (3.21) indicates the convergence behavior of the collocation solution $\hat{u}_h$, which is further detailed in the subsequent theorem.

Theorem 3.4. *Suppose that the functions $f(t)$ and $k(t, s) \in C^\nu$ in HOFIE* (2.1) *with* $1 \leq \nu \leq m$. *Then, the estimation of the error associated with the numerical method outlined in Eqs* (2.10) *and* (2.11) *is given by*

$$\max_{t\in X_h} |u(t) - \hat{u}_h(t)| \leq C \begin{cases} \max\left\{M_\nu \min\left\{h^\nu, \dfrac{h^{\nu-1}}{\omega}, \dfrac{h^{\nu-2}}{\omega^2}\right\}, \dfrac{1}{\omega^2}\right\}, & \textit{for } \nu \geq 2 \textit{ and } c_1 = 0, c_m = 1, \\ \max\left\{M_\nu \min\left\{h^\nu, \dfrac{h^{\nu-1}}{\omega}\right\}, \dfrac{1}{\omega^2}\right\}, & \textit{otherwise,} \end{cases}$$

where C is a constant independent of h and ω.

Remark 3.1. *According to Theorem 3.4, when M_ν remains bounded regardless of ω, the method under consideration demonstrates an asymptotic order of* 1, *which can potentially rise to 2 when $\nu \geq 2$ and* $c_1 = 0$, $c_m = 1$. *Consequently, as ω becomes larger, the GQ collocation method is likely to produce accurate results, with improved numerical precision. Additionally, for a fixed ω, the method converges as step length h approaches* 0.

4. Numerical illustrations

In this section, we provide numerical examples to showcase the proposed method's efficiency and accuracy. Our intention is to ascertain that the error e_h has an asymptotic order of α, contingent upon the absolute error being scaled by ω^α, i.e., $\omega^\alpha|e_h|$ remains bounded as $\omega \gg 1$, and the method converges with order ν as $h \to 0$. For numerical comparison, we report the maximum absolute errors for different vales of ω and N in each example.

For clarity, in the examples provided in this section, we will show the maximum absolute errors and classical convergence orders with, respectively, ϵ and ν such that

$$\epsilon_{N} := \max|e_h|, \quad \nu := \log_2(\epsilon_{N}/\epsilon_{2N}).$$

Moreover, the TQC, SQC, and GQC notations will be used to denote the trapezoidal quadrature collocation, Simpson quadrature collocation, and the generalized quadrature collocation methods, respectively. Mathematica software was utilized for the execution of all numerical computations.

Example 4.1. *Consider the following HOFIE:*

$$u(t) = f(t) - \frac{1}{2}\int_0^1 e^{i\omega(t-\tau)}u(\tau)d\tau, \quad \omega \gg 1, \tag{4.1}$$

where the function $f(t)$ is selected such that the integral equation's exact solution is

$$u(t) = e^t.$$

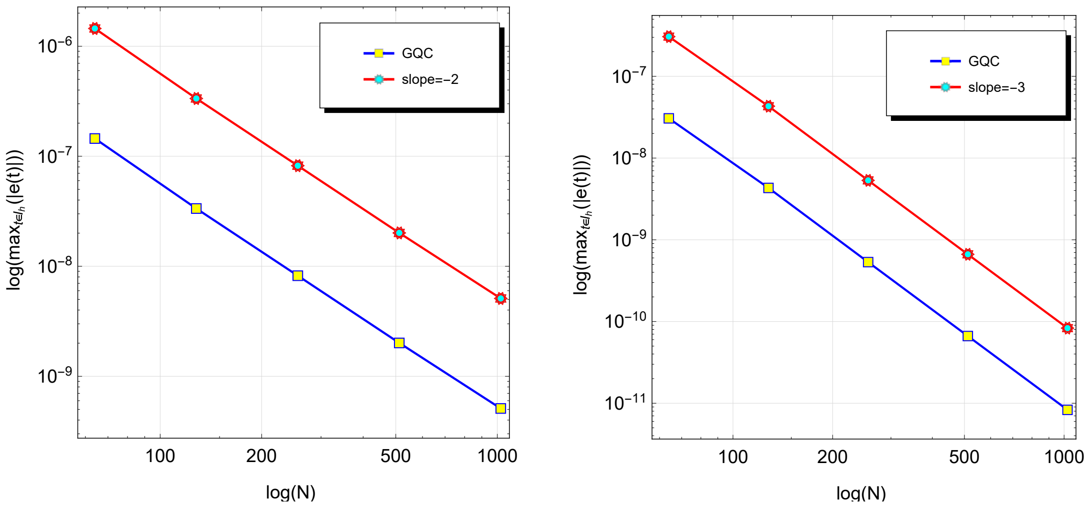
We implemented the proposed method with $m = 2$ for solving (4.1). *The results, including the maximum absolute errors for different values of N and ω, for parameters $c_1 = 0,\ c_2 = 1$ and $c_1 = \frac{1}{3},\ c_2 = 1$ are reported in Tables 1 and 2, respectively. In order to illustrate the classical order for the parameters $c_1 = 0,\ c_2 = 1$ and $c_1 = \frac{1}{3},\ c_2 = 1$, we set $\omega = 100$ and presented the maximum absolute errors for $N =$ $64, 128, 256, 512, 1024$ in Figure 1. The figures illustrate the associated classical orders, accompanied by a slope line. For a direct observation of the asymptotic order concerning the parameters $c_1 = \frac{1}{3},\ c_2 = 1$ and $c_1 = 0,\ c_2 = 1$, we provided a graphical representation of the absolute errors scaled by ω and ω^2, respectively, with $N = 2$, illustrated in Figure 2. The observations in Tables 1 and 2 and Figures 1 and 2 confirm that the method behaves as predicted in terms of its order.*

Table 1. The maximum absolute errors with $c_1 = 0, c_2 = 1$ in Example 4.1.

	$\omega = 100$		$\omega = 200$		$\omega = 400$	
N	ϵ	ν	ϵ	ν	ϵ	ν
2^6	1.45e − 07	−	1.23e − 07	−	4.49e − 06	−
2^7	3.35e − 08	2.11	2.26e − 08	2.44	2.15e − 08	7.71
2^8	8.21e − 09	2.03	5.22e − 09	2.11	3.95e − 09	2.44
2^9	2.01e − 09	2.01	1.28e − 09	2.03	9.13e − 10	2.11
2^{10}	5.10e − 10	2.00	3.18e − 10	2.01	2.23e − 10	2.03

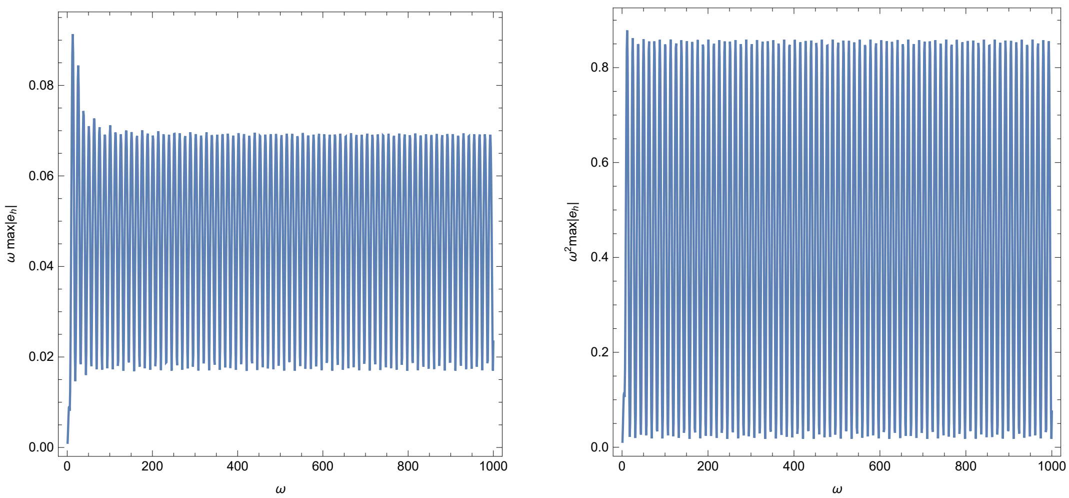
Table 2. The maximum absolute errors with $c_1 = \frac{1}{3}, c_2 = 1$ in Example 4.1.

	$\omega = 100$		$\omega = 200$		$\omega = 400$	
N	ϵ	ν	ϵ	ν	ϵ	ν
2^6	3.60e − 08	–	5.56e − 08	–	5.74e − 06	–
2^7	4.31e − 09	3.06	5.61e − 09	3.33	9.88e − 09	9.18
2^8	5.33e − 10	3.01	6.70e − 10	3.06	9.81e − 10	3.33
2^9	6.64e − 11	3.00	8.29e − 11	3.01	1.17e − 10	3.07
2^{10}	8.30e − 12	3.00	1.03e − 11	3.01	1.45e − 11	3.01



(a) $c_1 = 0, c_2 = 1$ (b) $c_1 = \frac{1}{3}, c_2 = 1$

Figure 1. The classical orders with $c_1 = \frac{1}{3}, c_2 = 1$ and $c_1 = 0, c_2 = 1$ for Example 4.1.



(a) $c_1 = \frac{1}{3}, c_2 = 1$ (b) $c_1 = 0, c_2 = 1$

Figure 2. Graph of the asymptotic order with $N = 2,\ m = 2$ in Example 4.1.

To show the superiority, we compared the GQC developed here with the TQC given in [28]. The results are listed in Table 3, which show that both methods converge as the step length decreases. However, the new quadrature method yields more precise numerical results.

Table 3. Comparison of GQC and TQC with $\omega = 50$, $m = 2$, and collocation parameters $c_1 = 0, c_2 = 1$ in Example 4.1.

Method	N				
	2^6	2^7	2^8	2^9	2^{10}
TQC	6.07e − 04	1.51e − 04	3.76e − 05	9.40e − 06	2.35e − 06
GQC	2.47e − 07	6.05e − 08	1.50e − 08	3.76e − 09	9.39e − 10

Example 4.2. *As a final example, we consider the following HOFIE:*

$$u(t) = f(t) + \int_0^1 \cos(\omega(t+\tau))u(\tau)\mathrm{d}\tau, \quad t \in I := [0,1], \tag{4.2}$$

where the function $f(t)$ is chosen in such a way that the exact solution to this problem is

$$u(t) = \frac{\cos(\omega t) + \omega^2 t}{\omega^2}.$$

This example is selected to demonstrate how the proposed method handles HOFIE with an oscillatory solution. Similar to the previous example, we employed the proposed method for solving Eq (4.2) *and reported the errors for several values of ω and N in Tables 4 and 5. We also plotted in Figure 3 the errors embedded in Table 5. The findings validated the expected classical order for the method discussed. In addition, to show the asymptotic order for the parameters $c_1 = \frac{1}{3}$, $c_2 = 1$ and $c_1 = 0$, $c_2 = 1$, we plotted the graph of the absolute errors multiplied by ω and ω^2, respectively, with $N = 2$ in Figure 4.*

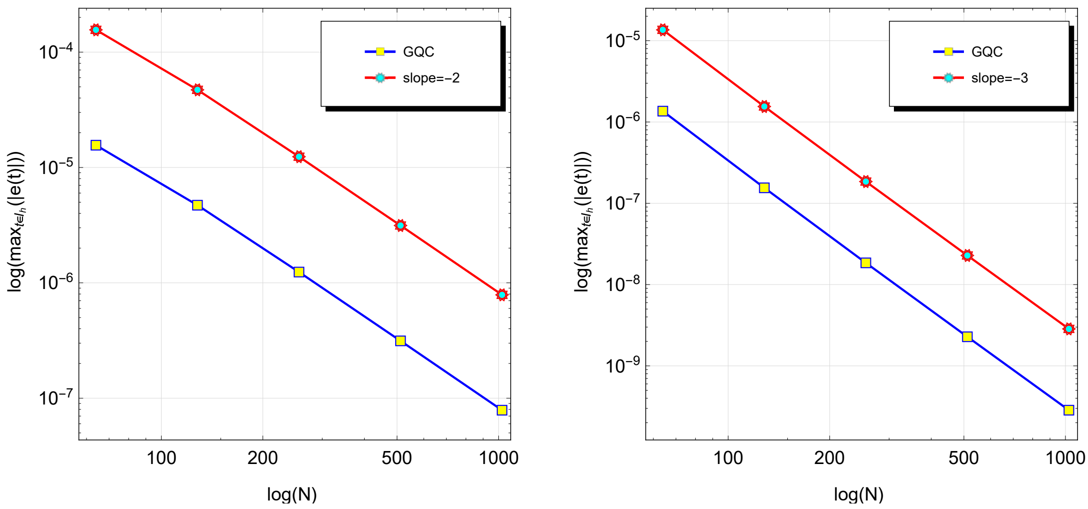
The results obtained in this example imply that our method is also applicable for certain FIEs with highly oscillatory solutions.

Table 4. The maximum absolute errors with $c_1 = 0, c_2 = 1$ in Example 4.2.

	$\omega = 100$		$\omega = 200$		$\omega = 400$	
N	ϵ	ν	ϵ	ν	ϵ	ν
2^6	1.56e − 05	–	8.23e − 06	–	3.12e − 06	–
2^7	4.71e − 06	1.73	3.93e − 06	1.07	2.54e − 06	0.29
2^8	1.24e − 06	1.92	1.18e − 06	1.73	9.89e − 07	1.36
2^9	3.14e − 07	1.98	3.11e − 07	1.92	2.97e − 07	1.73
2^{10}	7.87e − 08	1.82	7.88e − 08	1.98	7.82e − 08	1.92

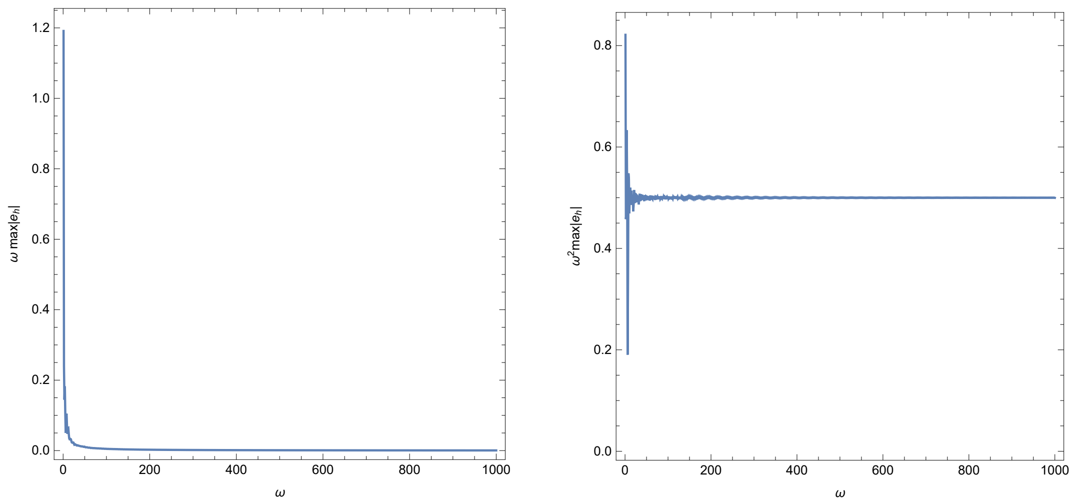
Table 5. The maximum absolute errors with $c_1 = \frac{1}{3}, c_2 = 1$ in Example 4.2.

	$\omega = 100$		$\omega = 200$		$\omega = 400$	
N	ϵ	ν	ϵ	ν	ϵ	ν
2^6	1.36e − 06	–	4.16e − 06	–	2.84e − 06	–
2^7	1.55e − 07	3.13	3.50e − 07	3.57	6.02e − 07	2.24
2^8	1.85e − 08	3.07	3.89e − 08	3.17	8.90e − 08	2.76
2^9	2.28e − 09	3.02	4.64e − 09	3.07	9.92e − 09	3.17
2^{10}	2.85e − 10	3.00	5.73e − 10	3.02	1.17e − 09	3.08



(a) $c_1 = 0, c_2 = 1$ (b) $c_1 = \frac{1}{3}, c_2 = 1$

Figure 3. The classical orders with $c_1 = \frac{1}{3}, c_2 = 1$ and $c_1 = 0, c_2 = 1$ for Example 4.2.



(a) $c_1 = \frac{1}{3}, c_2 = 1$ (b) $c_1 = 0, c_2 = 1$

Figure 4. Graph of the asymptotic order with $N = 2,\ m = 2$ in Example 4.2.

In line with the previous example, we showed the superiority of the GQC method by comparing it with both the TQC and SQC methods. The results are listed in Table 6. We can see that the proposed method provides more accurate numerical results.

Table 6. Comparison of GQC, TQC, and SQC with $\omega = 700$, $m = 2$, and collocation parameters $c_1 = 0, c_2 = 1$ in Example 4.2.

	N					
Method	2^4	2^5	2^6	2^7	2^8	2^9
TQC	6.55e − 01	1.68e − 03	1.11e − 02	1.32e − 02	1.26e − 03	2.82e − 04
SQC	1.06e − 01	1.16e − 02	1.22e − 02	2.31e − 03	3.46e − 05	1.84e − 06
GQC	1.02e − 06	1.01e − 06	1.01e − 06	1.01e − 06	6.66e − 07	2.60e − 07

5. Conclusions

In summary, this paper presented a robust collocation method utilizing a GQ rule to address FIEs with highly oscillatory trigonometric kernels. We analyzed the convergence of the proposed method, showing that it achieves both a classical and an asymptotic order for high-frequency values. The convergence rate with respect to frequency suggests that an asymptotic order of two can be attained in some cases. In addition, the method demonstrates convergence for a fixed ω as the step length h tends toward 0. Numerical tests confirmed the method's efficiency and revealed that accuracy improves as the frequency increases. As mentioned earlier, the approximation of solutions to HOFIEs has been explored in [5, 22–24, 30, 34]. However, there are several limitations and drawbacks associated with the proposed methods, which we summarize as follows:

- Almost all of the proposed methods address the problem in a special case where $g(t) = t$.
- The method presented in [22] is difficult to implement, and its convergence analysis is complex. The numerical examples provided in [24] focus solely on problems with non-oscillatory solutions, excluding oscillatory cases. Additionally, the method in [23] does not include an examination of convergence or error analysis.

In contrast, the implementation of our method is straightforward and computationally efficient. It applies to a general function g in the absence of stationary points. Furthermore, our method is both accurate and efficient for approximating HOFIEs with both oscillatory and non-oscillatory solutions, making it a versatile choice for all types of solutions. This technique can also be easily extended to most classes of integral equations, including those involving functions g with stationary points. As a result, the method proposed in this paper surpasses other methods in terms of implementation, simplicity, and comprehensiveness.

Author contributions

Adil Owaid Jhaily: Methodology, Investigation; Saeed Sohrabi: Writing–review & editing, Writing–original draft, Validation, Supervision, Software, Investigation; Hamid Ranjbar: Validation, Formal analysis, Software, Investigation. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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