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Research article

On relationships between vector variational inequalities and optimization problems using convexificators on the Hadamard manifold

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Abstract: This study extended a fundamental idea about convexificators to the Hadamard manifolds. The mean value theorem for convexificators on the Hadamard manifold was also derived. An important characterization for the bounded convexificators to have ∂^*_*-geodesic convexity was derived and the monotonicity of the bounded convexificators was explored. Additionally, a convexificator-based vector variational inequality problem on the Hadamard manifold was examined. Furthermore, the necessary and sufficient conditions for vector optimization problems in terms of the Stampacchia and Minty-type partial vector variational inequality problems (∂^*_*-VVIPs) were derived.

Keywords: geodesic convexity; monotonicity; ∂^*_*-VVIP; VOP; Hadamard manifold
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1. Introduction

Giannessi [11] defined variational inequality problems (VIPs) in vector form in 1980 and demonstrated the connections between effective solutions to differential convex vector optimization problems and Minty vector variational inequalities. Since then, a great deal of research has been done on the relationships between nonsmooth vector variational inequalities and nonsmooth vector optimization problems, see [1, 9, 19]. In 1994, Demyanov [6] proposed the concept of convexificators in order to generalize upper convex and lower concave approximations. Later, Demyanov and Jeyakumar [7, 8] evaluated convexificators for positively homogeneous and locally Lipschitz functions. Furthermore, Jeyakumar and Luc [14] defined non-compact convexificators and presented several calculus rules for calculating convexificators. For more details, one can see [6] and the references therein. Laha et al. [16] studied the convexity for vector valued functions in terms of convexificators and the monotonicity of the corresponding convexificators. They [16] also formulated the vector variational inequality problems (VVIPs) of Stampacchia [27] and Minty [18]-type using convexificators on Euclidean spaces.

Furthermore, several authors have laid focus on the extension of the methods and techniques developed on Euclidean spaces to Riemannian manifolds. For more details, see: [1, 2, 10, 17, 28]. And in particularly on the Hadamard manifolds, one can see [5, 22, 23, 29]. Nemeth [22] extended the VIP on the Hadamard manifolds and studied their existence. Later, Chen et al. [5] showed the relations between VVIPs and vector optimization problems (VOPs) on the Hadamard manifolds. Furthermore, Chen [4] studied the existence results of VVIPs on the Hadamard manifolds and Jayswal et al. [13] investigated it on Riemannian manifolds with some appropriate conditions. Later, Singh et al. [26] discussed the existence of nonsmooth vector variational inequality problems (NVVIPs) on the Hadamard manifold by using the bifunction.

Convexificators are a concept that has been utilized recently to extend a variety of findings in nonsmooth analysis and optimization, see [6, 10–12, 16, 19]. From an optimization and application perspective, the descriptions of the optimality conditions in terms of convexificators yield more precise results because, in general, convexificators are closed sets, unlike the well-known subdifferentials, which are convex and compact. This study aims to bridge these gaps by extending the theory of convexificators to the Hadamard manifolds, deriving new versions of the mean value theorem, and investigating the monotonicity and geodesic convexity of bounded convexificators. Furthermore, the work provides a rigorous formulation and analysis of convexificator-based vector variational inequality problems (VVIPs) and establishes the necessary and sufficient conditions for vector optimization problems on the Hadamard manifolds. These results not only advance the mathematical theory but also open new pathways for solving complex problems in applied fields where non-Euclidean geometries are essential.

Motivated by the above work, we extend the concept of convexificators to the Hadamard manifold and discuss several relations for the monotonicity of $\partial^*_* f$ and ∂^*_*-convexity. Furthermore, we prove the mean value theorem using convexificators on the Hadamard manifold and extend the concept of VVIPs to the Hadamard manifold. Additionally, we use it as a tool for finding the solution of VOPs.

2. Preliminaries

For the purpose of comprehending the fundamental ideas of this work, some definitions, theorems, and results pertaining to Riemannian manifolds are reviewed in this section. For more study on Riemannian manifolds, see [3, 24, 25, 28].

Let $\mathbb{R}^m$ be an m-dimensional Euclidean space and $\mathbb{R}^m_+$ be its non-negative orthant.

Let $p = (p_1, p_2, ..., p_m)$ and $q = (q_1, q_2, ..., q_m)$ be the two vectors in $\mathbb{R}^m$. Then,

$$\begin{aligned} p \leqq q &\Leftrightarrow p_l \leq q_l \quad \text{for } l = 1, 2, ..., m \quad \Leftrightarrow p - q \in -\mathbb{R}^m_+; \\ p \leq q &\Leftrightarrow p_l \leq q_l \quad \text{for} \quad l = 1, 2, ..., m \quad \text{and} \quad p \neq q \quad \Leftrightarrow p - q \in -\mathbb{R}^m_+; \\ p < q &\Leftrightarrow p_l < q_l \quad \text{for} \quad l = 1, 2, ..., m \quad \Leftrightarrow p - q \in -int\, \mathbb{R}^m_+. \end{aligned}$$

Definition 2.1. [14] Let $\Psi : \mathbb{R}^m \longrightarrow \mathbb{R} \cup \{+\infty\}$ be such that for $p \in \mathbb{R}^m$, $\Psi(p)$ is finite. The lower and upper Dini derivative of Ψ at p in the given direction of $w \in \mathbb{R}^m$ are defined, respectively, as follows:

$$\Psi^{-}(p, w) := \liminf_{t \downarrow 0} \frac{\Psi(p + tw) - \Psi(p)}{t},$$

$$\Psi^{+}(p, w) := \limsup_{t \downarrow 0} \frac{\Psi(p + tw) - \Psi(p)}{t}.$$

Definition 2.2. [14] Let $\Psi : \mathbb{R}^m \longrightarrow \mathbb{R} \cup \{+\infty\}$ be such that for $p \in \mathbb{R}^m$, $\Psi(p)$ is finite. Then, the function Ψ is said to have:

(1) An upper convexificator $\partial^*\Psi(p) \subset \mathbb{R}^m$ at $p \in \mathbb{R}^m$, iff $\partial^*\Psi(p)$ is closed and for each $w \in \mathbb{R}^m$, one has

$$\Psi^{-}(p; w) \leq \sup_{\xi \in \partial^*\Psi(p)} \langle \xi, w \rangle .$$

(2) A lower convexificator $\partial_*\Psi(p) \subset \mathbb{R}^m$ at $p \in \mathbb{R}^m$, iff $\partial_*\Psi(p)$ is closed and for each $w \in \mathbb{R}^m$, one has

$$\Psi^{+}(p; w) \geq \inf_{\xi \in \partial_*\Psi(p)} \langle \xi, w \rangle .$$

(3) A convexificator $\partial^*_*\Psi(p) \subset \mathbb{R}^m$ at $p \in \mathbb{R}^m$, iff $\partial^*_*\Psi(p)$ is both the upper and lower convexificator of Φ at p.

Let $\mathcal{M}$ be an m-dimensional Riemannian manifold with Levi-civita (or Riemannian) connection ∇. The scalar product on $T_p\mathcal{M}$ with the norm $\|\cdot\|$ is denoted by $\langle \cdot, \cdot \rangle$.

For any $p, q \in \mathcal{M}$, let $\gamma_{pq} : [0, 1] \longrightarrow \mathcal{M}$ be a piece-wise smooth curve joining p to q. Then the arc length of $\gamma_{pq}(t)$ is:

$$L(\gamma_{pq}) := \int_0^1 \|\dot{\gamma}_{pq}(t)\| dt,$$

where $\dot{\gamma}_{pq}(t)$ is the tangent vector to the curve γ_{pq}.

A smooth curve γ_{pq} satisfying the conditions $\gamma_{pq}(0) = p$, $\gamma_{pq}(1) = q$, and $\nabla_{\dot{\gamma}_{pq}} \dot{\gamma}_{pq} = 0$ on $[0, 1]$ is called a geodesic on manifold. If we take two points $p, w \in \mathcal{M}$, $P_{w,p}$ denotes the parallel transport from $T_p\mathcal{M}$ to $T_w\mathcal{M}$.

By the Hopf-Rinow theorem, we know that, if any two points on $\mathcal{M}$ can be joined by a minimal geodesic, then $\mathcal{M}$ is a complete Riemannian manifold and the arc-length of the geodesic is called the Riemannian distance between p and q and it is defined as $d(p, q) = \inf_{\gamma_{pq}} L(\gamma_{pq})$.

Now, recall that a function $\Psi : \mathcal{M} \longrightarrow \mathbb{R}$ is said to be Lipschitz on the given subset $\mathcal{K}$ of $\mathcal{M}$ if $\exists$ $\lambda \geq 0$, such that

$$|\Psi(p) - \Psi(q)| \leq \lambda d(p, q), \quad \forall\ p, q \in \mathcal{K}.$$

A function $\Psi : \mathcal{M} \longrightarrow \mathbb{R}$ is said to be a locally Lipschitz function at point $p_o \in \mathcal{M}$, if $\exists\ \lambda(p_o) \geq 0$ such that the above inequality satisfies with $\lambda = \lambda(p_o)$ for any p, q in a neighborhood of p_o. Let us recall some basic definitions of the generalized derivative for locally Lipschitz function on $\mathcal{M}$.

Definition 2.3. [20] Let $\Psi : \mathcal{M} \longrightarrow \mathbb{R}$ be a locally Lipschitz function. Let $p, q \in \mathcal{M}$, the generalized directional derivative $\Psi^o(p; v)$ of Ψ at a point p in the direction $v \in T_p\mathcal{M}$ defined as

$$\Psi^\circ(p; v) = \limsup_{q \to p, t \downarrow 0, q \in \mathcal{M}} \frac{\Psi \circ \Phi^{-1}(\Phi(q) + td\Phi(p)(v)) - \Psi \circ \Phi^{-1}(\Phi(q))}{t},$$

where $\Phi : U \subseteq \mathcal{M} \longrightarrow \Phi(U) \subseteq \mathbb{R}^m$ is a homeomarphism, that is (U, Φ) is the chart about the point p.

Definition 2.4. [20] Let $\Psi : \mathcal{M} \longrightarrow \mathbb{R}$ be a locally Lipschitz function on Riemannian manifold. Then, the generalized gradient of Ψ at the point $q \in \mathcal{M}$ is the subset $\partial_c\Psi(q)$ of $T_q^*\mathcal{M} \cong T_q\mathcal{M}$ defined as

$$\partial_c\Psi(q) = \{\xi \in T_q\mathcal{M} : \Psi^\circ(q; v) \geq \langle \xi, q \rangle, \quad \forall v \in T_q\mathcal{M}\}.$$

Definition 2.5. [15] (Hadamard manifold): A complete, simply connected Riemannian manifold which has non-positive sectional curvature is called a Hadamard manifold, and we denote it by $\mathbb{H}$ throughout the paper.

Proposition 2.6. [21] *Let p be any point of the Hadamard manifold $\mathbb{H}$. Then, $\exp_p : T_p\mathbb{H} \longrightarrow \mathbb{H}$ is a diffeomorphism. For any $p, q \in \mathbb{H}$, there exists a unique minimal geodesic γ_{pq} joining p to q such that*

$$\gamma_{pq}(t) = \exp_p(t \exp_p^{-1} q), \quad \forall\, t \in [0, 1].$$

Definition 2.7. [28] A set $\mathcal{K} \subseteq \mathbb{H}$ is said to be geodesic convex (GC) if for any two points $p, q \in \mathcal{K}$, $\exp_x(t \exp_p^{-1} q) \in \mathcal{K}$.

Definition 2.8. [28] Suppose $\mathcal{K} \subseteq \mathbb{H}$ is a GC set. Then $\Psi : \mathcal{K} \longrightarrow \mathbb{R}$ is said to be a convex function if for every $p, q \in \mathcal{K}$,

$$\Psi(\exp_p t \exp_p^{-1} q) \leq t\Psi(p) + (1 - t)\Psi(q), \quad \forall\, t \in [0, 1].$$

Definition 2.9. [1] Let $\Psi : \mathbb{H} \longrightarrow \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function on $\mathbb{H}$ and p be a point where Ψ is finite.

(1) The Dini-lower directional derivative at point $p \in \mathbb{H}$ in the direction $v \in T_p\mathbb{H}$ is defined as

$$\Psi^-(p; v) := \liminf_{t \to 0^+} \frac{\Psi(\exp_p tv) - \Psi(p)}{t}.$$

(2) The Dini-upper directional derivative at point $p \in \mathbb{H}$ in the direction $v \in T_p\mathbb{H}$ is defined as

$$\Psi^+(p; v) := \limsup_{t \to 0^+} \frac{\Psi(\exp_p tv) - \Psi(p)}{t}.$$

As discussed in [1], for a fixed $s \in (0, 1)$, we take a point $w = \gamma_{pq}(s) = \exp_p(s \exp_p^{-1} q)$ on the geodesic $\gamma_{pq} : [0, 1] \longrightarrow H$, which divides the geodesic into two parts. The first part can be written as

$$\gamma_{wp}(t) = \gamma_{pq}(-st + s) = \exp_p(-st + s) \exp_p^{-1} q, \quad \forall \quad t \in [0, 1],$$

that is,

$$\exp_w(t \exp_w^{-1} p) = \exp_p(-st + s) \exp_p^{-1} q, \quad \forall \quad t \in [0, 1], \tag{2.1}$$

and the second part can be written as

$$\gamma_{wq} = \gamma_{pq}((1-s)t+s) = \exp_p(((1-s)t+s)\exp_p^{-1} q), \quad \forall \quad t \in [0,1],$$

that is,

$$\exp_w(t\exp_w^{-1} q) = \exp_p(((1-s)t+s)\exp_p^{-1} q), \quad \forall \quad t \in [0,1]. \tag{2.2}$$

From (2.1) and (2.2), we get

$$\exp_w^{-1} p = -sP_{w,p}\exp_p^{-1} q, \tag{2.3}$$

$$\exp_w^{-1} q = (1-s)P_{w,p}\exp_p^{-1} q. \tag{2.4}$$

Similarly, we have

$$\exp_w^{-1} p = sP_{w,q}\exp_q^{-1} p. \tag{2.5}$$

3. Convexity and monotonicity of convexificators

In this section, we first prove the mean value theorem for convexificators on the Hadamard manifold. We extend the notions of convexity and monotonicity of vector-valued functions using convexificators to the Riemannian manifold, particularly the Hadamard manifold, and establish some relations between them.

Definition 3.1. Let $\Psi : \mathbb{H} \longrightarrow \bar{\mathbb{R}}$ be an extended real-valued function, $p \in \mathbb{H}$, and $\Psi(p)$ is finite.

(1) The function Ψ is said to have an upper convexificator $\partial^*\Psi(p) \subset T_p\mathbb{H}$ at a point $p \in \mathbb{H}$, iff $\partial^*\Psi(p)$ is closed and for each $v \in T_p\mathbb{H}$,

$$\Psi^-(p;v) \le \sup_{\xi\in\partial^*\Psi(x)} \langle \xi; v\rangle .$$

(2) The function Ψ is said to have a lower convexificator $\partial_*\Psi(p) \subset T_p\mathbb{H}$ at point $p \in \mathbb{H}$, iff $\partial_*\Psi(p)$ is closed and for each $v \in T_p\mathbb{H}$,

$$\Psi^+(p;v) \ge \inf_{\xi\in\partial_*\Psi(p)} \langle \xi; v\rangle .$$

(3) The function Ψ is said to have a convexificator $\partial^*_*\Psi(p) \subset T_p\mathbb{H}$ at point $p \in \mathbb{H}$, iff $\partial^*_*\Psi(p)$ is both upper and lower convexificator of Ψ at p.

Theorem 3.2. *[Mean value theorem] Suppose $\mathcal{K}(\neq \phi) \subseteq \mathbb{H}$ is a GC set. Let $p, q \in \mathcal{K}$ and let $\Psi : \mathcal{K} \longrightarrow \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ be finite and continuous. Suppose that, for each $t \in (0,1)$, $z(t) := \exp_p(t\exp_p^{-1} q)$, $\partial^*\Psi(z)$, and $\partial_*\Psi(z)$ are respectively upper and lower convexificators of Ψ. Then, there exists $w(t) \in (p,q)$ and a sequence $\{\xi_k\} \subset co(\partial^*\Psi(w) \cup \partial_*\Psi(w))$ such that*

$$\Psi(q) - \Psi(p) = \lim_{k\to\infty} \left\langle \xi_k; P_{w,p}\exp_p^{-1} q \right\rangle ,$$

or

$$\Psi(q) - \Psi(p) = \left\langle \xi; P_{w,p}\exp_p^{-1} q \right\rangle .$$

Proof. Consider a function $\rho : [0,1] \longrightarrow \mathbb{R}$, such that

$$\rho(t) := \Psi(\exp_p t \exp_p^{-1} q) - \Psi(p) + t(\Psi(p) - \Psi(q)).$$

Here, ρ is continuous on $[0,1]$ and $\rho(0) = \rho(1) = 0$. Then, $\exists\, \mu \in (0,1)$ such that μ is the extremum point of ρ. Define

$$w(\mu) = \exp_p \mu \exp_p^{-1} q.$$

Without loss of generality, let μ be the minimal point of ρ, then using the necessary condition of a minimal point, for each $v \in \mathbb{R}$,

$$\rho_d^-(\mu; v) \geq 0,$$

since,

$$\rho_d^-(\mu; v) := \liminf_{k \to 0^+} \frac{\rho(\mu + kv) - \rho(\mu)}{k}.$$

Therefore, we have

$$\liminf_{k \to 0^+} \frac{\Psi(\exp_p(\mu + kv) \exp_p^{-1} q) - \Psi(\exp_p \mu \exp_p^{-1} q)}{k} + v(\Psi(p) - \Psi(q)) \geq 0,$$

since,

$$\exp_p(\mu + kv) \exp_p^{-1} q = \exp_p \left(-\mu \left(\frac{kv}{-\mu} \right) + \mu \right) \exp_p^{-1} q. \tag{3.1}$$

Now, suppose

$$\frac{kv}{-\mu} = \lambda \quad \text{(say)}.$$

Therefore, Eq (3.1) becomes

$$\begin{aligned} \exp_p(\mu + kv) \exp_p^{-1} q &= \exp_p (-\mu\lambda + \mu) \exp_p^{-1} q = \gamma_{wp}(\lambda) \\ &= \exp_w \lambda \exp_w^{-1} p \\ &= \exp_w k \left(\frac{v}{-\mu} \right) \exp_w^{-1} p. \end{aligned}$$

Hence, from the above inequality

$$\begin{aligned} &\liminf_{k \to 0^+} \frac{\Psi\left(\exp_w k \left(\frac{v}{-\mu} \exp_w^{-1} p \right)\right) - \Psi(\exp_p \mu \exp_p^{-1} q)}{k} + v\,(\Psi(p) - \Psi(q)) \geq 0, \\ &\liminf_{k \to 0^+} \frac{\Psi\,(\exp_w kv') - \Psi(w)}{k} + v\,(\Psi(p) - \Psi(q)) \geq 0, \\ &\Psi_d^-(w; v') + v(\Psi(p) - \Psi(q)) \geq 0, \end{aligned}$$

$$\Psi_d^-\left(w; \frac{v}{-\mu} \exp_w^{-1} p\right) + v(\Psi(p) - \Psi(q)) \geq 0.$$

We know that

$$-\frac{1}{\mu} \exp_w^{-1} p = P_{w,p} \exp_p^{-1} q.$$

This implies that

$$\Psi_d^-(w; vP_{w,p} \exp_p^{-1} q) \geq v(\Psi(q) - \Psi(p)).$$

Now, putting $v = 1$ and $v = -1$, respectively, we get

$$-\Psi_d^-(w; P_{w,p} \exp_p^{-1} q) \leq \Psi(q) - \Psi(p) \leq \Psi_d^-(w; P_{w,p} \exp_p^{-1} q),$$

since $\partial^*\Psi(w)$ is an upper convexificator of Ψ at w, and we have

$$\inf_{\xi \in \partial^*\Psi(w)} \left\langle \xi; P_{w,p} \exp_p^{-1} q \right\rangle \leq \Psi(q) - \Psi(p) \leq \sup_{\xi \in \partial^*\Psi(w)} \left\langle \xi; P_{w,p} \exp_p^{-1} q \right\rangle.$$

Then, this inequality follows that $\exists$ sequence $\{\xi_k\} \subset \text{co}(\partial^*\Psi)$ such that

$$\Psi(q) - \Psi(p) = \lim_{k \to 0} \left\langle \xi_k; P_{w,p} \exp_p^{-1} q \right\rangle$$

or

$$\Psi(q) - \Psi(p) = \left\langle \xi; P_{w,p} \exp_p^{-1} q \right\rangle$$

holds with some $\xi \in \text{co}(\partial^*\Psi(w) \cup \partial_*\Psi(w))$.

On the other hand, if μ is the maximal point of ρ, then using the same arguments as above, we get the conclusion. Hence,

$$\Psi(q) - \Psi(p) = \left\langle \xi; P_{w,p} \exp_p^{-1} q \right\rangle$$

holds with some $\xi \in \text{co}(\partial^*\Psi(w) \cup \partial_*\Psi(w))$. □

Definition 3.3. Suppose $\mathcal{K}(\neq \phi) \subseteq \mathbb{H}$ is a GC set and $\Psi : \mathcal{K} \longrightarrow \mathbb{R}^m$ is a function such that $\Psi_i : \mathcal{K} \longrightarrow \mathbb{R}$ are locally Lipschitz at $\bar{p} \in \mathcal{K} \subseteq \mathbb{H}$ and admit a bounded convexificator $\partial_*^*\Psi_i(\bar{p})$ at a point $\bar{p}$ for all $\forall\ i \in M = \{1, 2, ...m\}$. Then, Ψ is said to be:

(1) ∂_*^*-convex at point $\bar{p}$ over $\mathcal{K}$, iff for any $p \in \mathcal{K}$ and $\xi^* \in \partial_*^*\Psi(\bar{p})$, such that

$$\Psi(p) - \Psi(\bar{p}) \geqq \left\langle \xi^*; \exp_{\bar{p}}^{-1} p \right\rangle_m,$$

(2) strictly ∂_*^*-convex at point $\bar{p}$ over $\mathcal{K}$, iff for any $p \in \mathcal{K}$ and $\xi^* \in \partial_*^*\Psi(\bar{p})$,

$$\Psi(p) - \Psi(\bar{p}) > \left\langle \xi^*; \exp_{\bar{p}}^{-1} p \right\rangle_m,$$

where,

$$\xi^* := (\xi_1^*, \xi_2^*, ..., \xi^*),$$

$$\partial_*^*\Psi(\bar{p}) := \partial_*^*\Psi_1(\bar{p}) \times ... \times \partial_*^*\Psi_m(\bar{p}),$$

$$\left\langle \xi^*; \exp_{\bar{p}}^{-1} p \right\rangle_m := (\left\langle \xi_1^*; \exp_{\bar{p}}^{-1} p \right\rangle, \left\langle \xi_2^*; \exp_{\bar{p}}^{-1} p \right\rangle, ..., \left\langle \xi_m^*; \exp_{\bar{p}}^{-1} p \right\rangle).$$

Definition 3.4. Let $\Psi := (\Psi_1, \Psi_2, ..., \Psi_m) : \mathcal{K} \longrightarrow \mathbb{R}^m$ be a vector-valued function such that $\Psi_i : \mathcal{K} \longrightarrow \mathbb{R}$ are locally Lipschitz on $\mathcal{K} \subseteq \mathbb{H}$ and admit a bounded convexificator $\partial_*^*\Psi_i(p)$ for all $p \in \mathcal{K}$ and $\forall\, i \in M = \{1, 2, ...m\}$. Then, $\partial_*^*\Psi$ is said to be:

(1) monotone on $\mathcal{K}$, iff for any $p, q \in \mathcal{K}$, $\xi \in \partial_*^*\Psi(p)$, and $\zeta \in \partial_*^*\Psi(q)$, one has

$$\left\langle P_{q,p}\xi - \zeta; \exp_q^{-1} p \right\rangle_m \geqq 0;$$

(2) strictly monotone on $\mathcal{K}$, iff for any $p, q \in \mathcal{K}$, $\xi \in \partial_*^*\Psi(p)$, and $\zeta \in \partial_*^*\Psi(q)$, one has

$$\left\langle P_{q,p}\xi - \zeta; \exp_q^{-1} q \right\rangle_m > 0.$$

In the following theorem, we discuss an important characterization of ∂_*^*-convex functions in terms of monotonicity.

Theorem 3.5. *Suppose $\mathcal{K}(\neq \phi) \subseteq \mathbb{H}$ is a GC set and $\Psi : \mathcal{K} \longrightarrow \mathbb{R}^m$ be a function such that $\Psi_i : \mathcal{K} \longrightarrow \mathbb{R}$ are locally Lipschitz functions on $\mathcal{K}$ and admit bounded convexificators $\partial_*^*\Psi_i(p)$, $\forall\, p \in \mathcal{K}$ and $\forall\, i \in M = \{1, 2, ...m\}$. Then, Ψ is ∂_*^*-convex on $\mathcal{K}$ iff $\partial_*^*\Psi$ is monotone on $\mathcal{K}$.*

Proof. Suppose that Ψ is ∂_*^*-convex on $\mathcal{K}$. Then, for any $p, q \in \mathcal{K}$, $\xi \in \partial_*^*\Psi(p)$, and $\zeta \in \partial_*^*\Psi(q)$, one has

$$\Psi(p) - \Psi(q) \geqq \left\langle \zeta; \exp_q^{-1} p \right\rangle_m, \tag{3.2}$$

and

$$\Psi(q) - \Psi(p) \geqq \left\langle \xi; \exp_p^{-1} q \right\rangle_m. \tag{3.3}$$

Adding (3.2) and (3.3), we have

$$\left\langle P_{q,p}\xi - \zeta; \exp_q^{-1} p \right\rangle_m \geqq 0.$$

Hence, $\partial_*^*\Psi$ is monotone on $\mathcal{K}$.

For the converse, let $\partial_*^*\Psi$ be monotone on $\mathcal{K}$ and $z(\mu) := \exp_q(\mu \exp_q^{-1} p)\ \forall\, \mu \in [0, 1]$. By the geodesic convexity of $\mathcal{K}$, $z(\mu) \in \mathcal{K}$, $\forall\, \mu \in [0, 1]$. By Theorem 3.2, for $i \in M$, and $\hat{\mu} \in (0, 1)$, $\exists$ $\tilde{\mu}_i \in (0, \hat{\mu})$ and $\bar{\mu}_i \in (\hat{\mu}, 1)$ such that for $\tilde{\xi}_i \in co\partial_*^*\Psi_i(z(\tilde{\mu}_i))$ and $\bar{\xi}_i \in co\partial_*^*\Psi_i(z(\bar{\mu}_i))$,

$$\Psi_i(z(\hat{\mu})) - \Psi_i(z(0)) = \left\langle \tilde{\xi}_i; \exp_{z(0)}^{-1} z(\hat{\mu}) \right\rangle = \hat{\mu} \left\langle \tilde{\xi}_i; \exp_y^{-1} p \right\rangle,$$

and

$$\Psi_i(z(1)) - \Psi_i(z(\hat{\mu})) = \left\langle \bar{\xi}_i; \exp_{z(\hat{\mu})}^{-1} z(1) \right\rangle = (1-\hat{\mu}) \left\langle \bar{\xi}; exp_q^{-1} p \right\rangle.$$

By the monotonicity of $\partial_*^*\Psi$ on $\mathcal{K}$, for any $i \in M$ and $\zeta_i \in co\partial_*^*\Psi_i(q)$, it follows that

$$\Psi_i(z(\hat{\mu})) - \Psi_i(z(0)) \geq \hat{\mu} \left\langle \zeta_i; \exp_q^{-1} p \right\rangle,$$

$$\Psi_i(z(1)) - \Psi_i(z(\hat{\mu})) \geq (1-\hat{\mu}) \left\langle \zeta_i : \exp_q^{-1} p \right\rangle.$$

By adding the above inequalities, we get

$$\Psi_i(p) - \Psi_i(q) \geq \left\langle \zeta_i; \exp_q^{-1} p \right\rangle.$$

$\implies$ Ψ is ∂_*^*-convex on $\mathcal{K}$. □

Corollary 3.6. *Suppose $\mathcal{K}(\neq \phi) \subseteq \mathbb{H}$ is a GC set and let $\Psi : \mathcal{K} \longrightarrow \mathbb{R}^m$ be a vector-valued function such that $\Psi_i : \mathcal{K} \longrightarrow \mathbb{R}$ are locally Lipschitz functions on $\mathcal{K}$ and admit bounded convexificators $\partial_*^*\Psi(p)$ for any $p \in \mathcal{K}$ and $i \in M = \{1, 2, ...m\}$. Then, Ψ is strictly ∂_*^*-convex on $\mathcal{K}$ iff $\partial_*^*\Psi$ is strictly monotone on $\mathcal{K}$.*

Proposition 3.7. *Suppose $\mathcal{K}(\neq \phi) \subseteq \mathbb{H}$ is a GC set and let $\Psi : \mathcal{K} \longrightarrow \mathbb{R}^m$ be a function such that $\Psi_i : \mathcal{K} \longrightarrow \mathbb{R}$ are locally Lipschitz functions on $\mathcal{K}$ and admit a bounded convexificator $\partial_*^*\Psi(p)$ for any $p \in \mathcal{K}$ and $\forall\ i \in M$. If Ψ is ∂_*^*-convex on $\mathcal{K}$, then for any $p, q \in \mathcal{K}$ and $\mu \in [0, 1]$,*

$$\Psi(\exp_q \mu \exp_q^{-1} p) \leqq \Psi(q) + \mu(\Psi(p) - \Psi(q)).$$

Proof. Let $p, q \in \mathcal{K}$ and $z(\mu) := \exp_q \mu \exp_q^{-1} p$ for any $\mu \in [0, 1]$. By the geodesic convexity of $\mathcal{K}$, $z \in \mathcal{K}$. By the ∂_*^*-convexity of Ψ on $\mathcal{K}$, for any $\zeta \in \partial_*^*\Psi(z)$,

$$\Psi(p) - \Psi(z) \geqq \left\langle \zeta; \exp_z^{-1} p \right\rangle_m = (1-\mu) \left\langle \zeta; \exp_q^{-1} p \right\rangle_m, \tag{3.4}$$

and

$$\Psi(q) - \Psi(z) \geqq \left\langle \zeta; \exp_z^{-1} q \right\rangle_m = -\mu \left\langle \zeta; \exp_q^{-1} p \right\rangle_m. \tag{3.5}$$

From (3.4) and (3.5), we have

$$\Psi(z) \leqq \mu\Psi(p) + (1-\mu)\Psi(q),$$

that is,

$$\Psi(\exp_q \mu \exp_q^{-1} p) \leqq \Psi(q) + \mu(\Psi(p) - \Psi(q)).$$

□

Proposition 3.8. *Suppose $\mathcal{K}(\neq \phi) \subseteq \mathbb{H}$ is a GC set and let $\Psi : \mathcal{K} \longrightarrow \mathbb{R}^m$ be a function such that $\Psi_i : \mathcal{K} \longrightarrow \mathbb{R}$ are locally Lipschitz functions on $\mathcal{K}$ and for any $p \in \mathcal{K}$ admit a bounded convexificator $\partial_*^*\Psi(p)$, $\forall\ i \in M$. If Ψ is strictly ∂_*^*-convex on $\mathcal{K}$, then, for any $p, q \in \mathcal{K}$ and $\mu \in [0, 1]$,*

$$\Psi(\exp_q \mu \exp_q^{-1} p) < \Psi(q) + \mu(\Psi(p) - \Psi(q)).$$

Proof. The proof is analogous to Proposition 3.7. □

4. Vector variational inequality problems using convexificators

In this section, we consider the VVIP in terms of the convexificators on the Hadamard manifold and construct an example in support of the definition of convexificators. Moreover, we show the existence of Stampacchia ∂^*_*-VVI. Furthermore, we establish the relations among Stampacchia ∂^*_*-VVI, the Minty-type ∂^*_*-VVI, and VOP.

Suppose $\mathcal{K}(\neq \phi) \subseteq \mathbb{H}$ is a set and let $\Psi : \mathcal{K} \longrightarrow \mathbb{R}^m$ be a vector-valued function. We define:

Stampacchia ∂^*_*-VVI : Find $\bar{p} \in \mathcal{K}$, such that for any $q \in \mathcal{K}$, $\exists\, \xi \in \partial^*_*\Psi(\bar{p})$, and one has

$$\left\langle \xi; \exp^{-1}_{\bar{p}} q \right\rangle_m \notin -\mathbb{R}^m_+ \setminus \{0\},$$

or

$$\left(\left\langle \xi_1; \exp^{-1}_{\bar{p}} q \right\rangle, \left\langle \xi_2; \exp^{-1}_{\bar{p}} q \right\rangle, ..., \left\langle \xi_m; \exp^{-1}_{\bar{p}} q \right\rangle\right) \notin -\mathbb{R}^m_+ \setminus \{0\}.$$

Minty ∂^*_*-VVI : Find $\bar{p} \in \mathcal{K}$ such that for any $q \in \mathcal{K}$ and $\xi \in \partial^*_*\Psi(q)$, one has

$$\left\langle \xi; \exp^{-1}_{q} \bar{p} \right\rangle_m \notin \mathbb{R}^m_+ \setminus \{0\},$$

or

$$\left(\left\langle \xi_1; \exp^{-1}_{q} \bar{p} \right\rangle, \left\langle \xi_2; \exp^{-1}_{q} \bar{p} \right\rangle, ..., \left\langle \xi_m; \exp^{-1}_{q} \bar{p} \right\rangle\right) \notin \mathbb{R}^m_+ \setminus \{0\}.$$

In the following example, we show the existence of convexificators for the Hadamard manifolds and existence of a solution of the Stampacchia ∂^*_*-VVI.

Example 4.1. Let $\mathbb{H} = \{(p_1, p_2) \in \mathbb{R}^2 : p_1, p_2 > 0\}$ be a Hadamard manifold with the Riemannian metric $g_{i,j}(p_1, p_2) = (\frac{\delta_{i,j}}{p_i p_j})$ for $i = 1, 2$, where $\delta_{i,j}$ denotes the Kronecker delta. The geodesic passing at moment $t = 0$, through the point $p = (p_1, p_2)$, tangent to the vector $v = (v_1, v_2) \in T_p\mathbb{H}$ is given by

$$\gamma_v(t) = (p_1 e^{\frac{v_1}{p_1}t}, p_2 e^{\frac{v_2}{p_2}t}).$$

Consider the function $\Psi : \mathbb{H} \longrightarrow \mathbb{R}^2$ such that

$$\Psi(p) = (\Psi_1(p), \Psi_2(p)) = (|\ln p_1| + (\ln p_2)^2, (\ln p_1)^2 + |\ln p_2|).$$

Since, $\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t) = (p_1 e^{\frac{v_1}{p_1}t}, p_2 e^{\frac{v_2}{p_2}t})$ with the velocity vector $\gamma'_v(0) = (v_1, v_2) \in T_p\mathbb{H}$, for any $p \in \mathbb{H},\ v \in T_p\mathbb{H}$, and $t > 0$, from the triangle inequality, one has

$$\frac{\Psi_1(\exp_p tv) - \Psi_1(p)}{t} \leq \frac{|v_1|}{p_1} + \frac{v_2^2}{p_2^2}t + 2(\ln p_2)\frac{v_2}{p_2},$$

$$\frac{\Psi_1(\exp_p tv) - \Psi_1(p)}{t} \geq -\frac{|v_1|}{p_1} + \frac{v_2^2}{p_2^2}t + 2(\ln p_2)\frac{v_2}{p_2}.$$

Taking lim inf and lim sup as $t \to 0$, we have

$$\Psi_1^-(p; v) = \liminf_{t \to 0^+} \frac{\Psi_1(\exp_p tv) - \Psi_1(p)}{t} \leq \frac{|v_1|}{p_1} + 2(\ln p_2)\frac{v_2}{p_2},$$

$$\Psi_1^+(p;w) = \limsup_{t\to 0^+} \frac{\Psi_1(\exp_p tv) - \Psi_1(p)}{t} \geq -\frac{|v_1|}{p_1} + 2(\ln p_2)\frac{v_2}{p_2}.$$

Hence, the convexificators of Ψ_1 at p are given as follows:

$$\partial_*^*\Psi_1(p) = \begin{cases} \left\{\left(\frac{1}{p_1}, 2\frac{(\ln p_2)}{p_2}\right)\right\}, & p_1 > 1, \\ \left\{\left(1, 2\frac{(\ln p_2)}{p_2}\right), \left(-1, 2\frac{(\ln p_2)}{p_2}\right)\right\}, & p_1 = 1, \\ \left\{\left(-\frac{1}{p_1}, 2\frac{(\ln p_2)}{p_2}\right)\right\}, & 0 < p_1 < 1. \end{cases}$$

Similarly, for any $p \in \mathbb{H},\ v \in T_p\mathbb{H}$, and $t > 0$, from the triangle inequality, one has

$$\Psi_2^-(p;w) \leq 2(\ln p_1)\frac{v_1}{p_1} + \frac{|v_2|}{p_2},$$

$$\Psi_2^+(p;w) \geq 2(\ln p_1)\frac{v_1}{p_1} - \frac{|v_2|}{p_2}.$$

Hence, the convexificators of Ψ_2 at p are given as follows:

$$\partial_*^*\Psi_2(p) = \begin{cases} \left\{\left(2\frac{\ln p_1}{p_1}, \frac{1}{p_2}\right)\right\}, & p_2 > 1, \\ \left\{\left(2\frac{\ln p_1}{p_1}, 1\right), \left(2\frac{\ln p_1}{p_1}, -1\right)\right\}, & p_2 = 1, \\ \left\{\left(2\frac{\ln p_1}{p_1}, -\frac{1}{p_2}\right)\right\}, & 0 < p_2 < 1. \end{cases}$$

For any $q = (q_1, q_2) \in \mathbb{H}$ and $p = (1,1)$, $\xi_{11} := (1,0)$, $\xi_{12} := (-1,0) \in \partial_*^*\Psi_1(1,1)$, and $\xi_{21} := (0,1)$, and $\xi_{22} := (0,-1) \in \partial_*^*\Psi_2(1,1)$, and we have

$$\left\langle \xi_{11}; \exp_p^{-1} q \right\rangle = \ln q_1; \ \left\langle \xi_{12}; \exp_p^{-1} q \right\rangle = -\ln q_1,$$

$$\left\langle \xi_{21}; \exp_p^{-1} q \right\rangle = \ln q_2; \ \left\langle \xi_{22}; \exp_p^{-1} q \right\rangle = -\ln q_2,$$

which implies that, for any $q \in \mathbb{H}$, there exists $\xi \in \partial_*^*\Psi(p)$ such that

$$\left\langle \xi; \exp_p^{-1} q \right\rangle_2 \in \mathbb{R}_+^2.$$

Therefore, $p = (1,1)$ is a solution of the Stampacchia ∂_*^*-VVI.

In the following proposition, we discuss a relationship between the Stampacchia ∂_*^*-VVI and Minty ∂_*^*-VVI.

Proposition 4.2. *Suppose $\mathcal{K}(\neq \phi) \subseteq \mathbb{H}$ is a GC set and let $\Psi : \mathcal{K} \longrightarrow \mathbb{R}^m$ be a function such that $\Psi_i : \mathcal{K} \longrightarrow \mathbb{R}$ are locally Lipschitz functions on $\mathcal{K}$ and, for any $p \in \mathcal{K}$, admit a bounded convexificator $\partial_*^*\Psi_i(p)$ $\forall$, $i \in M = \{1, 2, ..., m\}$. Also, suppose that Ψ is ∂_*^*-convex on $\mathcal{K}$. If $\bar{p} \in \mathcal{K}$ is a solution of the Stampacchia ∂_*^*-VVIP, then $\bar{p}$ is also a solution of the Minty ∂_*^*-VVIP.*

Proof. Let $\bar{p}$ be a solution of the Stampacchia ∂_*^*-VVIP. Then, for any $q \in \mathcal{K}$, $\exists\, \xi \in \partial_*^*\Psi(\bar{p})$ such that

$$\left\langle \xi; \exp_{\bar{p}}^{-1} q \right\rangle_m \notin -\mathbb{R}_+^m \setminus \{0\}.$$

Since Ψ is ∂_*^*-convex on $\mathcal{K}$, by Theorem 3.5, $\partial_*^*\Psi$ is monotone over $\mathcal{K}$, which implies that for any $y \in \mathcal{K}$ and $\zeta \in \partial_*^*\Psi(y)$, we have

$$\left\langle \zeta; \exp_{q}^{-1} \bar{p} \right\rangle_m \notin \mathbb{R}_+^m \setminus \{0\}.$$

Hence, $\bar{p}$ is a solution of the Minty ∂_*^*-VVIP. □

Vector optimization problem (VOP): Let $\mathcal{K}(\neq \phi) \subseteq \mathbb{H}$ and $\Psi : \mathbb{H} \longrightarrow \mathbb{R}^m$ be a vector-valued function. We consider a vector optimization problem as follows:

$$min\ \Psi(p) = (\Psi_1(p), \Psi_2(x), ..., \Psi_m(p)),$$

$$such\ that\ p \in \mathcal{K},$$

where $\Psi_i : \mathcal{K} \longrightarrow \mathbb{R}$ are real-valued functions $\forall\, i \in M = \{1, 2, ..., m\}$.

Definition 4.3. A point $\bar{p} \in \mathcal{K}$ is said to be:

(1) an efficient solution of the VOP if

$$\Psi(q) - \Psi(\bar{p}) = (\Psi_1(q) - \Psi_1(\bar{p}), \Psi_2(q) - \Psi_2(\bar{p}), ..., \Psi_m(q) - \Psi_m(\bar{p})) \notin -\mathbb{R}_+^m \setminus \{0\}\ \forall\, q \in \mathcal{K};$$

(2) a weakly efficient solution of the VOP if

$$\Psi(q) - \Psi(\bar{p}) = (\Psi_1(q) - \Psi_1(\bar{p}), \Psi_2(q) - \Psi_2(\bar{p}), ..., \Psi_m(q) - \Psi_m(\bar{p})) \notin -int\mathbb{R}_+^m\ \forall\, q \in \mathcal{K}.$$

Remark: Efficient solution $\implies$ Weakly efficient solution.

The following theorem discusses a relationship between the Stampacchia ∂_*^*-VVIP and efficient solution of the VOP.

Theorem 4.4. *Suppose $\mathcal{K}(\neq \phi) \subseteq \mathbb{H}$ is a GC set and let $\Psi : \mathcal{K} \longrightarrow \mathbb{R}^m$ be a function such that $\Psi_i : \mathcal{K} \longrightarrow \mathbb{R}$ are locally Lipschitz functions at $\bar{p} \in \mathcal{K}$ and admit a bounded convexificators $\partial_*^*\Psi(\bar{p})$, $\forall\, i \in M = \{1, 2, ..., m\}$. Suppose that Ψ is ∂_*^*-convex at $\bar{p}$ over $\mathcal{K}$. If $\bar{p}$ is a solution of the Stampacchia ∂_*^*-VVIP, then $\bar{p}$ is also an efficient solution of the VOP.*

Proof. On the contrary, suppose $\bar{p}$ is not an efficient solution of the VOP. Then, $\exists\, \tilde{p}$ such that

$$\Psi(\tilde{p}) - \Psi(\bar{p}) \in -\mathbb{R}_+^m \setminus \{0\}.$$

By ∂_*^*-convexity of Ψ at $\bar{p}$ over $\mathcal{K}$, we have

$$\left\langle \xi; \exp_{\bar{p}}^{-1} \tilde{p} \right\rangle_m \in -\mathbb{R}_+^m \setminus \{0\}.$$

This contradicts the fact that $\bar{p}$ is a solution of the Stampacchia ∂_*^*-VVIP. □

In the following theorem, we study an important characterization of the Minty ∂_*^*-VVIP in terms of the VOP.

Theorem 4.5. *Suppose $\mathcal{K}(\neq \phi) \subseteq \mathbb{H}$ is a GC set and $\Psi : \mathcal{K} \longrightarrow \mathbb{R}^m$ be a function such that $\Psi_i : \mathcal{K} \longrightarrow \mathbb{R}$ are locally Lipschitz functions on $\mathcal{K}$ and for any $p \in \mathcal{K}$ admit a bounded convexificator $\partial_*^*\Psi(p)$, $\forall\, i \in M$. Suppose that Ψ is ∂_*^*-convex on $\mathcal{K}$. Then, $\bar{p}$ is a solution of the Minty ∂_*^*-VVIP iff $\bar{p} \in \mathcal{K}$ is an efficient solution of the VOP.*

Proof. On the contrary suppose that $\bar{p}$ is not an efficient solution of the VOP. Then, $\exists\, \tilde{p} \in \mathcal{K}$, such that

$$\Psi(\tilde{p}) - \Psi(\bar{p}) \in -\mathbb{R}^m_+ \setminus \{0\}. \tag{4.1}$$

By the geodesic convexity of $\mathcal{K}$, $p(\lambda) := \exp_{\bar{p}} \lambda \exp_{\bar{p}}^{-1} \tilde{p} \in \mathcal{K}$, for any $\lambda \in [0, 1]$.

Since, Ψ is ∂_*^*-convex on $\mathcal{K}$, by Proposition 3.7, we have

$$\Psi(\exp_{\bar{p}} \lambda \exp_{\bar{p}}^{-1} \tilde{p}) - \Psi(\bar{p}) \leqq \lambda(\Psi(\tilde{p}) - \Psi(\bar{p})),$$

or equivalently, for any $i \in M$ and $\lambda \in (0, 1)$, one has

$$\Psi_i(\exp_{\bar{p}} \lambda \exp_{\bar{p}}^{-1} \tilde{p}) - \Psi_i(\bar{p}) \leqq \lambda(\Psi_i(\tilde{p}) - \Psi_i(\bar{p})).$$

By Theorem 3.2, for any $i \in M$, $\exists\, \hat{\lambda}_i \in (0, \lambda)$, and $\hat{\xi}_i \in \text{co}\partial_*^*\Psi(p(\hat{\lambda}_i))$, we have

$$\Psi_i(\exp_{\bar{p}} \lambda \exp_{\bar{p}}^{-1} \tilde{p}) - \Psi_i(\bar{p}) = \left\langle \hat{\xi}_i; \lambda P_{p(\hat{\lambda}_i),\bar{p}} \exp_{\bar{p}}^{-1} \tilde{p} \right\rangle,$$

which implies that, for any $i \in M$, we have

$$\left\langle \hat{\xi}_i; P_{p(\hat{\lambda}_i),\bar{p}} \exp_{\bar{p}}^{-1} \tilde{p} \right\rangle \leq \Psi_i(\tilde{p}) - \Psi_i(\bar{p}). \tag{4.2}$$

Now, there are two possible cases:

Case(1): When $\hat{\lambda}_1 = \hat{\lambda}_2 = ... = \hat{\lambda}_m = \hat{\lambda}$. Multiplying both side of (4.2) by $\hat{\lambda}$, for any $i \in M$ and $\hat{\xi} \in \partial_*^*\Psi(p(\hat{\lambda}))$, one has

$$\left\langle \hat{\xi}_i; P_{p(\hat{\lambda}_i),\bar{p}} \exp_{\bar{p}}^{-1} p(\hat{\lambda}) \right\rangle \leq \hat{\lambda}(\Psi_i(\tilde{p}) - \Psi_i(\bar{p})).$$

From (4.1), some $p(\hat{\lambda}) \in \mathcal{K}$ and $\hat{\xi} \in \text{co}\partial_*^*\Psi(p(\hat{\lambda}))$, one has

$$\left\langle \hat{\xi}; \exp_{p(\hat{\lambda})}^{-1} \bar{p} \right\rangle_m \in \mathbb{R}^m_+ \setminus \{0\}.$$

This is a contradiction to the fact that $\bar{p}$ is a solution of the Minty ∂_*^*-VVI.

Case(2): When $\hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_m$ are not all equal. Without loss of generality, we take $\hat{\lambda}_1 \neq \hat{\lambda}_2$. Then, from (3.2), for some $\hat{\xi}_1 \in \text{co}\partial_*^*\Psi_1(p(\hat{\lambda}_1))$ and $\hat{\xi}_2 \in \text{co}\partial_*^*\Psi_2(p(\hat{\lambda}_2))$, one has

$$\left\langle \hat{\xi}_1; P_{p(\hat{\lambda}_1),\bar{p}} \exp_{\bar{p}}^{-1} \tilde{p} \right\rangle \leq \Psi_1(\tilde{p}) - \Psi_1(\bar{p}),$$

and

$$\left\langle \hat{\xi}_2; P_{p(\hat{\lambda}_2),\bar{p}} \exp_{\bar{p}}^{-1} \tilde{p} \right\rangle \leq \Psi_2(\tilde{p}) - \Psi_2(\bar{p}).$$

Since Ψ_1 and Ψ_2 are ∂^*_*-convex on $\mathcal{K}$, by Theorem 3.5, for any $\hat{\xi}_{12} \in \mathrm{co}\partial^*_*\Psi_1(p(\hat{\lambda}_1))$ and $\hat{\xi}_{21} \in \mathrm{co}\partial^*_*\Psi_2(p(\hat{\lambda}_2))$, one has

$$\left\langle \hat{\xi}_1 - \hat{\xi}_{12}; \exp^{-1}_{p(\hat{\lambda}_1)} p(\hat{\lambda}_2) \right\rangle \geq 0,$$

and

$$\left\langle \hat{\xi}_2 - \hat{\xi}_{21}, \exp^{-1}_{p(\hat{\lambda}_2)} p(\hat{\lambda}_1) \right\rangle \geq 0.$$

If $\hat{\lambda}_1 - \hat{\lambda}_2 > 0$, it follows that

$$\left\langle \hat{\xi}_{12}; P_{p(\hat{\lambda}_1),\bar{p}} \exp^{-1}_{\bar{p}} \tilde{p} \right\rangle \leq \Psi_1(\tilde{p}) - \Psi_1(\bar{p}).$$

If $\hat{\lambda}_2 - \hat{\lambda}_1 > 0$, it follows that

$$\left\langle \hat{\xi}_{21}; P_{x(\hat{\lambda}_2),\bar{p}} \exp^{-1}_{\bar{p}} \tilde{p} \right\rangle \leq \Psi_2(\tilde{p}) - \Psi_2(\bar{p}).$$

Therefore, for $\hat{\lambda}_1 \neq \hat{\lambda}_2$, setting $\hat{\lambda} := \{\hat{\lambda}_1, \hat{\lambda}_2\}$, for any $i = 1, 2$, $\exists\, \hat{\xi}_i \in \mathrm{co}\partial^*_*\Psi_i(p(\hat{\lambda}))$ such that

$$\left\langle \hat{\xi}_i; P_{p(\hat{\lambda}),\bar{p}} \exp^{-1}_{\bar{p}} \tilde{p} \right\rangle \leq \Psi_i(\tilde{p}) - \Psi_i(\bar{p}).$$

Continuing the above process, we get $\bar{\lambda} \in (0, \lambda)$ such that $\bar{\lambda} := \min\{\hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_m\}$ and $\bar{\xi}_i \in \mathrm{co}\partial^*_*\Psi_i(p(\bar{\lambda}))$, such that

$$\left\langle \bar{\xi}_i; P_{p(\bar{\lambda}),\bar{p}} \exp^{-1}_{\bar{p}} \tilde{p} \right\rangle \leq \Psi_i(\tilde{p}) - \Psi_i(\bar{p}), \quad \forall\, i \in M.$$

Multiplying the above inequality by $\bar{\lambda}$, one has

$$\left\langle \bar{\xi}_i; -\exp^{-1}_{p(\bar{\lambda})} \bar{p} \right\rangle \leq \bar{\lambda}(\Psi_i(\tilde{p}) - \Psi_i(\bar{p})).$$

By (4.1), for some $p(\bar{\lambda}) \in \mathcal{K}$ and $\bar{\lambda} := (\bar{\xi}_1, \bar{\xi}_2, ..., \bar{\xi}_m) \in \partial^*_*\Psi(p(\bar{\lambda}))$, one has

$$\left\langle \bar{\xi}; \exp^{-1}_{p(\bar{\lambda})} \bar{p} \right\rangle_m \in \mathbb{R}^m_+ \setminus \{0\}.$$

This contradicts the Minty ∂^*_*-VVI.

For the converse, suppose that $\bar{x}$ is not a solution of the Minty ∂^*_*-VVI. Then, $\exists\, \tilde{p} \in \mathcal{K}$ and $\xi \in \partial^*_*\Psi(\tilde{p})$ such that

$$\left\langle \xi; \exp^{-1}_{\tilde{p}} \bar{p} \right\rangle_m \in \mathbb{R}^m_+ \setminus \{0\}.$$

By ∂^*_*-convexity of Ψ on $\mathcal{K}$, we have

$$\Psi(\tilde{p}) - \Psi(\bar{p}) \in -\mathbb{R}^m_+ \setminus \{0\},$$

a contradiction to the fact that $\bar{p}$ is an efficient solution of the VOP. □

5. Weak vector variational inequalities using convexificators

In this section, we first consider the weak formulations of the Stampacchia and Minty ∂_*^*-VVIs and establish their relations with the weakly efficient solution of the VOP.

Stampacchia ∂_*^*-WVVI: Find $\bar{p} \in \mathcal{K}$ such that, for any $q \in \mathcal{K}$, $\exists\, \xi \in \partial_*^*\Psi(\bar{p})$,

$$\left\langle \xi; \exp_{\bar{p}}^{-1} q \right\rangle_m \notin -int\mathbb{R}_+^m.$$

Minty ∂_*^*-WVVI: Find $\bar{p} \in \mathcal{K}$ such that, for any $q \in \mathcal{K}$ and $\xi \in \partial_*^*\Psi(q)$,

$$\left\langle \xi; \exp_{q}^{-1} \bar{p} \right\rangle_m \notin int\mathbb{R}_+^m.$$

The following theorem demonstrates a necessary and sufficient condition for a point to be a weakly efficient solution of the VOP in terms of the Stampacchia ∂_*^*-WVVI.

Theorem 5.1. *Suppose $\mathcal{K}(\neq \phi) \subseteq \mathbb{H}$ is a GC set and $\Psi : \mathcal{K} \longrightarrow \mathbb{R}^m$ is a function such that $\Psi_i : \mathcal{K} \longrightarrow \mathbb{R}$ are locally Lipschitz at point $\bar{p} \in \mathcal{K}$ and admit a bounded convexificator $\partial_*^*\Psi_i(\bar{p})$, $\forall\, i \in M = \{1, 2, ..., m\}$. Also suppose that Ψ is ∂_*^*-convex on $\mathcal{K}$. Then, $\bar{p}$ is a weakly efficient solution of the VOP iff $\bar{p}$ is a solution of the Stampacchia ∂_*^*-WVVI.*

Proof. Suppose that $\bar{p}$ is a weakly efficient solution of the VOP. Then, form any $q \in \mathcal{K}$,

$$\Psi(q) - \Psi(p) \notin -int\mathbb{R}_+^m.$$

By the geodesic convexity of $\mathcal{K}$, for any $\lambda \in [0, 1]$ and $y \in \mathcal{K}$, $\exp_{\bar{x}} \lambda \exp_{\bar{p}}^{-1} q \in \mathcal{K}$, which implies that

$$\frac{\Psi(\exp_{\bar{p}} \lambda \exp_{\bar{p}}^{-1} q) - \Psi(\bar{p})}{\lambda} \notin -int\mathbb{R}_+^m.$$

Taking the limit inf as $\lambda \to 0^+$, we have

$$\liminf_{\lambda \to 0^+} \frac{\Psi(\exp_{\bar{p}} \lambda \exp_{\bar{p}}^{-1} q) - \Psi(\bar{p})}{\lambda} \notin -int\mathbb{R}_+^m,$$

$$\Psi^-(\bar{p}; \exp_{\bar{p}}^{-1} q) := (\Psi_1^-(\bar{p}; \exp_{\bar{p}}^{-1} q), \Psi_2^-(\bar{p}; \exp_{\bar{p}}^{-1} q), ..., \Psi_m^-(\bar{p}; \exp_{\bar{p}}^{-1} q)) \notin -int\mathbb{R}_+^m,\ \forall q \in \mathcal{K}.$$

Since, Ψ_i admits a bounded convexificator $\partial_*^*\Psi_i(\bar{p})$, $\forall i \in M$, for any $q \in \mathcal{K}$, $\exists\, \bar{\xi} \in \partial_*^*\Psi_i(\bar{p})$, such that

$$\left\langle \bar{\xi}; \exp_{\bar{p}}^{-1} q \right\rangle_m \notin -int\mathbb{R}_+^m.$$

Hence, $\bar{p}$ is a solution of the Stampacchia ∂_*^*-WVVI.

For the converse, suppose $\bar{p}$ is not a weakly efficient solution of the VOP. Then $\exists\, \tilde{p} \in \mathcal{K}$, such that

$$\Psi(\tilde{p}) - \Psi(\bar{p}) \in -int\mathbb{R}_+^m.$$

By the ∂_*^*-convexity of Ψ at $\bar{p}$ over $\mathcal{K}$, for any $\bar{\xi} \in \partial_*^*\Psi(\bar{p})$,

$$\left\langle \bar{\xi}; \exp_{\bar{p}}^{-1} p \right\rangle_m \in -int\mathbb{R}_+^m,$$

which is a contradiction to the fact that $\bar{p}$ is a solution of the Stampacchia ∂_*^*-WVVI. ◻

The following theorem gives the condition under which the Stampacchia ∂^*_*-WVVI and Minty ∂^*_*-WVVI become equivalent.

Theorem 5.2. *Suppose $\mathcal{K}(\neq \phi) \subseteq \mathbb{H}$ is a GC set and let $\Psi : \mathcal{K} \longrightarrow \mathbb{R}^m$ be a function such that $\Psi_i : K \longrightarrow \mathbb{R}$ are locally Lipschitz on $\mathcal{K}$ and admit bounded convexificator $\partial^*_*\Psi_i(\bar{p})$ for any $\bar{p} \in \mathcal{K}, \forall\ i \in M = \{1, 2, ..., m\}$. Also, suppose that Ψ is ∂^*_*-convex on $\mathcal{K}$. Then, $\bar{p}$ is solution of the Minty ∂^*_*-WVVI iff $\bar{p}$ is a solution of the Stampacchia ∂^*_*-WVVI.*

Proof. Suppose that $\bar{p}$ is a solution of the Minty ∂^*_*-WVVI, and consider any sequence $\{\lambda_k\} \subset (0, 1]$ such that $\lambda_k \to 0$ as $k \to \infty$. By the geodesic convexity of $\mathcal{K}$, for any $q \in \mathcal{K}$, $q_k = \exp_{\bar{p}} \lambda_k \exp^{-1}_{\bar{p}} q \in \mathcal{K}$, since $\bar{p}$ is the solution of the Minty ∂^*_*-WVVI, $\exists\ \xi_k \in \partial^*_*\Psi(q_k)$ and

$$\left\langle \xi_k; \exp^{-1}_{q_k} \bar{p} \right\rangle_m \notin int\mathbb{R}^m_+.$$

Since, Ψ_i are locally Lipschitz and admit bounded convexificators on $\mathcal{K}$ for all $i \in M$, there exists $d > 0$ such that $\|\xi_k\| \leq d$ which implies that the sequence $\{\xi_{k_i}\} \subset \partial^*_*\Psi_i(q_k)$ converges to ξ_i for all $i \in M$. For any $q \in \mathcal{K}$, the convexificator $\partial^*_*\Psi_i(q)$ is closed for all $i \in M$. It follows that $q_k \to q$ and $\xi_{k_i} \to \xi_i$ as $k \to \infty$ with $\xi_i \in \partial^*_*\Psi_i(\bar{p})$ for all $i \in M$. Therefore, for any $y \in \mathcal{K}$, $\exists\ \xi \in \partial^*_*\Psi_i(\bar{p})$ such that

$$\left\langle \xi; \exp^{-1}_{\bar{p}} q \right\rangle_m \notin -int\mathbb{R}^m_+.$$

Hence, $\bar{p}$ is a solution of the Stampacchia ∂^*_*-WVVI.

For the converse, suppose $\bar{p}$ is a solution of the Stampacchia ∂^*_*-WVVI. Then, for any $q \in \mathcal{K}$, $\exists\ \bar{\xi} \in \partial^*_*\Psi(\bar{p})$ such that

$$\left\langle \bar{\xi}; \exp^{-1}_{\bar{p}} q \right\rangle_m \notin -int\mathbb{R}^m_+.$$

Since, Ψ is ∂^*_*-convex on $\mathcal{K}$, by Theorem 3.5, we get that $\partial^*_*\Psi$ is monotone on $\mathcal{K}$, which implies

$$\left\langle \xi; \exp^{-1}_{q} \bar{p} \right\rangle_m \notin int\mathbb{R}^m_+$$

for any $q \in \mathcal{K}$ and $\xi \in \partial^*_*\Psi(q)$. Hence, $\bar{p}$ is a solution of the Minty ∂^*_*-WVVI. □

The following theorem gives the condition for a weakly efficient solution to be an efficient solution of the VOP.

Theorem 5.3. *Suppose $\mathcal{K}(\neq \phi) \subseteq \mathbb{H}$ is a GC set and $\Psi : \mathcal{K} \longrightarrow \mathbb{R}^m$ is a function such that $\Psi_i : \mathcal{K} \longrightarrow \mathbb{R}$ are local Lipschitz at $\bar{p} \in \mathcal{K}$ and admit the bounded convexificator $\partial^*_*\Psi_i(\bar{p})$, $\forall\ i \in M = \{1, 2, ..., m\}$. Also suppose that Ψ is strictly ∂^*_*-convex at $\bar{p}$ over $\mathcal{K}$. Then, $\bar{p}$ is an efficient solution of the VOP iff $\bar{p}$ is a weakly efficient solution of the VOP.*

Proof. Obviously, every efficient solution is also a weakly efficient solution of the VOP.

Conversely, suppose that $\bar{p}$ is a weakly efficient solution of the VOP but not an efficient solution of the VOP. Then, $\exists\ \tilde{p} \in \mathcal{K}$ such that

$$\Psi(\tilde{p}) - \Psi(\bar{p}) \in -int\mathbb{R}^m_+.$$

By strict ∂^*_*-convexity of Ψ at $\bar{p}$ over $\mathcal{K}$, for any $\bar{\xi} \in \partial^*_*\Psi(\bar{p})$, we have

$$\left\langle \bar{\xi}; \exp^{-1}_{\bar{p}} \tilde{p} \right\rangle_m \in -int\mathbb{R}^m_+,$$

which implies that $\bar{p}$ is not a solution of the Stampacchia ∂^*_*-WVVI. By Theorem 5.1, $\bar{p}$ is not a weakly efficient solution of the VOP. This contradiction leads to the results. □

6. Conclusions

In this paper, we have formulated the concept of convexificators for the Hadamard manifolds which are weaker version of the notion of sub-differentials. We proved the mean value theorem for them and discussed the characterizations of the ∂^*_*-convex functions in terms of monotonicity. Furthermore, we defined the Stampacchia ∂^*_*-VVI and Minty-type ∂^*_*-VVI using convexificators and by a non-trivial example showed their existence and also established the relationships between their solutions and efficient solutions of the VOP.

The results of this research are more precise as well as comprehensive than the comparable results previously published in the literature because convexificators were utilized. However, there is still a difficulty with the existence results of the ∂^*_*-VVI, which can be considered in the future. The results may be extended to Riemannian manifolds using some more assumptions. Furthermore, some related problems like fixed point problems, complementarity problem, and equilibrium problems can be explored in the future using the concept of convexificators.

Author contributions

Nagendra Singh: Conceptualization, Formal Analysis, Investigation, Methodology, Writing. Sunil Kumar Sharma: Funding acquisition, Investigations and results corrections, Review and editing. Akhlad Iqbal: Supervision, Visualization, Results corrections, Writing, Review and editing. Shahid Ali: Supervision, Review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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