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Research article

Construction and analysis of the quasi-ruled surfaces based on the quasi-focal curves in $\mathbb{R}^3$

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Abstract: This paper presents the concept of a quasi-ruled surface, which is a ruled surface generated by a base curve and a ruling, both of which are defined by the quasi-frame (q-frame). This study begins with the original curve defined by the q-frame, and then we focus on the focal curve of the original curve, which serves as the base curve of the ruled surface. We define the focal curve by the q-frame, so the terminology quasi-focal curve is used in this paper. This paper investigates the formation and properties of the quasi-ruled surface (QRS) using a quasi-focal curve (QFC) as the base curve (directrix). The ruling of the surface is expressed in terms of the q-frame associated with the QFC. A variety of QRS types are discussed in this study, including the osculating, normal, and rectifying types. In addition, the types of a quasi-tangent developable surface, a quasi-principal normal surface, and a quasi-binormal ruled surface will also be discussed. The geometric properties of these surfaces, such as the first and second fundamental quantities, Gaussian curvature, mean curvature, second Gaussian curvature, and second mean curvature, are described. The conditions for their developability and minimality are derived. Moreover, we provide an example that includes the study of geometric properties and clear visualizations of these novel types of QRS.

Keywords: focal curves; ruled surfaces; q-frame; quasi-focal curve; osculating type of quasi-ruled surface; rectifying type of quasi-ruled surface; quasi-tangent developable surface; quasi-principal normal ruled surface; quasi-binormal ruled surface
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1. Introduction

Ruled surfaces play a crucial role in differential geometry. They are characterized by the movement of generators, which are straight lines that produce the surface. Furthermore, a directrix (base curve) is any curve that crosses all of the generators (rulings). Understanding this concept is crucial for grasping the characteristics and applications of ruled surfaces in various geometric contexts.

Many researchers are interested in studying ruled surfaces according to different frames. Tuncer [1] used a novel technique to study ruled surfaces in $\mathbb{R}^3$. These surfaces were defined in terms of their rulings, base curve curvatures, shape operators, and Gauss curvatures.

In [2], the pitch, angle of pitch, and dual angle of pitch of the ruled surface in $\mathbb{R}^3$, corresponding to a closed curve on the dual unit sphere, were examined. The vectors of the Frenet and Bishop frames of the closed curve were also analyzed, resulting in a relationship between the dual angle of pitch and the pitch angle. In [3], a fundamental method was adopted to analyze the ruled surfaces, focusing on the most basic foliated submanifolds in $\mathbb{R}^3$. The structural functions of the ruled surfaces were specified. The geometric properties and kinematical characterizations of the non-developable ruled surfaces in $\mathbb{R}^3$ were investigated.

In [4], the ruled surfaces in $\mathbb{R}^3$ were studied using the base curves with the Bishop frame. These surfaces were characterized by their directrices, Bishop curvatures, shape operators, and Gauss curvatures. Masal [5] developed ruled surfaces created by type-2 Bishop vectors, distinguishing Gaussian curvature (GC) and mean curvature (MC), as well as integral invariants. The fundamental forms, geodesic curvatures, normal curvatures, and geodesic torsions were determined.

In [6], the Darboux frame was used to define the ruled surface and study its properties, including geodesic curvature, normal curvature, and geodesic torsion. In [7], parallel ruled surfaces with the Darboux frame in $\mathbb{R}^3$ were introduced, highlighting aspects such as developability, striction points, and distribution parameters. The Steiner rotation vector for such a kind of surface was determined, and the pitch length and angle of the parallel ruled surfaces associated with the Darboux frame were computed. In [8], a necessary and sufficient condition was established for a ruled surface to be the principal normal ruled surface of a space curve using the theories of ruled invariants in $\mathbb{R}^3$.

In [9], the ruled surfaces created by normal and binormal vectors throughout a timelike space curve utilizing a q-frame were explored in three-dimensional Minkowski space. The directional evolutions of quasi-principal normal and quasi-binormal ruled surfaces were investigated, employing their directrices. The geometric properties of the ruled surfaces were examined, including their inextensibility, minimality, and developability. In [10], the striction curve of a non-cylindrical ruled surface is considered to be the base curve, with its ruling represented as linear combinations of Frenet-Serret frame (FSF) vectors from the first ruled surface.

In [11], a novel family of ruled surfaces was constructed and studied via q-frame vectors, known as quasi-vectors. The features of these governed surfaces, such as the first and second fundamental forms, GC and MC, were determined. Furthermore, several geometric properties such as developability, minimality, striction curve, and distribution parameters were investigated. Senyurt et al. [12] introduced a new type of special ruled surface, where the construction of each surface is based on a Smarandache curve and a specified curve according to the FSF. The generator (ruling) is selected as the unit Darboux vector. The properties of those ruled surfaces were investigated using the first and second fundamental forms, as well as their corresponding curvatures.

The q-frames of the rational and polynomial Bezier curves were computed algorithmically in [13]. The frame was constructed even at singular points based on the curve's second derivative. This study provides an important improvement to computer-aided geometric design research.

Kaymanli et al. [14] derived ruled surfaces using a quasi-principal normal, and a quasi-binormal vectors along a spacelike curve in three-dimensional Minkowski space, leading to the formulation of the time evolution equations based on quasi-curvatures. Pal et al. [15] introduced a new type of ruled surfaces in $\mathbb{R}^3$, called ruled-like surfaces, which are generated by a base curve and a director curve. In addition, the properties of these surfaces, such as GC, MC, and the existence of Bertrand mates, were investigated.

Using the FSF in $\mathbb{R}^3$, Gaber et al. [16] investigated a family of ruled surfaces formed of circular helices (W-curves). The second mean curvature (SMC), and the second Gaussian curvature (SGC) formulas were obtained, the properties of the constructed ruled surfaces were described, and the conditions for minimal, flat, II-minimal, and II-flat surfaces were determined. In addition, the conditions for the base curves of these surfaces were classified as a geodesic curve, an asymptotic line, and a principal line.

In this work, we introduce a specified concept of QRS, which refers to a ruled surface whose base curve is defined by a q-frame, and the q-frame vectors of the base curve describe the ruling.

This study focuses on a directrix, which is the focal curve of the original curve. In [17], the focal curve given by the q-frame is defined as QFC.

The structure of this work is as follows: Section 2 provides background information on the fundamental ideas of curves and ruled surfaces in three-dimensional Euclidean space. Section 3 covers the construction of QRS from the QFC with specific geometric features. Section 4 provides techniques for constructing several innovative types of QRS, using a QFC as the base curve and influencing its ruling vector. Section 5 presents and visualizes novel types of QRS. Finally, we give a conclusion.

2. Geometry of curves in $\mathbb{R}^3$

In this section, we present some geometric concepts on curves in $\mathbb{R}^3$, defining the FSF, the q-frame, and their relationship. The construction of the QRS is based on specific concepts of the q-frame of the original curve, the quasi-focal curve (QFC). Therefore, it is important to highlight these concepts.

2.1. Curves defined by the Frenet-Serret frame FSF in $\mathbb{R}^3$

Consider a unit speed curve $\alpha : I \in \mathbb{R} \to \mathbb{R}^3$ with an arc length parameter s. Let $\mathfrak{F} = \{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be an orthogonal FSF at the point p_0 on the open curve, where $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are the unit tangent, unit principal normal, and unit binomial vectors, respectively. The FSF has the following characteristics [18]:

- $\langle \mathbf{T}, \mathbf{N}\rangle = \langle \mathbf{N}, \mathbf{B}\rangle = \langle \mathbf{T}, \mathbf{B}\rangle = 0$, $\langle \mathbf{T}, \mathbf{T}\rangle = \langle \mathbf{N}, \mathbf{N}\rangle = \langle \mathbf{B}, \mathbf{B}\rangle = 1$.
- $\mathbf{T} \wedge \mathbf{N} = \mathbf{B}$, $\mathbf{N} \wedge \mathbf{B} = \mathbf{T}$, and $\mathbf{B} \wedge \mathbf{T} = \mathbf{N}$.

Let $\kappa = \kappa(s)$ and $\tau = \tau(s)$ be the curvature and torsion of the open curve. Then, the Frenet equations are given by

$$\mathbf{T}' = \kappa\, \mathbf{N}, \quad \mathbf{N}' = -\kappa\, \mathbf{T} + \tau \mathbf{B}, \quad \mathbf{B}' = -\tau\, \mathbf{N},\ (\,)' = \frac{d}{ds}(\,). \tag{2.1}$$

2.2. Curves defined by q-frame in $\mathbb{R}^3$

The Frenet-Serret frame FSF loses effectiveness when the curvature of a curve is zero. To solve this issue, we use an alternative frame known as a q-frame, which is related to the equations of the Frenet-Serret frame. The q-frame offers several advantages, including the ability to be defined even in the absence of a tangent line. Additionally, the formation of the q-frame does not require the space curve to have a unit speed. Finally, the q-frame is easy to calculate.

Definition 1. *Let s represent the arc length along the curve $\alpha : I \in \mathbb{R} \to \mathbb{R}^3$ within the interval I. Assume that $\alpha(s)$ is a unit speed curve. Assume that $\{\boldsymbol{T}_{q\alpha}, \boldsymbol{N}_{q\alpha}, \boldsymbol{B}_{q\alpha}\}$ is the q-frame, where $\boldsymbol{T}_{q\alpha}$ is the unit quasi-tangent vector, $\boldsymbol{N}_{q\alpha}$ is the unit quasi-principal normal vector, and $\boldsymbol{B}_{q\alpha}$ is the unit quasi-binormal vector. The q-frame is defined as follows [19, 20]:*

$$\boldsymbol{T}_{q\alpha} = \alpha'(s), \quad \boldsymbol{N}_{q\alpha} = \frac{\boldsymbol{T}_{q\alpha} \wedge \boldsymbol{u}}{\|\boldsymbol{T}_{q\alpha} \wedge \boldsymbol{u}\|}, \quad \boldsymbol{B}_{q\alpha} = \boldsymbol{T}_{q\alpha} \wedge \boldsymbol{N}_{q\alpha}, \tag{2.2}$$

where $(\wedge)$ refers to the cross product and $\boldsymbol{u}$ represents the projection vector; for convenience, we select $\boldsymbol{u} = (0, 0, 1)$.

Definition 2. *Consider the q-frame $\{\boldsymbol{T}_{q\alpha}, \boldsymbol{N}_{q\alpha}, \boldsymbol{B}_{q\alpha}\}$ of the curve $\alpha(s)$ at a point p, alongside the FSF $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ at the same point p on the curve. Let θ represent the Euclidean angle between the principal normal vector $\boldsymbol{N}$ and the quasi-principal normal vector $\boldsymbol{N}_{q\alpha}$. The relation between the directional q-frames and the FSF is provided by [19, 20] as follows:*

$$\begin{bmatrix} \boldsymbol{T}_{q\alpha} \\ \boldsymbol{N}_{q\alpha} \\ \boldsymbol{B}_{q\alpha} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}. \tag{2.3}$$

Definition 3. *[19] The q-frame $\{\boldsymbol{T}_{q\alpha}, \boldsymbol{N}_{q\alpha}, \boldsymbol{B}_{q\alpha}\}$, where $\boldsymbol{T}_{q\alpha} = \boldsymbol{T}$, possesses the characteristics outlined below:*

$$\begin{aligned} \langle \boldsymbol{T}_{q\alpha}, \boldsymbol{T}_{q\alpha} \rangle &= \langle \boldsymbol{N}_{q\alpha}, \boldsymbol{N}_{q\alpha} \rangle = \langle \boldsymbol{B}_{q\alpha}, \boldsymbol{B}_{q\alpha} \rangle = 1, \\ \langle \boldsymbol{T}_{q\alpha}, \boldsymbol{N}_{q\alpha} \rangle &= \langle \boldsymbol{T}_{q\alpha}, \boldsymbol{B}_{q\alpha} \rangle = \langle \boldsymbol{N}_{q\alpha}, \boldsymbol{B}_{q\alpha} \rangle = 0, \\ \langle \boldsymbol{N}_{q\alpha}, \boldsymbol{N} \rangle &= \langle \boldsymbol{B}_{q\alpha}, \boldsymbol{B} \rangle = \cos\theta \,, \langle \boldsymbol{N}_{q\alpha}, \boldsymbol{B} \rangle = -\langle \boldsymbol{B}_{q\alpha}, \boldsymbol{N} \rangle = \sin\theta. \end{aligned} \tag{2.4}$$

Definition 4. *[20] The relation between the curvatures κ and τ of the curve α described by the FSF and the curvatures κ_1, κ_2, and κ_3 of the curve α described by the q-frame is established as follows:*

$$\kappa_1 = \kappa\cos\theta, \quad \kappa_2 = -\kappa\sin\theta, \quad \kappa_3 = d\theta + \tau. \tag{2.5}$$

This paper uses quasi-curvatures, referred to κ_1, κ_2, and κ_3, which are defined in the following manner [13]:

$$\begin{aligned} \kappa_1 &= \left\langle \boldsymbol{T}'_{q\alpha}, \boldsymbol{N}_{q\alpha} \right\rangle = \frac{-\det[\alpha', \alpha'', \boldsymbol{u}]}{\|\alpha' \wedge \boldsymbol{u}\|} \,, \; (\,)' = \frac{d}{ds}(\,), \\ \kappa_2 &= \left\langle \boldsymbol{T}'_{q\alpha}, \boldsymbol{B}_{q\alpha} \right\rangle = \frac{\langle \alpha', \boldsymbol{u} \rangle \langle \alpha', \alpha'' \rangle - \langle \alpha'', \boldsymbol{u} \rangle}{\|\alpha' \wedge \boldsymbol{u}\|}, \\ \kappa_3 &= -\left\langle \boldsymbol{B}'_{q\alpha}, \boldsymbol{N}_{q\alpha} \right\rangle = \frac{\langle \alpha', \boldsymbol{u} \rangle \det[\alpha', \alpha'', \boldsymbol{u}]}{\|\alpha' \wedge \boldsymbol{u}\|^2} \,. \end{aligned} \tag{2.6}$$

Lemma 1. *[19] Let s represent the arc length along the curve $\alpha : I \in \mathbb{R} \to \mathbb{R}^3$ within the interval I. Assume that $\alpha(s)$ is a unit speed curve. The derivatives of the q-frame $\{\boldsymbol{T}, \boldsymbol{N}_{q\alpha}, \boldsymbol{B}_{q\alpha}\}$ with respect to the arc length s satisfy the following equations:*

$$\begin{bmatrix} \alpha' \\ \boldsymbol{T}'_{q\alpha}(s) \\ \boldsymbol{N}'_{q\alpha}(s) \\ \boldsymbol{B}'_{q\alpha}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \kappa_1(s) & \kappa_2(s) \\ 0 & -\kappa_1(s) & 0 & \kappa_3(s) \\ 0 & -\kappa_2(s) & -\kappa_3(s) & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \boldsymbol{T}_{q\alpha}(s) \\ \boldsymbol{N}_{q\alpha}(s) \\ \boldsymbol{B}_{q\alpha}(s) \end{bmatrix}. \tag{2.7}$$

2.3. Focal curves based on the q-frame in $\mathbb{R}^3$

In this paper, we focus on studying focal curves by employing a q-frame, and we refer to them as quasi-focal curves QFC.

Definition 5. *[17] Let s represent the arc length along the curve $\alpha : I \in \mathbb{R} \to \mathbb{R}^3$ within the interval I. Assume that $\alpha(s)$ is a unit speed curve. Consider $\{\boldsymbol{T}_{q\alpha}, \boldsymbol{N}_{q\alpha}, \boldsymbol{B}_{q\alpha}\}$ be a quasi-frame for the original curve, and assume that F_α is its unit speed QFC, which is defined by*

$$F_\alpha(s) = \alpha(s) + \varphi_1(s)\boldsymbol{N}_{q\alpha} + \varphi_2(s)\boldsymbol{B}_{q\alpha}, \tag{2.8}$$

where the smooth functions φ_1, φ_2 are the quasi-focal curvatures. Here, we call the curve α the original curve.

Theorem 2. *Let s represent the parameter of the arc length along the curve $\alpha : I \in \mathbb{R} \to \mathbb{R}^3$ within the interval I. Assume that $\alpha(s)$ is a unit speed curve defined by the q-frame $\{\boldsymbol{T}_{q\alpha}, \boldsymbol{N}_{q\alpha}, \boldsymbol{B}_{q\alpha}\}$. Let $F_\alpha\left(s^F(s)\right)$ be a QFC for the curve α and assume that F_α is a unit speed curve defined by the q-frame $\left\{\boldsymbol{T}^F_q, \boldsymbol{N}^F_q, \boldsymbol{B}^F_q\right\}$. Let $s^F(s)$ be the QFC arc length parameter and assume that $s^F(s)$ is measured on the focal curve $F_\alpha(s^F(s))$ in the direction of increasing s on the curve α. The relation between the q-frame for the QFC $F_\alpha(s)$ and the q-frame for the original curve α is given by*

$$\begin{bmatrix} \boldsymbol{T}^F_q \\ \boldsymbol{N}^F_q \\ \boldsymbol{B}^F_q \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ -\varepsilon & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_{q\alpha} \\ \boldsymbol{N}_{q\alpha} \\ \boldsymbol{B}_{q\alpha} \end{bmatrix}, \quad \varepsilon = \pm 1. \tag{2.9}$$

Proof. Taking the s-derivative of (2.8) with respect to s, we have

$$\frac{dF_\alpha}{ds} = \alpha' + \varphi_1' \mathbf{N}_{q\alpha} + \varphi_1 \mathbf{N}'_{q\alpha} + \varphi_2' \mathbf{B}_{q\alpha} + \varphi_2 \mathbf{B}'_{q\alpha}. \tag{2.10}$$

Substituting from (2.7) into (2.10), we have

$$\frac{dF_\alpha}{ds} = \frac{dF_\alpha}{ds^F} \cdot \frac{ds^F}{ds} = (1 - \kappa_1\varphi_1 - \kappa_2\varphi_2)\, \mathbf{T}_{q\alpha} + (\varphi_1' - \kappa_3\varphi_2)\, \mathbf{N}_{q\alpha} + (\varphi_2' + \kappa_3\varphi_1)\, \mathbf{B}_{q\alpha}. \tag{2.11}$$

Since the QFC represents the centers of the tangential oscillating spheres, the components of $\mathbf{T}_{q\alpha}$ and $\mathbf{N}_{q\alpha}$ vanish. Then,

$$\begin{aligned} 1 - \kappa_1\varphi_1 - \kappa_2\varphi_2 &= 0, \\ \varphi_1' - \kappa_3\varphi_2 &= 0. \end{aligned} \tag{2.12}$$

Hence, we have

$$\frac{dF_\alpha}{ds^F}\cdot\frac{ds^F}{ds}=\left(\varphi_2'+\kappa_3\varphi_1\right)\mathbf{B}_{q\alpha}. \tag{2.13}$$

Since F_α is a unit speed curve, then $\|\frac{dF_\alpha}{ds^F}\|=1$. Define $\mathbf{T}_q^F=\frac{dF_\alpha}{ds^F}$ as the unit quasi-tangent vector of F_α. Then,

$$\mathbf{T}_q^F\cdot\frac{ds^F}{ds}=\left(\varphi_2'+\kappa_3\varphi_1\right)\mathbf{B}_{q\alpha}. \tag{2.14}$$

Taking the norm of the two sides of (2.14), then

$$\frac{ds^F}{ds}=|\varphi_2'+\kappa_3\varphi_1|.$$

Since s^F is measured on $F_\alpha(s^F(s))$ in the direction of increasing s on the curve $\alpha(s)$, then s^F is an increasing function of s. So, $\frac{ds^F}{ds}>0$, and then $\frac{ds^F}{ds}=\varphi_2'+\kappa_3\varphi_1$. Hence, we obtain the quasi-binormal vector for the QFC:

$$\mathbf{T}_q^F=\mathbf{B}_{q\alpha}. \tag{2.15}$$

Let $\mathbf{N}_q^F$ be the quasi-principal normal vector to F_α, where

$$\mathbf{N}_q^F=\frac{\mathbf{T}_q^F\wedge\mathbf{u}}{\|\mathbf{T}_q^F\wedge\mathbf{u}\|}=\frac{\mathbf{B}_{q\alpha}\wedge\mathbf{u}}{\|\mathbf{B}_{q\alpha}\wedge\mathbf{u}\|},\ \mathbf{u}=(1,0,0). \tag{2.16}$$

Assume that the quasi-tangent and the quasi-principal normal vectors for the curve α are defined by the following components:

$$\mathbf{T}_{q\alpha}=(t_1,t_2,t_3),\ \mathbf{N}_{q\alpha}=(n_1,n_2,n_3). \tag{2.17}$$

Then,

$$\mathbf{B}_{q\alpha}\wedge\mathbf{u}=\mathbf{T}_{q\alpha}\wedge\mathbf{N}_{q\alpha}\wedge\mathbf{u}=t_3\,\mathbf{N}_{q\alpha}. \tag{2.18}$$

Substituting from (2.18) into (2.16), we have

$$\mathbf{N}_q^F=\frac{t_3}{|t_3|}\mathbf{N}_{q\alpha}. \tag{2.19}$$

Hence,

$$\mathbf{N}_q^F=\varepsilon\,\mathbf{N}_{q\alpha},\ \varepsilon=\pm1. \tag{2.20}$$

Since $\mathbf{B}_q^F=\mathbf{T}_q^F\wedge\mathbf{N}_q^F$, then, by using (2.15) and (2.20), we obtain the quasi-binormal vector of the QFC as

$$\mathbf{B}_q^F=-\varepsilon\mathbf{T}_{q\alpha}.$$

Hence, the theorem holds. □

Remark 1. *Throughout this paper, we assume* $\varepsilon=1$. *Therefore,*

$$\boldsymbol{T}_q^F=\boldsymbol{B}_{q\alpha},\ \ \boldsymbol{N}_q^F=\boldsymbol{N}_{q\alpha},\ \ \boldsymbol{B}_q^F=-\boldsymbol{T}_{q\alpha}. \tag{2.21}$$

Lemma 3. *Consider a unit speed curve, $\alpha : I \longrightarrow \mathbb{R}^3$ defined by the q-frame $\{\boldsymbol{T}_{q\alpha}, \boldsymbol{N}_{q\alpha}, \boldsymbol{B}_{q\alpha}\}$ with arc length s. Let $F_\alpha\left(s^F(s)\right)$ be a QFC for the original curve α. Let $s^F(s)$ be the arc length parameter of the QFC and assume that $s^F(s)$ is measured on the focal curve $F_\alpha(s^F(s))$ in the direction of increasing s on the curve α. Let $\left\{\boldsymbol{T}_q^F, \boldsymbol{N}_q^F, \boldsymbol{B}_q^F\right\}$ be the q-frame for F_α. The q-frame of the quasi-focal curve F_α is constructed similarly to the q-frame of any curve by the following equations:*

$$\frac{d}{ds^F}\begin{bmatrix} \boldsymbol{T}_q^F \\ \boldsymbol{N}_q^F \\ \boldsymbol{B}_q^F \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1^F & \kappa_2^F \\ -\kappa_1^F & 0 & \kappa_3^F \\ -\kappa_2^F & -\kappa_3^F & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{T}_q^F \\ \boldsymbol{N}_q^F \\ \boldsymbol{B}_q^F \end{bmatrix}. \tag{2.22}$$

where κ_1^F, κ_2^F, and κ_3^F are the quasi-curvatures for the quasi-focal curve F_α, and they have the following relations with the quasi-curvatures of the original curve κ_1, κ_2, and κ_3:

$$\begin{aligned} \kappa_1^F &= \langle \frac{d\boldsymbol{T}_q^F}{ds^F}, \boldsymbol{N}_q^F \rangle = \frac{-\varepsilon\kappa_3}{|\varphi_2' + \kappa_3\varphi_1|}, \\ \kappa_2^F &= \langle \frac{d\boldsymbol{T}_q^F}{ds^F}, \boldsymbol{B}_q^F \rangle = \frac{\varepsilon\kappa_2}{|\varphi_2' + \kappa_3\varphi_1|}, \\ \kappa_3^F &= -\langle \frac{d\boldsymbol{B}_q^F}{ds^F}, \boldsymbol{N}_q^F \rangle = \frac{\kappa_1}{|\varphi_2' + \kappa_3\varphi_1|}. \end{aligned} \tag{2.23}$$

Theorem 4. *[17] Consider a unit speed curve $\alpha : I \longrightarrow \mathbb{R}^3$ with its QFC F_α. Then, the quasi-focal curvatures φ_1 and φ_2 are given by*

$$\begin{aligned} \varphi_1 &= e^{-\int \frac{\kappa_1\kappa_3}{\kappa_2}ds}\left(\int e^{\int \frac{\kappa_1\kappa_3}{\kappa_2}ds}\frac{\kappa_3}{\kappa_2}ds + C\right), \\ \varphi_2 &= \frac{1}{\kappa_2} - \frac{\kappa_1}{\kappa_2}e^{-\int \frac{\kappa_1\kappa_3}{\kappa_2}ds}\left(\int e^{\int \frac{\kappa_1\kappa_3}{\kappa_2}ds}\frac{\kappa_3}{\kappa_2}ds + C\right), \end{aligned} \tag{2.24}$$

where C is a constant of integration.

2.4. Ruled surfaces according to Frenet frame in $\mathbb{R}^3$

Definition 6. *[21] Let $\gamma(s) : \mathrm{I} \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve with an arc length parameter s. A ruled surface is a surface constructed by straight lines parametrized by $\gamma(s)$ and $\eta(s)$. It has the following parametrization:*

$$\Psi(s, v) = \gamma(s) + v\,\eta(s),$$

where $\gamma = \gamma(s) : I \subset \mathbb{R} \to \mathbb{R}^3$ is the directrix or base curve, and $\eta(s)$ represents a unit vector in the direction of the ruling of the ruled surface.

Definition 7. *[22] Let $\gamma(s) : \mathrm{I} \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve with the arc length parameter s, and let $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ be the Frenet frame of the curve at a point q. The ruled surface $\Psi(s, v) = I \times \mathbb{R} \to \mathbb{R}^3$ defined by*

$$\Psi(s, v) = \gamma(s) + v\,\eta(s),\ \ \eta(s) = \mu_1(s)\boldsymbol{T}(s) + \mu_2(s)\boldsymbol{N}(s),$$

is called the generalized osculating type ruled surface, where $\mu_1(s)$ and $\mu_2(s)$ are smooth functions ($\mu_1^2 + \mu_2^2 = 1$). The following cases can be given:

1. *If $\mu_1(s) = 0$ and $\mu_2(s) = \pm 1$, then the surface $\Psi(s, v)$ is a principal normal surface along the base curve.*

2. *If $\mu_2(s) = 0$ and $\mu_1(s) = \pm 1$, then the surface $\Psi(s, (v)$ is a tangent developable surface along the base curve.*

Definition 8. *[23] Let $\gamma(s) : \mathrm{I} \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve with the arc length parameter s with Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ at a point q on the base curve. The ruled surface $\Psi(s, v) : \mathrm{I} \times \mathbb{R} \to \mathbb{R}^3$ defined by*

$$\Psi(s, v) = \gamma(s) + v\eta(s), \quad \eta(s) = \mu_2(s)\boldsymbol{N}(s) + \mu_3(s)\boldsymbol{B}(s),$$

is called the generalized normal ruled surface, where μ_2, μ_3 are smooth functions of the arc length parameter s, and $\mu_2^2 + \mu_3^2 = 1$. The following cases can be given:

1. *If $\mu_2(s) = \pm 1$ and $\mu_3(s) = 0$, then the ruled surface $\Psi(s, v)$ is called the principal normal surface along the base curve $\gamma(s)$.*

2. *If $\mu_2(s) = 0$ and $\mu_3(s) = \pm 1$, then the ruled surface $\Psi(s, v)$ is called the binormal surface along the base curve $\gamma(s)$.*

Definition 9. *[24] Let $\gamma(s) : \mathrm{I} \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve with the arc length parameter s, with FSF $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ at a point q on the base cuve, and assume that κ and τ are the curvature and torsion of the curve. The ruled surface $\Psi(s, v) : I \times \mathbb{R} \to \mathbb{R}^3$ is determined as*

$$\Psi(s, v) = \gamma(s) + v\eta(s), \quad \eta(s) = \mu_1(s)\boldsymbol{T}(s) + \mu_3(s)\boldsymbol{B},$$

is called the generalized rectifying ruled surface, where $\mu_1(s)$, and $\mu_3(s)$ are smooth functions and $\mu_1^2 + \mu_3^2 = 1$.

Lemma 5. *[24] Let $\Psi(s, v)$ be a generalized rectifying ruled surface of the base curve $\gamma(s)$. Then:*

1. *If $\mu_1(s) = \frac{\tau(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}}$ and $\mu_3(s) = \frac{\kappa(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}}$, then the surface $\Psi(s, v)$ is a rectifying developable surface along the base curve $\gamma(s)$.*

2. *If $\mu_1(s) = 0$ and $\mu_3(s) = \pm 1$, then the surface $\Psi(s, v)$ is a binormal surface along the base curve $\gamma(s)$.*

3. *If $\mu_1(s) = \pm 1$ and $\mu_3(s) = 0$, then $\Psi(s, v)$ is the tangent developable surface along $\gamma(s)$.*

3. Construction of the quasi-ruled surface from the quasi-focal curve in $\mathbb{R}^3$

In this paper, we define a quasi-ruled surface QRS as a ruled surface generated by a base curve, which is described by the q-frame, and the ruling is defined by the q-frame of the base curve. We focus on the QFC of the original curve as a base curve (directrix) of the constructed QRS.
The QRS has the following parametrization:

$$\psi(s, v) = F_\alpha(s) + v\ \eta(s), \tag{3.1}$$

where F_α is a quasi-focal curve and it serves as the base curve (directrix), and the line passing through F_α is called the ruling of the surface $\psi(s, v)$ at F_α. The surface $\psi(s, v)$ has singular points at (s, v) if

$\psi_s \wedge \psi_v = 0$. Substituting from (2.8) into (3.1), the QRS can be expressed in terms of the original curve $\alpha(s)$ as

$$\psi(s, v) = \alpha(s) + \varphi_1 \mathbf{N}_{q_\alpha} + \varphi_2 \mathbf{B}_{q_\alpha} + v\, \eta(s), \tag{3.2}$$

where φ_1 and φ_2 are quasi-focal curvatures of α satisfying (2.12) and are given explicitly by Eq (2.24).

Definition 10. *[21] Consider the QRS that is defined by* (3.2)*. It has a unit normal vector field n_ψ which is defined by*

$$n_\psi = \frac{\psi_s \wedge \psi_v}{\|\psi_s \wedge \psi_v\|}. \tag{3.3}$$

where $\psi_s = \frac{\partial \psi(s,v)}{\partial s}$ and $\psi_v = \frac{\partial \psi(s,v)}{\partial v}$.

Definition 11. *[21] The geodesic curvature κ_g, normal curvature κ_n, and geodesic torsion τ_g of the QFC F_α on the surface ψ are defined as follows:*

$$\kappa_g = \langle n_\psi \wedge \boldsymbol{T}_q^F, (\boldsymbol{T}_q^F)' \rangle, \quad \kappa_n = \langle (\boldsymbol{T}_q^F)', n_\psi \rangle, \quad \tau_g = \langle n_\psi \wedge \frac{\partial n}{\partial s}, (\boldsymbol{T}_q^F)' \rangle, \quad (\,)' = \frac{d}{ds}(\,). \tag{3.4}$$

Definition 12. *[21] The curve F_α lying on the surface ψ is a geodesic curve, an asymptotic line, and a principal line if and only if $\kappa_g = 0$, $\kappa_n = 0$, and $\tau_g = 0$, respectively.*

Definition 13. *[21] The coefficients of the first fundamental form (CFFF) are defined as follows:*

$$\boldsymbol{g}_{11} = \langle \psi_s, \psi_s \rangle, \quad \boldsymbol{g}_{12} = \langle \psi_s, \psi_v \rangle, \quad \boldsymbol{g}_{22} = \langle \psi_v, \psi_v \rangle. \tag{3.5}$$

Also, the coefficients of the second fundamental form (CSFF) are defined as follows:

$$\boldsymbol{L}_{11} = \langle \psi_{ss}, n_\psi \rangle, \quad \boldsymbol{L}_{12} = \langle \psi_{sv}, n_\psi \rangle, \quad \boldsymbol{L}_{22} = \langle \psi_{vv}, n_\psi \rangle. \tag{3.6}$$

Definition 14. *[21] The Gaussian curvature GC, the mean curvature MC, and the distribution parameter are denoted, respectively, by $\boldsymbol{K}$, $\boldsymbol{H}$, and λ, where they are given by*

$$\boldsymbol{K} = \frac{\boldsymbol{L}_{11}\boldsymbol{L}_{22} - \boldsymbol{L}_{12}^2}{\boldsymbol{g}_{11}\boldsymbol{g}_{22} - \boldsymbol{g}_{12}^2}, \tag{3.7}$$

$$\boldsymbol{H} = \frac{\boldsymbol{g}_{11}\boldsymbol{L}_{22} - 2\boldsymbol{g}_{12}\boldsymbol{L}_{12} + \boldsymbol{g}_{22}\boldsymbol{L}_{11}}{2(\boldsymbol{g}_{11}\boldsymbol{g}_{22} - \boldsymbol{g}_{12}^2)}, \tag{3.8}$$

$$\lambda = \frac{\det(F'_\alpha, \eta, \eta')}{\|\eta'\|^2}, \ (\,)' = \frac{d}{ds}(\,). \tag{3.9}$$

Definition 15. *[25] The second mean curvature (SMC), denoted as $\boldsymbol{H}_{II}$, is defined for the QRS in three-dimensional Euclidean space $\mathbb{R}^3$ by*

$$\boldsymbol{H}_{II} = \boldsymbol{H} + \frac{1}{4}\Delta_{II} \log(|\boldsymbol{K}|), \tag{3.10}$$

where Δ_{II} stands for the Laplacian function. In explicit terms, we have

$$\boldsymbol{H}_{II} = \boldsymbol{H} + \frac{1}{2\sqrt{|\det(II)|}} \sum_{i,j} \frac{\partial}{\partial x^i}\Big(\sqrt{|\det(II)|}\, \boldsymbol{L}^{ij} \frac{\partial}{\partial x^j}(\ln \sqrt{|\boldsymbol{K}|}\,)\Big), \tag{3.11}$$

where $\boldsymbol{L}^{ij}$ is the inverse of $\boldsymbol{L}_{ij}$, and the indices i, j belong to $\{1, 2\}$. Let the parameters $\{x^1, x^2\}$ correspond to the coordinates $\{s, v\}$.

Definition 16. *[26] Let $\boldsymbol{K}_{II}$ stand for the SGC of the QRS in $\mathbb{R}^3$. It is defined by using Brioschi's formula by replacing the curvature tensor $\boldsymbol{L}_{11}, \boldsymbol{L}_{12}$, and $\boldsymbol{L}_{22}$ with the metric tensor components $\boldsymbol{g}_{11}, \boldsymbol{g}_{12}$, and $\boldsymbol{g}_{22}$, respectively:*

$$\boldsymbol{K}_{II} = \frac{1}{(\det(II))^2}\left(\begin{vmatrix} -\frac{1}{2}\boldsymbol{L}_{11,vv} + \boldsymbol{L}_{12,sv} - \frac{1}{2}\boldsymbol{L}_{22,ss} & \frac{1}{2}\boldsymbol{L}_{11,s} & \boldsymbol{L}_{12,s} - \frac{1}{2}\boldsymbol{L}_{11,v} \\ \boldsymbol{L}_{12,v} - \frac{1}{2}\boldsymbol{L}_{22,s} & \boldsymbol{L}_{11} & \boldsymbol{L}_{12} \\ \frac{1}{2}\boldsymbol{L}_{22,v} & \boldsymbol{L}_{12} & \boldsymbol{L}_{22} \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}\boldsymbol{L}_{11,v} & \frac{1}{2}\boldsymbol{L}_{22,s} \\ \frac{1}{2}\boldsymbol{L}_{11,v} & \boldsymbol{L}_{11} & \boldsymbol{L}_{12} \\ \frac{1}{2}\boldsymbol{L}_{22,s} & \boldsymbol{L}_{12} & \boldsymbol{L}_{22} \end{vmatrix} \right), \tag{3.12}$$

where $(\ )_{,v} = \frac{\partial}{\partial v}$, $(\ )_{,vv} = \frac{\partial^2}{\partial v^2}$, $(\ )_{,s} = \frac{\partial}{\partial s}$, $(\ )_{,ss} = \frac{\partial^2}{\partial s^2}$, and $(\ )_{,sv} = \frac{\partial^2}{\partial v \partial s}$. While the minimal surfaces are characterized by a vanishing SGC, $\mathbf{K}_{II} = 0$, the converse is not true: A surface with $\mathbf{K}_{II} = 0$ is not necessarily minimal.

Definition 17. *[21] A developable surface in $\mathbb{R}^3$ has a vanishing GC ($\boldsymbol{K} = 0$), while a minimal surface has a vanishing MC ($\boldsymbol{H} = 0$).*

Definition 18. *[27] A non-developable surface in $\mathbb{R}^3$ is called II-flat if the SGC, ($\boldsymbol{K}_{II} = 0$), and II-minimal if the SMC, ($\boldsymbol{H}_{II} = 0$).*

4. Methods and results

Let s represent the arc length along the curve $\alpha : I \in \mathbb{R} \to \mathbb{R}^3$ within the interval I. Assume that $\alpha(s)$ is a unit speed curve defined by the q-frame $\{\mathbf{T}_{q\alpha}, \mathbf{N}_{q\alpha}, \mathbf{B}_{q\alpha}\}$. Consider $F_\alpha\left(s^F(s)\right)$ to be a unit speed QFC for the original curve α, with arc length $s^F(s)$, and described by the q-frame $\{\mathbf{T}_q^F, \mathbf{N}_q^F, \mathbf{B}_q^F\}$. The QFC $F_\alpha\left(s^F(s)\right)$ is defined by (2.8), and the relation between the q-frame for the QFC and the q-frame of the original curve α is obtained by (2.21). In this section, we present some novel types of QRS constructed by the QFC as a base curve (directrix), and with the ruling that is given by the q-frame of F_α. We define the following novel types of QRS as follows:

1. The osculating type of quasi-ruled surface whose ruling lies in the osculating plane $\{\mathbf{T}_q^F, \mathbf{N}_q^F\}$ of the base curve F_α.
$$\psi_1(s, v) = F_\alpha(s) + v(\mu_1 \mathbf{T}_q^F + \mu_2 \mathbf{N}_q^F),\ \mu_1^2 + \mu_2^2 = 1.$$
2. The normal type of quasi-ruled surface whose ruling lies in the normal plane $\{\mathbf{N}_q^F, \mathbf{B}_q^F\}$ of the base curve F_α.
$$\psi_2(s, v) = F_\alpha(s) + v(\mu_2 \mathbf{N}_q^F + \mu_3 \mathbf{B}_q^F),\ \mu_2^2 + \mu_3^2 = 1.$$
3. The rectifying type of quasi-ruled surface whose ruling lies in the rectifying plane $\{\mathbf{T}_q^F, \mathbf{B}_q^F\}$ of the base curve F_α.
$$\psi_3(s, v) = F_\alpha(s) + v(\mu_1 \mathbf{T}_q^F + \mu_3 \mathbf{B}_q^F),\ \mu_1^2 + \mu_3^2 = 1.$$
4. The quasi-tangent developable surface whose ruling parallels the quasi-tangent vector of F_α.
$$\psi_4(s, v) = F_\alpha(s) + v\mathbf{T}_q^F.$$

5. The quasi-principal normal ruled surface whose ruling parallels the quasi-principal normal vector of F_α.

$$\psi_5(s,v) = F_\alpha(s) + v\mathbf{N}_q^F.$$

6. The quasi-binormal ruled surface whose ruling parallels the quasi-binormal vector of F_α.

$$\psi_6(s,v) = F_\alpha(s) + v\mathbf{B}_q^F.$$

4.1. Construction of the osculating type of quasi-ruled surfaces

Let $F_\alpha\left(s^F(s)\right)$ be the QFC of the original curve α. Assume that the ruling lies in the osculating plane $\{\mathbf{T}_q^F, \mathbf{N}_q^F\}$ of the base curve F_α. In this case, the constructed surface is called the osculating type of quasi-ruled surface, where

$$\eta(s) = \mu_1\mathbf{T}_q^F + \mu_2\mathbf{N}_q^F, \quad \mu_1^2 + \mu_2^2 = 1. \tag{4.1}$$

Substituting from (2.21) into (4.1), then

$$\eta(s) = \mu_2\mathbf{N}_{q_\alpha} + \mu_1\mathbf{B}_{q_\alpha}. \tag{4.2}$$

Substituting from (4.2) into (3.2), then we obtain the osculating type QRS:

$$\psi_1(s,v) = \alpha(s) + (\varphi_1 + v\mu_2)\mathbf{N}_{q_\alpha} + (\varphi_2 + v\mu_1)\mathbf{B}_{q_\alpha}. \tag{4.3}$$

Taking the first derivative of (4.3) with respect to s, we have

$$\psi_{1,s} = \alpha' + (\varphi_1' + v\mu_2')\mathbf{N}_{q_\alpha} + (\varphi_1 + v\mu_2)\mathbf{N}_{q_\alpha}' + (\varphi_2' + v\mu_1')\mathbf{B}_{q_\alpha} + (\varphi_2 + v\mu_1)\mathbf{B}_{q_\alpha}'. \tag{4.4}$$

Substituting from (2.7) into (4.4), then

$$\begin{aligned}\psi_{1,s} = {} & (1 - \kappa_1(\varphi_1 + v\mu_2) - \kappa_2(\varphi_2 + v\mu_1)\ )\mathbf{T}_{q_\alpha} + (\varphi_1' + v\mu_2' - \kappa_3(\varphi_2 + v\mu_1)\ )\mathbf{N}_{q_\alpha} \\ & + (\varphi_2' + v\mu_1' + \kappa_3(\varphi_1 + v\mu_2)\ )\mathbf{B}_{q_\alpha}.\end{aligned} \tag{4.5}$$

Using relation (2.12), we obtain

$$\psi_{1,s} = -v(\kappa_1\mu_2 + \kappa_2\mu_1)\mathbf{T}_{q_\alpha} + v(\mu_2' - \kappa_3\mu_1)\mathbf{N}_{q_\alpha} + \left(v(\mu_1' + \kappa_3\mu_2) + \varphi_2' + \kappa_3\varphi_1\right)\mathbf{B}_{q_\alpha}.$$

Choose

$$\begin{aligned}\xi_1 &= -(\kappa_1\mu_2 + \kappa_2\mu_1), \\ \xi_2 &= \mu_2' - \kappa_3\mu_1, \\ \xi_3 &= \mu_1' + \kappa_3\mu_2, \quad \mu_1\xi_3 + \mu_2\xi_2 = 0, \\ \xi_4 &= \varphi_2' + \kappa_3\varphi_1.\end{aligned} \tag{4.6}$$

Then,

$$\psi_{1,s} = v\xi_1\mathbf{T}_{q_\alpha} + v\xi_2\mathbf{N}_{q_\alpha} + (v\xi_3 + \xi_4)\mathbf{B}_{q_\alpha}. \tag{4.7}$$

Taking the first derivative of (4.3) with respect to v, we have

$$\psi_{1,v} = \mu_2\mathbf{N}_{q_\alpha} + \mu_1\mathbf{B}_{q_\alpha}\ . \tag{4.8}$$

By substituting from (4.7) and (4.8) into (3.5), we obtain the following lemma.

Lemma 6. *The CFFF of the osculating type of QRS are given by*

$$\boldsymbol{g}_{11} = v^2(\xi_1^2 + \xi_2^2 + \xi_3^2) + 2v\xi_3\xi_4 + \xi_4^2, \quad \boldsymbol{g}_{12} = \mu_1\xi_4, \quad \boldsymbol{g}_{22} = 1. \tag{4.9}$$

Lemma 7. *The normal vector n_{ψ_1} to the osculating type of QRS is given by*

$$\begin{aligned} n_{\psi_1} &= \frac{1}{\epsilon_1}\Big(((\mu_1\xi_2 - \mu_2\xi_3)v - \mu_2\xi_4)\boldsymbol{T}_{q_\alpha} - v\mu_1\xi_1\boldsymbol{N}_{q_\alpha} + v\mu_2\xi_1\boldsymbol{B}_{q_\alpha}\Big), \\ \epsilon_1 &= \Big(v^2(\xi_1^2 + \xi_2^2 + \xi_3^2) + 2v\xi_3\xi_4 + \mu_2^2\xi_4^2\Big)^{1/2}. \end{aligned} \tag{4.10}$$

Lemma 8. *Consider the osculating type of QRS that is defined by* (4.3). *Then, the second partial derivatives with respect to s and v are given by*

$$\begin{aligned} \psi_{1,ss} &= (\lambda_1 v - \kappa_2\xi_4)\boldsymbol{T}_{q_\alpha} + (\lambda_2 v - \kappa_3\xi_4)\boldsymbol{N}_{q_\alpha} + (\lambda_3 v + \xi_4')\boldsymbol{B}_{q_\alpha}, \\ \psi_{1,sv} &= \xi_1\boldsymbol{T}_{q_\alpha} + \xi_2\boldsymbol{N}_{q_\alpha} + \xi_3\boldsymbol{B}_{q_\alpha}, \ \psi_{1,vv} = 0, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} \lambda_1 &= \xi_1' - \kappa_1\xi_2 - \kappa_2\xi_3, \\ \lambda_2 &= \xi_2' + \kappa_1\xi_1 - \kappa_3\xi_3, \\ \lambda_3 &= \xi_3' + \kappa_2\xi_1 + \kappa_3\xi_2. \end{aligned} \tag{4.12}$$

Lemma 9. *The CSFF of the osculating type of QRS are given as*

$$\boldsymbol{L}_{11} = \frac{1}{\epsilon_1}(A_1v^2 + A_2v + \kappa_2\mu_2\xi_4^2), \ \boldsymbol{L}_{12} = -\frac{\mu_2\xi_1\xi_4}{\epsilon_1}, \ \boldsymbol{L}_{22} = 0, \tag{4.13}$$

where

$$\begin{aligned} A_1 &= \lambda_1(\mu_1\xi_2 - \mu_2\xi_3) - \lambda_2\mu_1\xi_1 + \lambda_3\mu_2\xi_1, \\ A_2 &= -\lambda_1\mu_2\xi_4 - \kappa_2\xi_4(\mu_1\xi_2 - \mu_2\xi_3) + \mu_1\kappa_3\xi_1\xi_4 + \mu_2\xi_1\xi_4'. \end{aligned} \tag{4.14}$$

Lemma 10. *The MC and GC for the osculating type of QRS are given directly by substituting from* (4.9) *and* (4.13) *into* (3.7) *and* (3.8)*:*

$$\boldsymbol{H} = \frac{1}{2\epsilon_1^3}\Big(A_1v^2 + A_2v + \mu_2\xi_4^2(\kappa_2 + 2\mu_1\xi_1)\Big), \ \boldsymbol{K} = -\frac{(\mu_2\xi_1\xi_4)^2}{\epsilon_1^4}. \tag{4.15}$$

4.2. Construction of the normal type of quasi-ruled surface

Let $F_\alpha\left(s^F(s)\right)$ be the QFC of the original curve α. Assume that the ruling lies in the normal plane $\{\mathbf{N}_q^F, \mathbf{B}_q^F\}$ of the base curve F_α. In this case, the constructed surface is called the normal type of quasi-ruled surface QRS, where

$$\eta(s) = \mu_2\mathbf{N}_q^F + \mu_3\mathbf{B}_q^F, \qquad \mu_2^2 + \mu_3^2 = 1. \tag{4.16}$$

Substituting from (2.21) into (4.16), then

$$\eta(s) = \mu_2\mathbf{N}_{q_\alpha} - \mu_3\mathbf{T}_{q_\alpha}. \tag{4.17}$$

Substituting from (4.17) into (3.2), we obtain the normal type of QRS, which has the following parametrization:

$$\psi_2(s,v) = \alpha(s) - v\mu_3\mathbf{T}_{q_\alpha} + (\varphi_1 + v\mu_2)\mathbf{N}_{q_\alpha} + \varphi_2\mathbf{B}_{q_\alpha}. \tag{4.18}$$

Taking the first derivative of (4.18) with respect to s, then

$$\begin{aligned}\psi_{2,s} = &(1 - \kappa_1(\varphi_1 + v\mu_2) - \kappa_2\varphi_2 - v\mu_3'\ )\mathbf{T}_{q_\alpha} + (\varphi_1' + v\mu_2' - \kappa_3\varphi_2 - v\kappa_1\mu_3\ )\mathbf{N}_{q_\alpha}\\ &+ (\varphi_2' + \kappa_3\varphi_1 + v(\mu_2\kappa_3 - \mu_3\kappa_2)\ )\mathbf{B}_{q_\alpha}.\end{aligned}$$

Using relation (2.12), we obtain

$$\psi_{2,s} = -v(\mu_3' + \mu_2\kappa_1)\mathbf{T}_{q_\alpha} + v(\mu_2' - \mu_3\kappa_1)\mathbf{N}_{q_\alpha} + (\varphi_2' + \kappa_3\varphi_1 + v(\mu_2\kappa_3 - \mu_3\kappa_2)\ )\mathbf{B}_{q_\alpha}.$$

Choose

$$\begin{aligned}\tilde{\xi}_1 &= -(\mu_3' + \mu_2\kappa_1),\\ \tilde{\xi}_2 &= \mu_2' - \mu_3\kappa_1, \quad \mu_2\tilde{\xi}_2 - \mu_3\tilde{\xi}_1 = 0,\\ \tilde{\xi}_3 &= \mu_2\kappa_3 - \mu_3\kappa_2,\\ \xi_4 &= \varphi_2' + \kappa_3\varphi_1.\end{aligned} \tag{4.19}$$

Then,

$$\psi_{2,s} = v\tilde{\xi}_1\mathbf{T}_{q_\alpha} + v\tilde{\xi}_2\mathbf{N}_{q_\alpha} + (v\tilde{\xi}_3 + \xi_4)\mathbf{B}_{q_\alpha}. \tag{4.20}$$

Taking the first derivative of (4.18) with respect to v, we have

$$\psi_{2,v} = -\mu_3\mathbf{T}_{q_\alpha} + \mu_2\mathbf{N}_{q_\alpha}. \tag{4.21}$$

By substituting from (4.20) and (4.21) into (3.5), we obtain the following lemma.

Lemma 11. *The CFFF of the normal type of QRS are given as*

$$\boldsymbol{g}_{11} = v^2(\tilde{\xi}_1^2 + \tilde{\xi}_2^2 + \tilde{\xi}_3^2) + 2v\tilde{\xi}_3\xi_4 + \xi_4^2, \quad \boldsymbol{g}_{12} = 0, \quad \boldsymbol{g}_{22} = 1. \tag{4.22}$$

Lemma 12. *The normal vector n_{ψ_2} to the normal type of QRS is given by*

$$\begin{aligned}n_{\psi_2} &= \frac{1}{\epsilon_2}\Big(- \mu_2(v\tilde{\xi}_3 + \xi_4)\boldsymbol{T}_{q_\alpha} - \mu_3(v\tilde{\xi}_3 + \xi_4)\boldsymbol{N}_{q_\alpha} + v(\mu_2\tilde{\xi}_1 + \mu_3\tilde{\xi}_2)\boldsymbol{B}_{q_\alpha}\Big),\\ \epsilon_2 &= \Big(v^2(\tilde{\xi}_1^2 + \tilde{\xi}_2^2 + \tilde{\xi}_3^2) + 2v\tilde{\xi}_3\xi_4 + \xi_4^2\Big)^{1/2}.\end{aligned} \tag{4.23}$$

Lemma 13. *Consider the normal type of QRS that is defined by* (4.18)*. Then, the second partial derivatives with respect to s and v are given by*

$$\begin{aligned}\psi_{2,ss} &= (\tilde{\lambda}_1 v - \kappa_2\xi_4)\boldsymbol{T}_{q_\alpha} + (\tilde{\lambda}_2 v - \kappa_3\xi_4)\boldsymbol{N}_{q_\alpha} + (\tilde{\lambda}_3 v + \xi_4')\boldsymbol{B}_{q_\alpha},\\ \psi_{2,sv} &= \tilde{\xi}_1\boldsymbol{T}_{q_\alpha} + \tilde{\xi}_2\boldsymbol{N}_{q_\alpha} + \tilde{\xi}_3\boldsymbol{B}_{q_\alpha},\ \psi_{2,vv} = 0,\end{aligned} \tag{4.24}$$

where

$$\begin{aligned}\tilde{\lambda}_1 &= \tilde{\xi}_1' - \kappa_1\tilde{\xi}_2 - \kappa_2\tilde{\xi}_3,\\ \tilde{\lambda}_2 &= \tilde{\xi}_2' + \kappa_1\tilde{\xi}_1 - \kappa_3\tilde{\xi}_3,\\ \tilde{\lambda}_3 &= \tilde{\xi}_3' + \kappa_2\tilde{\xi}_1 + \kappa_3\tilde{\xi}_2.\end{aligned} \tag{4.25}$$

Lemma 14. *The CSFF of the normal type of QRS are given as*

$$\boldsymbol{L}_{11} = \frac{1}{\epsilon_2}\Big(\tilde{A}_1 v^2 + \tilde{A}_2 v + (\kappa_2\mu_2 + \kappa_3\mu_3)\xi_4^2\Big),\ \boldsymbol{L}_{12} = -\frac{\xi_4}{\epsilon_2}(\mu_2\tilde{\xi}_1 + \mu_3\tilde{\xi}_2)\ ,\ \boldsymbol{L}_{22} = 0, \tag{4.26}$$

where

$$\begin{aligned}\tilde{A}_1 &= -(\tilde{\lambda}_1\mu_2 + \tilde{\lambda}_2\mu_3)\tilde{\xi}_3 + \tilde{\lambda}_3(\mu_2\tilde{\xi}_1 + \mu_3\tilde{\xi}_2)\ ,\\ \tilde{A}_2 &= -(\tilde{\lambda}_1\mu_2 + \tilde{\lambda}_2\mu_3)\xi_4 + (\mu_2\kappa_2 + \mu_3\kappa_3)\tilde{\xi}_3\xi_4 + (\mu_2\tilde{\xi}_1 + \mu_3\tilde{\xi}_2)\xi_4'\ .\end{aligned} \tag{4.27}$$

Lemma 15. *The MC and GC for the normal type of QRS are given directly by substituting from* (4.22) *and* (4.26) *into* (3.7) *and* (3.8).

$$\boldsymbol{H} = \frac{1}{2\epsilon_2^3}\Big(\tilde{A}_1 v^2 + \tilde{A}_2 v + (\mu_2\kappa_2 + \mu_3\kappa_3)\xi_4^2\Big),\ \boldsymbol{K} = -\frac{\xi_4^2}{\epsilon_2^4}(\mu_2\tilde{\xi}_1 + \mu_3\tilde{\xi}_2)^2. \tag{4.28}$$

4.3. Construction of the rectifying type of quasi-ruled surface

Let $F_\alpha\left(s^F(s)\right)$ be the QFC of the original curve α. Assume that the ruling lies in the rectifying plane $\{\mathbf{T}_q^F, \mathbf{B}_q^F\}$ of the base curve F_α. In this case, the constructed surface is called the rectifying type of quasi-ruled surface QRS, where

$$\eta(s) = \mu_1\mathbf{T}_q^F + \mu_3\mathbf{B}_q^F, \qquad \mu_1^2 + \mu_3^2 = 1. \tag{4.29}$$

Substituting from (2.21) into (4.29), then

$$\eta(s) = -\mu_3\mathbf{T}_{q_\alpha} + \mu_1\mathbf{B}_{q_\alpha}. \tag{4.30}$$

Substituting from (4.30) into (3.2), we obtain the rectifying type of QRS with the following parametrization:

$$\psi_3(s, v) = \alpha(s) - v\mu_3\mathbf{T}_{q_\alpha} + \varphi_1\mathbf{N}_{q_\alpha} + (\varphi_2 + v\mu_1)\mathbf{B}_{q_\alpha}. \tag{4.31}$$

Taking the first derivative of (4.31) with respect to s, then

$$\begin{aligned}\psi_{3,s} = {}& (1 - \kappa_1\varphi_1 - \kappa_2\varphi_2 - v(\mu_3' + \mu_1\kappa_2))\mathbf{T}_{q_\alpha} + (\varphi_1' - \kappa_3\varphi_2 - v(\mu_3\kappa_1 + \mu_1\kappa_3)\ )\mathbf{N}_{q_\alpha}\\ &+ (\varphi_2' + \kappa_3\varphi_1 + v(\mu_1' - \mu_3\kappa_2)\ )\mathbf{B}_{q_\alpha}.\end{aligned}$$

Using relation (2.12), we obtain

$$\psi_{3,s} = -v(\mu_3' + \mu_1\kappa_2)\mathbf{T}_{q_\alpha} - v(\mu_3\kappa_1 + \mu_1\kappa_3)\mathbf{N}_{q_\alpha} + (\varphi_2' + \kappa_3\varphi_1 + v(\mu_1' - \mu_3\kappa_2)\ )\mathbf{B}_{q_\alpha}.$$

Choose

$$\begin{aligned}\hat{\xi}_1 &= -(\mu_3' + \mu_1\kappa_2),\\ \hat{\xi}_2 &= -(\mu_3\kappa_1 + \mu_1\kappa_3),\\ \hat{\xi}_3 &= \mu_1' - \mu_3\kappa_2,\ \mu_1\hat{\xi}_3 - \mu_3\hat{\xi}_1 = 0,\\ \xi_4 &= \varphi_2' + \kappa_3\varphi_1.\end{aligned} \tag{4.32}$$

Then,

$$\psi_{3,s} = v\hat{\xi}_1\mathbf{T}_{q_\alpha} + v\hat{\xi}_2\mathbf{N}_{q_\alpha} + (v\hat{\xi}_3 + \xi_4)\mathbf{B}_{q_\alpha}. \tag{4.33}$$

Taking the first derivative of (4.31) with respect to v, we have

$$\psi_{3,v} = -\mu_3\mathbf{T}_{q_\alpha} + \mu_1\mathbf{B}_{q_\alpha}. \tag{4.34}$$

By substituting from (4.33) and (4.34) into (3.5), we obtain the following lemma.

Lemma 16. *The CFFF of the rectifying type of QRS are given as*

$$\boldsymbol{g}_{11} = v^2(\hat{\xi}_1^2 + \hat{\xi}_2^2 + \hat{\xi}_3^2) + 2v\hat{\xi}_3\xi_4 + \xi_4^2, \quad \boldsymbol{g}_{12} = \mu_1\xi_4, \quad \boldsymbol{g}_{22} = 1. \tag{4.35}$$

Lemma 17. *The normal vector n_{ψ_3} to the rectifying type of QRS is given by*

$$\begin{aligned} n_{\psi_3} &= \frac{1}{\epsilon_3}\Big(v\mu_1\hat{\xi}_2\boldsymbol{T}_{q_\alpha} - (v(\mu_1\hat{\xi}_1 + \mu_3\hat{\xi}_3) + \mu_3\xi_4)\boldsymbol{N}_{q_\alpha} + v\mu_3\hat{\xi}_2\boldsymbol{B}_{q_\alpha}\Big), \\ \epsilon_3 &= \Big(v^2(\hat{\xi}_1^2 + \hat{\xi}_2^2 + \hat{\xi}_3^2) + 2v\hat{\xi}_3\xi_4 + \mu_3^2\xi_4^2\Big)^{1/2}. \end{aligned} \tag{4.36}$$

Lemma 18. *Consider the rectifying type of QRS that is defined by* (4.31). *Then, the second partial derivatives with respect to s and v are given by*

$$\begin{aligned} \psi_{3,ss} &= (\hat{\lambda}_1 v - \kappa_2\xi_4)\boldsymbol{T}_{q_\alpha} + (\hat{\lambda}_2 v - \kappa_3\xi_4)\boldsymbol{N}_{q_\alpha} + (\hat{\lambda}_3 v + \xi_4')\boldsymbol{B}_{q_\alpha}, \\ \psi_{3,sv} &= \hat{\xi}_1\boldsymbol{T}_{q_\alpha} + \hat{\xi}_2\boldsymbol{N}_{q_\alpha} + \hat{\xi}_3\boldsymbol{B}_{q_\alpha}\ ,\ \psi_{3,vv} = 0, \end{aligned} \tag{4.37}$$

where

$$\begin{aligned} \hat{\lambda}_1 &= \hat{\xi}_1' - \kappa_1\hat{\xi}_2 - \kappa_2\hat{\xi}_3, \\ \hat{\lambda}_2 &= \hat{\xi}_2' + \kappa_1\hat{\xi}_1 - \kappa_3\hat{\xi}_3, \\ \hat{\lambda}_3 &= \hat{\xi}_3' + \kappa_2\hat{\xi}_1 + \kappa_3\hat{\xi}_2. \end{aligned} \tag{4.38}$$

Lemma 19. *The CSFF of the rectifying type of QRS are given as*

$$\boldsymbol{L}_{11} = \frac{1}{\epsilon_3}\Big(\hat{A}_1 v^2 + \hat{A}_2 v + \kappa_3\mu_3\xi_4^2\Big),\ \boldsymbol{L}_{12} = -\frac{\mu_3\hat{\xi}_2\xi_4}{\epsilon_3},\ \boldsymbol{L}_{22} = 0, \tag{4.39}$$

where

$$\begin{aligned} \hat{A}_1 &= (\hat{\lambda}_1\mu_1 + \hat{\lambda}_3\mu_3)\hat{\xi}_2 - \hat{\lambda}_2(\mu_1\hat{\xi}_1 + \mu_3\hat{\xi}_3), \\ \hat{A}_2 &= -(\mu_3\hat{\lambda}_2 + \mu_1\kappa_2\hat{\xi}_2)\xi_4 + \mu_3\hat{\xi}_2\xi_4' + \kappa_3\xi_4(\mu_1\hat{\xi}_1 + \mu_3\hat{\xi}_3). \end{aligned} \tag{4.40}$$

Lemma 20. *The MC and GC for the rectifying type of QRS are given directly by substituting from* (4.35) *and* (4.39) *into* (3.7) *and* (3.8)*:*

$$\boldsymbol{H} = \frac{1}{2\epsilon_3^3}\Big(\hat{A}_1 v^2 + \hat{A}_2 v + \mu_3\xi_4^2(\kappa_3 + 2\mu_1\hat{\xi}_2)\Big),\ \ \boldsymbol{K} = -\frac{(\mu_3\hat{\xi}_2\xi_4)^2}{\epsilon_3^4}. \tag{4.41}$$

4.4. Construction of the quasi-tangent developable surface

Let $F_\alpha\left(s^F(s)\right)$ be the QFC of the original curve α. Assume that the ruling parallels the quasi-tangent vector $\mathbf{T}_q^F$ of the base curve F_α, so

$$\eta(s) = \mathbf{T}_q^F = \mathbf{B}_{q_\alpha}. \tag{4.42}$$

Substituting from (4.42) into (3.2), we obtain the quasi-tangent developable surface as follows:

$$\psi_4(s, v) = \alpha(s) + \varphi_1 \mathbf{N}_{q_\alpha} + (\varphi_2 + v)\mathbf{B}_{q_\alpha}. \tag{4.43}$$

Taking the first derivative of (4.43) with respect to s, we have

$$\psi_{4,s} = \alpha' + \varphi_1' \mathbf{N}_{q_\alpha} + \varphi_1 \mathbf{N}_{q_\alpha}' + \varphi_2' \mathbf{B}_{q_\alpha} + (\varphi_2 + v)\mathbf{B}_{q_\alpha}'. \tag{4.44}$$

Substituting from (2.7) into (4.44), then

$$\psi_{4,s} = (1 - \kappa_1\varphi_1 - \kappa_2(\varphi_2 + v))\mathbf{T}_{q_\alpha} + (\varphi_1' - \kappa_3(\varphi_2 + v))\mathbf{N}_{q_\alpha} + (\varphi_2' + \kappa_3\varphi_1)\mathbf{B}_{q_\alpha}. \tag{4.45}$$

Using relation (2.12), we obtain

$$\psi_{4,s} = -v\kappa_2 \mathbf{T}_{q_\alpha} - v\kappa_3 \mathbf{N}_{q_\alpha} + (\varphi_2' + \kappa_3\varphi_1)\mathbf{B}_{q_\alpha}.$$

Choose

$$\xi_4 = \varphi_2' + \kappa_3\varphi_1, \quad \text{where} \quad \varphi_2' = \left(\frac{1 - \kappa_1\varphi_1}{\kappa_2}\right)'. \tag{4.46}$$

Then, we have

$$\psi_{4,s} = -v\kappa_2 \mathbf{T}_{q_\alpha} - v\kappa_3 \mathbf{N}_{q_\alpha} + \xi_4 \mathbf{B}_{q_\alpha}. \tag{4.47}$$

Taking the first partial derivative of (4.43) with respect to v, we have

$$\psi_{4,v} = \mathbf{B}_{q_\alpha}. \tag{4.48}$$

Lemma 21. *The CFFF for the quasi-tangent developable surface are given by*

$$\boldsymbol{g}_{11} = \xi_4^2 + v^2(\kappa_2^2 + \kappa_3^2), \quad \boldsymbol{g}_{12} = \xi_4, \quad \boldsymbol{g}_{22} = 1. \tag{4.49}$$

Lemma 22. *The normal vector n_{ψ_4} to the quasi-tangent developable surface is given by*

$$n_{\psi_4} = \frac{-\kappa_3 \boldsymbol{T}_{q_\alpha} + \kappa_2 \, \boldsymbol{N}_{q_\alpha}}{\sqrt{\kappa_2^2 + \kappa_3^2}}. \tag{4.50}$$

Lemma 23. *The CSFF of the quasi-tangent developable surface are given as*

$$\boldsymbol{L}_{11} = \frac{v}{\sqrt{\kappa_2^2 + \kappa_3^2}}(\kappa_2'\kappa_3 - \kappa_2\kappa_3' - \kappa_1(\kappa_2^2 + \kappa_3^2)), \ \boldsymbol{L}_{12} = 0, \ \boldsymbol{L}_{22} = 0. \tag{4.51}$$

Proof. Taking the second partial derivatives of (4.47) and (4.48) with respect to s and v, we obtain

$$\begin{aligned} \psi_{4,ss} &= -\left(\xi_4\kappa_2 + v(\kappa_2' - \kappa_1\kappa_3)\right)\mathbf{T}_{q_\alpha} - \left(\xi_4\kappa_3 + v(\kappa_3' + \kappa_1\kappa_2)\right)\mathbf{N}_{q_\alpha} + \left(\xi_4' - v(\kappa_2^2 + \kappa_3^2)\right)\mathbf{B}_{q_\alpha}, \\ \psi_{4,sv} &= -\kappa_2 \mathbf{T}_{q_\alpha} - \kappa_3 \mathbf{N}_{q_\alpha}, \ \ \psi_{4,vv} = 0. \end{aligned} \tag{4.52}$$

Taking the inner product of (4.50) and (4.52) and substituting into (3.6), the lemma holds. □

Lemma 24. *The MC and GC for the quasi-tangent developable surface are given directly by substituting from* (4.49) *and* (4.51) *into* (3.7) *and* (3.8).

$$\boldsymbol{H} = \frac{\kappa_2'\kappa_3 - \kappa_2\kappa_3' - \kappa_1(\kappa_2^2 + \kappa_3^2)}{2v(\kappa_2^2 + \kappa_3^2)^{3/2}}, \ \boldsymbol{K} = 0. \tag{4.53}$$

4.5. Construction of the quasi-principal normal ruled surface

Let $F_\alpha\left(s^F(s)\right)$ be the QFC of the original curve α. Consider the case where the ruling parallels the quasi-principal normal vector N_q^F of the base curve F_α. Then,

$$\eta(s) = \mathbf{N}_q^F. \tag{4.54}$$

Substituting from (2.21) into (4.54), then

$$\eta(s) = \mathbf{N}_{q_\alpha}. \tag{4.55}$$

Substituting from (4.55) into (3.2), we obtain the quasi-principal normal ruled surface, which is given by

$$\psi_5(s, v) = \alpha(s) + (\varphi_1 + v)\mathbf{N}_{q_\alpha} + \varphi_2\mathbf{B}_{q_\alpha}. \tag{4.56}$$

Taking the first derivative of (4.56) with respect to s and using (2.7), then

$$\psi_{5,s} = (1 - \kappa_1(\varphi_1 + v) - \kappa_2\varphi_2)\mathbf{T}_{q_\alpha} + (\varphi_1' - \kappa_3\varphi_2)\mathbf{N}_{q_\alpha} + (\varphi_2' + \kappa_3\varphi_1 + v\kappa_3)\mathbf{B}_{q_\alpha}. \tag{4.57}$$

Using relation (2.12), we obtain

$$\psi_{5,s} = -v\kappa_1\mathbf{T}_{q_\alpha} + (v\kappa_3 + \xi_4)\mathbf{B}_{q_\alpha}, \ \xi_4 = \varphi_2' + \kappa_3\varphi_1. \tag{4.58}$$

Taking the first derivative of (4.56) with respect to v, we have

$$\psi_{5,v} = \mathbf{N}_{q_\alpha}. \tag{4.59}$$

By substituting from (4.58) and (4.59) into (3.5), we obtain the following lemma.

Lemma 25. *The CFFF of the quasi-principal normal ruled surface are given by*

$$\boldsymbol{g}_{11} = (\kappa_1^2 + \kappa_3^2)v^2 + 2v\kappa_3\xi_4 + \xi_4^2, \quad \boldsymbol{g}_{12} = 0, \quad \boldsymbol{g}_{22} = 1. \tag{4.60}$$

Lemma 26. *The normal vector n_{ψ_5} to the quasi-principal normal ruled surface is given by*

$$n_{\psi_5} = \frac{-(v\kappa_3 + \xi_4)\boldsymbol{T}_{q_\alpha} - v\kappa_1\boldsymbol{B}_{q_\alpha}}{\sqrt{v^2(\kappa_1^2 + \kappa_3^2) + 2v\kappa_3\xi_4 + \xi_4^2}}. \tag{4.61}$$

Lemma 27. *Consider the quasi-principal normal ruled surface that is defined by* (4.56). *Then, the second partial derivatives with respect to s and v are given by*

$$\begin{aligned} \psi_{5,ss} &= (- v(\kappa_1' + \kappa_2\kappa_3) - \kappa_2\xi_4)\boldsymbol{T}_{q_\alpha} - \left(v(\kappa_1^2 + \kappa_3^2) + \kappa_3\xi_4\right)\boldsymbol{N}_{q_\alpha} + \left(v(\kappa_3' - \kappa_1\kappa_2) + \xi_4'\right)\boldsymbol{B}_{q_\alpha}, \\ \psi_{5,sv} &= -\kappa_1\boldsymbol{T}_{q_\alpha} + \kappa_3\boldsymbol{B}_{q_\alpha}, \ \ \psi_{5,vv} = 0. \end{aligned} \tag{4.62}$$

Lemma 28. *The CSFF of the quasi-principal normal ruled surface are given as*

$$\begin{aligned}\boldsymbol{L}_{11} &= \frac{1}{\sqrt{v^2(\kappa_1^2+\kappa_3^2)+2v\kappa_3\xi_4+\xi_4^2}}\Big(v^2(\kappa_1'\kappa_3-\kappa_1\kappa_3'+\kappa_2(\kappa_1^2+\kappa_3^2))+v(\xi_4(\kappa_1'+2\kappa_2\kappa_3)-\kappa_1\xi_4')+\kappa_2\xi_4^2\Big),\\ \boldsymbol{L}_{12} &= \frac{\kappa_1\xi_4}{\sqrt{v^2(\kappa_1^2+\kappa_3^2)+2v\kappa_3\xi_4+\xi_4^2}},\ \boldsymbol{L}_{22}=0.\end{aligned} \tag{4.63}$$

Lemma 29. *The MC and GC for the quasi-principal normal ruled surface are given directly by substituting from* (4.60) *and* (4.63) *into* (3.7) *and* (3.8).

$$\begin{aligned}\boldsymbol{H} &= \frac{v^2(\kappa_1'\kappa_3-\kappa_1\kappa_3'+\kappa_2(\kappa_1^2+\kappa_3^2))+v(\xi_4(\kappa_1'+2\kappa_2\kappa_3)-\kappa_1\xi_4')+\kappa_2\xi_4^2}{2(v^2(\kappa_1^2+\kappa_3^2)+2v\kappa_3\xi_4+\xi_4^2)^{3/2}},\\ \boldsymbol{K} &= \frac{-(\kappa_1\xi_4)^2}{(v^2(\kappa_1^2+\kappa_3^2)+2v\kappa_3\xi_4+\xi_4^2)^2}.\end{aligned} \tag{4.64}$$

4.6. Construction of the quasi-binormal ruled surface

Let $F_\alpha\left(s^F(s)\right)$ be the QFC of the original curve α. Consider the case where the ruling parallels the quasi-binormal vector B_q^F of the base curve F_α. Then,

$$\eta(s) = \mathbf{B}_q^F. \tag{4.65}$$

Substituting from (2.21) into (4.65), then

$$\eta(s) = -\mathbf{T}_{q_\alpha}. \tag{4.66}$$

Substituting from (4.66) into (3.2), we obtain the quasi-binormal ruled surface as

$$\psi_6(s,v) = \alpha(s) - v\mathbf{T}_{q_\alpha} + \varphi_1\mathbf{N}_{q_\alpha} + \varphi_2\mathbf{B}_{q_\alpha}. \tag{4.67}$$

Taking the first derivative of (4.67) with respect to s and using (2.7), then

$$\psi_{6,s} = (1-\kappa_1\varphi_1-\kappa_2\varphi_2)\mathbf{T}_{q_\alpha} + (\varphi_1'-\kappa_3\varphi_2-v\kappa_1)\mathbf{N}_{q_\alpha} + (\varphi_2'+\kappa_3\varphi_1-v\kappa_2)\mathbf{B}_{q_\alpha}. \tag{4.68}$$

Using relation (2.12), we obtain

$$\psi_{6,s} = -v\kappa_1\mathbf{N}_{q_\alpha} + (\xi_4-v\kappa_2)\mathbf{B}_{q_\alpha},\ \xi_4 = \varphi_2'+\kappa_3\varphi_1. \tag{4.69}$$

Taking the first derivative of (4.67) with respect to v, we have

$$\psi_{6,v} = -\mathbf{T}_{q_\alpha}. \tag{4.70}$$

By substituting from (4.69) and (4.70) into (3.5), we obtain the following lemma.

Lemma 30. *The CFFF of the quasi-binormal ruled surface are given by*

$$\boldsymbol{g}_{11} = (\kappa_1^2+\kappa_2^2)v^2 - 2v\kappa_2\xi_4 + \xi_4^2,\quad \boldsymbol{g}_{12}=0,\quad \boldsymbol{g}_{22}=1. \tag{4.71}$$

Lemma 31. *The normal vector n_{ψ_6} to the quasi-binormal ruled surface is given by*

$$n_{\psi_6} = -\frac{(\xi_4 - v\kappa_2)\boldsymbol{N}_{q_\alpha} + v\kappa_1\boldsymbol{B}_{q_\alpha}}{\sqrt{v^2(\kappa_1^2+\kappa_2^2) + 2v\kappa_2\xi_4 + \xi_4^2}}. \tag{4.72}$$

Lemma 32. *Consider the quasi-binormal ruled surface that is defined by* (4.67). *Then, the second partial derivatives with respect to s and v are given by*

$$\begin{aligned}\psi_{6,ss} &= (v(\kappa_1^2+\kappa_2^2) - \kappa_2\xi_4)\boldsymbol{T}_{q_\alpha} - (v(\kappa_1' - \kappa_2\kappa_3) + \kappa_3\xi_4)\boldsymbol{N}_{q_\alpha} + (\xi_4' - v(\kappa_2' + \kappa_1\kappa_3))\boldsymbol{B}_{q_\alpha},\\ \psi_{6,sv} &= -\kappa_1\boldsymbol{N}_{q_\alpha} - \kappa_2\boldsymbol{B}_{q_\alpha},\ \psi_{6,vv} = 0,\end{aligned} \tag{4.73}$$

Lemma 33. *The CSFF of the quasi-binormal ruled surface are given as*

$$\begin{aligned}\boldsymbol{L}_{11} &= \frac{1}{\sqrt{v^2(\kappa_1^2+\kappa_2^2) + 2v\kappa_2\xi_4 + \xi_4^2}}\Big(v^2(\kappa_1\kappa_2' - \kappa_1'\kappa_2 + \kappa_3(\kappa_1^2+\kappa_2^2)) + v(\xi_4(\kappa_1' - 2\kappa_2\kappa_3) - \kappa_1\xi_4') + \kappa_3\xi_4^2\Big),\\ \boldsymbol{L}_{12} &= \frac{\kappa_1\xi_4}{\sqrt{v^2(\kappa_1^2+\kappa_2^2) + 2v\kappa_2\xi_4 + \xi_4^2}},\quad \boldsymbol{L}_{22} = 0.\end{aligned} \tag{4.74}$$

Lemma 34. *The MC and GC for the quasi-binormal ruled surface are given directly by substituting from* (4.71) *and* (4.74) *into* (3.7) *and* (3.8).

$$\begin{aligned}\boldsymbol{H} &= \frac{v^2(\kappa_1\kappa_2' - \kappa_1'\kappa_2 + \kappa_3(\kappa_1^2+\kappa_2^2)) + v(\xi_4(\kappa_1' - 2\kappa_2\kappa_3) - \kappa_1\xi_4') + \kappa_3\xi_4^2}{2(v^2(\kappa_1^2+\kappa_2^2) + 2v\kappa_2\xi_4 + \xi_4^2)^{3/2}},\\ \boldsymbol{K} &= \frac{-(\kappa_1\xi_4)^2}{(v^2(\kappa_1^2+\kappa_2^2) + 2v\kappa_2\xi_4 + \xi_4^2)^2}.\end{aligned} \tag{4.75}$$

Remark 2. *For the previous types of QRS, we obtained $\boldsymbol{L}_{22} = 0$. So, the SMC and SGC for these types of QRS are given by*

$$\begin{aligned}\boldsymbol{H}_{II} &= \boldsymbol{H} + \frac{1}{2\boldsymbol{L}_{12}}\left(2\frac{\partial}{\partial s}\left(\frac{\partial}{\partial v}\ln\sqrt{|\boldsymbol{K}|}\right) - \frac{\partial}{\partial v}\left(\frac{\boldsymbol{L}_{11}}{\boldsymbol{L}_{12}}\frac{\partial}{\partial v}\ln\sqrt{|\boldsymbol{K}|}\right)\right),\\ \boldsymbol{K}_{II} &= \frac{-1}{2(\boldsymbol{L}_{12})^3}\left(\boldsymbol{L}_{12}\left(2\boldsymbol{L}_{12,sv} - \boldsymbol{L}_{11,vv}\right) + \boldsymbol{L}_{12,v}\left(\boldsymbol{L}_{11,v} - 2\boldsymbol{L}_{12,s}\right)\right).\end{aligned} \tag{4.76}$$

5. Examples of novel types of quasi-ruled surfaces

Ruled surfaces can be created in different ways, depending on the type of base curve, ruling, or modification to the base curve frame. Methods like using the Frenet frame, Bishop frame, and q-frame can be used. The choice of the process depends on the practical application.

5.1. A mathematical analysis of the features of novel types of quasi-ruled surfaces

Example 1. *Consider a unit speed curve $\alpha(s)$ given as the original curve with the following parametrization:*

$$\alpha = \left(\frac{2}{3}(\cos s - 1), \frac{2}{3}\sin s, \frac{\sqrt{5}}{3}s\right), \tag{5.1}$$

where s represents the arc length along the curve α with FSF given by

$$\boldsymbol{T} = \left(\frac{-2}{3}\sin s, \frac{2}{3}\cos s, \frac{\sqrt{5}}{3}\right), \ \boldsymbol{N} = (-\cos s, -\sin s, 0)\,, \ \boldsymbol{B} = \left(\frac{\sqrt{5}}{3}\sin s, \frac{-\sqrt{5}}{3}\cos s, \frac{2}{3}\right). \tag{5.2}$$

The curvature and torsion κ, τ are given by

$$\kappa = \frac{2}{3}\,, \ \tau = \frac{\sqrt{5}}{3}.$$

The q-frame of the original curve α is given by

$$\boldsymbol{T}_{q\alpha} = (\frac{-2}{3}\sin s, \frac{2}{3}\cos s, \frac{\sqrt{5}}{3}), \ \boldsymbol{N}_{q\alpha} = (\cos s, \sin s, 0), \ \boldsymbol{B}_{q\alpha} = (\frac{-\sqrt{5}}{3}\sin s, \frac{\sqrt{5}}{3}\cos s, \frac{-2}{3}), \tag{5.3}$$

with quasi-curvatures $\kappa_1, \kappa_2, \kappa_3$ given by

$$\kappa_1 = \frac{-2}{3}, \ \kappa_2 = 0, \ \kappa_3 = \frac{\sqrt{5}}{3}.$$

Using (2.12)*, then $\varphi_1 = \frac{-3}{2}$, $\varphi_2 = 0$, and the QFC associated with the original curve is given by*

$$F_\alpha = \left(-\frac{1}{6}(4 + 5\cos s), -\frac{5}{6}\sin s, \frac{\sqrt{5}}{3}s\right), \tag{5.4}$$

Using (2.21)*, the q-frame of the QFC F_α is given by*

$$\boldsymbol{T}_q^F = (\frac{-\sqrt{5}}{3}\sin s, \frac{\sqrt{5}}{3}\cos s, \frac{-2}{3}), \ \boldsymbol{N}_q^F = (\cos s, \sin s, 0), \ \boldsymbol{B}_q^F = (\frac{2}{3}\sin s, \frac{-2}{3}\cos s, \frac{-\sqrt{5}}{3}). \tag{5.5}$$

Using (2.23)*, the quasi-curvatures for the QFC are given as*

$$\kappa_1^F = \frac{-2}{3}\,, \ \kappa_2^F = 0\,, \ \kappa_3^F = \frac{-4}{3\sqrt{5}}.$$

Now, we can construct new types of QRS as follows:

1. ***The osculating type of QRS:***
The osculating type of QRS has the following parametrization:

$$\psi_1(s, v) = \frac{1}{6}\Big(-\sqrt{10}v\sin s - (3\sqrt{2}v + 5)\cos s - 4, \ \sqrt{10}v\cos s - (3\sqrt{2}v + 5)\sin s, 2(\sqrt{5}s - \sqrt{2}v)\Big),$$

for $\eta(s) = \frac{1}{\sqrt{2}}(\boldsymbol{T}_q^F - \boldsymbol{N}_q^F)$, $\mu_1 = -\mu_2 = \frac{1}{\sqrt{2}}$. This surface is illustrated with Figure 1(a). The normal vector to the surface is

$$n_{\psi_1} = \frac{\Big(2\sqrt{5}\left(\sqrt{2}v + 3\right)\sin s + 6\sqrt{2}v\cos s, 6\sqrt{2}v\sin s - 2\sqrt{5}\left(\sqrt{2}v + 3\right)\cos s, -14\sqrt{2}v - 15\Big)}{3\sqrt{56v^2 + 60\sqrt{2}v + 45}}.$$

Lemma 35. *The CFFF and CSFF for the osculating type of QRS are given, respectively, by*

$$\boldsymbol{g}_{11} = \frac{7}{9}v^2 + \frac{5}{3\sqrt{2}}v + \frac{5}{4}, \quad \boldsymbol{g}_{12} = -\frac{\sqrt{5}}{2\sqrt{2}}, \quad \boldsymbol{g}_{22} = 1,$$
$$\boldsymbol{L}_{11} = \frac{2v\left(14v + 15\sqrt{2}\right)}{9\sqrt{56v^2 + 60\sqrt{2}v + 45}}, \quad \boldsymbol{L}_{12} = \frac{\sqrt{10}}{\sqrt{56v^2 + 60\sqrt{2}v + 45}}, \quad \boldsymbol{L}_{22} = 0.$$

Lemma 36. *The MC, GC, SMC, and SGC for the osculating type of QRS are given by*

$$\boldsymbol{H} = \frac{4\left(28v^2 + 30\sqrt{2}v + 45\right)}{\left(56v^2 + 60\sqrt{2}v + 45\right)^{3/2}}, \quad \boldsymbol{K} = -\frac{720}{\left(56v^2 + 60\sqrt{2}v + 45\right)^2}.$$

$$\boldsymbol{H}_{II} = \frac{8\left(5488v^4 + 11760\sqrt{2}v^3 + 25844v^2 + 14190\sqrt{2}v + 5085\right)}{45\left(56v^2 + 60\sqrt{2}v + 45\right)^{3/2}},$$
$$\boldsymbol{K}_{II} = \frac{28v\left(784v^3 + 1680\sqrt{2}v^2 + 2430v + 675\sqrt{2}\right) + 8100}{45\left(56v^2 + 60\sqrt{2}v + 45\right)^{3/2}}.$$

Lemma 37. *The geodesic curvature κ_g, normal curvature κ_n, and geodesic torsion τ_g of the QFC F_α on the surface ψ_1 are given, respectively, according to Eq* (3.4)*, as follows:*

$$\kappa_g = \frac{-5\left(2\sqrt{2}v + 3\right)}{3\sqrt{56v^2 + 60\sqrt{2}v + 45}}, \ \kappa_n = \frac{-2\sqrt{10}v}{3\sqrt{56v^2 + 60\sqrt{2}v + 45}}, \ \tau_g = \frac{-2\sqrt{5}v\left(28v + 15\sqrt{2}\right)}{9\left(56v^2 + 60\sqrt{2}v + 45\right)}.$$

2. ***The normal type of quasi-ruled surfaces:***
The normal type of quasi-ruled surface has the following parametrization:

$$\psi_2(s, v) = \frac{1}{6}\left(\sqrt{2}v(3\cos s - 2\sin s) - 5\cos s - 4,\ \sqrt{2}v(3\sin s + 2\cos s) - 5\sin s,\ \sqrt{5}\left(2s + \sqrt{2}v\right)\right),$$

for $\eta(s) = \frac{1}{\sqrt{2}}(\boldsymbol{N}_q^F - \boldsymbol{B}_q^F)$, $\mu_2 = -\mu_3 = \frac{1}{\sqrt{2}}$. *This surface is illustrated with Figure 1(b).*
The normal vector to the surface is

$$n_{\psi_2} = \frac{\sqrt{5}\left(-(4v + 6\sqrt{2})\sin s + \left(6v - 9\sqrt{2}\right)\cos s, (6v - 9\sqrt{2})\sin s + (4v + 6\sqrt{2})\cos s, \frac{6}{\sqrt{5}}\left(\frac{5}{\sqrt{2}} - \frac{13v}{3}\right)\right)}{6\sqrt{26v^2 - 30\sqrt{2}v + 45}}.$$

Lemma 38. *The CFFF and CSFF for the normal type of QRS are given, respectively, by*

$$\boldsymbol{g}_{11} = \frac{1}{36}\left(26v^2 - 30\sqrt{2}v + 45\right), \quad \boldsymbol{g}_{12} = 0, \quad \boldsymbol{g}_{22} = 1,$$
$$\boldsymbol{L}_{11} = -\frac{1}{18}\sqrt{65v^2 - 75\sqrt{2}v + \frac{225}{2}}, \quad \boldsymbol{L}_{12} = \frac{2\sqrt{5}}{\sqrt{26v^2 - 30\sqrt{2}v + 45}}, \quad \boldsymbol{L}_{22} = 0.$$

Lemma 39. *The MC, GC, SMC, and SGC for the normal type of QRS are given by*

$$\boldsymbol{H} = -\frac{\sqrt{10}}{2\sqrt{26v^2 - 30\sqrt{2}v + 45}}, \quad \boldsymbol{K} = -\frac{720}{\left(26v^2 - 30\sqrt{2}v + 45\right)^2},$$

$$\boldsymbol{H}_{II} = -\frac{13\left(26v^2 - 30\sqrt{2}v + 45\right) + 180}{36\sqrt{10}\sqrt{26v^2 - 30\sqrt{2}v + 45}}, \quad \boldsymbol{K}_{II} = -\frac{13\sqrt{10}}{720}\sqrt{26v^2 - 30\sqrt{2}v + 45},$$

Lemma 40. *The geodesic curvature κ_g, normal curvature κ_n, and geodesic torsion τ_g of the QFC F_α on the surface ψ_2 are given, respectively, according to Eq (3.4) as follows:*

$$\kappa_g = \frac{5\left(3\sqrt{2} - 2v\right)}{6\sqrt{26v^2 - 30\sqrt{2}v + 45}}, \; \kappa_n = \frac{5\left(3\sqrt{2} - 2v\right)}{6\sqrt{26v^2 - 30\sqrt{2}v + 45}}, \; \tau_g = \frac{20\sqrt{2}v+}{78v^2 - 90\sqrt{2}v + 135} - \frac{5}{18}.$$

3. ***The rectifying type of quasi-ruled surfaces:***
The rectifying type of quasi-ruled surface has the following parametrization:

$$\psi_3(s, v) = \frac{1}{6}\left(-\sqrt{2}\left(\sqrt{5} + 2\right)v\sin s - 5\cos s - 4, \; \sqrt{2}\left(\sqrt{5} + 2\right)v\cos s - 5\sin s, 2\sqrt{5}s + \sqrt{2}\left(\sqrt{5} - 2\right)v\right),$$

for $\eta(s) = \frac{1}{\sqrt{2}}(\boldsymbol{T}_q^F - \boldsymbol{B}_q^F)$, $\mu_1 = -\mu_3 = \frac{1}{\sqrt{2}}$. *This surface is illustrated with Figure 1(c).*
The normal vector to the surface is

$$n_{\psi_3} = \frac{\left(-2v\sin s - 9\sqrt{10}\cos s, 2v\cos s - 9\sqrt{10}\sin s, -2\left(4\sqrt{5} + 9\right)v\right)}{3\sqrt{8\left(4\sqrt{5} + 9\right)v^2 + 90}}.$$

Lemma 41. *The CFFF and CSFF for the rectifying type QRS are given, respectively, by*

$$\boldsymbol{g}_{11} = \frac{1}{18}\left(4\sqrt{5} + 9\right)v^2 + \frac{5}{4}, \quad \boldsymbol{g}_{12} = -\frac{\sqrt{5}}{2\sqrt{2}}, \quad \boldsymbol{g}_{22} = 1,$$

$$\boldsymbol{L}_{11} = \frac{-2\left(\sqrt{5} + 2\right)v^2 - 45\sqrt{5}}{18\sqrt{4\left(4\sqrt{5} + 9\right)v^2 + 45}}, \quad \boldsymbol{L}_{12} = \frac{2\sqrt{5} + 5}{\sqrt{8\left(4\sqrt{5} + 9\right)v^2 + 90}}, \quad \boldsymbol{L}_{22} = 0.$$

Lemma 42. *The MC, GC, SMC, and SGC for the rectifying type of QRS are given by*

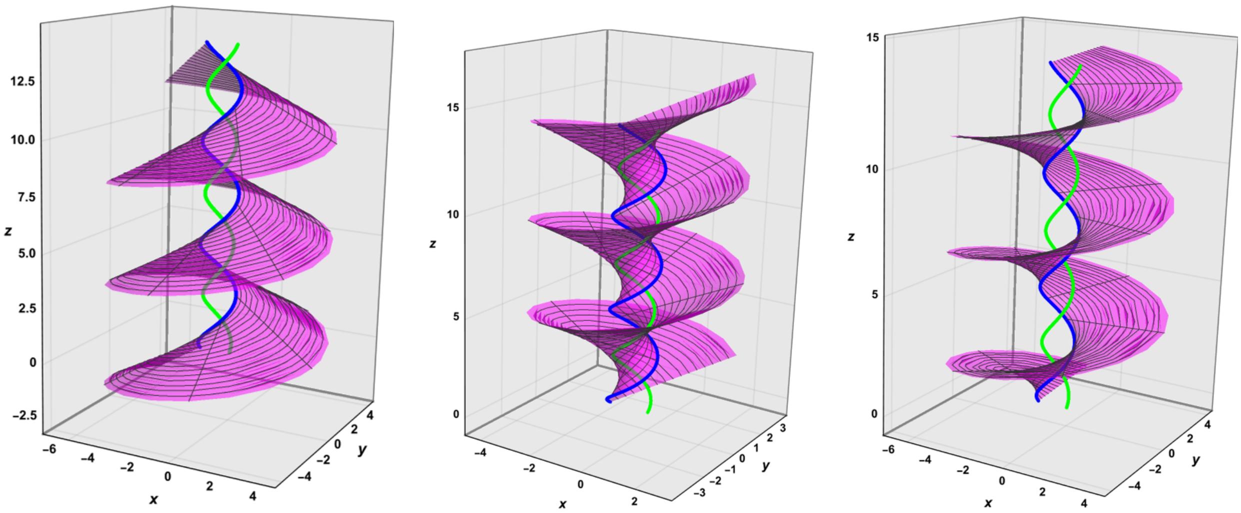
$$\boldsymbol{H} = \frac{360 - 8\left(\sqrt{5} + 2\right)v^2}{2\left(4\left(4\sqrt{5} + 9\right)v^2 + 45\right)^{3/2}}, \quad \boldsymbol{K} = -\frac{180\left(4\sqrt{5} + 9\right)}{\left(4\left(4\sqrt{5} + 9\right)v^2 + 45\right)^2},$$

$$\boldsymbol{H}_{II} = \frac{180 - 4\left(\sqrt{5} + 2\right)v^2}{\left(4\left(4\sqrt{5} + 9\right)v^2 + 45\right)^{3/2}} + \frac{-32\left(161\sqrt{5} + 360\right)v^4 + 360\left(143\sqrt{5} + 320\right)v^2 - 40500\left(\sqrt{5} + 2\right)}{45\sqrt{\frac{1}{\frac{4v^2}{5} - 36\sqrt{5} + 81}}\left(4\left(4\sqrt{5} + 9\right)v^2 + 45\right)^2},$$

$$\boldsymbol{K}_{II} = -\frac{4\left(4\left(17\sqrt{5} + 38\right)v^4 + 45\left(19\sqrt{5} + 42\right)v^2 - 2025\right)}{45\left(4\left(4\sqrt{5} + 9\right)v^2 + 45\right)^{3/2}}.$$

Lemma 43. *The geodesic curvature κ_g, normal curvature κ_n, and geodesic torsion τ_g of the QFC F_α on the surface ψ_3 are given, respectively, according to Eq* (3.4) *as follows:*

$$\kappa_g = \frac{-\left(2\sqrt{5}+5\right)\sqrt{2}v}{3\sqrt{4\left(4\sqrt{5}+9\right)v^2+45}}, \quad \kappa_n = \frac{5}{\sqrt{4\left(4\sqrt{5}+9\right)v^2+45}}, \quad \tau_g = \frac{5\sqrt{2}\left(4\sqrt{5}+9\right)v}{3\left(4\left(4\sqrt{5}+9\right)v^2+45\right)}.$$



(**a**) $\eta(s) = \frac{1}{\sqrt{2}}(\mathbf{T}_q^F - \mathbf{N}_q^F)$ (**b**) $\eta(s) = \frac{1}{\sqrt{2}}(\mathbf{N}_q^F - \mathbf{B}_q^F)$ (**c**) $\eta(s) = \frac{1}{\sqrt{2}}(\mathbf{T}_q^F - \mathbf{B}_q^F)$

Figure 1. Visualization of the osculating type, normal type, and rectifying type of QRS. The green curve represents the original curve, and the blue curve represents the quasi-focal curve for $s \in [0, 6\pi]$ and $v \in [0, 5]$.

4. ***The quasi-tangent developable surface***
The quasi-tangent developable surface has the following parametrization:

$$\psi_4(s,v) = \frac{1}{6}\left(-2\sqrt{5}v\sin s - 5\cos s - 4, 2\sqrt{5}v\cos s - 5\sin s, 2(\sqrt{5}s - 2v)\right),$$

for $\eta(s) = \boldsymbol{T}_q^F$. This surface is illustrated with Figure 2(a). The normal vector to the surface is

$$n_{\psi_4} = \frac{1}{3}(2\sin s, -2\cos s, -\sqrt{5}).$$

Lemma 44. *The CFFF and CSFF for the quasi-tangent developable surface are given, respectively, by*

$$\boldsymbol{g}_{11} = \frac{1}{36}(20\,v^2+45), \quad \boldsymbol{g}_{12} = -\frac{\sqrt{5}}{2}, \quad \boldsymbol{g}_{22} = 1,$$
$$\boldsymbol{L}_{11} = \frac{2\sqrt{5}}{9}\,v, \quad \boldsymbol{L}_{12} = 0, \quad \boldsymbol{L}_{22} = 0.$$

Lemma 45. *The MC and GC for the quasi-tangent developable surface are given by*

$$\boldsymbol{H} = \frac{1}{\sqrt{5}\,v}, \quad \boldsymbol{K} = 0.$$

Furthermore, the SMC and SGC are undefined.

Lemma 46. *The geodesic curvature κ_g, normal curvature κ_n, and geodesic torsion τ_g of the QFC F_α on the surface ψ_4 are given, respectively, according to Eq* (3.4) *as follows:*

$$\kappa_g = -\frac{\sqrt{5}}{3}, \qquad \kappa_n = 0, \qquad \tau_g = 0.$$

Hence, the QFC F_α, which is the base curve of the quasi-tangent developable surface ψ_4, is both an asymptotic line and a principal line at any point (s, v) on the surface.

5. ***The quasi-principal normal ruled surface***:
The quasi-principal normal ruled surface has the following parametrization:

$$\psi_5(s, v) = \frac{1}{6}\left((6v - 5)\cos(s) - 4, (6v - 5)\sin(s), 2\sqrt{5}s\right),$$

for $\eta(s) = \boldsymbol{N}_q^F$. This surface is illustrated with Figure 2(b). The normal vector to the surface is

$$n_{\psi_5} = \frac{\sqrt{5}}{\sqrt{12v(3v-5)+45}}\left(-2\sin s, 2\cos s, \frac{5-6v}{\sqrt{5}}\right).$$

Lemma 47. *The CFFF and CSFF for the quasi-principal normal ruled surface are given, respectively, by*

$$\boldsymbol{g}_{11} = \frac{1}{12}(12v^2 - 20v + 15), \quad \boldsymbol{g}_{12} = 0, \quad \boldsymbol{g}_{22} = 1,$$
$$\boldsymbol{L}_{11} = 0, \quad \boldsymbol{L}_{12} = \frac{2\sqrt{5}}{\sqrt{12v(3v-5)+45}}, \quad \boldsymbol{L}_{22} = 0.$$

Lemma 48. *The MC, GC, SMC, and SGC for the quasi-principal normal ruled surface are given by*

$$\boldsymbol{H} = 0, \quad \boldsymbol{K} = -\frac{80}{(4v(3v-5)+15)^2}, \ \boldsymbol{H}_{II} = 0, \ \boldsymbol{K}_{II} = 0.$$

Hence, the quasi-principal normal ruled surface is minimal, II flat, and II minimal.

Lemma 49. *The geodesic curvature κ_g, normal curvature κ_n, and geodesic torsion τ_g of the QFC F_α on the surface ψ_5 are given, respectively, according to Eq* (3.4) *as follows:*

$$\kappa_g = \frac{15 - 10v}{3\sqrt{12v(3v-5)+45}}, \ \kappa_n = 0, \ \tau_g = 0.$$

Hence, the QFC F_α, which is the base curve of the quasi-principal normal ruled surface ψ_5, is both an asymptotic line and a principal line at any point (s, v) on the surface.

6. ***The quasi-binormal ruled surface***:
The quasi-binormal ruled surface has the following parametrization:

$$\psi_6(s, v) = \frac{1}{6}\left(4v\sin s - 5\cos s - 4, -4v\cos s - 5\sin s, 2\sqrt{5}(s - v)\right),$$

for $\eta(s) = \boldsymbol{B}_q^F$*. This surface is illustrated with Figure 2(c). The normal vector to the surface is*

$$n_{\psi_6} = \frac{\sqrt{5}}{3\sqrt{16v^2+45}}\left(9\cos s - 4v\sin s, 4v\cos s + 9\sin s, -\frac{8v}{\sqrt{5}}\right).$$

Lemma 50. *The CFFF and CSFF for the quasi-binormal ruled surface are given, respectively, by*

$$\begin{aligned} \boldsymbol{g}_{11} &= \frac{1}{36}(16v^2+45), \quad \boldsymbol{g}_{12} = 0, \quad \boldsymbol{g}_{22} = 1, \\ \boldsymbol{L}_{11} &= \frac{\sqrt{5}}{18}\sqrt{16v^2+45}, \quad \boldsymbol{L}_{12} = \frac{2\sqrt{5}}{\sqrt{16v^2+45}}, \quad \boldsymbol{L}_{22} = 0. \end{aligned}$$

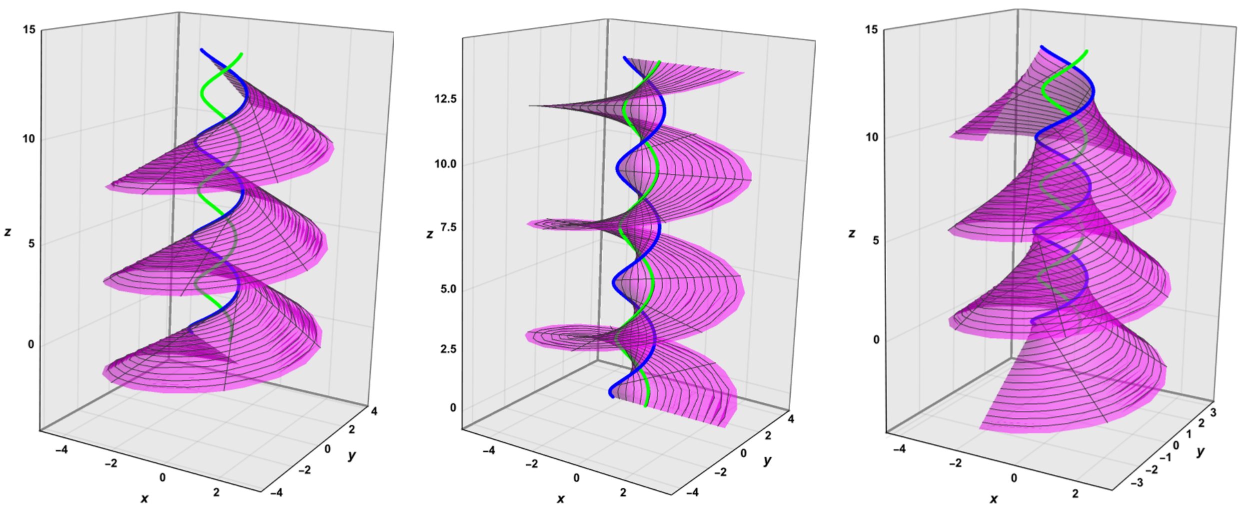
Lemma 51. *The MC, GC, SMC, and SGC for the quasi-binormal ruled surface are given by*

$$\boldsymbol{H} = \frac{\sqrt{5}}{\sqrt{16v^2+45}}, \ \boldsymbol{K} = -\frac{720}{(16v^2+45)^2}, \ \boldsymbol{H}_{II} = \frac{32v^2+135}{9\sqrt{5}\sqrt{16v^2+45}}, \ \boldsymbol{K}_{II} = \frac{\sqrt{16v^2+45}}{9\sqrt{5}}.$$

Lemma 52. *The geodesic curvature* κ_g*, normal curvature* κ_n*, and geodesic torsion* τ_g *of the QFC* F_α *on the surface* ψ_6 *are given, respectively, according to Eq* (3.4) *as follows:*

$$\kappa_g = 0, \qquad \kappa_n = -\frac{5}{\sqrt{16v^2+45}}, \qquad \tau_g = -\frac{40v}{48v^2+135}.$$

Hence, the QFC F_α*, which is the base curve of the quasi-binormal ruled surface* ψ_6*, is a geodesic curve at any point* (s, v) *on the surface.*



(d) $\eta(s) = \mathbf{T}_q^F$ **(e)** $\eta(s) = \mathbf{N}_q^F$ **(f)** $\eta(s) = \mathbf{B}_q^F$

Figure 2. Visualization of the quasi-tangent, quasi-principal normal, and quasi-binormal ruled surfaces. The green curve represents the original curve, and the blue curve represents the quasi-focal curve for $s \in [0, 6\pi]$, and $v \in [0, 5]$.

5.2. *Visualization of quasi-ruled surfaces*

Effective visual aids are crucial for understanding complex geometric constructs. This section provides detailed visualizations of the constructed QRS to enhance comprehension. These visualizations are created using Mathematica 13, a powerful tool for generating high-quality graphics in differential geometry.

6. Conclusions

This research presents a comprehensive study of quasi-ruled surfaces based on quasi-focal curves in 3-dimensional Euclidean space. The definitions of q-frame, quasi-focal curves, and quasi-ruled surfaces and their detailed analysis provide a new perspective on the construction of these surfaces in differential geometry.

In this work, we have introduced and defined several novel types of QRS based on the QFC as the base curve and utilized the q-frame of the QFC to describ the rulings. These novel types of QRS include:

- Osculating type of quasi-ruled surface: This type of QRS has the ruling lies in the osculating plane of the base curve QFC.
- Normal type of quasi-ruled surface: This type of QRS has a ruling that lies in the normal plane of the base curve QFC.
- Rectifying type of quasi-ruled surface: This type of QRS has a ruling that lies in the rectifying plane of the base curve QFC.
- Quasi-tangent developable surfaces: This type of QRS has a ruling that parallels the quasi-tangent vector of the QFC.
- Quasi-principal normal ruled surfaces: This type of QRS has a ruling that parallels the quasi-principal normal vector of the QFC.
- Quasi-binormal ruled surfaces: This type of QRS has a ruling that parallels the quasi-binormal vector of the QFC.

Some geometric properties are specified and analyzed for these types of QRS, including curvatures MC, GC, SMC, and SGC. These geometric properties contribute to the theoretical understanding of these surfaces. These novel types of quasi-ruled surfaces provide a rich framework for studying the geometric properties of surfaces constructed from the quasi-focal curves. Each type of QRS has unique characteristics based on the orientation of the ruling and the base curve. This classification allows for a deeper understanding of the intrinsic and extrinsic properties of these surfaces, which can be further explored in various applications, such as differential geometry, computer-aided design, and geometric modeling.

Abbreviations

The abbreviations used in this manuscript are illustrated by

CFFF	Coefficients of the first fundamental form
CSFF	Coefficients of the second fundamental form
CEFSF	Equations of Frenet-Serret frame
FSF	Frenet-Serret frame
GC	Gaussian curvature
MC	Mean curvature
QFC(s)	Quasi-focal curve(s)
q-frame	Quasi-frame
QRS	Quasi-ruled surface(s)
SMC	Second mean curvature
SGC	Second Gaussian curvature

Author contributions

Samah Gaber: Investigation, writing the original draft, writing the review, editing, software; Asmahan Essa Alajyan: Investigation, writing the original draft, writing the review, editing; Adel H. Sorour: Investigation, writing the original draft, writing the review, editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in creating this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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