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Research article

Novel efficient estimators of finite population mean in stratified random sampling with application

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Abstract: Unbiased estimators are valuable when no auxiliary information is available beyond the primary study variables. However, once auxiliary information is accessible, biased estimators with smaller Mean Square Error (MSE) often outperform unbiased estimators that have large variances. We sought to develop new estimators that incorporate a single auxiliary variable in stratified random sampling. This study contributes to the field by introducing two distinct families of estimators designed to estimate the finite population mean. We conducted a theoretical evaluation of the estimators' performance by examining bias and MSE derived under first-order approximation. Additionally, we established the theoretical conditions necessary for the proposed estimator families to exhibit superior performance compared with existing alternatives. Empirical and simulation-based studies demonstrated significant improvements in estimators over competing estimators for finite-population parameter estimation.

Keywords: unbiasedness; auxiliary; stratified random sampling; efficiency; mean square error
Mathematics Subject Classification: 62D

1. Introduction

The selection of biased and unbiased estimators has drawn considerable attention from researchers in the field of statistical estimation. However, researchers frequently employ biased estimators in scenarios with modest variations, ensuring their estimates closely resemble the underlying population parameter on average. These approaches typically result in greater variety, which reduces their usefulness in most cases. As more information becomes available, the estimation of scenario changes favors biased estimators with a lower mean square error (MSE), not with standing their bias. This feature increased the precision of the estimator. Using supplementary data with a strong relationship to the variable under study is a standard procedure in the field of survey sampling. This methodology frequently enhances the accuracy and dependability of the estimators during both the design and estimation phases. Selecting pertinent additional data with care can significantly reduce the mean square error (MSE) of the estimators used to estimate the population parameters. As ratio estimators may leverage the current link between the study and auxiliary variables, they have become popular in support of this goal. Ratio estimators are useful tools for increasing the accuracy of estimates when calculating the population total or average. Significant progress has been made in this sector as a result of numerous academics that have created a range of ratio and regression-based estimators, each based on a different transformation [1], significantly increasing the amount of knowledge in this field. Within the SSRS framework, some studies have presented estimators based on mixed ratio-type techniques. Koyuncu et al. [2] Examined the estimators developed by [1] within the context of SSRS. Moreover, Koyuncu et al. [3] provided a combined version of the SSRS estimator that had been put forth by [5]. Singh et al. [6] Proposed an extensive set of estimators that utilize supplemental data in the SSRS. Singh et al. [7] generated an extraordinarily effective set of estimators using the same SSRS architecture. Together with the references included in these publications, the [8] provided a thorough assessment.

Stratified sampling with auxiliary variables has diverse applications in physics, engineering, and environmental sciences. In physics, it enhances the particle density estimates in high-energy collisions, cosmological parameter estimates, and material property predictions. Engineering applications include reliability analysis, signal processing, and network-traffic estimation. Specific examples have revealed their utility in estimating ocean currents, predicting structural failures, and optimizing energy systems. These applications underscore the flexibility and potential of stratified sampling in improving the estimation accuracy and efficiency [9].

Our primary goal of this study, in the context of stratified random sampling, is to develop and evaluate efficient estimators that utilize only one additional variable. Furthermore, we describe and assess two novel groups of mean estimators for a finite population. Our investigation includes a thorough examination of their bias and MSE up to the first level of approximation, which yields useful insights into their performance.

Researchers constantly strive for progress in their respective domains. The proposed estimator is a significant improvement in the field of sampling methodology. The adoption of this strategy enhances the development of statistical methods, leading to a constant enhancement in the precision and reliability of estimating population parameters. The suggested estimator is specifically developed to offer improved accuracy in calculating the mean of a finite population. By strategically including a single auxiliary variable in each stratum, it optimizes the use of available information, leading to more precise estimations in comparison to current approaches.

In stratified random sampling, precise estimation of the finite population mean is crucial.

Estimators, such as the traditional stratified sampling estimator, ratio estimator, and regression estimator, often rely on simplistic assumptions or fail to effectively hitch the correlation between the study and auxiliary variable. Recently developed estimators, such as the generalized regression estimator and the exponential ratio estimator, offer enhancements, but still have limitations. To address these gaps, we introduce new estimators that incorporate a single auxiliary variable into stratified random sampling. These novel estimators aim to enhance the estimation accuracy and efficiency by better capturing the complex relationships between variables. The proposed estimators are significant, as they provide more reliable and precise estimates, especially in scenarios with non-linear relationships or non-normal distributions, thereby filling an important methodological gap in the survey sampling literature. Simulation studies and empirical evaluations demonstrate the superiority of the proposed estimators over existing ones, making them valuable tools for practitioners and researchers seeking improved estimation strategies.

2. Methodology

Let us take population of size N that comprises L strata (a group of homogenous units) such that $\sum_{h=1}^{L} N_h = N$ where N_h shows the h^{th} stratum size (h=1, 2,...,L). Let each stratum sampled n_h units through simple random sample without replacement (SRSWOR) scheme, such that, $\sum_{h=1}^{L} n_h = n$. Let us suppose that the i^{th} pair of the sample (y_{hi}, x_{hi}) represent the values of y (study variable) and x (auxiliary variable) on the i^{th} unit of the h^{th} stratum, where $i=1, 2, 3, \ldots N_h$.

To obtain the expressions for the Bias and *MSE* of the estimators, we supposed the various properties listed below to be true.

Suppose $\bar{y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_h = \bar{Y}(1+\varepsilon_0)$, $\bar{x}_{st} = \sum_{h=1}^{L} W_h \bar{x}_h = \bar{X}(1+\varepsilon_1)$ are the overall means of the study and auxiliary variables obtained through a stratified random sample, respectively. Thus the relative error terms ε_0 and ε_1 satisfies the below properties.

$E(\varepsilon_0) = E(\varepsilon_1) = 0$ and $E(\varepsilon_0^2) = C_y^2 = \sum_{h=1}^{L} w_h \lambda_h C_{yh}^2 = V_{20}$ $E(\varepsilon_1^2) = C_x^2 = \sum_{h=1}^{L} w_h \lambda_h C_{xh}^2 = V_{02}$

$$E(\varepsilon_0\varepsilon_1) = \sum_{h=1}^{L} w_h^2 \lambda_h \rho_{yxh} C_{yh} C_{xh} = V_{11}$$

where,

$C_{yh}^2 = \frac{S_{yh}^2}{\bar{Y}^2}$ and $C_{xh}^2 = \frac{S_{xh}^2}{\bar{X}^2}$ are population coefficient of variations of the study and auxiliary variables, respectively. $\lambda_h = \frac{1}{n_h} - \frac{1}{N_h}$ is the finite population correction (fpc), $w_h = \frac{N_h}{N}$ is the stratum weight, and $R = \frac{\bar{X}}{\bar{Y}}$.

3. Summary of some estimators

Several estimators have been devised to evaluate the finite population mean in the context of stratified random sampling with a single auxiliary variable. Researchers and statisticians have studied several approaches, each designed to utilize information contained in the auxiliary variable to improve the accuracy of the population parameters estimates. The estimators in this context are designed to address the complexities of finite population sampling, considering the stratified structure and the utilization of a single auxiliary variable as a valuable tool for more robust and reliable mean estimation [4]. The conventional estimator for the population mean in the context of stratified random sampling is an unbiased estimator and is defined as follows:

$$T_{st} = \sum_{h=1}^{L} w_h \overline{y}_h \,. \tag{3.1}$$

The formula for the variance of the conventional unbiased estimator is provided as:

$$MSE(T_{st}) = \overline{Y}^2 V_{20} \,. \tag{3.2}$$

Though the usual estimator is unbiased, its variance is large. Therefore, when auxiliary information X_i about the study variable Y_i is available, then the researchers in [10] suggest the traditional ratio estimator as

$$T_r = \overline{y}_{st} \frac{\overline{X}}{\overline{x}_{st}} \,. \tag{3.3}$$

The bias of Cochran's ratio estimator along with its *MSE* is given as:

$$Bias(T_r) = \overline{Y}(V_{02} - V_{11}) \,, \tag{3.4}$$

$$MSE(T_r) = \overline{Y}^2 (V_{20} + V_{02} - 2V_{11}) \,. \tag{3.5}$$

Bahl et al. [11] suggested an exponential ratio-type estimator. The functional form, Bias, and *MSE* of Bahl and Tuteja's estimators are as follows:

$$T_{BT} = \overline{y}_{st} \exp\left[\frac{\overline{X} - \overline{x}_{st}}{\overline{X} + \overline{x}_{st}} \right] \tag{3.6}$$

$$Bias(T_{BT}) = \overline{Y}(\frac{3}{8}V_{02} - \frac{1}{2}V_{11}) \tag{3.7}$$

$$MSE(T_{BT}) = \overline{Y}^2 (V_{20} + \frac{V_{02}}{4} - 2V_{11}) \,. \tag{3.8}$$

Based on the work of [12,13] a ratio estimator is introduced where the population coefficient of variation is known.

$$T_{KC} = \overline{y}_{st} \frac{(\overline{X} + C_x)}{(\overline{x}_{st} + C_x)} \,. \tag{3.9}$$

The estimator's first-order bias and *MSE* are discussed as follows:

$$Bias(T_{KC}) = \overline{Y}(\phi^2 V_{02} - \phi V_{11}) \tag{3.10}$$

$$MSE(T_{KC}) = \overline{Y}^2 (V_{20} + \phi^2 V_{02} - 2\phi V_{11}). \tag{3.11}$$

Where $\phi = \sum_{h=1}^{L} w_h \frac{\overline{X}_h}{(\overline{X} + C_x)}$.

Upadhyaya et al. [14] suggested a modified version of [15] by multiplying the coefficient of kurtosis by the mean of the auxiliary variable:

$$T_{US} = \overline{y}_{st} \sum_{h=1}^{L} w_h \frac{(\overline{X}_h \beta_{2h}(x) + C_{xh})}{(\overline{x}_h \beta_{2h}(x) + C_{xh})}. \tag{3.12}$$

The MSE and Bias of this estimator are:

$$Bias(T_{US}) = \overline{Y}(\theta V_{02} - V_{11}) \tag{3.13}$$

$$MSE(T_{US}) = \overline{Y}^2 (V_{20} + \theta^2 V_{02} - 2\theta V_{11}). \tag{3.14}$$

Here, $\theta = \frac{\overline{X}_h \beta_{2h}(x)}{\overline{X}_h \beta_{2h}(x) + C_{xh}}$.

A general family of estimators of the population mean was proposed by [16] in response to the work of [17].

$$T_{ch} = \overline{y}_{st} \left[\frac{a\overline{X} + b}{\alpha(a\overline{x}_{st} + b) + (1-\alpha)(a\overline{X} + b)} \right]^{\tau}. \tag{3.15}$$

Substituting different values of the constants *a, b, τ (=0, 1, -1),* and *α,* we obtain several estimators. The bias and *MSE* of the estimator are:

$$Bias(T_{ch}) = \overline{Y}\left(\frac{\tau(\tau+1)}{2}\alpha^2\pi^2 V_{02} - \alpha\tau\pi V_{11}\right) \tag{3.16}$$

$$MSE(T_{ch}) = \overline{Y}^2\left(V_{20} + \alpha^2\tau^2\pi^2 V_{02} - 2\alpha\tau\pi V_{11}\right) \tag{3.17}$$

$\pi = \frac{a\overline{X}}{a\overline{X} + b}$, $\alpha = \frac{V_{11}}{\pi\tau V_{02}}$ and

based on [18,19], introduced a class of exponential estimators for the population mean in the SRSWOR scheme is introduced.

$$T_O = \overline{y}_{st}\left[\alpha_1 \exp\left\{\frac{a\left(\overline{X} - \overline{x}_{st}\right)}{a\left(\overline{x}_{st} + \overline{X}\right) + 2b}\right\} + \alpha_2 \exp\left\{\frac{a\left(\overline{x}_{st} - \overline{X}\right)}{a\left(\overline{x}_{st} + \overline{X}\right) + 2b}\right\}\right]. \tag{3.18}$$

The estimator's MSE is obtained as,

$$MSE(T_O) = \frac{V_{00}V_{22} - V_{12}^2}{V_{00} + V_{22} - 2V_{12}} \tag{3.19}$$

where,

$$V_{00} = \sum_{h=1}^{L} w_h^2 \lambda_h \overline{Y}_h^2 \left[C_{yh}^2 + \frac{1}{4}\varphi C_{xh}^2 (\varphi - 4K) \right],$$

$$V_{22} = \sum_{h=1}^{L} w_h^2 \lambda_h \overline{Y}_h^2 \left[C_{yh}^2 + \frac{1}{4}\varphi C_{xh}^2 (\varphi + 4K) \right]$$

$$V_{12} = \sum_{h=1}^{L} w_h^2 \lambda_h \overline{Y}_h^2 (C_{yh}^2 - \frac{1}{4}\varphi^2 C_{xh}^2)$$

$$\varphi = \frac{a\overline{X}}{a\overline{X} + b} \quad and \quad K = \frac{\rho_h C_{yh}}{C_{xh}}.$$

Motivated by [20,21], proposed a ratio cum exponential type estimator is proposed, as follows:

$$T_G = \overline{y}_{st} \left[\frac{\overline{x}_{st}}{\overline{X}_{st}} \right]^{\alpha_2} \exp\left[\frac{\overline{X}_{st'} - \overline{x}_{st'}}{\overline{X}_{st'} + \overline{x}_{st'}} \right]. \tag{3.20}$$

Here, $\overline{x}_{st'} = \sum_{h=1}^{L} w_h \left(a_h \overline{x}_h + b_h \right)$ and $\overline{X}_{st'} = \sum_{h=1}^{L} w_h \left(a_h \overline{X}_h + b_h \right)$, and a_h and b_h are functions of the known parameters like coefficient of Kurtosis, coefficient of variations etc. of the auxiliary variable. The Bias and *MSE* of the above estimator are:

$$Bias\left(T_G\right) = \overline{Y}\left[\frac{1}{2}\left\{ \alpha_2\left(\alpha_2 - 1\right) - \theta^2 - 2\alpha_2\theta \right\} V_{02} + \left(\alpha_2 + \theta\right) V_{11} \right] \tag{3.21}$$

$$MSE(T_G) = \sum_{h=1}^{L} w_h^2 \lambda_h \left(S_{yh}^2 + (\alpha_2 - \theta)^2 S_{xh}^2 + 2(\alpha_2 - \theta) R S_{yxh} \right) \tag{3.22}$$

where $\theta = \frac{a\overline{X}}{2(a\overline{X} + b)}$ and α_2 are minimizing constant.

For $\alpha_{2(opt)} = \frac{\sum_{h=1}^{L} w_h^2 \lambda_h (\theta_h S_{xh} - R S_{yh})}{\sum_{h=1}^{L} S_{xh}}$ the optimum MSE converges to Regression estimator as,

$$MSE\left(T_G\right) = \overline{Y}^2 V_{20}\left(1 - \rho_c\right). \tag{3.23}$$

The factor ρ_c represents the aggregate correlation coefficient over all strata and is defined as,

$$\rho_c^2 = \frac{\left(\sum_{h=1}^{L} w_h^2 \lambda_h \rho_h S_{yh} S_{xh} \right)^2}{\sum_{h=1}^{L} w_h^2 \lambda_h S_{xh}^2 \sum_{h=1}^{L} w_h^2 \lambda_h S_{yh}^2}.$$

Motivated by [22,23] the following difference exponential ratio estimator are proposed:

$$T_{NK} = \left[k_1 \overline{y}_{st} + k_2 \left\{ \frac{\overline{x}_{st}}{\overline{X}} \right\}^{\gamma} \right] \exp\left[\frac{A_{st}\left(\overline{x}_{st} - \overline{X}\right)}{A_{st}\left(\overline{x}_{st} - \overline{X}\right) + 2B_{st}} \right]. \tag{3.24}$$

Here, A_{st}, B_{st}, and γ are the generalizing constants, and k_1 and k_2 are the minimizing constants. The Koyuncu estimator's first order of approximated Bias and MSE are given as:

$$BIAS\left(T_{NK}\right)=\bar{Y}\left(k_1-1\right)+k_1\bar{Y}\delta\left(\frac{3}{8}\delta V_{02}-\frac{1}{2}V_{11}\right)+k_2\left\{\frac{1}{2}\gamma\left(\gamma+\frac{3}{4}\delta^2-\delta-1\right)V_{02}\right\} \tag{3.25}$$

$$MSE\left(T_{NK}\right)=\bar{Y}^2+\bar{Y}^2k_1^2A+k_2^2B+k_1\bar{Y}^2C+k_2\bar{Y}D+k_1k_2\bar{Y}E\,. \tag{3.26}$$

Here, $\delta=\dfrac{A_{st}\bar{X}}{A_{st}\bar{X}+B_{st}}$. For optimum values of $k_1=\dfrac{DE-2BC}{4AB-E^2}$ and $k_2=\bar{Y}\dfrac{CE-2AD}{4AB-E^2}$ the MSE is given as:

$$MSE\left(T_{NK}\right)=\bar{Y}^2\left[1-\frac{AD^2+BC^2-CDE}{4AB-E^2}\right]. \tag{3.27}$$

Here, $A=1+V_{20}+\delta^2V_{02}-2\delta V_{11}$, $B=1+\left(2\gamma^2+\delta^2-\gamma-2\delta\gamma\right)V_{02}$, $C=\delta V_{11}-2-\dfrac{3}{4}\delta^2V_{02}$, $D=\left\{\delta\gamma-\dfrac{3}{4}\delta^2-\gamma\left(\gamma-1\right)\right\}V_{02}-2$ and $E=\left\{2\delta^2+\gamma^2-\gamma\left(2\delta+1\right)\right\}V_{02}+2+2\left(\gamma-\delta\right)V_{11}$.

Tiwari et al. [24] proposed the following difference cum ratio exponential estimator as

$$T_{TSS}=\left\{k_3\bar{y}_{st}+k_4\left(\bar{X}-\bar{x}_{st}\right)\right\}\left[\frac{a_{st}\bar{X}+b_{st}}{c_{st}\bar{x}_{st}+d_{st}}\right]^{\alpha_3}\left[\exp\left\{\frac{\bar{X}-\bar{x}_{st}}{\bar{X}+\bar{x}_{st}}\right\}\right]^{\beta}. \tag{3.28}$$

Here, a_{st}, b_{st}, c_{st}, and d_{st} are either known parameters or some functions of the parameters of X, α_3, and β which are the generalizing constants that can take values like (1, 0, -1) etc, and k_3 and k_4 are the minimizing constants. The estimator's bias and *MSE* are provided:

$$Bias\left(T_{TSS}\right)=\bar{Y}_{st}\left[\left(\mu_{st}k_3-1\right)+\mu_{st}\left\{\left(\left(\frac{\beta}{2}+\alpha_3\upsilon_{st}\right)k_4R+\left(\frac{\beta}{2}+\alpha_3\upsilon_{st}\right)\frac{k_3}{2}+\left(\frac{\beta}{2}+\alpha_3\upsilon_{st}\right)^2\frac{k_3}{2}\right)V_{20}-\left(\frac{\beta}{2}+\alpha_3\upsilon_{st}\right)k_3V_{11}\right\}\right]. \tag{3.29}$$

Here, $\mu_{st}=\left[\dfrac{a_{st}\bar{X}+b_{st}}{c_{st}\bar{X}+d_{st}}\right]^{\alpha_3}$ and $\upsilon_{st}=\left[\dfrac{c_{st}\bar{X}}{c_{st}\bar{X}+d_{st}}\right]$

$$MSE\left(T_{TSS}\right)=\bar{Y}_{st}^2\left[1+A_1k_3^2+B_1k_4^2-C_1k_3-2D_1k_4+2E_1k_3k_4\right]. \tag{3.30}$$

For $k_3=\dfrac{B_1C_1-2D_1E_1}{2\left(A_1B_1-E_1^2\right)}$ and $k_4=\dfrac{2A_1D_1-C_1E_1}{2\left(A_1B_1-E_1^2\right)}$, the lowest possible MSE is calculated as:

$$MSE\left(T_{TSS}\right)\cong\bar{Y}_{st}^2\left[1-\frac{B_1C_1^2+4A_1D_1^2-4C_1D_1E_1}{2\left(A_1B_1-E_1^2\right)}\right]. \tag{3.31}$$

Here, $A_1=\mu_{st}^2\left[1+V_{20}+\left(V_{02}-4V_{11}\right)\left(\dfrac{\beta}{2}+\alpha_3\upsilon_{st}\right)+2V_{02}\left(\dfrac{\beta}{2}+\alpha_3\upsilon_{st}\right)^2\right]$, $B_1=R^2\mu_{st}^2V_{02}$, $C_1=\mu_{st}\left[2+\left(V_{02}-2V_{11}\right)\left(\dfrac{\beta}{2}+\alpha_3\upsilon_{st}\right)+V_{02}\left(\dfrac{\beta}{2}+\alpha_3\upsilon_{st}\right)^2\right]$, $D_1=R\mu_{st}V_{02}\left(\dfrac{\beta}{2}+\alpha_3\upsilon_{st}\right)$ and $E_1=R\mu_{st}^2V_{11}$.

Javed et al. [25] proposed the following family of estimator estimators,

$$T_{MJ}=\left[\frac{\bar{y}_{st}}{2}\left\{\exp\left[\frac{\bar{x}_{st}-\bar{X}}{\bar{x}_{st}+\bar{X}}\right]+\exp\left[\frac{\bar{X}-\bar{x}_{st}}{\bar{x}_{st}+\bar{X}}\right]\right\}+k_5\bar{y}_{st}+k_6\left(\bar{X}-\bar{x}_{st}\right)\right]\exp\left[\frac{a_{st}\bar{X}+b_{st}}{a_{st}\bar{x}_{st}+b_{st}}-1\right]. \quad (3.32)$$

Here, the constants a, b are generalizing elements. The bias of the proposed estimator is given as,

$$Bias\left(T_{MJ}\right)=\bar{Y}_{st}\left[V_{02}\left(\frac{1+20\eta^2}{8}\right)-\eta V_{11}\right]+k_5\left(\bar{Y}_{st}+\frac{5\bar{Y}_{st}\eta^2V_{02}}{2}-\bar{Y}_{st}\eta V_{11}\right)+\bar{X}_{st}\eta k_6V_{02}. \quad (3.33)$$

For $k_5=-\frac{B_2C_2-2D_2E_2}{2\left(A_2B_2-E_2^2\right)}$ and $k_6=-\frac{R\left(2A_2D_2-C_2E_2\right)}{2\left(A_2B_2-E_2^2\right)}$ a minimal value for MSE is expressed as,

$$MSE(T_{MJ})_{\min}=\bar{Y}_{st}^2\left[\left(V_{20}+\eta^2V_{02}-2\eta V_{11}\right)-\frac{4A_2D_2^2+B_2C_2^2-4C_2D_2E_2}{4\left(A_2B_2-E_2^2\right)}\right]. \quad (3.34)$$

Here, $A_2=1+V_{20}+4\eta^2V_{02}$, $B_2=R^2V_{02}$, $C_2=V_{20}+\left(\frac{4\eta^2+1}{8}\right)V_{02}-3\eta V_{11}$, $D_2=V_{11}-\eta V_{02}$ and $E_2=V_{11}-2\eta V_{02}$.

4. Proposed estimator

The estimators suggested in this study represent significant improvements in the field of finite population estimation. In contrast to conventional unbiased estimators, which are appropriate when only the primary study variable is accessible, these innovative estimators designed to exploit the potential of supplementary information. By carefully including only one auxiliary variable in the estimation process, we achieved sophisticated equilibrium between bias and precision.

Two discrete estimator families were carefully designed and assessed using stratified random sampling. The aforementioned estimators were specifically designed to address the inherent difficulties associated with estimating the average of a determinate population. Consequently, they provided a novel approach for enhancing the accuracy of the estimation.

4.1. First proposed estimator

Muneer et al. [26] proposed the following regression-exponential-Ratio type estimator

$$T_S=\left[w_1\bar{y}-w_2\left(\bar{x}-\bar{X}\right)\right]\left[\alpha\left\{2-\exp\left(\frac{\bar{z}-\bar{Z}}{\bar{z}+\bar{Z}}\right)\right\}+\left(1-\alpha\right)\exp\left(\frac{\bar{Z}-\bar{z}}{\bar{z}+\bar{Z}}\right)\right]. \quad (4.1.a)$$

Here, w_1 and w_2 are minimizing constants and α takes values either 1 or 0 to have ratio exponential or product exponential estimators, respectively. Similarly, Shabbir et al. [28] proposed the below estimator

$$T_{SGO}=\left[w_3\bar{y}+w_4\right]\exp\left[\frac{u\left(\bar{X}-\bar{x}\right)}{u\left(\bar{X}+\bar{x}\right)+2v}\right]. \quad (4.1.b)$$

Here, w_3 and w_4 are the generalizing constants and u, v are some known suitably chosen parameters of the auxiliary variable or some real valued constants.

In light of the work of [26,28], we propose the following estimator:

$$T_{pro1} = \left(S_1\bar{y}_{st} + S_2\right)\left[1\left(\frac{\bar{X}_{st}}{\bar{x}_{st}}\right)\left\{2-\exp\left[\frac{u_{st}\left(\bar{x}_{st}-\bar{X}_{st}\right)}{u_{st}\left(\bar{X}_{st}+\bar{x}_{st}\right)+2v_{st}}\right]\right\}+\left(1-1\right)\left(\frac{\bar{x}_{st}}{\bar{X}_{st}}\right)\exp\left[\frac{u_{st}\left(\bar{X}_{st}-\bar{x}_{st}\right)}{u_{st}\left(\bar{X}_{st}+\bar{x}_{st}\right)+2v_{st}}\right]\right]. \quad (4.1.1)$$

Here, S_1 and S_2 are optimizing constants, whose values are obtained so that the MSE is minimum, 1 can take values from 0 to 1 and the generalizing constants u and v are to be replaced by the values of the population parameters or some function of the parameters of the supplementary variable.

$$T_{pro1} = \left(S_1\bar{Y}_{st}\left(1+\varepsilon_0\right)+S_2\right)\left[1\left(\frac{\bar{X}_{st}}{\bar{X}_{st}\left(1+\varepsilon_1\right)}\right)\left\{2-\exp\left[\frac{u_{st}\left(\bar{X}_{st}\left(1+\varepsilon_1\right)-\bar{X}_{st}\right)}{u_{st}\left(\bar{X}_{st}+\bar{X}_{st}\left(1+\varepsilon_1\right)\right)+2v_{st}}\right]\right\}+\left(1-1\right)\left(\frac{\bar{X}_{st}\left(1+\varepsilon_1\right)}{\bar{X}_{st}}\right)\exp\left[\frac{u_{st}\left(\bar{X}_{st}-\bar{X}_{st}\left(1+\varepsilon_1\right)\right)}{u_{st}\left(\bar{X}_{st}+\bar{X}_{st}\left(1+\varepsilon_1\right)\right)+2v_{st}}\right]\right]. \quad (4.1.2)$$

After simplification and application of different series, the proposed estimator is converted to the following form:

$$T_{pro1} = \left[S_1\bar{Y}_{st}\left(1+\varepsilon_0\right)+S_2\right]\left[1+\vartheta_1\varepsilon_1-\vartheta_2\varepsilon_1^2\right]. \quad (4.1.3)$$

Here, $\vartheta_1 = 1-\frac{\eta}{2}-21$ and $\vartheta_2 = 1+\left(1-\frac{1}{2}\right)\eta+\frac{1}{8}\left(3-21\right)\eta^2$ and $\eta = \frac{u\bar{X}_{st}}{u\bar{X}_{st}+v}$

$$T_{pro1} = S_1\bar{Y}_{st}\left(1+\varepsilon_0+\vartheta_1\varepsilon_1+\vartheta_1\varepsilon_0\varepsilon_1-\vartheta_2\varepsilon_1^2\right)+S_2\left(1+\vartheta_1\varepsilon_1-\vartheta_2\varepsilon_1^2\right). \quad (4.1.4)$$

Now, subtracting $\bar{Y}_{st}$ from both sides, we have:

$$T_{pro1}-\bar{Y}_{st} = S_1\bar{Y}_{st}\left(1+\varepsilon_0+\vartheta_1\varepsilon_1+\vartheta_1\varepsilon_0\varepsilon_1-\vartheta_2\varepsilon_1^2\right)+S_2\left(1+\vartheta_1\varepsilon_1-\vartheta_2\varepsilon_1^2\right)-\bar{Y}_{st}. \quad (4.1.5)$$

When we apply expectation to both sides of the previous equation, we get the following bias expression:

$$Bias\left(T_{pro1}\right) = S_1\bar{Y}_{st}\left(1+\vartheta_1V_{11}-\vartheta_2V_{02}\right)+S_2\left(1-\vartheta_2V_{02}\right)-\bar{Y}_{st}. \quad (4.1.6)$$

To obtain the MSE expression, we take the square of both sides of the equation

$$\begin{aligned}\left(T_{pro1}-\bar{Y}_{st}\right)^2 = {} & S_1^2\bar{Y}_{st}^2\left(1+\varepsilon_0^2+\left(\vartheta_1^2-2\vartheta_2\right)\varepsilon_1^2+4\vartheta_1\varepsilon_0\varepsilon_1\right)+S_2^2\left(1+\left(\vartheta_1^2-2\vartheta_2\right)\varepsilon_1^2\right)\\ & +\bar{Y}_{st}^2-2S_1\bar{Y}_{st}^2\left(1-\vartheta_2\varepsilon_1^2+\vartheta_1\varepsilon_0\varepsilon_1\right)-2S_2\bar{Y}_{st}\left(1-\vartheta_2\varepsilon_1^2\right)\\ & +2S_1S_2\bar{Y}_{st}\left(1+\left(\vartheta_1^2-2\vartheta_2\right)\varepsilon_1^2+2\vartheta_1\varepsilon_0\varepsilon_1\right)\end{aligned}. \quad (4.1.7)$$

After taking expectation the MSE expression obtained as,

$$\begin{aligned}MSE\left(T_{pro1}\right) = {} & S_1^2\bar{Y}_{st}^2\left(1+V_{20}+\left(\vartheta_1^2-2\vartheta_2\right)V_{02}+4\vartheta_1V_{11}\right)+S_2^2\left(1+\left(\vartheta_1^2-2\vartheta_2\right)V_{02}\right)\\ & +\bar{Y}_{st}^2-2S_1\bar{Y}_{st}^2\left(1-\vartheta_2V_{02}+\vartheta_1V_{11}\right)-2S_2\bar{Y}_{st}\left(1-\vartheta_2V_{02}\right)\\ & +2S_1S_2\bar{Y}_{st}\left(1+\left(\vartheta_1^2-2\vartheta_2\right)V_{02}+2\vartheta_1V_{11}\right)\end{aligned} \quad (4.1.8)$$

$$MSE\left(T_{pro1}\right) = S_1^2\bar{Y}_{st}^2A_{pr}+S_2^2B_{pr}+\bar{Y}_{st}^2-2S_1\bar{Y}_{st}^2C_{pr}-2s_2\bar{Y}_{st}D_{pr}+2S_1S_2\bar{Y}_{st}E_{pr}. \quad (4.1.9)$$

Here, $A_{pr} = 1+V_{20}+\left(\vartheta_1^2-2\vartheta_2\right)V_{02}+4\vartheta_1V_{11}$, $B_{pr} = 1+\left(\vartheta_1^2-2\vartheta_2\right)V_{02}$, $C_{pr} = 1-\vartheta_2V_{02}+\vartheta_1V_{11}$,

$D_{pr} = 1 - \vartheta_2 V_{02}$ and $E_{pr} = 1 + \left(\vartheta_1^2 - 2\vartheta_2\right) V_{02} + 2\vartheta_1 V_{11}$.

Now, let differentiate the *MSE* equation to obtain the values of S_1 and S_2 to have minimum *MSE*.

$\frac{\partial MSE\left(T_{pro1}\right)}{\partial S_1} = 0$ and $\frac{\partial MSE\left(T_{pro1}\right)}{\partial S_2} = 0$. So, we obtain:

$$S_1 \bar{Y}_{st}^2 A_{pr} + S_2 \bar{Y}_{st} E_{pr} - \bar{Y}_{st}^2 C_{pr} = 0 \tag{4.1.10}$$

$$S_1 \bar{Y}_{st} E_{pr} + S_2 B_{pr} - \bar{Y}_{st} D_{pr} = 0 . \tag{4.1.11}$$

Solving Eqs (4.1.10) and (4.1.11), we gain the following optimal values of $S_1 = \frac{B_{pr} C_{pr} - D_{pr} E_{pr}}{A_{pr} B_{pr} - E_{pr}^2}$ and $S_2 = \frac{\bar{Y}_{st}\left(A_{pr} D_{pr} - C_{pr} E_{pr}\right)}{A_{pr} B_{pr} - E_{pr}^2}$. With these values, the minimum *MSE* adopts the below form:

$$MSE\left(T_{pro1}\right)_{\min} \cong \bar{Y}_{st}^2 \left\{1 - \frac{A_{pr} D_{pr}^2 + B_{pr} C_{pr}^2 - 2 C_{pr} D_{pr} E_{pr}}{A_{pr} B_{pr} - E_{pr}^2}\right\}. \tag{4.1.12}$$

4.2. Second proposed estimator

Taking some insights from the work of [26–28], we propose the following class of estimators.

$$T_{pro} = \left(T_1 \bar{y}_{st} + T_2\right)\left[1\left(\frac{\bar{X}_{st}}{\bar{x}_{st}}\right) + \left(1-1\right)\left(\frac{\bar{x}_{st}}{\bar{X}_{st}}\right)\right] \exp\left[\frac{u_{st}\left(\bar{X}_{st} - \bar{x}_{st}\right)}{u_{st}\left(\bar{X}_{st} + \bar{x}_{st}\right) + 2v_{st}}\right]. \tag{4.2.1}$$

The values of optimizing constants T_1 and T_2 are obtained so that the MSE is minimum. The difference equation up-to first order of approximation of the proposed estimator in terms of errors is expressed as

$$T_{pro2} - \bar{Y}_{st} = \left[\left(T_1 - 1\right)\bar{Y}_{st} + T_1 \bar{Y}_{st}\left(\varepsilon_0 - \delta_1 \varepsilon_1 + \delta_2 \varepsilon_1^2 - \delta_1 \varepsilon_0 \varepsilon_1\right) + T_2\left(1 - \delta_1 \varepsilon_1 + \delta_2 \varepsilon_1^2\right)\right]. \tag{4.2.2}$$

After taking the expectation the bias of the suggested estimator is given as,

$$Bias\left(T_{pro2}\right) = (T_1 - 1)\bar{Y}_{st} + T_1 \bar{Y}_{st}(\delta_2 V_{02} - \delta_1 V_{11}) + T_2(1 + \delta_2 V_{02}) . \tag{4.2.3}$$

Here, $\delta_1 = \frac{1}{2}\eta + 21 - 1$ and $\delta_2 = 1 + \frac{1}{2}\eta\left(21 - 1\right) + \frac{3}{8}\eta^2$.

Squaring both sides of the above (4.2.2) difference equation and using first order of approximation, we have,

$$E\left(T_{pro2} - \bar{Y}_{st}\right)^2 = \begin{bmatrix} \left(T_1 - 1\right)^2 \bar{Y}_{st}^2 + T_1^2 \bar{Y}_{st}^2\left(V_{20} + \left(\delta_1^2 + 2\delta_2\right)V_{02} - 4\delta_1 V_{11}\right) \\ + T_2^2\left(1 + \left(\delta_1^2 + 2\delta_2\right)V_{02}\right) - 2T_1 \bar{Y}_{st}^2\left(\delta_2 V_{02} - \delta_1 V_{11}\right) \\ - 2T_2 \bar{Y}_{st}\left(1 + \delta_2 V_{02}\right) + 2T_1 T_2 \bar{Y}_{st}\left(1 + \left(\delta_1^2 + 2\delta_2\right)V_{02} - 2\delta_1 V_{11}\right) \end{bmatrix} \tag{4.2.4}$$

or

$$MSE\left(T_{pro2}\right)=\begin{bmatrix}\left(T_1-1\right)^2\bar{Y}_{st}^2+T_1^2\bar{Y}_{st}^2\left(V_{20}+\left(\delta_1^2+2\delta_2\right)V_{02}-4\delta_1V_{11}\right)\\+T_2^2\left(1+\left(\delta_1^2+2\delta_2\right)V_{02}\right)-2T_1\bar{Y}_{st}^2\left(\delta_2V_{02}-\delta_1V_{11}\right)\\-2T_2\bar{Y}_{st}\left(1+\delta_2V_{02}\right)+2T_1T_2\bar{Y}_{st}\left(1+\left(\delta_1^2+2\delta_2\right)V_{02}-2\delta_1V_{11}\right)\end{bmatrix} \tag{4.2.5}$$

or

$$MSE\left(T_{pro2}\right)=\left[\left(T_1-1\right)^2\bar{Y}_{st}^2+T_1^2\bar{Y}_{st}^2A_p+T_2^2B_p-2T_1\bar{Y}_{st}^2C_p-2T_2\bar{Y}_{st}D_p+2T_1T_2\bar{Y}_{st}E_p\right]. \tag{4.2.6}$$

Here, $A_p=\left(V_{20}+\left(\delta_1^2+2\delta_2\right)V_{02}-4\delta_1V_{11}\right)$, $B_p=\left(1+\left(\delta_1^2+2\delta_2\right)V_{02}\right)$, $C_p=\left(\delta_2V_{02}-\delta_1V_{11}\right)$, $D_p=\left(1+\delta_2V_{02}\right)$ and $E_p=\left(1+\left(\delta_1^2+2\delta_2\right)V_{02}-2\delta_1V_{11}\right)$.

For optimum values of $T_1=-\left[\frac{B_pC_p-D_pE_p+B_p}{A_pB_p-E_p^2+B_p}\right]$ and $T_2=\left[\frac{\bar{Y}\left(A_pD_p-C_pE_p+D_p-E_p\right)}{A_pB_p-E_p^2+B_p}\right]$, the least possible value of the MSE up to the first order of approximation is shown as

$$MSE\left(T_{pro2}\right)_{\min}\cong\bar{Y}_{st}^2\left\{1-\frac{A_pD_p^2+B_pC_p^2-2C_pD_pE_p+B_p+2B_pC_p-D_p^2-2D_pE_p}{A_pB_p-E_p^2+B_p}\right\}. \tag{4.2.7}$$

5. Efficiency comparison

In this section, we define the conditions that must be met for the suggested estimators to outperform the currently used estimating methods in terms of efficiency.

5.1. Conditions for the first proposed estimator

Condition (i)

By comparing (3.2) and (4.1.12), $MSE\left(T_{pro1}\right)\leq MSE\left(T_{st}\right)$ if

$$\left[\left(V_{20}-1\right)+\Re_1\right]>0. \tag{5.1.1}$$

Here, $\Re_1=\frac{A_{pr}D_{pr}^2+B_{pr}C_{pr}^2-2C_{pr}D_{pr}E_{pr}}{A_{pr}B_{pr}-E_{pr}^2}$

Condition (ii)

By comparing (3.5) and (4.1.12), $MSE\left(T_{pro1}\right)\leq MSE\left(T_r\right)$ if

$$\left[V_{20}+V_{02}-2V_{11}-1+\Re_1\right]>0. \tag{5.1.2}$$

Condition (iii)

By comparing (3.8) and (4.1.12), $MSE\left(T_{pro1}\right)\leq MSE\left(T_{SD}\right)$ if

$$\left(V_{20}+\frac{1}{4}V_{02}-V_{11}\right)-1+\Re_1>0. \tag{5.1.3}$$

Condition (iv)

By comparing (3.11) and (4.1.12), $MSE\left(T_{pro1}\right) \leq MSE\left(T_{BT}\right)$ if

$$\left(V_{20}+\phi^2 V_{02}-2\phi V_{11}\right)-1+\Re_1>0. \tag{5.1.4}$$

Condition (v)

By comparing (3.14) and (4.1.12), $MSE\left(T_{pro1}\right) \leq MSE\left(T_{US}\right)$ if

$$\left(V_{20}+\theta^2 V_{02}-2\theta V_{11}\right)-1+\Re_1>0. \tag{5.1.5}$$

Condition (vi)

By comparing (3.17) and (4.1.12), $MSE\left(T_{pro1}\right) \leq MSE\left(T_{Ch}\right)$ if

$$\left(V_{20}+\alpha^2 g^2 \pi^2 V_{02}-2\alpha g \pi V_{11}\right)-1+\Re_1>0. \tag{5.1.6}$$

Condition (vii)

By comparing (3.20) and (4.1.12), $MSE\left(T_{pro1}\right) \leq MSE\left(T_{O}\right)$ if

$$\left[\frac{V_{00}V_{22}-V_{12}^2}{V_{00}+V_{22}-2V_{12}}\right]-\bar{Y}^2\left[1-\Re_1\right]>0. \tag{5.1.7}$$

Condition (viii)

By comparing (3.24) and (4.1.12), $MSE\left(T_{pro1}\right) \leq MSE\left(T_{G}\right)$ if

$$\left[V_{20}\left(1-\rho_{st}\right)-1+\Re_1\right]>0. \tag{5.1.8}$$

Condition (ix)

By comparing (3.28) and (4.1.12), $MSE\left(T_{pro1}\right) \leq MSE\left(T_{NK}\right)$ if

$$\Re_1-\frac{AD^2+BC^2-CDE}{4AB-E^2}>0. \tag{5.1.9}$$

Condition (x)

By comparing (3.32) and (4.1.12), $MSE\left(T_{pro1}\right) \leq MSE\left(T_{TSS}\right)$ if

$$\Re_1-\left[\frac{B_1C_1^2+4A_1D_1^2-4C_1D_1E_1}{2\left(A_1B_1-E_1^2\right)}\right]>0. \tag{5.1.10}$$

Condition (xi)

By comparing (3.35) and (4.1.12) $MSE\left(T_{pro1}\right) \leq MSE\left(T_{MJ}\right)$ if

$$\left(V_{20}+\eta^2 V_{02}-2\eta V_{11}\right)+\Re_1-\left[\frac{4A_2D_2^2+B_2C_2^2-4C_2D_2E_2}{4\left(A_2B_2-E_2^2\right)}\right]>0. \tag{5.1.11}$$

5.2. Conditions for the second proposed estimator

Condition (i)

By comparing (3.2) and (4.2.7), $MSE\left(T_{pro2}\right) \le MSE\left(T_{st}\right)$ if

$$\left[\left(V_{20}-1\right)+\frac{\Re_2}{\Re_3}\right]>0. \tag{5.2.1}$$

Where, $\Re_2 = A_p B_p^2 + B_p C_p^2 - 2C_p D_p E_p + B_p + 2B_p C_p - D_p^2 - 2D_p E_p$ and $\Re_3 = A_p B_p - E_p^2 + B_p$

Condition (ii)

By comparing (3.5) and (4.2.7), $MSE\left(T_{pro2}\right) \le MSE\left(T_{r}\right)$ if

$$\left[V_{20}+V_{02}-2V_{11}-1+\frac{\Re_2}{\Re_3}\right]>0. \tag{5.2.2}$$

Condition (iii)

By comparing (3.8) and (4.2.7), $MSE\left(T_{pro2}\right) \le MSE\left(T_{SD}\right)$ if

$$\left(V_{20}+\frac{1}{4}V_{02}-V_{11}\right)-1+\frac{\Re_2}{\Re_3}>0. \tag{5.2.3}$$

Condition (iv)

By comparing (3.11) and (4.2.7), $MSE\left(T_{pro2}\right) \le MSE\left(T_{BT}\right)$ if

$$\left(V_{20}+\phi^2 V_{02}-2\phi V_{11}\right)-1+\frac{\Re_2}{\Re_3}>0. \tag{5.2.4}$$

Condition (v)

By comparing (3.14) and (4.2.7), $MSE\left(T_{pro2}\right) \le MSE\left(T_{US}\right)$ if

$$\left(V_{20}+\theta^2 V_{02}-2\theta V_{11}\right)-1+\frac{\Re_2}{\Re_3}>0. \tag{5.2.5}$$

Condition (vi)

By comparing (3.17) and (4.2.7), $MSE\left(T_{pro2}\right) \le MSE\left(T_{Ch}\right)$ if

$$\left(V_{20}+\alpha^2 g^2 \pi^2 V_{02}-2\alpha g \pi V_{11}\right)-1+\frac{\Re_2}{\Re_3}>0. \tag{5.2.6}$$

Condition (vii)

By comparing (3.20) and (4.2.7), $MSE\left(T_{pro2}\right) \le MSE\left(T_{O}\right)$ if

$$\left[\frac{V_{00}V_{22}-V_{12}^2}{V_{00}+V_{22}-2V_{12}}\right]-\bar{Y}^2\left[1-\frac{\Re_2}{\Re_3}\right]>0. \tag{5.2.7}$$

Condition (viii)

By comparing (3.24) and (4.2.7), $MSE\left(T_{pro2}\right) \le MSE\left(T_G\right)$ if

$$\left[V_{20}\left(1-\rho_{st}\right)-1+\frac{\Re_2}{\Re_3}\right]>0. \tag{5.2.8}$$

Condition (ix)

By comparing (3.28) and (4.2.7), $MSE\left(T_{pro2}\right) \le MSE\left(T_{NK}\right)$ if

$$\frac{\Re_2}{\Re_3}-\frac{AD^2+BC^2-CDE}{4AB-E^2}>0. \tag{5.2.9}$$

Condition (x)

By comparing (3.32) and (4.2.7), $MSE\left(T_{pro2}\right) \le MSE\left(T_{TSS}\right)$ if

$$\frac{\Re_2}{\Re_3}-\left[\frac{B_1C_1^2+4A_1D_1^2-4C_1D_1E_1}{2\left(A_1B_1-E_1^2\right)}\right]>0. \tag{5.2.10}$$

Condition (xi)

By comparing (3.35) and (4.2.7), $MSE\left(T_{pro2}\right) \le MSE\left(T_{MJ}\right)$ if

$$\left(V_{20}+\eta^2V_{02}-2\eta V_{11}\right)+\frac{\Re_2}{\Re_3}-\left[\frac{4A_2D_2^2+B_2C_2^2-4C_2D_2E_2}{4\left(A_2B_2-E_2^2\right)}\right]>0. \tag{5.2.11}$$

The above theorems are important for the development of conditions under which the novel estimators outperform the suggested estimators. If these conditions hold, then the novelty of the estimators is guaranteed. In other words, these assumptions are related to the efficiency of the proposed estimator.

6. Numerical comparison

To check the performance of the proposed estimator relative to the classical estimator, the following data sets were considered (see Table 1).

Data I: (source: [29])

(The two strata are **Stratum 1**: Rawalpindi, Lahore, Sargodha and Gujranwala. **Stratum 2**: Sahiwal, Faisalabad, D.G Khan, Multan and Bahawalpur)

Y: In 2012 division's wise employment level.

X: in 2012 division's wise quantity of registered factories.

Data II: (source: [29])

Y: in 2012 division's wise enrollment of students.

X: in 2012 divisions wise the count of Govt schools.

Data III: (source: [17]). The dataset has information on the apple production amount (Y) and the number of apple trees (X) in 854 villages in Turkey in the year 1999. The data is categorized into strata based on the region of Turkey.

Data IV: (source: [2]) The study contains the number of instructors as study variable and the number of

students as supplementary variable in schools for 923 districts in six regions in Turkey in 2007. (1: Aegean 2: Black Sea 3: Central Anatolia 4: East and Southeast Anatolia 5: Marmara 6: Mediterranean)
Data V: (source: [30]). The main variable pertains to the number of wet days, whereas the auxiliary variable refers to the total number of sunshine hours.

Table 1. Summary statistics of all the data sets.

Data	Stratum	N_h	n_h	$\bar{Y}_h$	$\bar{X}_h$	S_{yh}	S_{xh}	ρ_{yxh}	C_{yh}	C_{xh}
I	1	18	8	85572.11	414.5556	248216	521.68	0.3473	2.9007	1.2584
	2	18	8	19293.61	257	37979.33	365.70	0.9796	1.9685	1.423
II	1	18	8	162979.3	962.0556	255887.7	307.95	0.1447	1.5701	0.3202
	2	18	8	134458	1146.722	50235.82	469.93	0.787	0.3736	0.4098
III	1	10	4	149.7	1630	102.17	13.470	-0.779	0.063	0.09
	2	10	4	102.6	2036	103.26	12.610	-0.503	0.050	0.122
IV	1	127	31	703.74	20804.59	883.835	30486.75	0.937	1.256	1.465
	2	117	21	413	9211.79	644.922	15180.77	0.996	1.562	1.648
	3	103	29	573.17	14309.3	1033.467	27549.70	0.291	1.803	1.925
	4	170	38	424.66	9478.85	810.585	18218.93	0.983	1.909	1.922
	5	205	22	267.03	5569.95	403.654	8497.776	0.989	1.512	1.526
	6	201	39	393.84	12997.59	711.723	23094.14	0.965	1.807	1.777
V	1	106	9	1536	127	49189	6425	0.82	4.18	2.02
	2	106	17	2212	117	57461	11552	0.86	5.22	2.1
	3	94	38	9384	103	160757	29907	0.9	3.19	2.22
	4	171	67	5588	170	285603	28643	0.99	5.13	3.84
	5	204	7	967	205	45403	2390	0.71	2.47	1.75
	6	173	2	404	201	18794	946	0.89	2.34	1.91

Table 2 shows the MSE of all the estimators selected from the [25], along with the proposed estimators under stratified random sampling with a single supplementary variable. The first, second, and third populations consisted of two strata, each with summary information mentioned. The fourth and fifth populations consisted of six strata each. MSE results were obtained for the proposed estimators for three different values of the generalizing constants u and v. In the first estimator, $u=1$ and $v=0$, and no transformation is applied. In the second estimator, u=1 and $v=Cx$, while the third value had the proposed estimators $u=\rho_{yx}$ and $v=C_x$. Furthermore, in the first three populations, the value of the generalizing constant α was 0.5, while in the fourth population, it was α=0.65. In the fifth population, α=0.40, and the suggested estimators were compared. It was apparent that the *MSEs* of the proposed estimators (Tpor1 and Tpro2) were less than those of all competing estimators in this study. In addition, the use of transformation further decreased the *MSE* values of the suggested estimator.

Table 2. *MSEs* of the estimators of finite population means in real data.

Estimators	Data-I	Data-II	Data-III	Data-IV	Data-V
T_0	1094699971	1180635530	8.877803	2229.266	674045.7
T_R	951308503	1137894915	18.79581	727.6426	159151.3
T_{BT}	969403894	1134267624	13.08039	934.847	341011.7
T_G	951118894	1137875148	18.79535	727.5525	159165.3
T_{NK}	951118894	1137875148	18.79535	727.6346	159151.7
T_{Reg}	3788003666	4524594622	1295.744	24760.6	9975257
T_{Ch}	941649845	1129698712	4.945649	403.8754	107055.4
T_{SD}	945428869	1129700285	4.952633	675.4922	121796.6
T_O	51053306	22296777	0.170545	457.1203	38205.42
T_{TSS}	843775561	1086123241	4.944853	403.2811	106561.4
T_{MJ}	855382199	1086209302	4.944776	403.5506	106547.6
${}^{0.5}_{1,0}T_{pro1}$	38240012	16382473	0.358289	112.3608	8942.898
${}^{0.5}_{1,C_x}T_{pro1}$	38009353	16371323	0.358265	112.3313	8941.515
${}^{0.5}_{\rho,C_x}T_{pro1}$	37582179	16325464	0.358324	112.2752	8941.243
${}^{0.5}_{1,0}T_{pro2}$	42105942	16465112	0.358323	110.2066	8459.403
${}^{0.5}_{1,C_x}T_{pro2}$	41844854	16453885	0.358299	110.1783	8458.088
${}^{0.5}_{\rho,C_x}T_{pro2}$	41353334	16407580	0.358359	110.1219	8457.812

The entries in Table 3 and Figure 1 represent the *PREs* of the estimators for the population mean in the stratified random sampling WOR scheme, in the presence of an auxiliary variable. PREs were obtained relative to the classical estimator of the mean. In all five populations, the efficiencies of the proposed estimators were higher than those of all the listed estimators. In addition, the use of transformation (by applying different parameter values for *u* and *v*) further enhanced the efficiency of the estimator. As in the given case, *Tproi(1)* did not undergo transformations. In *Tproi(2)*, *u=1* and *v=Cx*, and in *Tproi(3), u=rho* and *v=Cx (i=1,2)*. A visual display of the *PREs* relative to each dataset is shown in Figure 1. Each of the five lines compare the *PREs* of the estimators in different datasets. It is obvious that among the five lines, the height of the graph was maximum for the last six entries *(Tpro1(1) to Tpor2(3), proposed estimators)* compared to the rest of the existing estimators. Hence, the graphical display of *PREs* supports the claim that the proposed estimators are significantly more efficient than the existing estimators of the finite population mean in stratified random sampling with single auxiliary information.

Table 3. *PREs* of the estimators relative to usual estimator for real data.

Estimators	Data-I	Data-II	Data-III	Data-IV	Data-V
T_0	100	100	100	100	100
T_R	115.0731	103.7561	47.2329	306.3683	423.5251
T_{BT}	112.9251	104.0879	67.87107	238.4632	197.6606
T_G	115.096	103.7579	47.23404	306.4062	423.4879
T_{NK}	115.096	103.7579	47.23404	306.3717	423.5241
$T_{\mathrm{Re}g}$	28.89913	26.09373	0.685151	9.00328	6.757176
T_{Ch}	116.2534	104.5089	179.5073	551.9688	629.623
T_{SD}	115.7887	104.5087	179.2542	330.021	553.4193
T_O	2144.229	5295.095	5205.549	487.6761	1764.267
T_{TSS}	129.7383	108.7018	179.5362	552.7822	632.5421
T_{MJ}	127.9779	108.6932	179.539	552.4131	632.6243
${}^{0.5}_{1,0}T_{pro1}$	2862.708	7206.699	2477.835	1984.024	7537.217
${}^{0.5}_{1,C_x}T_{pro1}$	2880.081	7211.607	2478.002	1984.545	7538.383
${}^{0.5}_{\rho,C_x}T_{pro1}$	2912.817	7231.865	2477.591	1985.537	7538.612
${}^{0.5}_{1,0}T_{pro2}$	2599.871	7170.529	2477.597	2022.807	7968.005
${}^{0.5}_{1,C_x}T_{pro2}$	2616.092	7175.421	2477.763	2023.326	7969.244
${}^{0.5}_{\rho,C_x}T_{pro2}$	2647.187	7195.671	2477.352	2024.363	7969.504

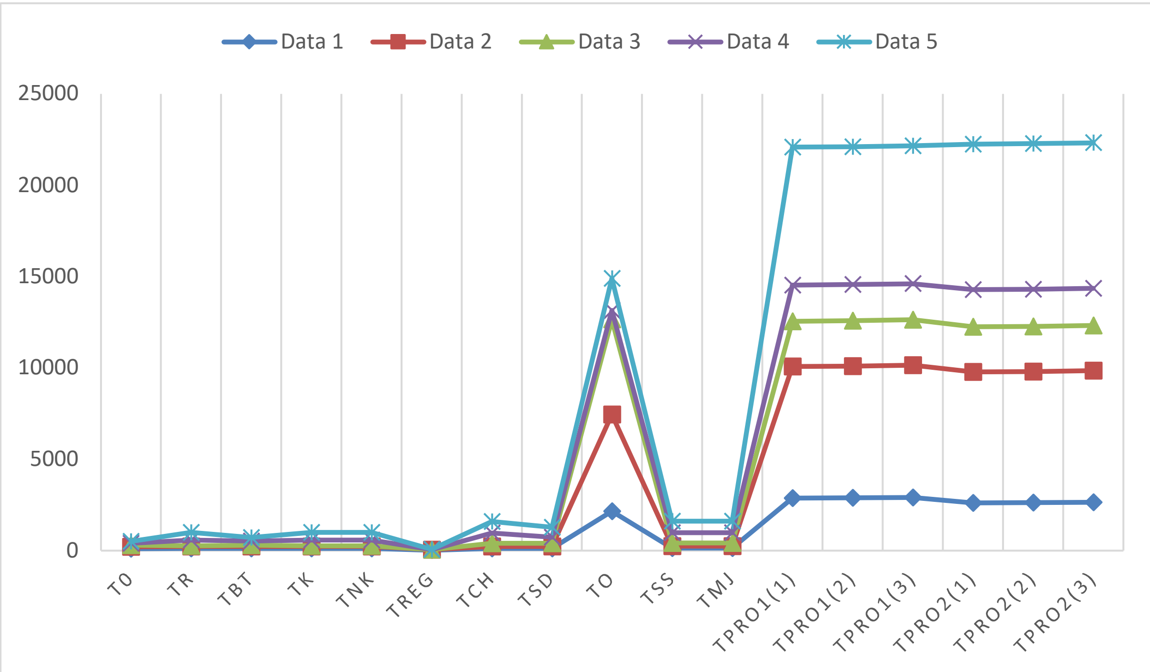

Figure 1. PREs of the estimators in real data.

7. Simulation study

In the section, we conducted a simulation study of both the established and newly introduced estimators to assess the stability of these estimators across random samples. We began with a stratified population of *N=1000* units, from which a sample of *n=100* pairs of values *(y, x)* were selected. This population comprised two strata with sizes $N_1=600$ and $N_2=400$. By employing proportional allocation, we extracted samples of size $n_1=60\%$ and $n_2=40\%$ of the total sample size *(n)* from these respective strata. The mean vectors and covariance matrices are expressed as follows (see Table 4):

Table 4. Strata summary statistics.

Stratum	N	n	μ	Σ
1	600	60%	$\begin{bmatrix}3 & 8\end{bmatrix}$	$\begin{bmatrix}5 & 4\\4 & 4\end{bmatrix}$
2	400	40%	$\begin{bmatrix}6 & 2\end{bmatrix}$	$\begin{bmatrix}3 & 2\\2 & 1.5\end{bmatrix}$

Here, *MSE* and *PRE* values for the estimators were carried out using the following steps in R software.
Step-1: Simple random samples without replacement (SRSWOR) of different sizes *n=10, 20, 50, 100, 200*. were drawn from the target population. For each sample size, a loop of 10,000 times was caried out and allowed R-studio to compute the estimator values at each iteration.
Step-2: For each sample, the values of the existing and suggested estimators were calculated separately by taking the average of all iterations.
Step-3: Using the values obtained in Step-2 the MSE of the estimators is obtained.
Step-4: PRE of the estimators is obtained using the following formula:

$$pre(T_i)=\frac{Var\left(T_0\right)}{MSE\left(T_i\right)}\times 100$$ Where, T_i replaces different estimators.

Table 5 presents the simulation results for the MSEs of the estimators with respect to the usual estimators for various sample sizes. By exploring the table, we can see that the MSEs of the suggested estimators are smaller than those of other estimators. Furthermore, our estimator is stable with respect to sample size, and as the sample size increases, the MSE of the estimator also decreases. Hence, our suggested estimators are the best among all competing estimators under study.

Table 5. *MSEs* of the estimators of population mean through simulations.

Estimator	Sample Size (n)				
	10	20	50	100	200
T_0	0.57167885	0.23228891	0.097460	0.04408674	0.02228120
T_R	0.2800072	0.10384171	0.03535013	0.01578904	0.00729696
T_P	3.46848296	1.42177894	0.59069627	0.26428811	0.13505578
T_{BT}	0.07554437	0.02890564	0.01200708	0.00584716	0.00253814
T_{SD}	0.16541336	0.06573896	0.0225091	0.01076952	0.00461603
T_{US}	0.23223990	0.08851816	0.02989319	0.01389308	0.00621069
T_{CH}	0.28000722	0.10384171	0.03535013	0.01578904	0.00729696
T_O	0.02673456	0.01189527	0.00481989	0.00252548	0.00098261
T_K	0.06359180	0.02652505	0.01157819	0.00575902	0.00251554
T_{TSS}	0.16076453	0.02538855	0.00588480	0.00266663	0.00101421
T_{MJ}	0.07674410	0.02874377	0.01196348	0.00584770	0.00253646
${}_{1,0}^{0.5}T_{pro1}$	0.013249273	0.00518138	0.00217518	0.00100506	0.00033924
${}_{1,C_x}^{0.5}T_{pro1}$	0.013210254	0.00507809	0.00215045	0.00083683	0.00033277
${}_{\rho,C_x}^{0.5}T_{pro1}$	0.013109884	0.00503650	0.00211079	0.00078796	0.00033036
${}_{1,0}^{0.5}T_{pro2}$	0.0138653	0.00547664	0.00243563	0.00108857	0.00036112
${}_{1,C_x}^{0.5}T_{pro2}$	0.0132086	0.00539962	0.0023895	0.0009051	0.00035744
${}_{\rho,C_x}^{0.5}T_{pro2}$	0.0130670	0.00526565	0.00235865	0.00086015	0.00034784

Table 6 shows the simulation results of the different estimators with respect to the usual estimators for the various sample sizes. By exploring the table, we can see that the PREs of the suggested estimators are higher than those of all rival estimators. Furthermore, the suggested estimators are stable with respect to sample size, and as the size of the sample increases, efficiency also increases. Hence, our proposed estimator is superior to all the competing estimators under study. The visual display of the PREs is shown in Figure 2, where each line shows the PREs distribution with a different sample size. Upon examination of the graph, we decided that in each of the samples, the height of the line was the maximum for the last six values (Tpro1(1) to Tpro2(3), the proposed estimators). Hence, the graphical display of the simulation results supports the superiority of the proposed estimators.

Table 6. *PREs* of the estimators of population mean relative to usual estimator in simulated data.

	Sample Size (n)				
Estimator	10	20	50	100	200
T_0	100	100.00	100.00	100	100.000
T_R	204.1658	223.6952	275.6999	287.3062	305.3491
T_P	16.4821	16.33791	16.49922	16.68181	16.49777
T_{BT}	756.7458	803.6109	811.6894	773.0501	877.8541
T_{SD}	345.6062	353.3505	432.9804	424.0894	482.6918
T_{US}	246.1588	262.4195	326.0283	326.8191	358.7558
T_{CH}	204.1658	223.6952	275.6999	287.3062	305.3491
T_O	2138.351	1952.783	2022.043	1735.336	2267.554
T_K	898.982	875.7341	841.7566	785.2975	885.7426
T_{TSS}	355.6001	914.9358	1656.135	1626.34	2196.898
T_{MJ}	744.9157	808.1366	814.6483	773.5519	878.4379
${}_{1,0}^{0.5}T_{pro1}$	4314.7941	4483.149	4480.539	4386.484	6631.513
${}_{1,C_x}^{0.5}T_{pro1}$	4432.0027	4574.340	4532.065	5304.581	6692.862
${}_{\rho,C_x}^{0.5}T_{pro1}$	4656.5601	4612.107	4617.214	5718.092	6744.618
${}_{1,0}^{0.5}T_{pro2}$	4123.084	4241.449	4001.426	4049.964	6229.81
${}_{1,C_x}^{0.5}T_{pro2}$	4237.098	4301.946	4078.654	4904.424	6324.287
${}_{\rho,C_x}^{0.5}T_{pro2}$	4405.5421	4411.404	4132.013	5238.144	6405.531

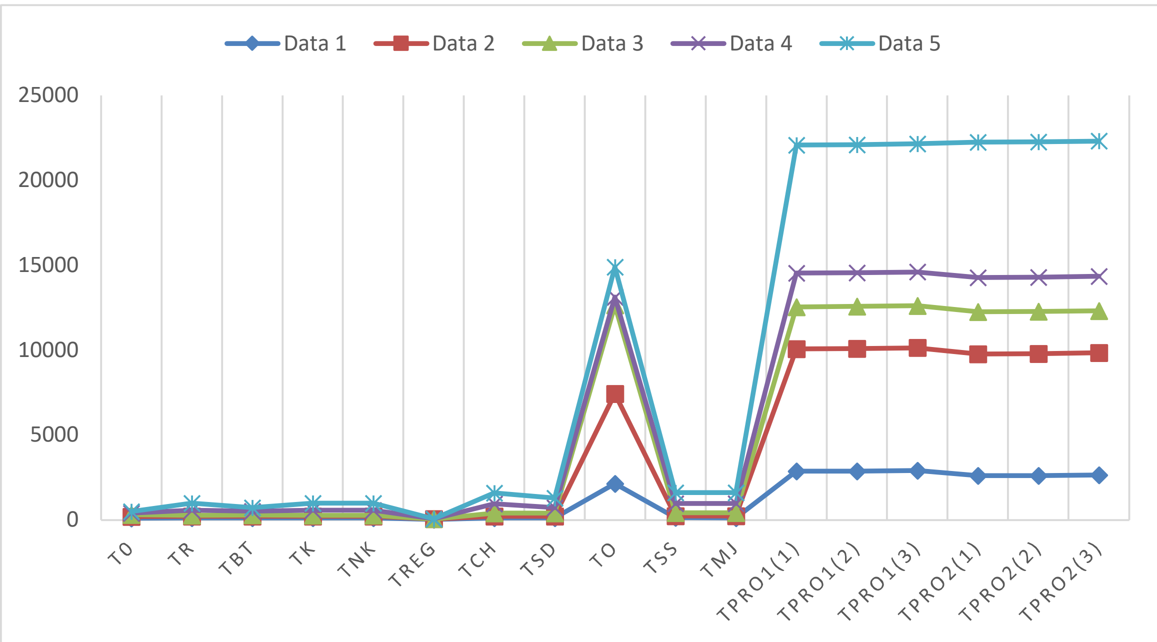

Figure 2. *PREs* of the estimators in simulated data.

8. Discussion

We provide two new families of estimators in the context of stratified random sampling that are intended to enhance population mean estimation by utilizing a single auxiliary variable. Eq (4.1.1) formalizes the study in [26,28], which has a major effect on the construction of the first family of estimators. Equation (4.2.1) provides the mathematical representation of the second family of estimators, which is also developed based on the findings published in [28,31]. Expressions for the bias and mean squared error (MSE) of both estimator families were derived by a comprehensive theoretical study that took into account their first-order approximations. The statistical features of these formulations are explained in depth in Eqs (4.1.6), (4.1.12), (4.2.3), and (4.2.7).

To determine the relative efficiency of our proposed estimators compared to existing methods, we established performance criteria based on MSE minimization and precision improvement. Specifically, Eqs (5.1.1)–(5.1.11) delineated the necessary conditions under which the first estimator family achieved superior performance relative to conventional estimators. Likewise, a corresponding set of conditions was identified for the second estimator family, ensuring its enhanced efficiency over competing methods. These are those situations that are necessary for the proposed estimators to be efficient relative to the estimators mentioned under study.

Both real and simulated datasets were used to thoroughly assess the suggested estimators' effectiveness. The MSE and percentage relative efficiency (PRE) values calculated for real-world data are shown in Tables 2 and 3, which also show how the estimator performs differently for varying values of auxiliary variables, represented by u and v. Among these tables, one can observe that the MSE values of the last six estimators, the proposed one, have small values relative to all the other estimators shown in the table. Similarly, the PRE values for the proposed last six estimators are larger than all the competing estimators for the population mean given in the tables. The findings support the suggested estimators' statistical superiority by showing a constant trend of producing lower MSEs and higher PREs than traditional estimators for the five data sets. The observed patterns indicate that the suggested methodologies provide more accurate estimates of the population mean, thereby reducing estimation errors and enhancing efficiency.

Furthermore, the robustness of these findings was confirmed through extensive simulation studies, with Tables 5 and 6 summarizing the outcomes. The performance of the estimators was tested under five sample sizes: 10, 20, 50, 100, and 200. In all sample sizes used in the simulation studies, both families of estimators exhibited small mean squared errors (MSEs) and large values of percent relative efficiencies (PREs) compared to all competing estimators for the population mean. Furthermore, the tables confirm that the proposed estimators are less variable across sample sizes in terms of MSEs and PREs. These simulated results align closely with the empirical observations, further validating the performance advantages of our proposed estimator families. Notably, the trends in simulated data mirror those observed in real-world datasets, suggesting the generalizability of our approach across population structures.

Visual representations of the PRE values are provided in Figures 1 and 2, which illustrate the efficiency comparisons between our estimators and existing alternatives. Each line in Figure 1 represents a distinct population, while each line in Figure 2 corresponds to a different sample size. In both figures, the graph lines for the proposed families reach the highest points, indicating superior percent relative efficiencies (PREs). A consistent upward trend is evident, demonstrating that the proposed families consistently achieve higher PRE values across scenarios. This graphical evidence strongly supports our conclusion that the new estimator families outperform traditional approaches in terms of precision and reliability.

By analyzing the summary statistics in Table 1 and the percent relative efficiencies (PREs) in Table 3, we observe the following patterns: In the first three datasets, the correlation coefficients for all strata are comparatively smaller than those in the last two datasets. Additionally, the PRE values for the first estimator are higher in the first three datasets compared to the second estimator. Conversely, the PREs for the second estimator are higher than those of the first estimator in the last two datasets.

Based on these findings, we conclude that the first family of estimators performs more efficiently when the correlation coefficients for all or some of the datasets are relatively small. On the other hand, the second family of estimators performs more efficiently when all or most strata have larger correlation coefficients.

Overall, our research offers a substantial contribution to the field of sampling methodology by introducing efficient estimators that optimize the use of an auxiliary variable in stratified random sampling. The proposed approaches not only enhance estimation accuracy but also provide a more reliable alternative to existing techniques. Researcher can extend this work by exploring the application of these estimators in more complex sampling frameworks or integrating additional auxiliary variables to further refine precision levels. The methodological advancements presented in this study pave the way for improved sampling strategies in statistical analysis, benefiting empirical research and practical data collection applications.

9. Conclusions

We introduced two new exponential estimators that can be used to calculate the population mean when stratified random sampling is applied with a single auxiliary variable. We also found formulas for the first-order bias and the MSE of the new estimators. Furthermore, a demanding criterion was established to identify the circumstances in which the proposed estimators outperformed traditional and existing alternatives. We performed a thorough comparison of the MSEs and PREs of our newly designed estimators with those of other approaches. We conducted a comprehensive review, including both simulated experiments and real-world datasets, to improve the robustness and usefulness of our findings. The empirical findings from this investigation consistently confirm the superiority and effectiveness of the proposed estimator families when compared to all the other estimators examined in this study.

We emphasize the significant advancements made by our cutting-edge exponential-type estimators and highlight their improved performance and efficacy in the difficult field of stratified random sampling using a single auxiliary variable.

The proposed estimators account for the nonlinear relationships between the study and auxiliary variables, in contrast to traditional estimators. Additionally, the proposed estimators are a hybrid of regression, ratio, product, and exponential functions to obtain more accurate results. Furthermore, the proposed estimators can adapt to multiple distributions, making them more versatile.

The first limitation of the suggested estimators is that although they can handle nonlinear relations, they assume specific functional forms between the variables. Violation of this assumption may affect the performance of the estimator. Adding more parameters to the suggested estimators increases complexity and computational requirements.

In conclusion, researchers can develop more effective variables in light of the suggested estimators to cope with nonresponse problems. Researchers can extend the proposed estimators to scenarios of multiple auxiliaries and examine the proposed estimators in other sampling designs and population scenarios.

Author contributions

Khazan Sher: Conceptualization, project administration, writing original draft, writing–review and editing; Muhammad Ameeq, Basem A. Alkhaleel, Sidra Naz: Investigation, writing original draft, writing–review and editing; Muhammad Muneeb Hassan, Olyan Albalawi: Project administration, investigation, writing original draft, writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

Reference

1. J. Shabbir, S. Gupta, A new estimator of population mean in stratified sampling, *Commun. Stat.-Theor. M.*, **35** (2006), 1201–1209. https://doi.org/10.1080/03610920600629112
2. N. Koyuncu, C. Kadilar, Family of estimators of population mean using two auxiliary variables in stratified random sampling, *Commun. Stat.-Theor. M.*, **38** (2009), 2398–2417. https://doi.org/10.1080/03610920802562723
3. N. Koyuncu, C. Kadilar, On improvement in estimating population mean in stratified random sampling, *J. Appl. Stat.*, **37** (2010), 999–1013. https://doi.org/10.1080/02664760903002675
4. N. Ali, I. Ahmad, M. Hanif, U. Shahzad, Robust-regression-type estimators for improving mean estimation of sensitive variables by using auxiliary information, *Commun. Stat.-Theor. M.*, **50** (2021), 979–992. https://doi.org/10.1080/03610926.2019.1645857
5. S. Gupta, J. Shabbir, On improvement in estimating the population mean in simple random sampling, *J. Appl. Stat.*, **35** (2008), 559–566. https://doi.org/10.1080/02664760701835839
6. H. P. Singh, G. K. Vishwakarma, A general procedure for estimating the population mean in stratified sampling using auxiliary information, *Metron*, **68** (2010), 47–65. https://doi.org/10.1007/BF03263523
7. H. P. Singh, R. S. Solanki, Efficient ratio and product estimators in stratified random sampling, *Commun. Stat.-Theor. M.*, **42** (2013), 1008–1023. https://doi.org/10.1080/03610926.2011.592257
8. R. S. Solanki, H. P. Singh, An improved estimation in stratified random sampling, *Commun. Stat.-Theor. M.*, **45** (2016), 2056–2070. https://doi.org/10.1080/03610926.2013.826367
9. J. H. Schuenemeyer, L. J. Drew, *Statistics for earth and environmental scientists*, John Wiley & Sons, 2011. https://doi.org/10.1002/9780470650707
10. W. G. Cochran, *Sampling techniques*, John Wiley & Sons, 1977.

11. S. Bahl, R. Tuteja, Ratio and product type exponential estimators, *J. Inform. Optim. Sci.*, **12** (1991), 159–164. https://doi.org/10.1080/02522667.1991.10699058
12. B. V. S. Sisodia, V. K. Dwivedi, Modified ratio estimator using coefficient of variation of auxiliary variable, *J.-Indian Soc. Agr. Stat.*, 1981.
13. C. Kadilar, H. Cingi, Ratio estimators in stratified random sampling, *Biometrical J.*, **45** (2003), 218–225. https://doi.org/10.1002/bimj.200390007
14. L. N. Upadhyaya, H. P. Singh, Use of transformed auxiliary variable in estimating the finite population mean, *Biometrical J.*, **41** (1999), 627–636. https://doi.org/10.1002/(SICI)1521-4036(199909)41:5<627::AID-BIMJ627>3.0.CO;2-W
15. B. V. S. Sisodia, V. K. Dwivedi, Modified ratio estimator using coefficient of variation of auxiliary variable, *J.-Indian Soc. Agr. Stat.*, 1981.
16. M. K. Chaudhary, R. Singh, R. K. Shukla, M. Kumar, F. Smarandache, A family of estimators for estimating population mean in stratified sampling under nonresponse, *Pak. J. Stat. Oper. Res.*, 2009. 47–54. https://doi.org/10.18187/pjsor.v5i1.153
17. C. Kadilar, H. Cingi, Improvement in estimating the population mean in simple random sampling, *Appl. Math. Lett.*, **19** (2006), 75–79. https://doi.org/10.1016/j.aml.2005.02.039
18. R. Singh, Improved exponential estimator in stratified random sampling, *Pak. J. Stat. Oper. Res.*, 2009, 67–82. https://doi.org/10.18187/pjsor.v5i2.118
19. A. C. Onyeka, A class of product-type exponential estimators of the population mean in simple random sampling scheme, *Stat. Transit. New Seri.*, **14** (2013), 189–200. https://doi.org/10.59170/stattrans-2013-012
20. H. P. Singh, R. Tailor, S. Singh, J. M. Kim, A modified estimator of population mean using power transformation, *Stat. Pap.*, **49** (2008), 37–58. https://doi.org/10.1007/s00362-006-0371-2
21. Ö. Z. E. L. Gamze, Modified exponential type estimator for population mean using auxiliary variables in stratified random sampling, *Alphanumeric J.*, **3** (2015), 49–56. https://doi.org/10.17093/aj.2015.3.2.5000145484
22. N. Koyuncu, Efficient estimators of population mean using auxiliary attributes, *Appl. Math. Comput.*, **218** (2012), 10900–10905. https://doi.org/10.1016/j.amc.2012.04.050
23. N. Koyuncu, Improved exponential type estimators for finite population mean in stratified random sampling, *Pak. J. Stat. Oper. Res.*, 2016, 429–441. https://doi.org/10.18187/pjsor.v12i3.1172
24. K. K. Tiwari, S. Bhougal, S. Kumar, A general class of estimators in stratified random sampling, *Commun. Stat.-Simul. C.*, **52** (2023), 442–452. https://doi.org/10.1080/03610918.2020.1859537
25. M. Javed, M. Irfan, S. H. Bhatti, R. Onyango, A simulation-based study for progressive estimation of population mean through traditional and nontraditional measures in stratified random sampling, *J. Math.*, 2021, 1–16. https://doi.org/10.1155/2021/9038126
26. S. Muneer, J. Shabbir, A. Khalil, Estimation of finite population mean in simple random sampling and stratified random sampling using two auxiliary variables, *Commun. Stat.-Theor. M.*, **46** (2017), 2181–2192. https://doi.org/10.1080/03610926.2015.1035394
27. A. Haq, M. Usman, M. Khan, Estimation of finite population variance under stratified random sampling, *Commun. Stat.-Simul. Comput.,* **52** (2023), 6193–6209. https://doi.org/10.1080/03610918.2021.2009866
28. J. Shabbir, S. Gupta, R. Onyango, On using the conventional and nonconventional measures of the auxiliary variable for mean estimation, *Math. Probl. Eng.*, 2021, 1–13. https://doi.org/10.1155/2021/3741620

29. U. Daraz, J. Shabbir, H. Khan, Estimation of finite population mean by using minimum and maximum values in stratified random sampling, *J. Mod. Appl. Stat. Method.*, **17** (2018), 20. https://doi.org/10.22237/jmasm/1532007537
30. C. Kadilar, H. Cingi, A new ratio estimator in stratified random sampling, *Commun. Stat.-Theor. Meth.*, **34** (2005), 597–602. https://doi.org/10.1081/STA-200052156
31. S. Choudhury, B. K. Singh, An efficient class of dual to product-cum-dual to ratio estimators of finite population mean in sample surveys, *Global J. Sci. Front. Res.*, **12** (2012), 25–33.

Appendix A

a. Development of the Bias and MSE of the first family of estimators

Rewriting the first family of estimators

$$T_{pro1}=\left(S_1\bar{y}_{st}+S_2\right)\left[1\left(\frac{\bar{X}_{st}}{\bar{x}_{st}}\right)\left\{2-\exp\left[\frac{u_{st}\left(\bar{x}_{st}-\bar{X}_{st}\right)}{u_{st}\left(\bar{X}_{st}+\bar{x}_{st}\right)+2v_{st}}\right]\right\}+(1-1)\left(\frac{\bar{x}_{st}}{\bar{X}_{st}}\right)\exp\left[\frac{u_{st}\left(\bar{X}_{st}-\bar{x}_{st}\right)}{u_{st}\left(\bar{X}_{st}+\bar{x}_{st}\right)+2v_{st}}\right]\right]. \quad (4.1.1)$$

In terms of error, the estimator could be written as

$$T_{pro1}=\left(S_1\bar{Y}_{st}\left(1+\varepsilon_0\right)+S_2\right)\left[1\left(\frac{\bar{X}_{st}}{\bar{X}_{st}\left(1+\varepsilon_1\right)}\right)\left\{2-\exp\left[\frac{u_{st}\left(\bar{X}_{st}\left(1+\varepsilon_1\right)-\bar{X}_{st}\right)}{u_{st}\left(\bar{X}_{st}+\bar{X}_{st}\left(1+\varepsilon_1\right)\right)+2v_{st}}\right]\right\}+(1-1)\left(\frac{\bar{X}_{st}\left(1+\varepsilon_1\right)}{\bar{X}_{st}}\right)\exp\left[\frac{u_{st}\left(\bar{X}_{st}-\bar{X}_{st}\left(1+\varepsilon_1\right)\right)}{u_{st}\left(\bar{X}_{st}+\bar{X}_{st}\left(1+\varepsilon_1\right)\right)+2v_{st}}\right]\right]$$

$$T_{pro1}=\left(S_1\bar{Y}_{st}\left(1+\varepsilon_0\right)+S_2\right)\left[1\left(1+\varepsilon_1\right)^{-1}\left\{2-\exp\left[\frac{u_{st}\bar{X}_{st}\varepsilon_1}{u_{st}\left(2\bar{X}_{st}+\bar{X}_{st}\varepsilon_1\right)+2v_{st}}\right]\right\}+(1-1)\left(1+\varepsilon_1\right)\exp\left[\frac{-u_{st}\bar{X}_{st}\varepsilon_1}{u_{st}\left(2\bar{X}_{st}+\bar{X}_{st}\varepsilon_1\right)+2v_{st}}\right]\right]$$

$$T_{pro1}=\left(S_1\bar{Y}_{st}\left(1+\varepsilon_0\right)+S_2\right)\left[1\left(1+\varepsilon_1\right)^{-1}\left\{2-\exp\left[\frac{\eta\varepsilon_1}{2}\left(1+\frac{\eta\varepsilon_1}{2}\right)^{-1}\right]\right\}+(1-1)\left(1+\varepsilon_1\right)\exp\left[-\frac{\eta\varepsilon_1}{2}\left(1+\frac{\eta\varepsilon_1}{2}\right)^{-1}\right]\right].$$

Expanding the above Taylor series up to first order of approximation, we have

$$T_{pro1}=\left(S_1\bar{Y}_{st}\left(1+\varepsilon_0\right)+S_2\right)\left[1\left(1+\varepsilon_1\right)^{-1}\left\{2-\exp\left[\frac{\eta\varepsilon_1}{2}\left(1-\frac{\eta\varepsilon_1}{2}+\frac{\eta^2\varepsilon_1^2}{4}-...\right)\right]\right\}+(1-1)\left(1+\varepsilon_1\right)\exp\left[-\frac{\eta\varepsilon_1}{2}\left(1-\frac{\eta\varepsilon_1}{2}+\frac{\eta^2\varepsilon_1^2}{4}-...\right)\right]\right]$$

$$T_{pro1}=\left(S_1\bar{Y}_{st}\left(1+\varepsilon_0\right)+S_2\right)\left[1\left(1+\varepsilon_1\right)^{-1}\left\{2-\exp\left[\frac{\eta\varepsilon_1}{2}-\frac{\eta^2\varepsilon_1^2}{4}+...\right]\right\}+(1-1)\left(1+\varepsilon_1\right)\exp\left[-\frac{\eta\varepsilon_1}{2}+\frac{\eta^2\varepsilon_1^2}{4}-...\right]\right].$$

After applying the exponential series, we obtain the below expression

$$T_{pro1}=\left(S_1\bar{Y}_{st}\left(1+\varepsilon_0\right)+S_2\right)\left[1\left(1-\varepsilon_1+\varepsilon_1^2-...\right)\left\{2-\left(1+\frac{\eta\varepsilon_1}{2}-\frac{\eta^2\varepsilon_1^2}{8}+...\right)\right\}+(1-1)\left(1+\varepsilon_1\right)\left(1-\frac{\eta\varepsilon_1}{2}+\frac{3\eta^2\varepsilon_1^2}{8}+...\right)\right]$$

$$T_{pro1}=\left(S_1\bar{Y}_{st}\left(1+\varepsilon_0\right)+S_2\right)\left[1\left(1-\varepsilon_1+\varepsilon_1^2-...\right)\left(1-\frac{\eta\varepsilon_1}{2}+\frac{\eta^2\varepsilon_1^2}{8}+...\right)+(1-1)\left(1+\varepsilon_1\right)\left(1-\frac{\eta\varepsilon_1}{2}+\frac{3\eta^2\varepsilon_1^2}{8}+...\right)\right]$$

$$T_{pro1}=\left(S_1\bar{Y}_{st}\left(1+\varepsilon_0\right)+S_2\right)\left[1\left(1-\varepsilon_1-\frac{\eta\varepsilon_1}{2}+\varepsilon_1^2+\frac{\eta\varepsilon_1^2}{2}+\frac{\eta^2\varepsilon_1^2}{8}\right)+(1-1)\left(1+\varepsilon_1-\frac{\eta\varepsilon_1}{2}-\frac{\eta\varepsilon_1^2}{2}+\frac{3\eta^2\varepsilon_1^2}{8}\right)\right]$$

$$T_{pro1}=\left(S_1\bar{Y}_{st}\left(1+\varepsilon_0\right)+S_2\right)\left[1+\left(1-\frac{\eta}{2}-21\right)\varepsilon_1-\left\{1+\left(1-\frac{1}{2}\right)\eta+\frac{\eta^2}{8}(3-21)\right\}\varepsilon_1^2\right]$$

$$T_{pro1}=\left(S_1\bar{Y}_{st}\left(1+\varepsilon_0\right)+S_2\right)\left[1+\vartheta_1\varepsilon_1-\vartheta_2\varepsilon_1^2\right].$$

By subtracting $\bar{Y}_{st}$ from both sides, we have

$$T_{pro1}-\bar{Y}_{st}=S_1\bar{Y}_{st}\left(1+\varepsilon_0+\vartheta_1\varepsilon_1+\vartheta_1\varepsilon_0\varepsilon_1-\vartheta_2\varepsilon_1^2\right)+S_2\left(1+\vartheta_1\varepsilon_1-\vartheta_2\varepsilon_1^2\right)-\bar{Y}_{st}.$$

When we apply expectation to both sides of the previous equation, we get the following bias expression:

$$Bias\left(T_{pro1}\right)=S_1\bar{Y}_{st}\left(1+\vartheta_1V_{11}-\vartheta_2V_{02}\right)+S_2\left(1-\vartheta_2V_{02}\right)-\bar{Y}_{st}. \quad (4.1.6)$$

To obtain the MSE expression, we take the square of both sides of the equation,

$$\left(T_{pro1}-\bar{Y}_{st}\right)^2=S_1^2\bar{Y}_{st}^2\left(1+\varepsilon_0+\vartheta_1\varepsilon_1+\vartheta_1\varepsilon_0\varepsilon_1-\vartheta_2\varepsilon_1^2\right)^2+S_2^2\left(1+\vartheta_1\varepsilon_1-\vartheta_2\varepsilon_1^2\right)^2+\bar{Y}_{st}^2+2S_1S_2\bar{Y}_{st}\left(1+\varepsilon_0+\vartheta_1\varepsilon_1+\vartheta_1\varepsilon_0\varepsilon_1-\vartheta_2\varepsilon_1^2\right)\left(1+\vartheta_1\varepsilon_1-\vartheta_2\varepsilon_1^2\right)$$
$$-2S_1\bar{Y}_{st}^2\left(1+\varepsilon_0+\vartheta_1\varepsilon_1+\vartheta_1\varepsilon_0\varepsilon_1-\vartheta_2\varepsilon_1^2\right)-2S_2\bar{Y}_{st}\left(1+\vartheta_1\varepsilon_1-\vartheta_2\varepsilon_1^2\right)$$

$$\begin{aligned}\left(T_{pro1}-\bar{Y}_{st}\right)^2=&\,S_1^2\bar{Y}_{st}^2\left(1+\varepsilon_0^2+\vartheta_1^2\varepsilon_1^2+2\varepsilon_0+2\vartheta_1\varepsilon_1+2\vartheta_1\varepsilon_0\varepsilon_1-2\vartheta_2\varepsilon_1^2+2\vartheta_1\varepsilon_0\varepsilon_1\right)+S_2^2\left(1+\vartheta_1^2\varepsilon_1^2+2\vartheta_1\varepsilon_1-2\vartheta_2\varepsilon_1^2\right)+\bar{Y}_{st}^2\\&+2S_1S_2\bar{Y}_{st}\left(1+\varepsilon_0+\vartheta_1\varepsilon_1+\vartheta_1\varepsilon_0\varepsilon_1-\vartheta_2\varepsilon_1^2+\vartheta_1\varepsilon_1+\vartheta_1\varepsilon_0\varepsilon_1+\vartheta_1^2\varepsilon_1^2-\vartheta_2\varepsilon_1^2\right)\\&-2S_1\bar{Y}_{st}^2\left(1+\varepsilon_0+\vartheta_1\varepsilon_1+\vartheta_1\varepsilon_0\varepsilon_1-\vartheta_2\varepsilon_1^2\right)-2S_2\bar{Y}_{st}\left(1+\vartheta_1\varepsilon_1-\vartheta_2\varepsilon_1^2\right)\end{aligned}$$

$$\begin{aligned}\left(T_{pro1}-\bar{Y}_{st}\right)^2=&\,S_1^2\bar{Y}_{st}^2\left(1+2\varepsilon_0+2\vartheta_1\varepsilon_1+\varepsilon_0^2+\left(\vartheta_1^2-2\vartheta_2\right)\varepsilon_1^2+4\vartheta_1\varepsilon_0\varepsilon_1\right)+S_2^2\left(1+2\vartheta_1\varepsilon_1+\left(\vartheta_1^2-2\vartheta_2\right)\varepsilon_1^2\right)\\&+\bar{Y}_{st}^2-2S_1\bar{Y}_{st}^2\left(1+\varepsilon_0+\vartheta_1\varepsilon_1-\vartheta_2\varepsilon_1^2+\vartheta_1\varepsilon_0\varepsilon_1\right)-2S_2\bar{Y}_{st}\left(1+\vartheta_1\varepsilon_1-\vartheta_2\varepsilon_1^2\right)\\&+2S_1S_2\bar{Y}_{st}\left(1+\varepsilon_0+2\vartheta_1\varepsilon_1+\left(\vartheta_1^2-2\vartheta_2\right)\varepsilon_1^2+2\vartheta_1\varepsilon_0\varepsilon_1\right)\end{aligned}. \quad (4.1.7)$$

After taking expectation, the MSE expression is obtained as:

$$\begin{aligned}MSE\left(T_{pro1}\right)=&\,S_1^2\bar{Y}_{st}^2\left(1+V_{20}+\left(\vartheta_1^2-2\vartheta_2\right)V_{02}+4\vartheta_1V_{11}\right)+S_2^2\left(1+\left(\vartheta_1^2-2\vartheta_2\right)V_{02}\right)\\&+\bar{Y}_{st}^2-2S_1\bar{Y}_{st}^2\left(1-\vartheta_2V_{02}+\vartheta_1V_{11}\right)-2S_2\bar{Y}_{st}\left(1-\vartheta_2V_{02}\right)\\&+2S_1S_2\bar{Y}_{st}\left(1+\left(\vartheta_1^2-2\vartheta_2\right)V_{02}+2\vartheta_1V_{11}\right)\end{aligned}. \quad (4.1.8)$$

Or

$$MSE\left(T_{pro1}\right)=S_1^2\bar{Y}_{st}^2A_{pr}+S_2^2B_{pr}+\bar{Y}_{st}^2-2S_1\bar{Y}_{st}^2C_{pr}-2S_2\bar{Y}_{st}D_{pr}+2S_1S_2\bar{Y}_{st}E_{pr}. \quad (4.1.9)$$

Here, $A_{pr}=1+V_{20}+\left(\vartheta_1^2-2\vartheta_2\right)V_{02}+4\vartheta_1V_{11}$, $B_{pr}=1+\left(\vartheta_1^2-2\vartheta_2\right)V_{02}$, $C_{pr}=1-\vartheta_2V_{02}+\vartheta_1V_{11}$, $D_{pr}=1-\vartheta_2V_{02}$ and $E_{pr}=1+\left(\vartheta_1^2-2\vartheta_2\right)V_{02}+2\vartheta_1V_{11}$.

Now, let us differentiate the *MSE* equation to obtain the values of S_1 and S_2 to have minimum *MSE.*

$$\frac{\partial MSE\left(T_{pro1}\right)}{\partial S_1}=0 \Rightarrow 2S_1\bar{Y}_{st}^2A_{pr}+2S_2\bar{Y}_{st}E_{pr}-2\bar{Y}_{st}^2C_{pr}=0$$

$$S_1\bar{Y}_{st}^2A_{pr}+S_2\bar{Y}_{st}E_{pr}-\bar{Y}_{st}^2C_{pr}=0 \quad (4.1.10)$$

$$\frac{\partial MSE\left(T_{pro1}\right)}{\partial S_2}=0 \Rightarrow 2S_2B_{pr}+2S_1\bar{Y}_{st}E_{pr}-2\bar{Y}_{st}D_{pr}=0$$

$$S_1\bar{Y}_{st}E_{pr}+S_2B_{pr}-\bar{Y}_{st}D_{pr}=0. \quad (4.1.11)$$

Solving Eq (4.1.10) for *S1,* we have

$$S_1 = \frac{-S_2 \bar{Y}_{st} E_{pr} + \bar{Y}_{st}^2 C_{pr}}{\bar{Y}_{st}^2 A_{pr}}. \tag{4.1.1a}$$

Solving Eq (4.1.11) for S_2, we have

$$S_2 = \frac{\bar{Y}_{st} D_{pr} - S_1 \bar{Y}_{st} E_{pr}}{B_{pr}}. \tag{4.1.2a}$$

By putting Eq (4.1.2a) in Eq (4.1.1a) we have

$$S_1 = \frac{-\left(\bar{Y}_{st} D_{pr} - S_1 \bar{Y}_{st} E_{pr}\right) \bar{Y}_{st} E_{pr} + \bar{Y}_{st}^2 B_{pr} C_{pr}}{\bar{Y}_{st}^2 A_{pr} B_{pr}}$$

$$S_1 = \frac{S_1 \bar{Y}_{st}^2 E_{pr}^2 - \bar{Y}_{st}^2 D_{pr} E_{pr} + \bar{Y}_{st}^2 B_{pr} C_{pr}}{\bar{Y}_{st}^2 A_{pr} B_{pr}}$$

$$S_1 = \frac{S_1 E_{pr}^2 + B_{pr} C_{pr} - D_{pr} E_{pr}}{A_{pr} B_{pr}}$$

$$S_1 - \frac{S_1 E_{pr}^2}{A_{pr} B_{pr}} = \frac{B_{pr} C_{pr} - D_{pr} E_{pr}}{A_{pr} B_{pr}}$$

$$\frac{S_1 \left(A_{pr} B_{pr} - E_{pr}^2\right)}{A_{pr} B_{pr}} = \frac{B_{pr} C_{pr} - D_{pr} E_{pr}}{A_{pr} B_{pr}}$$

$$S_1 = \frac{B_{pr} C_{pr} - D_{pr} E_{pr}}{A_{pr} B_{pr} - E_{pr}^2}. \tag{4.1.3a}$$

Now, to obtain a value for S2, we put the value from Eq (4.1.3a) in (4.1.2a)

$$S_2 = \frac{\bar{Y}_{st} D_{pr} - \left(\frac{B_{pr} C_{pr} - D_{pr} E_{pr}}{A_{pr} B_{pr} - E_{pr}^2}\right) \bar{Y}_{st} E_{pr}}{B_{pr}}$$

$$S_2 = \frac{\bar{Y}_{st} A_{pr} B_{pr} D_{pr} - \bar{Y}_{st} D_{pr} E_{pr}^2 - \bar{Y}_{st} B_{pr} C_{pr} E_{pr} + \bar{Y}_{st} D_{pr} E_{pr}^2}{B_{pr}\left(A_{pr} B_{pr} - E_{pr}^2\right)}$$

$$S_2 = \frac{\bar{Y}_{st}\left(A_{pr} D_{pr} - C_{pr} E_{pr}\right)}{A_{pr} B_{pr} - E_{pr}^2}. \tag{4.1.4a}$$

With these values from Eqs (4.1.3a) and (4.1.4a), the minimum *MSE* adopts the below form,

$$MSE\left(T_{pro1}\right) = \left(\frac{B_{pr} C_{pr} - D_{pr} E_{pr}}{A_{pr} B_{pr} - E_{pr}^2}\right)^2 \bar{Y}_{st}^2 A_{pr} + \left(\frac{\bar{Y}_{st}\left(A_{pr} D_{pr} - C_{pr} E_{pr}\right)}{A_{pr} B_{pr} - E_{pr}^2}\right)^2 B_{pr} + \bar{Y}_{st}^2 - 2\left(\frac{B_{pr} C_{pr} - D_{pr} E_{pr}}{A_{pr} B_{pr} - E_{pr}^2}\right) \bar{Y}_{st}^2 C_{pr}$$
$$-2\left(\frac{\bar{Y}_{st}\left(A_{pr} D_{pr} - C_{pr} E_{pr}\right)}{A_{pr} B_{pr} - E_{pr}^2}\right) \bar{Y}_{st} D_{pr} + 2\left(\frac{B_{pr} C_{pr} - D_{pr} E_{pr}}{A_{pr} B_{pr} - E_{pr}^2}\right)\left(\frac{\bar{Y}_{st}\left(A_{pr} D_{pr} - C_{pr} E_{pr}\right)}{A_{pr} B_{pr} - E_{pr}^2}\right) \bar{Y}_{st} E_{pr}$$

$$MSE\left(T_{pro1}\right)=\bar{Y}_{st}^{2}\left[1+\frac{\begin{array}{l}A_{pr}B_{pr}^{2}C_{pr}^{2}+A_{pr}D_{pr}^{2}E_{pr}^{2}-2A_{pr}B_{pr}C_{pr}D_{pr}E_{pr}+A_{pr}^{2}B_{pr}D_{pr}^{2}+B_{pr}C_{pr}^{2}E_{pr}^{2}-2A_{pr}B_{pr}C_{pr}D_{pr}E_{pr}\\-2A_{pr}B_{pr}^{2}C_{pr}^{2}+2A_{pr}B_{pr}C_{pr}D_{pr}E_{pr}+2B_{pr}C_{pr}^{2}E_{pr}^{2}-2C_{pr}D_{pr}E_{pr}^{3}-2A_{pr}^{2}B_{pr}D_{pr}^{2}+2A_{pr}B_{pr}C_{pr}D_{pr}E_{pr}\\+2A_{pr}D_{pr}^{2}E_{pr}^{2}-2C_{pr}D_{pr}E_{pr}^{3}+2A_{pr}B_{pr}C_{pr}D_{pr}E_{pr}-2B_{pr}C_{pr}^{2}E_{pr}^{2}-2A_{pr}D_{pr}^{2}E_{pr}^{2}+2C_{pr}D_{pr}E_{pr}^{3}\end{array}}{\left(A_{pr}B_{pr}-E_{pr}^{2}\right)^{2}}\right]$$

$$MSE\left(T_{pro1}\right)=\bar{Y}_{st}^{2}\left[1+\frac{-A_{pr}B_{pr}^{2}C_{pr}^{2}+A_{pr}D_{pr}^{2}E_{pr}^{2}-A_{pr}^{2}B_{pr}D_{pr}^{2}+B_{pr}C_{pr}^{2}E_{pr}^{2}-2C_{pr}D_{pr}E_{pr}^{3}+2A_{pr}B_{pr}C_{pr}D_{pr}E_{pr}}{\left(A_{pr}B_{pr}-E_{pr}^{2}\right)^{2}}\right]$$

$$MSE\left(T_{pro1}\right)=\bar{Y}_{st}^{2}\left[1+\frac{-A_{pr}B_{pr}\left(A_{pr}D_{pr}^{2}+B_{pr}C_{pr}^{2}-2C_{pr}D_{pr}E_{pr}\right)+E_{pr}^{2}\left(A_{pr}D_{pr}^{2}+B_{pr}C_{pr}^{2}-2C_{pr}D_{pr}E_{pr}\right)}{\left(A_{pr}B_{pr}-E_{pr}^{2}\right)^{2}}\right]$$

$$MSE\left(T_{pro1}\right)=\bar{Y}_{st}^{2}\left[1+\frac{-\left(A_{pr}B_{pr}-E_{pr}^{2}\right)\left(A_{pr}D_{pr}^{2}+B_{pr}C_{pr}^{2}-2C_{pr}D_{pr}E_{pr}\right)}{\left(A_{pr}B_{pr}-E_{pr}^{2}\right)^{2}}\right]$$

$$MSE\left(T_{pro1}\right)_{\min}\cong\bar{Y}_{st}^{2}\left\{1-\frac{A_{pr}D_{pr}^{2}+B_{pr}C_{pr}^{2}-2C_{pr}D_{pr}E_{pr}}{A_{pr}B_{pr}-E_{pr}^{2}}\right\}. \quad (4.1.12)$$

b. Development of the Bias and MSE of the second family of estimators

Rewriting the Eq (4.2.1), we have,

$$T_{st(Pro)}=\left(T_{1}\bar{y}_{st}+T_{2}\right)\left[1\left(\frac{\bar{X}_{st}}{\bar{x}_{st}}\right)+\left(1-1\right)\left(\frac{\bar{x}_{st}}{\bar{X}_{st}}\right)\right]\exp\left[\frac{u_{st}\left(\bar{X}_{st}-\bar{x}_{st}\right)}{u_{st}\left(\bar{X}_{st}+\bar{x}_{st}\right)+2v_{st}}\right]. \quad (4.2.1)$$

In terms of errors, the above equation could be written as:

$$T_{pro2}=\left(T_{1}\bar{Y}_{st}\left(1+\varepsilon_{0}\right)+T_{2}\right)\left[1\left(\frac{\bar{X}_{st}}{\bar{X}_{st}\left(1+\varepsilon_{1}\right)}\right)+\left(1-1\right)\left(\frac{\bar{X}_{st}\left(1+\varepsilon_{1}\right)}{\bar{X}_{st}}\right)\right]\exp\left[\frac{u_{st}\left(\bar{X}_{st}-\bar{X}_{st}\left(1+\varepsilon_{1}\right)\right)}{u_{st}\left(\bar{X}_{st}+\bar{X}_{st}\left(1+\varepsilon_{1}\right)\right)+2v_{st}}\right]$$

$$T_{pro2}=\left(T_{1}\bar{Y}_{st}\left(1+\varepsilon_{0}\right)+T_{2}\right)\left[1\left(1+\varepsilon_{1}\right)^{-1}+\left(1-1\right)\left(1+\varepsilon_{1}\right)\right]\exp\left[\frac{-u_{st}\bar{X}_{st}\varepsilon_{1}}{u_{st}\left(2\bar{X}_{st}+\bar{X}_{st}\varepsilon_{1}\right)+2v_{st}}\right]$$

$$T_{pro2}=\left(T_{1}\bar{Y}_{st}\left(1+\varepsilon_{0}\right)+T_{2}\right)\left[1\left(1+\varepsilon_{1}\right)^{-1}+\left(1-1\right)\left(1+\varepsilon_{1}\right)\right]\exp\left[-\frac{\eta\varepsilon_{1}}{2}\left(1+\frac{\eta\varepsilon_{1}}{2}\right)^{-1}\right]. \quad (4.2.1b)$$

Expanding the above Taylor series up to first order of approximation, we have:

$$T_{pro2}=\left(T_{1}\bar{Y}_{st}\left(1+\varepsilon_{0}\right)+T_{2}\right)\left[1\left(1+\varepsilon_{1}\right)^{-1}+\left(1-1\right)\left(1+\varepsilon_{1}\right)\right]\exp\left[-\frac{\eta\varepsilon_{1}}{2}\left(1-\frac{\eta\varepsilon_{1}}{2}+\frac{\eta^{2}\varepsilon_{1}^{2}}{4}-...\right)\right]$$

$$T_{pro2}=\left(T_{1}\bar{Y}_{st}\left(1+\varepsilon_{0}\right)+T_{2}\right)\left[1\left(1+\varepsilon_{1}\right)^{-1}+\left(1-1\right)\left(1+\varepsilon_{1}\right)\right]\exp\left[-\frac{\eta\varepsilon_{1}}{2}+\frac{\eta^{2}\varepsilon_{1}^{2}}{4}-...\right]. \quad (4.2.2b)$$

After applying exponential series, we obtain the below expression:

$$T_{pro2}=\left(T_{1}\bar{Y}_{st}\left(1+\varepsilon_{0}\right)+T_{2}\right)\left[1\left(1-\varepsilon_{1}+\varepsilon_{1}^{2}-...\right)+\left(1-1\right)\left(1+\varepsilon_{1}\right)\right]\left(1-\frac{\eta\varepsilon_{1}}{2}+\frac{3\eta^{2}\varepsilon_{1}^{2}}{8}+...\right)$$

$$T_{pro2}=\left(T_1\overline{Y}_{st}\left(1+\varepsilon_0\right)+T_2\right)\left[1+\varepsilon_1-2l\,\varepsilon_1+1l\,\varepsilon_1^2\right]\left(1-\frac{\eta\varepsilon_1}{2}+\frac{3\eta^2\varepsilon_1^2}{8}+...\right)$$

$$T_{pro2}=\left(T_1\overline{Y}_{st}\left(1+\varepsilon_0\right)+T_2\right)\left[1+\varepsilon_1-2l\,\varepsilon_1+1l\,\varepsilon_1^2-\frac{\eta\varepsilon_1}{2}-\frac{\eta\varepsilon_1^2}{2}+\eta l\,\varepsilon_1^2+\frac{3\eta^2\varepsilon_1^2}{8}\right]$$

$$T_{pro2}=\left(T_1\overline{Y}_{st}\left(1+\varepsilon_0\right)+T_2\right)\left[1-\left(\frac{\eta}{2}+2l-1\right)\varepsilon_1+\left\{1+\frac{\eta}{2}(2l-1)+\frac{3\eta^2}{8}\right\}\varepsilon_1^2\right]$$

$$T_{pro2}=\left[T_1\overline{Y}_{st}+T_1\overline{Y}_{st}\left(\varepsilon_0-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2-\delta_1\varepsilon_0\varepsilon_1\right)+T_2\left(1-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2\right)\right]. \tag{4.2.3b}$$

The difference equation up-to first order of approximation of the proposed estimator in terms of errors is expressed as

$$T_{pro2}-\overline{Y}_{st}=\left[\left(T_1-1\right)\overline{Y}_{st}+T_1\overline{Y}_{st}\left(\varepsilon_0-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2-\delta_1\varepsilon_0\varepsilon_1\right)+T_2\left(1-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2\right)\right]. \tag{4.2.2}$$

After taking the expectation, the bias of the suggested estimator is given as:

$$Bias\left(T_{pro2}\right)=(T_1-1)\overline{Y}_{st}+T_1\overline{Y}_{st}(\delta_2V_{02}-\delta_1V_{11})+T_2(1+\delta_2V_{02}) \tag{4.2.3}$$

where $\delta_1=\frac{1}{2}\eta+2l-1$ and $\delta_2=1+\frac{1}{2}\eta\left(2l-1\right)+\frac{3}{8}\eta^2$.

Squaring both sides of the above (49) difference equation and using first order of approximation, we have,

$$\left(T_{pro2}-\overline{Y}_{st}\right)^2=\begin{bmatrix}\left(T_1-1\right)^2\overline{Y}_{st}^2+T_1^2\overline{Y}_{st}^2\left(\varepsilon_0-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2-\delta_1\varepsilon_0\varepsilon_1\right)^2+T_2^2\left(1-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2\right)^2\\+2T_1\left(T_1-1\right)\overline{Y}_{st}^2\left(\varepsilon_0-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2-\delta_1\varepsilon_0\varepsilon_1\right)+2T_2\left(T_1-1\right)\overline{Y}_{st}\left(1-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2\right)\\+2T_1T_2\overline{Y}_{st}\left(\varepsilon_0-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2-\delta_1\varepsilon_0\varepsilon_1\right)\left(1-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2\right)\end{bmatrix}$$

$$\left(T_{pro2}-\overline{Y}_{st}\right)^2=\begin{bmatrix}\left(T_1-1\right)^2\overline{Y}_{st}^2+T_1^2\overline{Y}_{st}^2\left(\varepsilon_0^2+\delta_1^2\varepsilon_1^2-2\delta_1\varepsilon_0\varepsilon_1\right)\\+T_2^2\left(1+\delta_1^2\varepsilon_1^2-2\delta_1\varepsilon_1+2\delta_2\varepsilon_1^2\right)+2\left(T_1^2-T_1\right)\overline{Y}_{st}^2\left(\varepsilon_0-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2-\delta_1\varepsilon_0\varepsilon_1\right)\\+2\left(T_1T_2-T_2\right)\overline{Y}_{st}\left(1-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2\right)+2T_1T_2\overline{Y}_{st}\left(\varepsilon_0-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2-\delta_1\varepsilon_0\varepsilon_1-\delta_1\varepsilon_0\varepsilon_1+\delta_1^2\varepsilon_1^2\right)\end{bmatrix}$$

$$\left(T_{pro2}-\overline{Y}_{st}\right)^2=\begin{bmatrix}\left(T_1-1\right)^2\overline{Y}_{st}^2+T_1^2\overline{Y}_{st}^2\left(2\varepsilon_0-2\delta_1\varepsilon_1+\varepsilon_0^2+2\delta_2\varepsilon_1^2+\delta_1^2\varepsilon_1^2-2\delta_1\varepsilon_0\varepsilon_1-2\delta_1\varepsilon_0\varepsilon_1\right)\\+T_2^2\left(1+\delta_1^2\varepsilon_1^2-2\delta_1\varepsilon_1+2\delta_2\varepsilon_1^2\right)-2T_1\overline{Y}_{st}^2\left(\varepsilon_0-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2-\delta_1\varepsilon_0\varepsilon_1\right)\\-2T_2\overline{Y}_{st}\left(1-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2\right)+2T_1T_2\overline{Y}_{st}\left(1+\varepsilon_0-\delta_1\varepsilon_1-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2+\delta_2\varepsilon_1^2-\delta_1\varepsilon_0\varepsilon_1-\delta_1\varepsilon_0\varepsilon_1+\delta_1^2\varepsilon_1^2\right)\end{bmatrix}$$

$$\left(T_{pro2}-\overline{Y}_{st}\right)^2=\begin{bmatrix}\left(T_1-1\right)^2\overline{Y}_{st}^2+T_1^2\overline{Y}_{st}^2\left(2\varepsilon_0-2\delta_1\varepsilon_1+\varepsilon_0^2+\left(\delta_1^2+2\delta_2\right)\varepsilon_1^2-4\delta_1\varepsilon_0\varepsilon_1\right)\\+T_2^2\left(1-2\delta_1\varepsilon_1+\left(\delta_1^2+2\delta_2\right)\varepsilon_1^2\right)-2T_1\overline{Y}_{st}^2\left(\varepsilon_0-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2-\delta_1\varepsilon_0\varepsilon_1\right)\\-2T_2\overline{Y}_{st}\left(1-\delta_1\varepsilon_1+\delta_2\varepsilon_1^2\right)+2T_1T_2\overline{Y}_{st}\left(1+\varepsilon_0-2\delta_1\varepsilon_1+\left(\delta_1^2+2\delta_2\right)\varepsilon_1^2-2\delta_1\varepsilon_0\varepsilon_1\right)\end{bmatrix} \tag{4.2.4b}$$

$$E\left(T_{pro}-\overline{Y}_{st}\right)^{2}=\begin{bmatrix}\left(T_{1}-1\right)^{2}\overline{Y}_{st}^{2}+T_{1}^{2}\overline{Y}_{st}^{2}\left(V_{20}+\left(\delta_{1}^{2}+2\delta_{2}\right)V_{02}-4\delta_{1}V_{11}\right)\\+T_{2}^{2}\left(1+\left(\delta_{1}^{2}+2\delta_{2}\right)V_{02}\right)-2T_{1}\overline{Y}_{st}^{2}\left(\delta_{2}V_{02}-\delta_{1}V_{11}\right)\\-2T_{2}\overline{Y}_{st}\left(1+\delta_{2}V_{02}\right)+2T_{1}T_{2}\overline{Y}_{st}\left(1+\left(\delta_{1}^{2}+2\delta_{2}\right)V_{02}-2\delta_{1}V_{11}\right)\end{bmatrix} \tag{4.2.4}$$

or

$$MSE\left(T_{pro2}\right)=\begin{bmatrix}\left(T_{1}-1\right)^{2}\overline{Y}_{st}^{2}+T_{1}^{2}\overline{Y}_{st}^{2}\left(V_{20}+\left(\delta_{1}^{2}+2\delta_{2}\right)V_{02}-4\delta_{1}V_{11}\right)\\+T_{2}^{2}\left(1+\left(\delta_{1}^{2}+2\delta_{2}\right)V_{02}\right)-2T_{1}\overline{Y}_{st}^{2}\left(\delta_{2}V_{02}-\delta_{1}V_{11}\right)\\-2T_{2}\overline{Y}_{st}\left(1+\delta_{2}V_{02}\right)+2T_{1}T_{2}\overline{Y}_{st}\left(1+\left(\delta_{1}^{2}+2\delta_{2}\right)V_{02}-2\delta_{1}V_{11}\right)\end{bmatrix} \tag{4.2.5}$$

or

$$MSE\left(T_{pro2}\right)=\left[\left(T_{1}-1\right)^{2}\overline{Y}_{st}^{2}+T_{1}^{2}\overline{Y}_{st}^{2}A_{p}+T_{2}^{2}B_{p}-2T_{1}\overline{Y}_{st}^{2}C_{p}-2T_{2}\overline{Y}_{st}D_{p}+2T_{1}T_{2}\overline{Y}_{st}E_{p}\right] \tag{4.2.6}$$

where $A_{p}=\left(V_{20}+\left(\delta_{1}^{2}+2\delta_{2}\right)V_{02}-4\delta_{1}V_{11}\right)$, $B_{p}=\left(1+\left(\delta_{1}^{2}+2\delta_{2}\right)V_{02}\right)$, $C_{p}=\left(\delta_{2}V_{02}-\delta_{1}V_{11}\right)$, $D_{p}=\left(1+\delta_{2}V_{02}\right)$ and $E_{p}=\left(1+\left(\delta_{1}^{2}+2\delta_{2}\right)V_{02}-2\delta_{1}V_{11}\right)$.

To obtain the values of T_1 and T_2, we differentiate the Eq (4.2.6) w.r.t as below:

$$\frac{\partial MSE}{\partial T_{1}}=0\Rightarrow\quad 2\left(T_{1}-1\right)\overline{Y}_{st}^{2}+2T_{1}\overline{Y}_{st}^{2}A_{p}-2\overline{Y}_{2}^{2}C_{p}+2T_{2}\overline{Y}_{st}E_{p}=0$$

$$\frac{\partial MSE}{\partial T_{2}}=0\Rightarrow\quad 2T_{2}B_{p}-2\overline{Y}_{st}D_{p}+2T_{1}\overline{Y}_{st}E_{p}=0$$

$$\left(T_{1}-1\right)\overline{Y}_{st}+T_{1}\overline{Y}_{st}A_{p}-\overline{Y}_{st}C_{p}+T_{2}E_{p}=0$$

$$T_{2}B_{p}-\overline{Y}_{st}D_{p}+T_{1}\overline{Y}_{st}E_{p}=0$$

$$T_{1}=\frac{\overline{Y}_{st}+\overline{Y}_{st}C_{p}-T_{2}E_{p}}{\left(1+A_{p}\right)\overline{Y}_{st}} \tag{4.2.5b}$$

$$T_{2}=\frac{\overline{Y}_{st}D_{p}-T_{1}\overline{Y}_{st}E_{p}}{B_{p}}. \tag{4.2.6b}$$

Putting Eq (4.2.6b) in (4.2.5b), we have,

$$T_{1}=\frac{\overline{Y}_{st}+\overline{Y}_{st}C_{p}-\left(\dfrac{\overline{Y}_{st}D_{p}-T_{1}\overline{Y}_{st}E_{p}}{B_{p}}\right)E_{p}}{\left(1+A_{p}\right)\overline{Y}_{st}}$$

$$T_{1}=\frac{\overline{Y}_{st}B_{p}+\overline{Y}_{st}B_{p}C_{p}-\overline{Y}_{st}D_{p}E_{p}+T_{1}\overline{Y}_{st}E_{p}^{2}}{\left(1+A_{p}\right)B_{p}\overline{Y}_{st}}$$

$$\frac{T_1\left(A_pB_p+B_p-E_p^2\right)}{A_pB_p+B_p}=\frac{B_p+B_pC_p-D_pE_p}{A_pB_p+B_p}$$

$$T_1=\frac{B_pC_p-D_pE_p+B_p}{A_pB_p-E_p^2+B_p}. \tag{4.2.7b}$$

With this value of T_1, Eq (4.2.6b) adopts the following form

$$T_2=\frac{\overline{Y}_{st}D_p-\left(\frac{B_pC_p-D_pE_p+B_p}{A_pB_p-E_p^2+B_p}\right)\overline{Y}_{st}E_p}{B_p}$$

$$T_2=\frac{\overline{Y}_{st}\left(A_pB_pD_p-D_pE_p^2+D_pB_p-B_pC_pE_p+D_pE_p^2-B_pE_p\right)}{B_p\left(A_pB_p-E_p^2+B_p\right)}$$

$$T_2=\frac{\overline{Y}_{st}\left(A_pD_p-C_pE_p+D_p-E_p\right)}{A_pB_p-E_p^2+B_p}. \tag{4.2.8b}$$

The least possible value of the MSE is obtained by utilizing Eqs (4.2.7b) and (4.2.8b)

$$MSE\left(T_{pro2}\right)=\left[\begin{array}{l}\left(\frac{B_pC_p-D_pE_p+B_p}{A_pB_p-E_p^2+B_p}-1\right)^2\overline{Y}_{st}^2+\left(\frac{B_pC_p-D_pE_p+B_p}{A_pB_p-E_p^2+B_p}\right)^2\overline{Y}_{st}^2A_p+\left(\frac{\overline{Y}_{st}\left(A_pD_p-C_pE_p+D_p-E_p\right)}{A_pB_p-E_p^2+B_p}\right)^2B_p\\ -2\left(\frac{B_pC_p-D_pE_p+B_p}{A_pB_p-E_p^2+B_p}\right)\overline{Y}_{st}^2C_p-2\left(\frac{\overline{Y}_{st}\left(A_pD_p-C_pE_p+D_p-E_p\right)}{A_pB_p-E_p^2+B_p}\right)\overline{Y}_{st}D_p\\ +2\left(\frac{B_pC_p-D_pE_p+B_p}{A_pB_p-E_p^2+B_p}\right)\left(\frac{\overline{Y}_{st}\left(A_pD_p-C_pE_p+D_p-E_p\right)}{A_pB_p-E_p^2+B_p}\right)\overline{Y}_{st}E_p\end{array}\right]$$

$$MSE\left(T_{pro2}\right)_{\min}\cong\overline{Y}_{st}^2\left[\frac{\left(\begin{array}{l}B_p^2C_p^2+D_p^2E_p^2+A_p^2B_p^2+E_p^4-2B_pC_pD_pE_p-2A_pB_p^2C_p+2B_pC_pE_p^2+2A_pB_pD_pE_p-2D_pE_p^3\\ -2A_pB_pE_p^2+A_pB_p^2C_p^2+A_pD_p^2E_p^2+A_pB_p^2-2A_pB_pC_pD_pE_p+2A_pB_p^2C_p-2A_pB_pD_pE_p+A_p^2B_pD_p^2\\ +B_pC_p^2E_p^2+B_pD_p^2+B_pE_p^2-2A_pB_pC_pD_pE_p+2A_pB_pD_p^2-2A_pB_pD_pE_p-2B_pC_pD_pE_p+2B_pC_pE_p^2\\ -2A_pB_p^2C_p^2+2A_pB_pC_pD_pE_p-2A_pB_p^2C_p+B_pC_p^2E_p^2-2C_pD_pE_p^3+2B_pC_pE_p^2-2B_p^2C_p^2+2B_pC_pD_pE_p\\ -2B_p^2C_p-2A_p^2B_pD_p^2+2A_pB_pC_pD_pE_p-2A_pB_pD_p^2+2A_pB_pD_pE_p+2A_pD_p^2E_p^2-2C_pD_pE_p^3+2D_p^2E_p^2\\ -2D_pE_p^3-2A_pB_pD_p^2+2B_pC_pD_pE_p-2B_pD_p^2+2B_pD_pE_p+2A_pB_pC_pD_pE_p-2B_pC_p^2E_p^2+2B_pC_pD_pE_p\\ -2B_pC_pE_p^2-2A_pD_p^2E_p^2+2C_pD_pE_p^3-2D_p^2E_p^2+2D_pE_p^3+2A_pB_pD_pE_p-2B_pC_pE_p^2+2B_pD_pE_p-2B_pE_p^2\end{array}\right)}{\left(A_pB_p-E_p^2+B_p\right)^2}\right]$$

or

$$MSE\left(T_{pro}\right)_{\min}\cong\overline{Y}_{st}^2\left\{1-\frac{A_pD_p^2+B_pC_p^2-2C_pD_pE_p+B_p+2B_pC_p-D_p^2-2D_pE_p}{A_pB_p-E_p^2+B_p}\right\}. \tag{4.2.7}$$

Appendix B

Section 5.1

$$MSE\left(T_{pro1}\right)_{\min} \le MSE(T_{st})$$

$$\bar{Y}^2\left\{1-\frac{A_{pr}D_{pr}^2+B_{pr}C_{pr}^2-2C_{pr}D_{pr}E_{pr}}{A_{pr}B_{pr}-E_{pr}^2}\right\}\le \bar{Y}^2V_{20}$$

$$1-\Re_1 \le V_{20} \quad \text{where} \quad \Re_1=\frac{A_{pr}D_{pr}^2+B_{pr}C_{pr}^2-2C_{pr}D_{pr}E_{pr}}{A_{pr}B_{pr}-E_{pr}^2}$$

$$\left(V_{20}-1\right)+\Re_1 \ge 0.$$

Eq (5.1.1)

$$MSE\left(T_{pro1}\right)_{\min} \le MSE(T_{r})$$

$$\bar{Y}^2\left\{1-\Re_1\right\}\le \bar{Y}^2(V_{20}+V_{02}-2V_{11})$$

$$\left\{1-\Re_1\right\}\le (V_{20}+V_{02}-2V_{11})$$

$$\left[V_{20}+V_{02}-2V_{11}-1+\Re_1\right]>0.$$

Eq (5.1.2)

$$MSE\left(T_{pro1}\right)_{\min} \le MSE(T_{BT})$$

$$\bar{Y}^2\left\{1-\Re_1\right\}\le \bar{Y}^2(V_{20}+\frac{V_{02}}{4}-2V_{11})$$

$$\left\{1-\Re_1\right\}\le (V_{20}+\frac{V_{02}}{4}-2V_{11})$$

$$\left(V_{20}+\frac{1}{4}V_{02}-V_{11}\right)-1+\Re_1>0.$$

Eq (5.1.3)

$$MSE\left(T_{pro1}\right)_{\min} \le MSE(T_{KC})$$

$$\bar{Y}^2\left\{1-\Re_1\right\}\le \bar{Y}^2(V_{20}+\phi^2V_{02}-2\phi V_{11})$$

$$\left\{1-\Re_1\right\}\le (V_{20}+\phi^2V_{02}-2\phi V_{11})$$

$$\left(V_{20}+\phi^2V_{02}-2\phi V_{11}\right)-1+\Re_1>0.$$

Eq (5.1.4)

$$MSE\left(T_{pro1}\right)_{\min} \le MSE(T_{US})$$

$$\bar{Y}^2\left\{1-\Re_1\right\}\le \bar{Y}^2(V_{20}+\theta^2V_{02}-2\theta V_{11})$$

$$\left\{1-\Re_1\right\}\le (V_{20}+\theta^2V_{02}-2\theta V_{11})$$

$$\left(V_{20}+\theta^2V_{02}-2\theta V_{11}\right)-1+\Re_1>0. \qquad \text{Eq (5.1.5)}$$

$$MSE\left(T_{pro1}\right)_{\min}\le MSE(T_{ch})$$

$$\bar{Y}^2\left\{1-\Re_1\right\}\le\bar{Y}^2\left(V_{20}+\alpha^2\tau^2\pi^2V_{02}-2\alpha\tau\pi V_{11}\right)$$

$$\left\{1-\Re_1\right\}\le\left(V_{20}+\alpha^2\tau^2\pi^2V_{02}-2\alpha\tau\pi V_{11}\right)$$

$$\left(V_{20}+\alpha^2g^2\pi^2V_{02}-2\alpha g\pi V_{11}\right)-1+\Re_1>0. \qquad \text{Eq (5.1.6)}$$

$$MSE\left(T_{pro1}\right)_{\min}\le MSE(T_O)$$

$$\bar{Y}^2\left\{1-\Re_1\right\}\le\frac{V_{00}V_{22}-V_{12}^2}{V_{00}+V_{22}-2V_{12}}$$

$$\left[\frac{V_{00}V_{22}-V_{12}^2}{V_{00}+V_{22}-2V_{12}}\right]-\bar{Y}^2\left[1-\Re_1\right]>0. \qquad \text{Eq (5.1.7)}$$

$$MSE\left(T_{pro1}\right)_{\min}\le MSE\left(T_G\right)$$

$$\bar{Y}^2\left\{1-\Re_1\right\}\le\bar{Y}^2V_{20}\left(1-\rho_c\right)$$

$$\left\{1-\Re_1\right\}\le V_{20}\left(1-\rho_c\right)$$

$$\left[V_{20}\left(1-\rho_{st}\right)-1+\Re_1\right]>0. \qquad \text{Eq (5.1.8)}$$

$$MSE\left(T_{pro1}\right)_{\min}\le MSE\left(T_{NK}\right)$$

$$\bar{Y}^2\left\{1-\Re_1\right\}\le\bar{Y}^2\left[1-\frac{AD^2+BC^2-CDE}{4AB-E^2}\right]$$

$$\left\{1-\Re_1\right\}\le\left[1-\frac{AD^2+BC^2-CDE}{4AB-E^2}\right]$$

$$\Re_1-\frac{AD^2+BC^2-CDE}{4AB-E^2}>0. \qquad \text{Eq (5.1.9)}$$

$$MSE\left(T_{pro1}\right)_{\min}\le MSE\left(T_{TSS}\right)$$

$$\bar{Y}^2\left\{1-\Re_1\right\}\le\bar{Y}^2\left[1-\frac{B_1C_1^2+4A_1D_1^2-4C_1D_1E_1}{2\left(A_1B_1-E_1^2\right)}\right]$$

$$\left\{1-\Re_1\right\}\le\left[1-\frac{B_1C_1^2+4A_1D_1^2-4C_1D_1E_1}{2\left(A_1B_1-E_1^2\right)}\right]$$

$$\Re_1 - \left[\frac{B_1 C_1^2 + 4 A_1 D_1^2 - 4 C_1 D_1 E_1}{2\left(A_1 B_1 - E_1^2 \right)} \right] > 0 \,.$$

Eq (5.1.10)

$$MSE\left(T_{pro1}\right)_{\min} \le MSE(T_{MJ})_{\min}$$

$$\bar{Y}^2 \left\{1 - \Re_1\right\} \le \bar{Y}^2 \left[\left(V_{20} + \eta^2 V_{02} - 2\eta V_{11}\right) - \frac{4 A_2 D_2^2 + B_2 C_2^2 - 4 C_2 D_2 E_2}{4\left(A_2 B_2 - E_2^2\right)} \right]$$

$$\left\{1 - \Re_1\right\} \le \left[\left(V_{20} + \eta^2 V_{02} - 2\eta V_{11}\right) - \frac{4 A_2 D_2^2 + B_2 C_2^2 - 4 C_2 D_2 E_2}{4\left(A_2 B_2 - E_2^2\right)} \right]$$

$$\left(V_{20} + \eta^2 V_{02} - 2\eta V_{11}\right) + \Re_1 - \left[\frac{4 A_2 D_2^2 + B_2 C_2^2 - 4 C_2 D_2 E_2}{4\left(A_2 B_2 - E_2^2\right)} \right] > 0 \,.$$

Eq (5.1.11)

Section 5.2

$$MSE\left(T_{pro2}\right)_{\min} \le MSE(T_{st})$$

$$\bar{Y}^2 \left\{ 1 - \frac{A_p D_p^2 + B_p C_p^2 - 2 C_p D_p E_p + B_p + 2 B_p C_p - D_p^2 - 2 D_p E_p}{A_p B_p - E_p^2 + B_p} \right\} \le \bar{Y}^2 V_{20}$$

$$1 - \frac{\Re_2}{\Re_3} \le V_{20}$$

$$\left[\left(V_{20} - 1\right) + \frac{\Re_2}{\Re_3} \right] > 0 \,.$$

Eq (5.2.1)

Where, $\Re_2 = A_p B_p^2 + B_p C_p^2 - 2 C_p D_p E_p + B_p + 2 B_p C_p - D_p^2 - 2 D_p E_p$ and $\Re_3 = A_p B_p - E_p^2 + B_p$

$$MSE\left(T_{pro2}\right)_{\min} \le MSE(T_r)$$

$$\bar{Y}^2 \left\{ 1 - \frac{\Re_2}{\Re_3} \right\} \le \bar{Y}^2 (V_{20} + V_{02} - 2V_{11})$$

$$\left\{ 1 - \frac{\Re_2}{\Re_3} \right\} \le (V_{20} + V_{02} - 2V_{11})$$

$$\left[V_{20} + V_{02} - 2V_{11} - 1 + \frac{\Re_2}{\Re_3} \right] > 0 \,.$$

Eq (5.2.2)

$$MSE\left(T_{pro2}\right)_{\min} \le MSE(T_{BT})$$

$$\bar{Y}^2 \left\{ 1 - \frac{\Re_2}{\Re_3} \right\} \le \bar{Y}^2 (V_{20} + \frac{V_{02}}{4} - 2V_{11})$$

$$\left\{1-\frac{\Re_2}{\Re_3}\right\} \le (V_{20}+\frac{V_{02}}{4}-2V_{11})$$

$$\left(V_{20}+\frac{1}{4}V_{02}-V_{11}\right)-1+\frac{\Re_2}{\Re_3}>0. \quad \text{Eq (5.2.3)}$$

$$MSE\left(T_{pro2}\right)_{\min} \le MSE(T_{KC})$$

$$\bar{Y}^2\left\{1-\frac{\Re_2}{\Re_3}\right\} \le \bar{Y}^2(V_{20}+\phi^2V_{02}-2\phi V_{11})$$

$$\left\{1-\frac{\Re_2}{\Re_3}\right\} \le (V_{20}+\phi^2V_{02}-2\phi V_{11})$$

$$\left(V_{20}+\phi^2V_{02}-2\phi V_{11}\right)-1+\frac{\Re_2}{\Re_3}>0. \quad \text{Eq (5.2.4)}$$

$$MSE\left(T_{pro2}\right)_{\min} \le MSE(T_{US})$$

$$\bar{Y}^2\left\{1-\frac{\Re_2}{\Re_3}\right\} \le \bar{Y}^2(V_{20}+\theta^2V_{02}-2\theta V_{11})$$

$$\left\{1-\frac{\Re_2}{\Re_3}\right\} \le (V_{20}+\theta^2V_{02}-2\theta V_{11})$$

$$\left(V_{20}+\theta^2V_{02}-2\theta V_{11}\right)-1+\frac{\Re_2}{\Re_3}>0. \quad \text{Eq (5.2.5)}$$

$$MSE\left(T_{pro2}\right)_{\min} \le MSE(T_{ch})$$

$$\bar{Y}^2\left\{1-\frac{\Re_2}{\Re_3}\right\} \le \bar{Y}^2\left(V_{20}+\alpha^2\tau^2\pi^2V_{02}-2\alpha\tau\pi V_{11}\right)$$

$$\left\{1-\frac{\Re_2}{\Re_3}\right\} \le \left(V_{20}+\alpha^2\tau^2\pi^2V_{02}-2\alpha\tau\pi V_{11}\right)$$

$$\left(V_{20}+\alpha^2g^2\pi^2V_{02}-2\alpha g\pi V_{11}\right)-1+\frac{\Re_2}{\Re_3}>0. \quad \text{Eq (5.2.6)}$$

$$MSE\left(T_{pro2}\right)_{\min} \le MSE(T_O)$$

$$\bar{Y}^2\left\{1-\frac{\Re_2}{\Re_3}\right\} \le \frac{V_{00}V_{22}-V_{12}^2}{V_{00}+V_{22}-2V_{12}}$$

$$\left[\frac{V_{00}V_{22}-V_{12}^{2}}{V_{00}+V_{22}-2V_{12}}\right]-\bar{Y}^{2}\left[1-\frac{\Re_{2}}{\Re_{3}}\right]>0\,. \quad \text{Eq (5.2.7)}$$

$$MSE\left(T_{pro2}\right)_{\min}\leq MSE\left(T_{G}\right)$$

$$\bar{Y}^{2}\left\{1-\frac{\Re_{2}}{\Re_{3}}\right\}\leq\bar{Y}^{2}V_{20}\left(1-\rho_{c}\right)$$

$$\left\{1-\frac{\Re_{2}}{\Re_{3}}\right\}\leq V_{20}\left(1-\rho_{c}\right)$$

$$\left[V_{20}\left(1-\rho_{st}\right)-1+\frac{\Re_{2}}{\Re_{3}}\right]>0\,. \quad \text{Eq (5.2.8)}$$

$$MSE\left(T_{pro2}\right)_{\min}\leq MSE\left(T_{NK}\right)$$

$$\bar{Y}^{2}\left\{1-\frac{\Re_{2}}{\Re_{3}}\right\}\leq\bar{Y}^{2}\left[1-\frac{AD^{2}+BC^{2}-CDE}{4AB-E^{2}}\right]$$

$$\left\{1-\frac{\Re_{2}}{\Re_{3}}\right\}\leq\left[1-\frac{AD^{2}+BC^{2}-CDE}{4AB-E^{2}}\right]$$

$$\frac{\Re_{2}}{\Re_{3}}-\frac{AD^{2}+BC^{2}-CDE}{4AB-E^{2}}>0\,. \quad \text{Eq (5.2.9)}$$

$$MSE\left(T_{pro2}\right)_{\min}\leq MSE\left(T_{TSS}\right)$$

$$\bar{Y}^{2}\left\{1-\frac{\Re_{2}}{\Re_{3}}\right\}\leq\bar{Y}^{2}\left[1-\frac{B_{1}C_{1}^{2}+4A_{1}D_{1}^{2}-4C_{1}D_{1}E_{1}}{2\left(A_{1}B_{1}-E_{1}^{2}\right)}\right]$$

$$\left\{1-\frac{\Re_{2}}{\Re_{3}}\right\}\leq\left[1-\frac{B_{1}C_{1}^{2}+4A_{1}D_{1}^{2}-4C_{1}D_{1}E_{1}}{2\left(A_{1}B_{1}-E_{1}^{2}\right)}\right]$$

$$\frac{\Re_{2}}{\Re_{3}}-\left[\frac{B_{1}C_{1}^{2}+4A_{1}D_{1}^{2}-4C_{1}D_{1}E_{1}}{2\left(A_{1}B_{1}-E_{1}^{2}\right)}\right]>0\,. \quad \text{Eq (5.2.10)}$$

$$MSE\left(T_{pro2}\right)_{\min}\leq MSE\left(T_{MJ}\right)_{\min}$$

$$\bar{Y}^{2}\left\{1-\frac{\Re_{2}}{\Re_{3}}\right\}\leq\bar{Y}^{2}\left[\left(V_{20}+\eta^{2}V_{02}-2\eta V_{11}\right)-\frac{4A_{2}D_{2}^{2}+B_{2}C_{2}^{2}-4C_{2}D_{2}E_{2}}{4\left(A_{2}B_{2}-E_{2}^{2}\right)}\right]$$

$$\left\{1-\frac{\Re_{2}}{\Re_{3}}\right\}\leq\left[\left(V_{20}+\eta^{2}V_{02}-2\eta V_{11}\right)-\frac{4A_{2}D_{2}^{2}+B_{2}C_{2}^{2}-4C_{2}D_{2}E_{2}}{4\left(A_{2}B_{2}-E_{2}^{2}\right)}\right]$$

$$\left(V_{20}+\eta^2 V_{02}-2\eta V_{11}\right)+\frac{\Re_2}{\Re_3}-\left[\frac{4A_2D_2^2+B_2C_2^2-4C_2D_2E_2}{4\left(A_2B_2-E_2^2\right)}\right]>0\,.$$

Eq (5.2.11)