AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(3): 5444–5455.
DOI:10.3934/math.2025251
Received: 20 January 2025
Revised: 03 March 2025
Accepted: 05 March 2025
Published: 11 March 2025

Research article

On multiple solutions for an elliptic problem involving Leray–Lions operator, Hardy potential and indefinite weight with mixed boundary conditions

Khaled Kefi[1,*] **and Mohammed M. Al-Shomrani**[2,*]

[1] Center for Scientific Research and Entrepreneurship, Northern Border University, Arar 73213, Saudi Arabia

[2] Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia

* **Correspondence:** Email: khaled_kefi@yahoo.fr, malshamrani@kau.edu.sa.

Abstract: This paper concentrates on establishing the existence of multiple weak solutions for a specific type of elliptic equations that involve a Hardy potential and have mixed boundary conditions. The main goal of the study is to establish an existence result of at least three different weak solutions thanks to variational techniques, Hardy inequality, and a particular theorem called the Bonanno–Marano type three critical points theorem.

Keywords: two parameters; elliptic equations; mixed boundary conditions; $p(x)$-Laplacian; Hardy term
Mathematics Subject Classification: 35J15, 35J20, 35J25

1. Introduction

Partial differential equations (PDEs) play a fundamental role in modeling various physical and biological phenomena. In particular, elliptic equations have wide-ranging applications in fields such as physics, engineering, and biology. The study of elliptic equations with Hardy potential has gained significant attention due to their relevance in understanding the behavior of solutions in singular domains and their connection to critical exponent problems.

In [3], the authors study the existence of three distinct nonzero solutions to a mixed nonlinear differential problem involving the p-Laplacian, even in the presence of a nonhomogeneous term in the Neumann boundary condition. Specifically, they consider the following boundary value problem:

$$\begin{cases}-\Delta_p w + q(x)|w|^{p-2}w = \lambda g(x,w) & \text{in } \Omega,\\ w = 0 & \text{on } \Gamma_1,\\ |\nabla w|^{p-2}\frac{\partial w}{\partial \nu} = \mu h(w) & \text{on } \Gamma_2,\end{cases}$$

where $\Omega \subset \mathbb{R}^N$ (with $N \geq 3$) is a bounded, nonempty open set with a C^1-smooth boundary. The sets Γ_1 and Γ_2 are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega$, with their intersection $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \Sigma$, where Σ is a smooth $(N-2)$-dimensional submanifold of $\partial\Omega$. The function $q \in L^\infty(\Omega)$ is bounded below by a positive constant, denoted $q_0 := \operatorname{essinf}_\Omega q > 0$. The operator $\Delta_p w$ is defined as $\operatorname{div}(|\nabla w|^{p-2}\nabla w)$, with $p > N$. The function $g : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $h : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function. The parameters λ and μ are real numbers with $\lambda > 0$ and $\mu \geq 0$, and ν represents the outer unit normal to $\partial\Omega$. Recent contributions to this type of problem can be found in [13].

Motivated by the above problem, our paper focuses on a specific class of elliptic equations with a Hardy potential, mixed boundary conditions, and two parameters, particularly in the case where the exponents are variables. The equation involves a nonlinear term that exhibits power-law growth, which introduces additional complexity and challenges in the analysis. The presence of the Hardy potential, characterized by the term $\frac{\vartheta(x)}{|x|^s}|u|^{s-2}u$, where s is a constant that belongs in $(1, N)$, reflects the singularity of the domain at the origin and affects the behavior of the solutions.

To establish the existence of multiple weak solutions, variational methods and critical point theory are employed. Variational methods involve minimizing certain functionals associated with the equation, while critical point theory analyzes the existence of critical points of these functionals. The Hardy inequality, which provides a fundamental estimate for functions with Hardy potential, is also utilized in the analysis.

In this manuscript, we are concerned by existence and multiplicity for the following problem:

$$\begin{cases}-\operatorname{div}\mathbf{A}(x,\nabla u) + \dfrac{\vartheta(x)}{|x|^s}|u|^{s-2}u = \lambda f(x,u) & \text{in } \Omega,\\ u = 0 & \text{over } \Gamma_1,\\ \mathbf{A}(x,\nabla u)\cdot\nu = \mu g(x,u) & \text{over } \Gamma_2.\end{cases} \tag{1.1}$$

Here, $\mathbf{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a mapping that extends the $p(x)$-Laplacian operator, and Ω represents an open bounded subset in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. The vector field ν denotes the outward normal vectors on the boundary $\partial\Omega$.

Furthermore, there are two smooth submanifolds, Γ_1 and Γ_2, of dimension $N-1$, which are parts of the boundary $\partial\Omega$. These submanifolds are distinct and do not intersect. Their union $\overline{\Gamma_1} \cup \overline{\Gamma_2}$ covers the entire boundary, while their intersection $\overline{\Gamma_1} \cap \overline{\Gamma_2}$ forms a submanifold of dimension $N-2$ within the boundary $\partial\Omega$, the real parameters λ and μ are respectively positive and nonnegative, $\vartheta(x) \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega}\vartheta(x) > 0, \vartheta_0 = \operatorname{ess\,sup}_{x\in\Omega}\vartheta(x)$, $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$(\mathbf{f}) : c_f\tilde{f}(x)|u|^{q(x)-1} \leq f(x,u) \leq \bar{c_f}\tilde{f}(x)|u|^{q(x)-1},$$

where, c_f, $\bar{c_f}$ are two positive constants, $\tilde{f}(x) \in L^{\alpha(x)}(\Omega)$, $N < \alpha(x)$ for all $x \in \Omega$. The functions $q(x)$ and $p(x)$ belongs to $C(\overline{\Omega})$ and satisfies the following:

$$1 < s < q^- := \inf_{x\in\overline{\Omega}} q(x) \leq \sup_{x\in\overline{\Omega}} q(x) < \sup_{x\in\overline{\Omega}} p(x) < N.$$

Moreover, we assume that there exists $\bar{c_g}$, a positive constant such that the following hold:

$$(\mathbf{g}) : 0 \leq g(x,u) \leq \bar{c_g}\tilde{g}(x)|u|^{r(x)-1}, \ \forall (x,u) \in \Gamma_2 \times \mathbb{R},$$

where $1 < \sup_{x\in\partial\Omega} r(x) = r^+ < p^- := \inf_{x\in\overline{\Omega}} p(x)$, $\tilde{g}(x) \in L^{\gamma(x)}(\Gamma_2), \gamma(x) > \dfrac{N-1}{p(x)-1}$ for all $x \in \Omega$, we denote by $p^\partial(x) = \frac{(N-1)p(x)}{N-p(x)}$, if $p(x) < N$ and $p^\partial(x) = +\infty$ otherwise.

This work considers a nonlinear boundary value problem that involves a variable exponent $p(x)$-Laplacian operator with singular and possibly sign-changing nonlinearities. Specifically, the function $\tilde{f}(x)$ may change sign and exhibit singularities within the domain Ω, which directly influences the nonlinearity $f(x,u)$ due to the multiplicative factor $\tilde{f}(x)$. Similarly, $\tilde{g}(x)$, which appears in the boundary condition, may also be singular. This introduces significant challenges in both the mathematical analysis and the solution existence theory, as singularities often complicate the proof of existence and regularity of solutions.

The main objective of this paper is to demonstrate that for any positive λ that belongs in the interval $[A_\delta, B_d]$ (see Theorem 3.7, for more details), there exists $\bar{\mu}(\lambda)$ such that for any $\mu \in [0, \bar{\mu}(\lambda)]$, problem (1.1), possesses at least three weak solutions.

Such problems, where both the nonlinearity and the boundary terms involve singular functions that can change sign, have not been extensively treated in the literature, especially in the context of the $p(x)$-Laplacian with mixed boundary conditions. Most classical models either assume the nonlinearities to be continuous and sign-preserving or restrict the functions from exhibiting singularities. The introduction of these features and singularities in both the nonlinearities and boundary terms adds a layer of complexity that requires new techniques to establish the existence of multiple solutions.

Our manuscript is structured as follows: in the next section, we present some preliminaries and auxiliary results, and the last section is dedicated to our main result.

2. Variational structure

In what follows, denotes by $C_+(\overline{\Omega}) := \left\{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for all } x \in \overline{\Omega}\right\}$, for any $h \in C_+(\overline{\Omega})$, we define the infimum and supremum of h over $\overline{\Omega}$ as

$$h^- := \inf_{x\in\overline{\Omega}} h(x), \quad h^+ := \sup_{x\in\overline{\Omega}} h(x).$$

Moreover, let $\mathcal{L}(\Omega)$, the space of all measurable functions from Ω into $\mathbb{R}$.

Let $p(x)$ be a function that belongs in $C_+(\overline{\Omega})$; we define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ (see [7], for further details) as

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} : u \in \mathcal{L}(\Omega) \text{ and } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\}.$$

For any $u \in L^{p(x)}(\Omega)$, the Luxemburg norm is defined as:

$$|u|_{p(x)} := \inf\left\{\eta > 0 : \int_\Omega \left|\frac{u(x)}{\eta}\right|^{p(x)} dx \leq 1\right\}.$$

For two functions $u_1 \in L^{p(x)}(\Omega)$ and $u_2 \in L^{p'(x)}(\Omega)$, where $p'(x)$ is the conjugate exponent of $p(x)$, the following Hölder-type inequality holds (see [12], Theorem 2.1)

$$\left|\int_\Omega u_1(x)u_2(x)\,dx\right| \leq \left(\frac{1}{p^-}+\frac{1}{p'^-}\right)|u_1|_{p(x)}|u_2|_{p'(x)}. \tag{2.1}$$

For $h \in C_+(\overline{\Omega})$, put $[\tau]^h := \max\{\tau^{h^-}, \tau^{h^+}\}$, $[\tau]_h := \min\{\tau^{h^-}, \tau^{h^+}\}$, then, it is easy to show the following (see [1], for further details).

Remark 2.1.

$$\begin{aligned}
&(1) \quad [\tau]^{\frac{1}{h}} = \max\left\{\tau^{\frac{1}{h^-}}, \tau^{\frac{1}{h^+}}\right\},\\
&(2) \quad [\tau]_{\frac{1}{h}} = \min\left\{\tau^{\frac{1}{h^-}}, \tau^{\frac{1}{h^+}}\right\},\\
&(3) \quad [\tau]_{\frac{1}{h}} = a \Longleftrightarrow \tau = [a]^h,\\
&(4) \quad [\tau]^{\frac{1}{h}} = a \Longleftrightarrow \tau = [a]_h,\\
&(5) \quad [\tau]_h[\alpha]_h \leq [\tau\alpha]_h \leq [\tau\alpha]^h \leq [\tau]^h[\alpha]^h.
\end{aligned}$$

In what follows, we revisit these essential propositions

Proposition 2.1. *([7]) Let w that belongs in $L^{q(x)}(\Omega)$, we have the following:*

$$[|w|_{q(x)}]_q \leq \int_\Omega |w(x)|^{q(x)}\,dx \leq [|w|_{q(x)}]^q.$$

Moreover, the following result holds:

Proposition 2.2. *([8]) Let $h_1(x)$, $h_2(x) \in C_+(\overline{\Omega})$, and assume that $h_2(x) \leq h_1(x)$ a.e. in Ω. Then, we have*

$$L^{h_1(x)}(\Omega) \hookrightarrow L^{h_2(x)}(\Omega),$$

furthermore, there exists a positive constant $c_{h_2} > 0$ such that for all $u \in L^{h_1(x)}(\Omega)$, the following inequality is satisfied:

$$|u|_{h_2(x)} \leq c_{h_2}|u|_{h_1(x)}.$$

The Sobolev space $W^{1,p(x)}(\Omega)$, with a variable exponent, is given by

$$W^{1,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : \nabla u \in L^{p(x)}(\Omega)\right\}.$$

The norm on the space $W^{1,p(x)}(\Omega)$ is given by

$$\|u\|_{1,p(x)} = |\nabla u|_{p(x)}.$$

Let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, equipped with the equivalent norm $\|u\|_{1,p(x)} = |\nabla u|_{p(x)}$. Now, let $\tilde{X} := W_0^{1,p(x)}(\Omega)$, equipped with the norm

$$\|u\| := |\nabla u|_{p(x)} = \inf\left\{\xi > 0 : \int_\Omega \left|\frac{\nabla u}{\xi}\right|^{p(x)} dx \leq 1\right\}.$$

We define the so-called modular on $\tilde{X}$ as a map $\rho_{p(x)}$ from $\tilde{X}$ to $\mathbb{R}$ as

$$\rho_{p(x)}(u) = \int_{\Omega} |\nabla u|^{p(x)}\, dx.$$

Note that, this map fulfils same properties as Proposition 2.1. Specifically, we have

Proposition 2.3. *For any $u \in L^{q(x)}(\Omega)$, the following conditions are satisfied.*

- If $|\nabla u|_{q(x)} < 1$ (or $= 1$, or > 1), then $\rho_{q(x)}(u) < 1$ (or $= 1$, or > 1), respectively.
- The inequality $\left[|\nabla u|_{q(x)}\right]_q \leq \rho_{q(x)}(u) \leq \left[|\nabla u|_{q(x)}\right]^q$ holds.

Proposition 2.4. *([6]) Suppose $h_1, h_2 \in \mathcal{L}(\Omega)$ satisfying $h_1 \in L^\infty(\Omega)$ and $1 \leq h_1(x)h_2(x) \leq \infty$ for almost every $x \in \Omega$. Let $w \in L^{h_2(x)}(\Omega)$ be a non-zero function. In this case, we have the following inequalities:*

$$[|w|_{h_1(x)h_2(x)}]_{h_1} \leq \||w|^{h_1(x)}|_{h_2(x)} \leq [|w|_{h_1(x)h_2(x)}]^{h_1}.$$

The space $\tilde{X}$, as defined above, is a separable Banach space and also reflexive.

The critical Sobolev exponent $p^*(x)$ is expressed as:

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \geq N. \end{cases}$$

From Proposition 2.2, it follows that if $q(x) \leq p(x)$ almost everywhere in Ω, then the following embedding holds:

$$W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,q(x)}(\Omega).$$

In particular, we have the embedding:

$$\tilde{X} \hookrightarrow W_0^{1,p^-}(\Omega).$$

Furthermore, the space $\tilde{X}$ is continuously embedded in $L^{\alpha(x)}(\Omega)$ for any $\alpha \in C_+(\overline{\Omega})$ such that $\alpha(x) \leq p^*(x)$ almost everywhere on Ω. This leads to the following inequality:

$$|u|_{\alpha(x)} \leq c_\alpha |\nabla u|_{p(x)}, \tag{2.2}$$

where c_α is a positive constant.

3. Existence and multiplicity

In the whole manuscript, we assume the following:

Let $\mathcal{A} : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function with a continuously differentiable derivative given by $\mathbf{A}(x, \xi) = \frac{\partial}{\partial \xi}\mathcal{A}(x, \xi)$. Suppose that there exists a positive constant $\bar{k}$ such that the following inequality holds for all $u, v \in \tilde{X}$:

$$(A) \quad \mathbf{A}(x, u + v) \leq \bar{k}\left(\mathbf{A}(x, u) + \mathbf{A}(x, v)\right), \quad \forall x \in \overline{\Omega}.$$

Additionally, assume that $\mathcal{A}$ satisfies the following conditions:

(A1) $\mathcal{A}(x,0)=0,\ \mathcal{A}(x,\xi)=\mathcal{A}(x,-\xi)$ for all $x\in\Omega,\xi\in\mathbb{R}^N$.
(A2) $\mathcal{A}$ is strictly convex in $\mathbb{R}^N$ for all $x\in\Omega$.
(A3) There exists a constant $c_1>0$ such that

$$|\mathbf{A}(x,\xi)|\leqslant c_1\left(\mathcal{B}(x)+|\xi|^{p(x)-1}\right)$$

for almost every $(x,\xi)\in\Omega\times\mathbb{R}^N$, where $\mathcal{B}(x)\in L^{\frac{p(x)}{p(x)-1}}(\Omega)$ and $p(x)\in C_+(\overline{\Omega})$.

(A4)

$$c_2|\xi|^{p(x)}\geq\mathbf{A}(x,\xi)\cdot\xi\geq p(x)\mathcal{A}(x,\xi)\geq|\xi|^{p(x)},$$

for a.e. $x\in\Omega$ and all $\xi\in\mathbb{R}^N$ and for some constant $c_2>0$.

Define the functional $\mathcal{I}_\lambda$ from $\tilde{X}$ into $\mathbb{R}$ by $\mathcal{I}_{\lambda,\mu}(u):=\Phi(u)-\lambda\Psi(u)$, where

$$\Phi(u):=\int_\Omega\mathcal{A}(x,\nabla u(x))dx+\frac{1}{s}\int_\Omega\frac{\vartheta(x)|u(x)|^s}{|x|^s}dx,$$

$$\Psi(u):=\int_\Omega F(x,u(x))dx+\frac{\mu}{\lambda}\int_{\Gamma_2}G(x,u)d\sigma,$$

$F(x,u)=\displaystyle\int_0^u f(x,t)dt$ and $G(x,u)=\displaystyle\int_0^u g(x,t)dt,\ \forall(x,u)\in\Omega\times\mathbb{R}$.
$u\in\tilde{X}\backslash\{0\}$ is said to be a weak solution of the problem (1.1), if $\mathcal{I}'_{\lambda,\mu}(u)[v]=0$, for all $v\in\tilde{X}$.

The main tool used to achieve our results is the Bonanno-Marano type three critical points theorem, derived from the findings in [2].

Theorem 3.1. *[2, Theorem 3.6] Let Y be a reflexive Banach space over $\mathbb{R}$, and let $F:Y\to\mathbb{R}$ be a coercive functional that satisfies the following properties:*

- F is continuously Gâteaux differentiable.
- F is sequentially weakly lower semi-continuous.
- Suppose that the Gâteaux derivative of F admits a continuous inverse on Y^*.

Moreover, let $G:Y\to\mathbb{R}$ be a continuously Gâteaux differentiable functional with a compact Gâteaux derivative.

Additionally, suppose the following:

(a_0) $\inf_Y F=F(0)=G(0)=0$,

and suppose that there exist constants $d>0$ and $\overline{y}\in Y$ such that $F(\overline{y})>d$, and the following conditions hold:

(a_1) $\frac{\sup_{F(y)<d}G(y)}{d}<\frac{G(\overline{y})}{F(\overline{y})}$,
(a_2) For any $\lambda\in\Lambda_d:=\left(\frac{F(\overline{y})}{G(\overline{y})},\frac{d}{\sup_{F(y)\leq d}G(y)}\right)$, the functional $I_\lambda:=F-\lambda G$ is coercive.

Then, for every $\lambda\in\Lambda_d$, the functional $F-\lambda G$ has at least three distinct critical points in Y.

The following lemma provides the Hardy–Rellich inequality, which we recall here (see [5]).

Lemma 3.2. *Let* $1 < t < N$ *and* $u \in W^{1,t}(\Omega)$*. Then, the following inequality holds:*

$$\int_\Omega \frac{|u(x)|^t}{|x|^t}\,dx \le \frac{1}{\mathcal{H}_t}\int_\Omega |\nabla u(x)|^t\,dx,$$

where $\mathcal{H}_t$ is defined by

$$\mathcal{H}_t := \left(\frac{N-t}{t}\right)^t.$$

Now, we state the following remark:

Remark 3.3. *The following property holds:*

$$\hat{K}\left([|\nabla u|_{\tilde{p}(x)}]^p + |\nabla u|^s_{p(x)}\right) \ge \Phi(u) \ge \frac{1}{p^+}\left[|\nabla u|_{p(x)}\right]_p,$$

where

$$\hat{K} = \max\left\{\frac{c_2}{p^-}, c_s^s\frac{|\vartheta|_\infty}{s\mathcal{H}_s}\right\}.$$

Proof. By Assumption $(A4)$, Proposition 2.3, Lemma 3.2, and Proposition 2.2, since $1 < s < N$, one has

$$\begin{aligned}
\frac{1}{p^+}[|\nabla u|_{p(x)}]_p &\le \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx,\\
&\le \Phi\,(u)\\
&\le \int_\Omega \frac{c_2}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \vartheta(x)\frac{|u(x)|^s}{s|x|^s}dx,\\
&\le \frac{c_2}{p^-}\int_\Omega |\nabla u|^{p(x)}dx + \frac{|\vartheta|_\infty}{s}\int_\Omega \frac{|u(x)|^s}{|x|^s}dx,\\
&\le \frac{c_2}{p^-}\int_\Omega |\nabla u|^{p(x)}dx + \frac{|\vartheta|_\infty}{s\mathcal{H}_s}\int_\Omega |\nabla u|^s dx.\\
&\le \frac{c_2}{p^-}[|\nabla u|_{p(x)}]^p + \frac{c_s^s}{s\mathcal{H}_s}|\vartheta|_\infty|\nabla u|^s_{p(x)},\\
&\le \hat{K}\Big([|\nabla u|_{\tilde{p}(x)}]^p + |\nabla u|^s_{p(x)}\Big),
\end{aligned}$$

where $\hat{K} = \max\left\{\frac{c_2}{p^-}, c_s^s\frac{|\vartheta|_\infty}{s\mathcal{H}_s}\right\}$, this ends the proof. □

It is clear from assertions (**f**), (Ai), (i = 1, 2, 3, 4), and Lemma 3.2 that the functional $\Phi : \tilde{X} \to \mathbb{R}$ is well defined, convex, and sequentially weakly lower semicontinuous, and of class C^1 in $\tilde{X}$ with

$$\Phi'(u)(v) = \int_\Omega \mathbf{A}(x,\nabla u)\cdot\nabla v dx + \int_\Omega \frac{\vartheta(x)|u(x)|^{s-2}u(x)}{|x|^s}v dx.$$

Moreover, we have the following:

Proposition 3.1. *Under conditions* $(A_1)-(A_5)$*, the functional* $\Phi' : \tilde{X} \to \tilde{X}^*$ *is uniformly monotone and has a continuous inverse in* $\tilde{X}^*$*.*

Proof. As the proof is almost identical to the proof of Proposition 3.1 in [10], we choose to omit it here. □

Proposition 3.2. *Under conditions* (**f**) *and* (**g**), Ψ *is well defined and of class* C^1, *furthermore, for all* $u \in \tilde{X}$, $\Psi'(u)$ *is compact from* $\tilde{X}$ *into* $\tilde{X}^*$.

Proof. We mention that, the condition $\gamma(x) > \frac{N-1}{p(x)-1}$ for all $x \in \Omega$, assures that $1 < \gamma'(x)r(x) < \frac{(N-1)p(x)}{N-p(x)}$, for all $x \in \partial\Omega$, where $\gamma'(x)$ is the conjugate exponent of $\gamma(x)$, so $\tilde{X}$ is embedded in $L^{\gamma'(x)r(x)}(\partial\Omega)$, (See [9], Corollary 2.2). Moreover, $\alpha'(x)q(x) < p^*(x)$, for all $x \in \Omega$, where, $\alpha'(x)$ is the conjugate exponent of $\alpha(x)$, one has $\tilde{X}$ is embedded in $L^{\alpha'(x)q(x)}(\Omega)$. In what follows, let $c_{\alpha'q}$ be the continuous embedding constant of $\tilde{X} \hookrightarrow L^{\alpha'(x)q(x)}(\Omega)$ and $c_{\gamma'r}$, the one of $\tilde{X} \hookrightarrow L^{\gamma'(x)r(x)}(\partial\Omega)$. Furthermore, given that f fulfills the requirements stated in (**f**), and g satisfies the conditions specified in (**g**), Ψ is well defined; in fact, by using Hölder inequality (2.1) and Proposition 2.4, one has

$$\begin{aligned}
|\Psi(u)| : &= |\int_\Omega F(x,u(x))dx + \frac{\mu}{\lambda}\int_{\Gamma_2} G(x,u)d\sigma|, \\
|\Psi(u)| &\leq \frac{1}{q^-}\int_\Omega |\tilde{f}(x)||u|^{q(x)}dx + \frac{\mu}{\lambda r^-}\int_{\Gamma_2} |\tilde{g}(x)||u|^{r(x)}d\sigma, \\
&\leq \frac{1}{q^-}|\tilde{f}(x)|_{\alpha(x)}||u|^{q(x)}|_{\alpha'(x)} + \frac{\mu}{\lambda r^-}|\tilde{g}(x)|_{L^{\gamma(x)}(\Gamma_2)}||u|^{r(x)}|_{L^{\gamma'(x)}(\Gamma_2)}, \\
&\leq \frac{1}{q^-}|\tilde{f}(x)|_{\alpha(x)}[|u|_{\alpha'(x)q(x)}]^q + \frac{\mu}{\lambda r^-}|\tilde{g}(x)|_{\gamma(x)}[|u|_{L^{\gamma'(x)r(x)}(\Gamma_2)}]^r.
\end{aligned}$$

Moreover, by (2.2), we obtain:

$$|\Psi(u)| \leq \frac{1}{q^-}|\tilde{f}(x)|_{\alpha(x)}[c_{\alpha'q}|\nabla u|_{p(x)}]^q + \frac{\mu}{\lambda r^-}|\tilde{g}(x)|_{\gamma(x)}[c_{\gamma'r}|\nabla u|_{p(x)}]^r. \tag{3.1}$$

Therefore, Ψ is well-defined. Moreover, Ψ is of class C^1 and has as a derivative

$$\Psi'(u)[v] := \int_\Omega f(x,u)vdx + \frac{\mu}{\lambda}\int_{\Gamma_2} g(x,u)vd\sigma,$$

and $\Psi'(u) : \tilde{X} \to \tilde{X}^*$ is compact (see [11] for further details). □

Now, we are ready to present our main result. For this purpose, let us denote by, $D(x) := \sup\{D > 0 \mid B(x,D) \subseteq \Omega\}$; for all $x \in \Omega$, here B is a ball centered at x and of radius D. It is easy to see that there exists $x^0 \in \Omega$ such that $B(x^0,R) \subseteq \Omega$, where $R = \sup_{x\in\Omega} D(x)$.

In what remains of the paper, m denotes the value $\frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})}$, with Γ being the Gamma function, and assume that $\tilde{f}$ satisfies this extra assumption

$$\tilde{f}(x) := \begin{cases} \leq 0, & x \in \Omega \setminus B(x^0,R), \\ \geq \tilde{f}_0, & x \in B(x^0,\frac{R}{2}), \\ > 0, & x \in B(x^0,R) \setminus B(x^0,\frac{R}{2}), \end{cases} \tag{3.2}$$

where $\tilde{f}_0$ is a positive constant.

Theorem 3.4. *Assume that assumptions* (**f**), (**g**), (A), $(A1)-(A4)$ *hold, furthermore, suppose that there exist* $d, \delta > 0$, *where*

$$\frac{1}{p^+}\Big[\frac{2\delta}{R}\Big]_p m\left(R^N - \left(\frac{R}{2}\right)^N\right) = d.$$

Put

$$A_\delta := \frac{(2^N-1)\hat{K}\Big((\frac{2\delta}{R})^s + [\frac{2\delta}{R}]^p)\Big)}{\frac{c_f\tilde{f}_0}{q^+}[\delta]_q},$$

and

$$B_d := \frac{d}{\frac{(p^+)^{\frac{q^+}{p^-}}}{q^-}|\tilde{f}(x)|_{\alpha(x)}[c_{\alpha' q}]^q[[d]^{\frac{1}{p}}]^q + \frac{\mu(p^+)^{\frac{r^+}{p^-}}}{\lambda r^-}|\tilde{g}(x)|_{\gamma(x)}[c_{\gamma' r}]^r[[d]^{\frac{1}{p}}]^r},$$

then for any $\lambda \in [A_\delta, B_d\,]$, when $\mu \in \left[0, \frac{r^- d - \lambda\frac{(p^+)^{\frac{q^+}{p^-}}}{q^-}\left[c_{\alpha' q}\right]^q|\tilde{f}(x)|_{\alpha(x)}\left[[d]^{\frac{1}{p}}\right]^q}{(p^+)^{\frac{r^+}{p^-}}\left[c_{\gamma' r}\right]^r|\tilde{g}(x)|_{\gamma(x)}\left[[d]^{\frac{1}{p}}\right]^r}\right]$, problem (1.1) admits at least three weak solutions.

Proof. It should be noted that Φ and Ψ satisfy the regularity assumptions stated in Theorem 3.1. We will now demonstrate that conditions (a_1) and (a_2) are met. For this purpose, let

$$\frac{1}{p^+}\Big[\frac{2\delta}{R}\Big]_p m\left(R^N - \left(\frac{R}{2}\right)^N\right) = d,$$

and let $w \in X$ such that

$$w(x) := \begin{cases} 0 & x \in \Omega \backslash B\left(x^0, R\right), \\ \frac{2\delta}{R}\left(R - \left|x - x^0\right|\right) & x \in B\left(x^0, R\right) \backslash B\left(x^0, \frac{R}{2}\right), \\ \delta & x \in B\left(x^0, \frac{R}{2}\right). \end{cases}$$

Then, by utilizing Remark 3.3, we have

$$\begin{aligned} &\frac{1}{p^+}\Big[\frac{2\delta}{R}\Big]_p m\left(R^N - \left(\frac{R}{2}\right)^N\right) \\ &\quad < \Phi(w) \\ &\quad \leq m\left(R^N - \left(\frac{R}{2}\right)^N\right)\hat{K}\Big((\frac{2\delta}{R})^s + [\frac{2\delta}{R}]^p)\Big). \end{aligned}$$

Thus, we have $d < \Phi(w)$. Furthermore, by using assumption (**f**) and (3.2), it follows that

$$\Psi(w) \geq \int_\Omega \frac{c_f}{q(x)}\tilde{f}(x)|w|^{q(x)}dx \geq \int_{B(x^0, \frac{R}{2})} \frac{c_f}{q(x)}\tilde{f}(x)|w|^{q(x)}dx \geq \frac{mc_f\tilde{f}_0}{q^+}(\frac{R}{2})^N[\delta]_q,$$

which yields to $\dfrac{\Psi(w)}{\Phi(w)} \geq \dfrac{\frac{c_f\tilde{f}_0}{q^+}[\delta]_q}{(2^N-1)\hat{K}\Big((\frac{2\delta}{R})^s + [\frac{2\delta}{R}]^p)\Big)}$.

Now, for each $u \in \Phi^{-1}(]-\infty, d])$, one has

$$\frac{1}{p^+}[|\nabla u|_{p(x)}]_p \leq d. \tag{3.3}$$

Therefore, by (4) in Remark 2.1, one has

$$|\nabla u|_{p(x)} \leq \left[p^+\Phi(u)\right]^{\frac{1}{p}} < \left[p^+ d\right]^{\frac{1}{p}}.$$

Moreover, by inequality (3.1), we obtain:

$$|\Psi(u)| \leq \frac{1}{q^-}|\tilde{f}(x)|_{\alpha(x)}[c_{\alpha' q}|\nabla u|_{p(x)}]^q + \frac{\mu}{\lambda r^-}|\tilde{g}(x)|_{\gamma(x)}[c_{\gamma' r}|\nabla u|_{p(x)}]^r,$$

which gives

$$\begin{aligned}\sup_{\Phi(u)<d} \Psi(u) &\leq \frac{1}{q^-}|\tilde{f}(x)|_{\alpha(x)}\Big[c_{\alpha' q}\Big[p^+ d\Big]^{\frac{1}{p}}\Big]^q + \frac{\mu}{\lambda r^-}|\tilde{g}(x)|_{\gamma(x)}\Big[c_{\gamma' r}\Big[p^+ d\Big]^{\frac{1}{p}}\Big]^r. \\ &\leq \frac{(p^+)^{\frac{q^+}{p^-}}}{q^-}|\tilde{f}(x)|_{\alpha(x)}[c_{\alpha' q}]^q[[d]^{\frac{1}{p}}]^q + \frac{\mu(p^+)^{\frac{r^+}{p^-}}}{\lambda r^-}|\tilde{g}(x)|_{\gamma(x)}[c_{\gamma' r}]^r[[d]^{\frac{1}{p}}]^r.\end{aligned}$$

By

$$\mu < \frac{r^- d - \lambda\frac{(p^+)^{\frac{q^+}{p^-}}}{q^-}\left[c_{\alpha' q}\right]^q|\tilde{f}(x)|_{\alpha(x)}\left[[d]^{\frac{1}{p}}\right]^q}{(p^+)^{\frac{r^+}{p^-}}\left[c_{\gamma' r}\right]^r|\tilde{g}(x)|_{\gamma(x)}\left[[d]^{\frac{1}{p}}\right]^r},$$

we obtain:

$$\begin{aligned}\frac{1}{d}\sup_{\Phi(u)<d} \Psi(u) &\leq \frac{1}{d}\left\{\frac{(p^+)^{\frac{q^+}{p^-}}}{q^-}|\tilde{f}(x)|_{\alpha(x)}[c_{\alpha' q}]^q[[d]^{\frac{1}{p}}]^q + \frac{\mu(p^+)^{\frac{r^+}{p^-}}}{\lambda r^-}|\tilde{g}(x)|_{\gamma(x)}[c_{\gamma' r}]^r[[d]^{\frac{1}{p}}]^r\right\} \\ &< \frac{1}{\lambda}.\end{aligned}$$

We now proceed to establish that the energy functional $\mathcal{I}_{\lambda,\mu}$ is coercive for all $\lambda > 0$. By applying inequality (3.1) once more, we obtain:

$$|\Psi(u)| \leq \frac{1}{q^-}|\tilde{f}(x)|_{\alpha(x)}[c_{\alpha' q}|\nabla u|_{p(x)}]^q + \frac{\mu}{\lambda r^-}|\tilde{g}(x)|_{\gamma(x)}[c_{\gamma' r}|\nabla u|_{p(x)}]^r.$$

For $|\nabla u|_{p(x)} > 1$, we obtain:

$$\Phi(u) - \lambda\Psi(u) \geq \frac{1}{p^+}|\nabla u|_{p(x)}^{p^-} - \frac{\lambda}{q^-}|\tilde{f}(x)|_{\alpha(x)}[c_{\alpha' q}|\nabla u|_{p(x)}]^q - \frac{\mu}{r^-}|\tilde{g}(x)|_{\gamma(x)}[c_{\gamma' r}|\nabla u|_{p(x)}]^r.$$

Using the fact that $p^- > q^+$ and $p^- > r+ := \sup_{x\in\partial\Omega} r(x)$, we obtain the conclusion. Last, considering the fact that

$$\bar{\Lambda}_d := (A_\delta, B_d) \subseteq \left(\frac{\Phi(w)}{\Psi(w)}, \frac{d}{\sup_{\Phi(u)<d}\Psi(u)}\right),$$

according to Theorem 3.1, $\Phi - \lambda\Psi$ admits at least three critical points in $\tilde{X}$, which represent weak solutions of problem (1.1) for any $\lambda \in \bar{\Lambda}_d$. □

Author contributions

Khaled Kefi: Conceptualization, methodology, writing–original draft, supervision; Mohammed M. Al-Shomrani: Conceptualization, methodology, writing–original draft, supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA, for funding this research work through the project number NBU-FPEJ-2025-1706-04.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. G. Bonanno, A. Chinnì, V. D. Rădulescu, Existence of two non-zero weak solutions for a $p(\cdot)$-biharmonic problem with Navier boundary conditions, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur.*, **34** (2023), 727–743. https://doi.org/10.4171/RLM/1025
2. G. Bonanno, S. A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Appl. Anal.*, **89** (2010), 1–10. https://doi.org/10.1080/00036810903397438
3. G. Bonanno, G. D'Aguì, A. Sciammetta, Nonlinear elliptic equations involving the p-Laplacian with mixed Dirichlet-Neumann boundary conditions, *Opuscula Math.*, **39** (2019), 159–174. https://doi.org/10.7494/OpMath.2019.39.2.159
4. E. Colorado, I. Peral, Semilinear elliptic problems with mixed Dirichlet–Neumann boundary conditions, *J. Funct. Anal.*, **199** (2003), 468–507. https://doi.org/10.1016/S0022-1236(02)00101-5
5. E. B. Davies, A. M. Hinz, Explicit constants for Rellich inequalities in $L^p(\Omega)$, *Math. Z.*, **227** (1998), 511–523. https://doi.org/10.1007/PL00004389
6. D. E. Edmunds, J. Rákosník, Sobolev embeddings with variable exponent, *Studia Math.*, **143** (2000), 267–293. https://doi.org/10.4064/sm-143-3-267-293
7. X. L. Fan, Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Anal.-Theor.*, **52** (2003), 1843–1852. https://doi.org/10.1016/S0362-546X(02)00150-5
8. X. Fan, D. Zhao, On the generalized Orlicz-Sobolev space $W^{k,p(x)}(\Omega)$, *Journal of Gansu Education College*, **12** (1998), 1–6.

9. X. L. Fan, Boundary trace embedding theorems for variable exponent Sobolev spaces, *J. Math. Anal. Appl.*, **339** (2008), 1395–1412. https://doi.org/10.1016/j.jmaa.2007.08.003
10. K. Kefi, Existence and multiplicity of triple weak solutions for a nonlinear elliptic problem with fourth-order operator and Hardy potential, *AIMS Mathematics*, **9** (2024), 17758–17773. https://doi.org/10.3934/math.2024863
11. K. Kefi, N. Irzi, M. M. Al-Shomrani, Existence of three weak solutions for fourth-order Leray–Lions problem with indefinite weights, *Complex Var. Elliptic*, **68** (2023), 1473–1484. https://doi.org/10.1080/17476933.2022.2056887
12. O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czech. Math. J.*, **41** (1991), 592–618. https://doi.org/10.21136/CMJ.1991.102493
13. J. Liu, Z. Q. Zhao, Generalized solutions for singular double-phase elliptic equations under mixed boundary conditions, *Nonlinear Anal.-Model.*, **29** (2024), 1051–1061. https://doi.org/10.15388/namc.2024.29.37845
14. J. Simon, Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$, In: *Journées d'analyse non linéaire*, Berlin: Springer, 1978, 205–227. https://doi.org/10.1007/BFb0061807
15. E. Zeidler, *Nonlinear functional analysis and its applications II/B: nonlinear monotone operators*, New York: Springer, 1990. https://doi.org/10.1007/978-1-4612-0981-2

AIMS Press