Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(3): 5353–5372.
DOI: 10.3934/math.2025247
Received: 08 October 2024
Revised: 27 January 2025
Accepted: 10 February 2025
Published: 10 March 2025

Research article

Some convergence results on proximal contractions with application to nonlinear fractional differential equation

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Abstract: In this manuscript, we investigated the coincidence points, best proximity points, and fixed-points results endowed with F-contraction within the realm of suprametric spaces. The proximal point results obtained in this work show that our investigation is not purely theoretical; fundamental findings were supplemented with concrete examples that demonstrate their practical ramifications. Furthermore, this paper focuses on boundary value problems (BVPs) related to nonlinear fractional differential equations of order $2 < \varpi \leq 3$. By cleverly translating the BVP into an integral equation, we obtained conditions that confirm the existence and uniqueness of fixed points under $(\mathscr{F}_\tau)_{F_\mathscr{P}}$-contraction. A relevant part of this work is the approximation of the Green's function, which is critical in proving the existence and uniqueness of solutions. Our work not only adds to the current body of knowledge but also provides strong approaches for dealing with hard mathematical problems in the field of fractional differential equations.

Keywords: suprametric space; coincidence point; best proximity point; fixed point; Green function; fractional boundary value problem
2020 Mathematics Subject Classification: 34C28, 47H10, 34K37

1. Introduction

After Stefan Banach's pioneering work in fixed-point theory [1], other scholars continued advancing the field, expanding both its theoretical foundation and practical applications. Kazimierz Kuratowski [2], often in collaboration with Banach, played a crucial role in further developing the theory. Their partnership laid the groundwork for significant progress. Additionally, Stanislaw Saks [3] and Grzegorz Rozenblyum [4] made major contributions, with Saks exploring the topological aspects of fixed-point sets and Rozenblyum focusing on multi-valued mappings. Together, these scholars enriched the field, offering deeper insights and transforming our understanding of fixed-point phenomena. Their collective efforts shaped the modern state of fixed-point theory, building on Banach's legacy. Fast forward to 1989, when Bakhtin introduced the concept of b-metric space, followed by Czerwik in 1993, who further developed this idea [5, 6]. In 2017, T. Kamran took this a step further by proposing the notion of an extended b-metric space [7].

In 2022, Berzig laid the groundwork for suprametric spaces, introducing fundamental concepts that sparked future exploration in the area [8]. By 2023, Berzig delved further, examining nonlinear contractions within b-suprametric spaces and revealing the intricate behavior of such contractions through his arXiv preprint [9]. Around the same time, S. K. Panda made impactful advancements, expanding the concept to extended suprametric spaces and establishing connections with Stone-type theorems [10]. In 2024, Berzig continued to push boundaries, presenting fixed-point results in generalized suprametric spaces, adding new layers to the understanding of fixed-points in this context [11]. This evolving research timeline underscores the rapid development and broadening scope of suprametric theory.

Fixed-point theory serves as a vital instrument for resolving the equation $\mathbb{K}\hbar = \hbar$, where $\mathbb{K}$ denotes a mapping defined on a subset of a metric space, linear space, or topological vector space. In 1997, Sadiq Basha et al. introduced the concept of best proximity pairs [12, 13]. Not every mapping guarantees a fixed point. For example, mappings like non-self-mappings between disjoint sets or those with strict conditions often do not have fixed points. However, in some cases, we can find approximate fixed-points by meeting certain criteria. This idea is used in best approximation theory. If a non-self-mapping $\mathbb{K} : \Phi \to \Psi$ does not have a fixed-point, we can still find an element $\hbar$ that minimizes the distance to $\mathbb{K}\hbar$. This is where best approximation and best proximity point theorems are useful. A best proximity point theorem focuses on minimizing the function $\hbar \mapsto \mathscr{S}(\hbar, \mathbb{K}\hbar)$, which measures the error in approximating $\mathbb{K}\hbar = \hbar$. The theorem states that $\mathscr{S}(\hbar, \mathbb{K}\hbar) \geq \mathscr{S}(\Phi, \Psi)$ for all $\hbar \in \Phi$, and it ensures an optimal solution where $\mathscr{S}(\hbar, \mathbb{K}\hbar) = \mathscr{S}(\Phi, \Psi)$. Best proximity point theorems are generalizations of fixed-point theorems. When $\mathbb{K}$ is a self-mapping, these theorems reduce to fixed-point results.

Following this, Eldred et al. developed a technique to identify best proximity points for mappings $\mathbb{K}$ within the framework of uniformly convex Banach spaces [14]. Subsequently, Baria et al. presented the notion of cyclic Meir-Keeler contractions [15], while Kikkawa et al. explored relationships between Kannan mappings and contractions [16]. Anuradha et al. pioneered the idea of proximal pointwise contraction [17]. In a noteworthy contribution, Suzuki et al. introduced the concept of property UC [28], which Abkar and his collaborators later used to demonstrate the convergence and existence of best proximity points for asymptotic cyclic contractions possessing the UC property [18]. Basha et al. further established best proximity point theorems aimed at achieving globally optimal approximate solutions [19,20]. In a different vein, Samet et al. introduced the notion of α-admissible mappings [21], while Jleli et al. initiated the study of α- proximal admissible mappings [22].

The classical best approximation theorem, attributed to Fan [23], asserts that in a Hausdorff locally convex topological vector space $\mathscr{U}$ equipped with a seminorm p, if Φ is a nonempty compact convex subset, then for any continuous mapping $\mathbb{K} : \Phi \to \mathscr{U}$, there exists an element $\hbar$ in Φ such that the distance $\mathscr{S}(\hbar, \mathbb{K}\hbar) = \mathscr{S}(\mathbb{K}\hbar, \Phi)$. Expanding on this foundational concept, Komal et al. [24] explored generalized Geraghty proximal cyclic contractions, unveiling coincidence best proximity point results within complete metric spaces. Likewise, Latif et al. [25] introduced a partially ordered metric space framework and derived coincidence best proximity point results for $\mathscr{F}$-weak contractive mappings. One foundational work investigates proximinal retracts and establishes essential theorems related to best proximity pairs, providing critical insights into the interplay between proximity mappings by Kirk et al. in [26]. Expanding on this foundation, a notable theorem addresses best proximity points for weakly contractive non-self-mappings, illustrating the nuanced behavior of such mappings and their implications in nonlinear analysis by Sankar Raj [27]. Further contributions include the demonstration of best proximity points in metric spaces with the property UC, which underscores the importance of specific structural conditions for the existence of these points by Suzuki et al. in [28]. Additionally, the study by Vetro and Suzuki in [29] presents multiple existence theorems for weak contractions, contributing to the broader framework of fixed-point theorems by considering various types of contraction mappings. The exploration of proximity points and multivalued mappings has been a significant area of research in recent years. In 2021, Saleem et al. provided valuable results on coincidence best proximity points, which contributed to the development of proximity theory in mathematical analysis [30]. Building on this, Zahid et al. (2024) studied multivalued proximal contractions and their applications in integral equations, expanding the theoretical framework of proximity point theory [31]. In a similar context, Younis, Ahmad, and Shahid (2024) investigated best proximity points for multivalued mappings and their relevance to equations of motion, further extending the applicability of these concepts [32]. Ahmad et al. (2024) also advanced the study by focusing on proximal contractions for multivalued mappings, with a particular emphasis on their use in 2D Volterra integral equations [33]. Additionally, Zahid et al. (2024) explored mathematical models of fractals through proximal F-iterated function systems, establishing a connection between fractal geometry and the theory of proximity points [34]. These contributions highlight the growing significance and application of proximity point theory in both pure and applied mathematics. In 2024 D. Chalishajar [35], investigated the existence, uniqueness, exponential stability, trajectory controllability, and optimal control of fractional neutral stochastic differential systems (FNSDSs), employing bounded integral contractors, weaker Lipschitz continuity, and stochastic methods, eliminating the need for an inverse controllability operator while relaxing the Lipschitz condition. Key results include stability with Poisson jumps, T-controllability, and optimal control for higher-order FNSDSs, supported by numerical examples and applications to stochastic Kelvin–Voigt and Maxwell models, extending prior works on nonlinear Lipschitz operators. In the same year, D. Chalishajar [36] analyzed optimal controls for stochastic integro-differential equations in Hilbert space, demonstrating a unique parameter variation formula using the Leray-Schauder alternative. They also investigated the existence of optimal control for a Lagrange problem, validated with a theoretical and mechanical example of an ethanol-fueled engine.

Motivated by the rich literature on coincidence best proximity point theorems, this work sets out to advance the field by developing new results for $(\mathscr{F}_\tau)_{C_\mathscr{P}}$-proximal contractions and rational type

$(\mathscr{F}_{R_\tau})_{C_\mathscr{P}}$-proximal contraction. Building upon these foundational ideas, we derive best proximity point results and further extend these to fixed-point theorems under the same framework. To solidify these findings, illustrative examples are provided, which reinforce the theoretical contributions and offer practical insights. Additionally, the manuscript ventures into the realm of boundary value problems (BVPs) for nonlinear fractional differential equations, specifically for orders in the range $2 < \varpi \leq 3$. By skillfully reformulating the BVP into an integral equation, conditions are established under which a unique fixed-point can be identified. This process leverages the power of $(\mathscr{F}_\tau)_{F_\mathscr{P}}$-contractions within the innovative structure of suprametric spaces. A key aspect of this exploration is the estimation of Green's function, which plays a pivotal role in verifying both the existence and uniqueness of solutions. The approach presented here not only advances current mathematical understanding but also broadens the applicability of these methods in solving complex differential equations.

2. Basic concepts

Let $(\mathscr{U},\mathscr{S})$ be a suprametric space. Let Φ and Ψ be two nonempty subsets of $\mathscr{U}$. Define

$$\begin{aligned} \Phi_0 &= \{\hbar \in \Phi : \mathscr{S}(\hbar,\rho) = \mathscr{S}(\Phi,\Psi) \text{ for some } \rho \in \Psi\}, \\ \Psi_0 &= \{\rho \in \Psi : \mathscr{S}(\hbar,\rho) = \mathscr{S}(\Phi,\Psi) \text{ for some } \hbar \in \Phi\}, \end{aligned}$$

where

$$\mathscr{S}(\Phi,\Psi) = \inf\{\mathscr{S}(\hbar,\rho) : \hbar \in \Phi, \rho \in \Psi\} \text{ (distance of a set } \Phi \text{ to a set } \Psi).$$

In 2022, M. Barzig [8] introduced the concept of suprametric space in the following manner:

Definition 1. *[8] Consider $\mathscr{S} : \mathscr{U} \times \mathscr{U} \to [0,\infty)$ as a mapping on a nonempty set $\mathscr{U}$ that satisfies the following conditions:*

(1) $\mathscr{S}(\hbar,\rho) \geq 0$,

(2) $\mathscr{S}(\hbar,\rho) = 0 \Leftrightarrow \hbar = \rho$,

(3) $\mathscr{S}(\hbar,\rho) = \mathscr{S}(\rho,\hbar)$,

(4) $\mathscr{S}(\hbar,z) \leq \mathscr{S}(\hbar,\rho) + \mathscr{S}(\rho,z) + \eth\mathscr{S}(\hbar,\rho)\mathscr{S}(\rho,z)$;

for all $\hbar,\rho,z \in \mathscr{U}$ with $\eth \geq 0$, then the pair $(\mathscr{U},\mathscr{S})$ is referred to as a suprametric space.

Example 1. *Suppose $\mathscr{U} = \{1,2,3,4,5,6,7,8,9,10\}$ and a distance function is defined as*

$$\mathscr{S}(\hbar,\rho) = \begin{cases} 0, & \text{if } \hbar = \rho, \\ |\hbar-\rho|^2 & \text{otherwise.} \end{cases}$$

Through straightforward calculations, we find that for $\eth = 2$, the space $(\mathscr{U},\mathscr{S})$ is a suprametric space.

Definition 2. *[8] Suppose $(\mathscr{U},\mathscr{S})$ denotes a suprametric space.*

(i) The sequence $\{\hbar_p\} \to \hbar \in \mathscr{U}$, if for $\varepsilon > 0$, there is certain positive N_ε under the condition that $\mathscr{S}(\hbar_p,\hbar) < \varepsilon$ for each $p \geq N_\varepsilon$. It can be expressed in the form of

$$\lim_{p\to\infty} \hbar_p = \hbar.$$

(ii) The sequence $\{\hbar_p\}$ is referred to as a Cauchy sequence if, for every $\varepsilon > 0$, there exists a natural number $\varepsilon > 0$ such that $\mathscr{S}(\hbar_p, \hbar_q) < \varepsilon$ for all $q, p \geq N_\varepsilon$.

(iii) It is termed complete if every Cauchy sequence converges in $\mathscr{U}$.

(iv) If $\mathscr{S}$ is continuous, every convergent sequence possesses a unique limit.

In 2012, D. Wardowski [37] proposed a new class of contractive mappings, significantly extending fixed-point theory in complete metric spaces. The study presents fixed-point results for these mappings and explores their practical applications in mathematical modeling.

Definition 3. *[37] Let $\mathscr{F} : \mathbb{R}^+ \to \mathbb{R}$, be a mapping satisfying:*

F1 If $\hbar < \rho$, then $\mathscr{F}(\hbar) < \mathscr{F}(\rho)$, for all $\hbar, \rho \in \mathbb{R}^+$,

F2 For any sequence $\{\hbar_p\}_{p\in\mathbb{N}}$ of positive numbers $\lim_{p\to\infty} \hbar_p = 0$, holds if and only if $\lim_{p\to\infty} \mathscr{F}(\hbar_p) = -\infty$,

F3 There exists a constant $k \in (0,1)$ in such a way that $\lim_{\hbar\to 0^+} \hbar^k \mathscr{F}(\hbar) = 0$.

Since $\mathscr{F}^*$ represent the set of all functions that meet the previously stated conditions.

Example 2. *Some functions that satisfied all the above properties for $t > 0$ such that:*

i $\mathscr{F}(t) = \ln(t) \implies \mathscr{S}(\mathbb{K}\hbar, \mathbb{K}\rho) \leq \exp^{-\tau} \mathscr{S}(\hbar, \rho)$,

ii $\mathscr{F}(t) = \ln(t) + t \implies \frac{\mathscr{S}(\mathbb{K}\hbar,\mathbb{K}\rho)}{\mathscr{S}(\hbar,\rho)} \exp^{(} \mathscr{S}(\mathbb{K}\hbar, \mathbb{K}\rho) - \mathscr{S}(\hbar, \rho)) \leq \exp^{-\tau}$,

iii $\mathscr{F}(t) = \ln(t^2 + t) \implies \frac{\mathscr{S}(\mathbb{K}\hbar,\mathbb{K}\rho)(1+\mathscr{S}(\mathbb{K}\hbar,\mathbb{K}\rho))}{\mathscr{S}(\hbar,\rho)(1+\mathscr{S}(\hbar,\rho))} \leq \exp^{-\tau}$.

In 2006, Bhaskar and Lakshmikantham [40] introduced fixed-point theorems in partially ordered metric spaces, emphasizing their applications in nonlinear analysis and expanding the theoretical understanding of such spaces. Later, in 2013, Abbas et al. [41] explored fixed and periodic points of generalized contractions, offering significant contributions to fixed-point theory in metric spaces. Subsequently, in 2014, Batra et al [42] provided insights into $\mathscr{F}$-contractions on metric spaces equipped with a graph, highlighting their utility in computational mathematics. In the same year, Batra et al. [43] extended this work by developing a coincidence point theorem for F-contractions using an altered distance. Additionally, Cosentino et al. [44], introduced fixed-point results for $\mathscr{F}$-contractive mappings of the Hardy-Rogers type, further advancing the field with their innovative approaches. In 2019, Hammad et al. [45] presented a coupled fixed-point method for solving functional and nonlinear integral equations. In 2020, Hammad et al. [46] investigated generalized almost (s,q)-Jaggi F-contractions on b-metric-like spaces, resulting in novel fixed-point results. Later, Hammad et al. [47] worked to expand multi-valued F-contractions in metric-like spaces, exhibiting applicability in dynamic systems. These publications advanced both the theoretical and practical aspects of fixed-point approaches.

Definition 4. *[27] Let (Φ, Ψ) denote a pair of nonempty subsets within a metric space, ensuring that Φ_0 is also nonempty. The pair (Φ, Ψ) possesses the $\mathscr{P}$-property if and only if*

$$\left.\begin{array}{l} \mathscr{S}(\hbar_1, \rho_1) = \mathscr{S}(\Phi, \Psi) \\ \mathscr{S}(\hbar_2, \rho_2) = \mathscr{S}(\Phi, \Psi) \end{array}\right\} \textit{implies } \mathscr{S}(\hbar_1, \hbar_2) = \mathscr{S}(\rho_1, \rho_2),$$

where $\hbar_1, \hbar_2 \in \Phi$ and $\rho_1, \rho_2 \in \Psi$.

Definition 5. *[38] Let $\mathscr{CB}(\mathscr{U})$ represent the set of closed and bounded subsets within $\mathscr{U}$. The Pompeiu-Hausdorff metric H, generated by the metric $\mathscr{S}$, is defined by*

$$H(\Phi,\Psi) = \max\{\sup_{a\in\Phi} D(a,\Psi), \sup_{b\in\Psi} D(b,\Phi)\},$$

for $\Phi,\Psi \subseteq \mathscr{CB}(\mathscr{U})$, where

$$\mathscr{D}(a,\Psi) = \inf\{\mathscr{S}(a,b) : b \in \Psi\}.$$

Throughout this manuscript, we will adopt the following notation:

$$\mathscr{S}^*(a,b) = \mathscr{S}(a,b) - \mathscr{S}(\Phi,\Psi),$$

for $a \in \Phi$ and $b \in \Psi$.

3. Coincidence point results

In this section, we investigate the results pertaining to coincidence and best proximity points within the context of a complete suprametric space.

Definition 6. *Consider $\mathbb{K} : \Phi \to \Psi$ and $\mathscr{B} : \Phi \to \Phi$ are mappings, where (Φ,Ψ) are nonempty closed subsets of a suprametric space $(\mathscr{U},\mathscr{S})$ that fulfills the following conditions*

$$\mathscr{S}(\mathscr{B}\hbar,\mathbb{K}\hbar) = \mathscr{S}(\Phi,\Psi),$$

then $\hbar \in \Phi$ is referred to as a coincidence best proximity point of the mapping pair $(\mathscr{B},\mathbb{K})$.

Remark 1. *The results regarding coincidence best proximity points serve as a generalization of both best proximity point results and fixed-point results. Specifically, if we set $\mathscr{B} = I_\Phi$, then every coincidence best proximity point transforms into a best proximity point for the mapping $\mathbb{K}$. Furthermore, when this mapping is a self-mapping, the best proximity point is effectively equivalent to a fixed-point.*

Definition 7. *The pair of mappings $(\mathscr{B},\mathbb{K})$ is referred to as $(\mathscr{F}_\tau)_{C_\mathscr{P}}$-proximal contraction if there exists a $\tau \in (0,+\infty)$ for all $u,v,\hbar,\rho$ in Φ that fulfills the following conditions:*

$$\begin{array}{c}\mathscr{S}(\mathscr{B}u,\mathbb{K}\hbar) = \mathscr{S}(\Phi,\Psi)\\ \mathscr{S}(\mathscr{B}v,\mathbb{K}\rho) = \mathscr{S}(\Phi,\Psi)\end{array}$$

implies that

$$\tau + \mathscr{F}(\mathscr{S}(\mathbb{K}\hbar,\mathbb{K}\rho)) \le \mathscr{F}(\mathscr{S}(\hbar,\rho)). \tag{3.1}$$

Theorem 1. *Let Φ and Ψ be a nonempty closed subsets of a complete suprametric space $(\mathscr{U},\mathscr{S})$ that fulfills the $\mathscr{P}$-property, with Φ_0 being nonempty. Given continuous mappings $\mathbb{K} : \Phi \to \Psi$ and $\mathscr{B} : \Phi \to \Phi$ is one-to-one continuous with $\mathbb{K}(\Phi_0) \subseteq \Psi_0$ and $\Phi_0 \subseteq \mathscr{B}(\Phi_0)$. If the pair $(\mathscr{B},\mathbb{K})$ meets the conditions of $(\mathscr{F}_\tau)_{C_\mathscr{P}}$-proximal contraction. Then, $(\mathscr{B},\mathbb{K})$ concedes a coincidence best proximity point.*

Proof. Let $\hbar_0$ be an arbitrary element in Φ_0. Since $\mathbb{K}(\Phi_0) \subseteq \Psi_0$ and $\Phi_0 \subseteq \mathscr{B}(\Phi_0)$, it follows that there exists an element $\hbar_1$ in Φ_0 such that

$$\mathscr{S}(\mathscr{B}\hbar_1, \mathbb{K}\hbar_0) = \mathscr{S}(\Phi, \Psi).$$

Furthermore, since $\mathbb{K}\hbar_1$ belongs to $\mathbb{K}(\Phi_0)$, which is included in Ψ_0, and given that Φ_0 is a subset of $\mathscr{B}(\Phi_0)$, it can be concluded that there exists an element $\hbar_2$ in Φ_0 such that

$$\mathscr{S}(\mathscr{B}\hbar_2, \mathbb{K}\hbar_1) = \mathscr{S}(\Phi, \Psi).$$

Likewise, for $\hbar_{p-1} \in \Phi_0$ with $\mathbb{K}(\Phi_0) \subseteq \Psi_0$, there exists an element $\hbar_p \in \Phi_0$ such that

$$\mathscr{S}(\mathscr{B}\hbar_p, \mathbb{K}\hbar_{p-1}) = \mathscr{S}(\Phi, \Psi). \tag{3.2}$$

If we pick a sequence $\{\hbar_p\}$ that works, we can find a point $\hbar_{p+1}$ in Φ_0 such that

$$\mathscr{S}(\mathscr{B}\hbar_{p+1}, \mathbb{K}\hbar_p) = \mathscr{S}(\Phi, \Psi) \tag{3.3}$$

for any positive integer p. Utilizing the $\mathscr{P}$-property, $\mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_{p+1}) = \mathscr{S}(\mathbb{K}\hbar_p, \mathbb{K}\hbar_{p-1})$. Given that the pair $(\mathscr{B}, \mathbb{K})$ satisfies the conditions of $(\mathscr{F}_\tau)_{C_\mathscr{P}}$-proximal contraction, it follows from Eqs (3.2) and (3.3) that

$$\mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_{p+1}))) \leq \mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_{p-1}, \mathscr{B}\hbar_p))) - \tau. \tag{3.4}$$

Consequently,

$$\mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_{p+1}) \leq \mathscr{S}(\mathscr{B}\hbar_{p-1}, \mathscr{B}\hbar_p).$$

Consequently, the sequence $\{(\mathscr{B}\hbar_p, \mathscr{B}\hbar_{p+1})\}$ is monotonically decreasing and bounded from below. Thus, there exists a nonnegative real number λ such that

$$\lim_{p\to\infty} \mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_{p+1}) = \lambda \geq 0. \tag{3.5}$$

Let us assume that $\lim_{p\to\infty} \mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_{p+1}) = \lambda > 0$. Consequently, by applying Eqs (3.4) and (3.5), we deduce that

$$\mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_{p+1}))) \leq \mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_{p-2}, \mathscr{B}\hbar_{p-1}))) - 2\tau.$$

Proceeding in this manner, we obtain

$$\mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_{p+1}))) \leq \mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_0, \mathscr{B}\hbar_1))) - n\tau. \tag{3.6}$$

Consequently, the preceding equation yields the result that

$$\lim_{p\to\infty} \mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_{p+1}))) = -\infty.$$

Applying Eq (F2) from Definition 3, we obtain

$$\lim_{p\to\infty} (\mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_{p+1})) = 0. \tag{3.7}$$

Reapplying Eq (F3) from Definition 3, we find that there exists a real number $k \in (0,1)$, such that

$$\begin{aligned}\lim_{\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})\to 0}((\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})))^k\mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1}))) &= 0,\\ \lim_{p\to\infty}((\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})))^k\mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1}))) &= 0. \end{aligned} \tag{3.8}$$

If we use Eq (3.6), we can see that

$$\mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1}))) \leq \mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_0,\mathscr{B}\hbar_1))) - n\tau$$

$$\begin{aligned}\Rightarrow ((\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})))^k\mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1}))) &- \mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_0,\mathscr{B}\hbar_1)))\\ &\leq -((\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})))^k n\tau.\end{aligned}$$

If we let $\wp_p = \mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})$, then the preceding inequality can be expressed as follows:

$$(\wp_p)^k(\mathscr{F}(\wp_p) - \mathscr{F}(\wp_0)) \leq -(\wp_p)^k n\tau.$$

Applying Eqs (3.7) and (3.8) and taking $\lim_{p\to\infty}$, we conclude

$$\lim_{p\to\infty}(\wp_p)^k(\mathscr{F}(\wp_p) - \mathscr{F}(\wp_0)) \leq \lim_{p\to\infty} -(\wp_p)^k n\tau \leq 0.$$

Consequently, the preceding inequality yields the result

$$\lim_{p\to\infty} p(\wp_p)^k = 0. \tag{3.9}$$

The equation above implies that for any $\varepsilon > 0$, there exists $p_1 \in N$ such that

$$\begin{aligned}\left|p(\wp_p)^k - 0\right| &< \varepsilon, \text{ for all } p \geq p_1\\ \left|p(\wp_p)^k\right| &< \varepsilon\\ (\wp_p) &< \frac{\varepsilon}{p^{\frac{1}{k}}}.\end{aligned} \tag{3.10}$$

Now, we need to demonstrate that $\{\mathscr{B}\hbar_p\}$ forms a Cauchy sequence for all natural numbers $p,q \in \mathbb{N}$, we have

$$\begin{aligned}\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_q) &\leq \mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1}) + \mathscr{S}(\mathscr{B}\hbar_{p+1},\mathscr{B}\hbar_q) + \eth\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathscr{B}\hbar_q)\\ &\leq \mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1}) + [1 + \eth\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})]\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathscr{B}\hbar_q)\\ &\leq \mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1}) + [1 + \eth\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathscr{B}\hbar_{p+2})][\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathscr{B}\hbar_{p+2}) + \mathscr{S}(\mathscr{B}\hbar_{p+2},\mathscr{B}\hbar_q)\\ &+ \eth\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathscr{B}\hbar_{p+2}) + \mathscr{S}(\mathscr{B}\hbar_{p+2},\mathscr{B}\hbar_q)]\\ &= \mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1}) + \mathscr{S}(\mathscr{B}\hbar_{p+1},\mathscr{B}\hbar_{p+2}) + \mathscr{S}(\mathscr{B}\hbar_{p+2},\mathscr{B}\hbar_q)\\ &+ \eth\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathscr{B}\hbar_{p+2})\mathscr{S}(\mathscr{B}\hbar_{p+2},\mathscr{B}\hbar_q)\\ &+ \eth\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathscr{B}\hbar_{p+2}) + \eth\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})\mathscr{S}(\mathscr{B}\hbar_{p+2},\mathscr{B}\hbar_q)\end{aligned}$$

$$
\begin{aligned}
&+ \quad \eth^2 \mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_{p+1}) \mathscr{S}(\mathscr{B}\hbar_{p+1}, \mathscr{B}\hbar_{p+2}) \mathscr{S}(\mathscr{B}\hbar_{p+2}, \mathscr{B}\hbar_q) \\
&\leq \quad \mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_{p+1}) + \mathscr{S}(\mathscr{B}\hbar_{p+1}, \mathscr{B}\hbar_{p+2})[1+\eth\mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_{p+1})] \\
&+ \quad [1+\eth\mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_{p+1})][1+\eth\mathscr{S}(\mathscr{B}\hbar_{p+1}, \mathscr{B}\hbar_{p+2})]\mathscr{S}(\mathscr{B}\hbar_{p+2}, \mathscr{B}\hbar_q) \\
&\leq \quad \wp_p + \wp_{p+1}[1+\eth\wp_p] + [1+\eth\wp_p][1+\eth\wp_{p+1}]\mathscr{S}(\mathscr{B}\hbar_{p+2}, \mathscr{B}\hbar_q).
\end{aligned}
$$

By iterating this procedure, we ultimately arrive at

$$
\mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_q) \leq \wp_p \sum_{i=0}^{q-1} \mathscr{S}_i \prod_{j=0}^{i-1} [1+\eth\wp_{p+j}].
$$

If we use Eq (3.10), we can see that

$$
\mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_q) \leq \frac{\varepsilon}{p^{\frac{1}{k}}} \sum_{i=0}^{q-1} \frac{\varepsilon}{i^{\frac{1}{k}}} \prod_{j=0}^{i-1} \left[1+\eth\left(\frac{\varepsilon}{(p+j)^{\frac{1}{k}}}\right)\right].
$$

Given that $k \in (0,1)$, which follows that

$$
\mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_q) \leq \frac{\varepsilon}{p^{\frac{1}{k}}} \sum_{i=0}^{q-1} \frac{\varepsilon}{i^{\frac{1}{k}}} \prod_{j=0}^{i-1} \left[1+\eth\left(\frac{\varepsilon}{p^{\frac{1}{k}}}\right)\right]. \tag{3.11}
$$

Let us assume that,

$$
Q_i = \frac{\varepsilon}{i^{\frac{1}{k}}} \prod_{j=0}^{i-1} \left[1+\eth\left(\frac{\varepsilon}{p^{\frac{1}{k}}}\right)\right]. \tag{3.12}
$$

Utilizing the ratio test on Eq (3.12), it follows that

$$
\lim_{i\to\infty} \left|\frac{Q_{i+1}}{Q_i}\right| < 1, \text{ since } k \in (0,1).
$$

Taking $\lim_{p\to\infty}$ then Eq (3.11) transforms into

$$
\lim_{p\to\infty} \mathscr{S}(\mathscr{B}\hbar_p, \mathscr{B}\hbar_q) = 0.
$$

Since $\{\mathscr{B}\hbar_p\}$ forms a Cauchy sequence and $(\mathscr{U}, \mathscr{S})$ is a complete suprametric space, the sequence $\{\mathscr{B}\hbar_p\}$ converges to a point $\hbar^*$ in Φ_0. Consequently, the sequence $\{\hbar_p\}$ also converges to $\hbar^*$, as Φ is closed and Φ_0 is a subset of Φ. Given the continuity of the mappings $(\mathscr{B}, \mathbb{K})$, it follows that

$$
\mathscr{S}(\mathscr{B}\hbar^*, \mathbb{K}\hbar^*) = \mathscr{S}(\Phi, \Psi).
$$

Thus, $\hbar^*$ qualifies as a coincidence best proximity point for the pair of mappings $(\mathscr{B}, \mathbb{K})$. □

Definition 8. *A mapping $\mathbb{K} : \Phi \to \Psi$ is known as $(\mathscr{F}_\tau)_{\Psi_\mathscr{P}}$-proximal contraction if there exists a $\tau \in (0, +\infty)$ for all $u, v, \hbar, \rho$ in Φ that fulfills the following conditions,*

$$
\begin{aligned}
\mathscr{S}(u, \mathbb{K}\hbar) &= \mathscr{S}(\Phi, \Psi) \\
\mathscr{S}(v, \mathbb{K}\rho) &= \mathscr{S}(\Phi, \Psi)
\end{aligned}
$$

implies that

$$
\tau + \mathscr{F}(\mathscr{S}(\mathbb{K}\hbar, \mathbb{K}\rho)) \leq \mathscr{F}(\mathscr{S}(\hbar, \rho)). \tag{3.13}
$$

Corollary 1. *Let Φ and Ψ be the nonempty closed subsets of a complete suprametric space $(\mathscr{U},\mathscr{S})$ that fulfills the $\mathscr{P}$-property, with Φ_0 being nonempty. Given a continuous mapping $\mathbb{K}:\Phi\to\Psi$ with $\mathbb{K}(\Phi_0)\subseteq\Psi_0$. If $\mathbb{K}$ meets the conditions of $(\mathscr{F}_\tau)_{\Psi_{\mathscr{P}}}$-proximal contraction, then $\mathbb{K}$ concedes a best proximity point.*

Proof. If we take identity mapping $\mathscr{B}=I_\Phi$ ($\mathscr{B}$ is identity on Φ), the remaining proof is the same as of Theorem 1.

□

We will now discuss the rational-type proximal contraction, which involves rational expressions to guarantee coincidence and best proximity points in a structured way.

Definition 9. *The pair of mapping $(\mathscr{B},\mathbb{K})$ is said to be rational type $(\mathscr{F}_{R_\tau})_{C_{\mathscr{P}}}$-proximal contraction, where $\mathbb{K}:\Phi\to\Psi$ and $\mathscr{B}:\Phi\to\Phi$ if there exists a $\tau\in(0,+\infty)$ for all $u,v,\hbar,\rho$ in Φ that fulfills the following conditions,*

$$\mathscr{S}(\mathscr{B}u,\mathbb{K}\hbar)=\mathscr{S}(\Phi,\Psi)$$
$$\mathscr{S}(\mathscr{B}v,\mathbb{K}\rho)=\mathscr{S}(\Phi,\Psi)$$

implies that

$$\tau+\mathscr{F}(\mathscr{S}(\mathbb{K}\hbar,\mathbb{K}\rho))\leq\mathscr{F}(\mathscr{A}(u,v,\hbar,\rho)), \tag{3.14}$$

where

$$\begin{aligned}\mathscr{A}(u,v,\hbar,\rho) \;=\; & \max\{\mathscr{S}(\mathscr{B}\hbar,\mathscr{B}\rho),\mathscr{S}^*(\mathscr{B}v,Tu),\\ & \mathscr{S}(\mathscr{B}v,\mathbb{K}\hbar)-\mathscr{S}(\mathscr{B}u,\mathbb{K}\hbar)-\eth\mathscr{S}(\mathscr{B}v,\mathscr{B}u)\mathscr{S}(\mathscr{B}\rho,\mathbb{K}\hbar),\\ & \mathscr{S}(\mathscr{B}u,\mathbb{K}\rho)-\mathscr{S}(\mathscr{B}u,\mathbb{K}\hbar)-\eth\mathscr{S}(\mathscr{B}u,\mathscr{B}v)\mathscr{S}(\mathscr{B}v,\mathbb{K}\rho)\}.\end{aligned}$$

Theorem 2. *Let Φ and Ψ be a nonempty closed subsets of a complete suprametric space $(\mathscr{U},\mathscr{S})$ that fulfill the $\mathscr{P}$-property, with Φ_0 being nonempty. Given continuous mappings $\mathbb{K}:\Phi\to\Psi$ and $\mathscr{B}:\Phi\to\Phi$ is one-to-one continuous with $\mathbb{K}(\Phi_0)\subseteq\Psi_0$ and $\Phi_0\subseteq\mathscr{B}(\Phi_0)$. If the pair $(\mathscr{B},\mathbb{K})$ meets the conditions of rational type $(\mathscr{F}_{R_\tau})_{C_{\mathscr{P}}}$-proximal contraction. Consequently, the pair $(\mathscr{B},\mathbb{K})$ admits a coincidence best proximity point.*

Proof. By virtue of Theorem 1, we arrive at

$$\begin{aligned}\mathscr{S}(\mathscr{B}\hbar_p,\mathbb{K}\hbar_{p-1}) &= \mathscr{S}(\Phi,\Psi)\\ \mathscr{S}(\mathscr{B}\hbar_{p+1},\mathbb{K}\hbar_p) &= \mathscr{S}(\Phi,\Psi),\end{aligned}$$

for any positive integer p. Employing the $\mathscr{P}$-property $\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})=\mathscr{S}(\mathbb{K}\hbar_p,\mathbb{K}\hbar_{p-1})$. Given that the pair $(\mathscr{B},\mathbb{K})$ satisfies the conditions of $(\mathscr{F}_{R_\tau})_{C_{\mathscr{P}}}$-proximal contraction, from 3.14 we conclude

$$\mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})))\leq\mathscr{F}(\mathscr{A}(\hbar_p,\hbar_{p+1},\hbar_{p-1},\hbar_p))-\tau,$$

where

$$\begin{aligned}\mathscr{A}(\hbar_p,\hbar_{p+1},\hbar_{p-1},\hbar_p) \;=\; & \max\{\mathscr{S}(\mathscr{B}\hbar_{p-1},\mathscr{B}\hbar_p),\mathscr{S}^*(\mathscr{B}\hbar_{p+1},\mathbb{K}\hbar_p)\\ & \mathscr{S}(\mathscr{B}\hbar_{p+1},\mathbb{K}\hbar_{p-1})-\mathscr{S}(\mathscr{B}\hbar_p,\mathbb{K}\hbar_{p-1})-\eth\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathscr{B}\hbar_p)\mathscr{S}(\mathscr{B}\hbar_p,\mathbb{K}\hbar_{p-1}),\end{aligned}$$

$$
\begin{aligned}
& \mathscr{S}(\mathscr{B}\hbar_p,\mathbb{K}\hbar_p)-\mathscr{S}(\mathscr{B}\hbar_p,\mathbb{K}\hbar_{p-1})-\eth\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathbb{K}\hbar_p)\} \\
\leq\ & \max\{\mathscr{S}(\mathscr{B}\hbar_{p-1},\mathscr{B}\hbar_p),\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathbb{K}\hbar_p)-\mathscr{S}(\Phi,\Psi),\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathscr{B}\hbar_p) \\
+\ & \mathscr{S}(\mathscr{B}\hbar_p,\mathbb{K}\hbar_{p-1})+\eth\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathbb{K}\hbar_p)-\mathscr{S}(\mathscr{B}\hbar_p,\mathbb{K}\hbar_{p-1}) \\
-\ & \eth\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathbb{K}\hbar_p),\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})+\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathbb{K}\hbar_p) \\
+\ & \eth\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathbb{K}\hbar_p)-\mathscr{S}(\mathscr{B}\hbar_p,\mathbb{K}\hbar_{p-1}) \\
-\ & \eth\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathbb{K}\hbar_p)\} \\
=\ & \max\{\mathscr{S}(\mathscr{B}\hbar_{p-1},\mathscr{B}\hbar_p),0,\mathscr{S}(\mathscr{B}\hbar_{p+1},\mathscr{B}\hbar_p),\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})\} \\
=\ & \max\{\mathscr{S}(\mathscr{B}\hbar_{p-1},\mathscr{B}\hbar_p),\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})\}.
\end{aligned}
$$

If $\max\{\mathscr{S}(\mathscr{B}\hbar_{p-1},\mathscr{B}\hbar_p),\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})\}=\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})$, which results in a contradiction. Consequently, we get

$$\mathscr{F}((\mathscr{S}(\mathscr{B}\hbar_p,\mathscr{B}\hbar_{p+1})))\leq\mathscr{F}(\mathscr{S}(\mathscr{B}\hbar_{p-1},\mathscr{B}\hbar_p))-\tau.$$

In accordance with Theorem 1, we can establish that $(\mathscr{B},\mathbb{K})$ constitutes a pair of continuous mappings, which indicates that

$$\mathscr{S}(\mathscr{B}\hbar^*,\mathbb{K}\hbar^*)=\mathscr{S}(\Phi,\Psi).$$

Consequently, $\hbar^*$ serves as a coincidence best proximity point for the mapping pair $(\mathscr{B},\mathbb{K})$. □

Definition 10. *A mapping $\mathbb{K}:\Phi\to\Psi$ is referred to as rational type $(\mathscr{F}_{R_\tau})_{\Psi_{\mathscr{P}}}$-proximal contraction if there exists a $\tau\in(0,+\infty)$ for all $u,v,\hbar,\rho$ in Φ that meets the following criteria,*

$$
\begin{aligned}
\mathscr{S}(u,\mathbb{K}\hbar)=\mathscr{S}(\Phi,\Psi)\\
\mathscr{S}(v,\mathbb{K}\rho)=\mathscr{S}(\Phi,\Psi)
\end{aligned}
$$

implies that

$$\tau+\mathscr{F}(\mathscr{S}(\mathbb{K}\hbar,\mathbb{K}\rho))\leq\mathscr{F}(\mathscr{A}(u,v,\hbar,\rho)),\tag{3.15}$$

where

$$
\begin{aligned}
\mathscr{A}(u,v,\hbar,\rho)\ =\ & \max\{\mathscr{S}(\hbar,\rho),\mathscr{S}^*(v,Tu),\\
& \mathscr{S}(v,\mathbb{K}\hbar)-\mathscr{S}(u,\mathbb{K}\hbar)-\eth\mathscr{S}(v,u)\mathscr{S}(\rho,\mathbb{K}\hbar),\\
& \mathscr{S}(u,\mathbb{K}\rho)-\mathscr{S}(u,\mathbb{K}\hbar)-\eth\mathscr{S}(u,v)\mathscr{S}(v,\mathbb{K}\rho)\}.
\end{aligned}
$$

Corollary 2. *Let Φ and Ψ be the nonempty closed subsets of a complete suprametric space $(\mathscr{U},\mathscr{S})$ that fulfill the $\mathscr{P}$-property, with Φ_0 being nonempty. Given a continuous mapping $\mathbb{K}:\Phi\to\Psi$ with $\mathbb{K}(\Phi_0)\subseteq\Psi_0$. If $\mathbb{K}$ meets the conditions of $(\mathscr{F}_{R_\tau})_{\Psi_{\mathscr{P}}}$-proximal contraction, then $\mathbb{K}$ concedes a best proximity point.*

Proof. If we take identity mapping $\mathscr{B}=I_\Phi$ ($\mathscr{B}$ is identity on Φ), the remaining proof is the same as of Theorem 2.

□

Example 3. *Let $\mathscr{U} = \{1,2,3,4,5,6\}$ and distance function is defined as*

$$\mathscr{S}(\hbar,\rho) = \begin{cases} 0, & \text{if } \hbar = \rho, \\ |\hbar-\rho|^2 & \text{otherwise.} \end{cases}$$

By simple calculation, we obtain $\eth = 2$, and $(\mathscr{U},\mathscr{S})$ is a suprametric space. Assume that $\Phi = \{2,4,6\}$ and $\Psi = \{1,3,5\}$ are two nonempty subsets of suprametric space with $\mathscr{S}(\Phi,\Psi) = 1$. Clearly, the $\mathscr{P}$-property satisfied with $\Phi = \Phi_0$ and $\Psi = \Psi_0$. Define $\mathbb{K} : \Phi \to \Psi$ such that

$$\mathbb{K}\hbar = \begin{cases} 1 & \text{for } \hbar = \{2,4\} \\ 3 & \text{when } \hbar = 6, \end{cases}$$

and $\mathscr{B} : \Phi \to \Phi$ in such a way that

$$\mathscr{B}\hbar = \begin{cases} 2 & \text{for } \hbar = 4, \\ 4 & \text{if } \hbar = 6, \\ 6 & \text{when } \hbar = 2. \end{cases}$$

The analysis clearly indicates that $\mathbb{K}(\Phi_0) \subseteq \Psi_0$ and $\Phi_0 \subseteq \mathscr{B}(\Phi_0)$. The pair of mappings $(\mathscr{B},\mathbb{K})$ meets the criteria for a $(\mathscr{F}_\tau)_{C_\mathscr{P}}$-proximal contraction 3.1 for all $u,v,\hbar,\rho \in \Phi$. Since

$$\begin{aligned} \mathscr{S}(\mathscr{B}4,\mathbb{K}2) &= \mathscr{S}(\Phi,\Psi), \\ \mathscr{S}(\mathscr{B}4,\mathbb{K}6) &= \mathscr{S}(\Phi,\Psi), \end{aligned}$$

where $u = v = 4$, $\hbar = 2$ and $\rho = 6$. Through basic calculations, we arrive at the conclusion that

$$\begin{aligned} \tau + \mathscr{F}(\mathscr{S}(\mathbb{K}2,\mathbb{K}6)) &\leq \mathscr{F}(\mathscr{S}(2,6)), \\ \tau + \mathscr{F}(4) &\leq \mathscr{F}(16), \end{aligned}$$

for $\tau = 1.30$ and $\mathscr{F}(t) = \ln t$, all the requirements of Theorem 1 are fulfilled. Thus, 4 and 6 serve as the coincidence points for the pair of mappings $(\mathscr{B},\mathbb{K})$.

4. Fixed-point results

In this section, we discuss fixed-point results for single-valued mappings. By setting $\Phi = \Psi = \mathscr{U}$, Theorems 1 and 2 establish the existence of fixed points within the framework of a complete suprametric space.

Definition 11. *A self-mapping $\mathbb{K} : \mathscr{U} \to \mathscr{U}$ is referred to as a $(\mathscr{F}_\tau)_{F_\mathscr{P}}$-contraction if there exists a $\tau \in (0,+\infty)$ for all $\hbar,\rho \in \mathscr{U}$ such that,*

$$\tau + \mathscr{F}(\mathscr{S}(\mathbb{K}\hbar,\mathbb{K}\rho)) \leq \mathscr{F}(\mathscr{S}(\hbar,\rho)). \tag{4.1}$$

Corollary 3. *Suppose $\mathbb{K} : \mathscr{U} \to \mathscr{U}$ is a mapping on a nonempty subset of a complete suprametric space $(\mathscr{U},\mathscr{S})$ satisfying the requirements of $(\mathscr{F}_\tau)_{F_\mathscr{P}}$-contraction. Under these circumstances, the mapping $\mathbb{K}$ guarantees the existence of a fixed-point.*

Proof. If we take $\Phi = \Psi = \mathscr{U}$, the remaining proof is the same as of Theorem 1. □

Example 4. *Let* $\mathscr{U} = \{1,2,3,4,5,6\}$ *and distance function is defined as*

$$\mathscr{S}(\hbar,\rho) = \begin{cases} 0, & \text{if } \hbar = \rho, \\ |\hbar-\rho|^2 & \text{otherwise.} \end{cases}$$

By simple calculation, we obtain $\eth = 2$*, and* $(\mathscr{U},\mathscr{S})$ *is a suprametric space. Define* $\mathbb{K} : \mathscr{U} \to \mathscr{U}$ *such that*

$$\mathbb{K}\hbar = \begin{cases} 1 & \text{for } \hbar = \{1,2\} \\ 2 & \text{when } \hbar = 6. \end{cases}$$

The pair of mappings $(\mathscr{B},\mathbb{K})$ *satisfies* $(\mathscr{F}_\tau)_{F_\mathscr{P}}$*-contraction* 4.1 *for all* $\hbar,\rho \in \mathscr{U}$. *Through fundamental calculations, we can conclude that*

$$\begin{aligned} \tau + \mathscr{F}(\mathscr{S}(\mathbb{K}2,\mathbb{K}6)) &\leq \mathscr{F}(\mathscr{S}(2,6)), \\ \tau + \mathscr{F}(1) &\leq \mathscr{F}(16), \end{aligned}$$

for $\tau = 2.70$ *and* $\mathscr{F}(t) = \ln t$. *Since all the conditions of Corollary* 3 *are met, we can conclude that* 1 *is a fixed-point of the mapping* $\mathbb{K}$.

Definition 12. *A self-mapping* $\mathbb{K} : \mathscr{U} \to \mathscr{U}$ *is known as* $(\mathscr{F}_{R_\tau})_{F_\mathscr{P}}$*-contraction if there exists a* $\tau \in (0,+\infty)$ *for all* $\hbar,\rho \in \mathscr{U}$ *such that,*

$$\tau + \mathscr{F}(\mathscr{S}(\mathbb{K}\hbar,\mathbb{K}\rho)) \leq \mathscr{F}(\mathscr{A}(\hbar,\rho)), \tag{4.2}$$

where

$$\begin{aligned} \mathscr{A}(\hbar,\rho) = \max\{&\mathscr{S}(\hbar,\rho), \mathscr{S}(\rho,\mathbb{K}\hbar), \\ &\mathscr{S}(\rho,\mathbb{K}\hbar) - \eth\mathscr{S}(\rho,\hbar)\mathscr{S}(\rho,\mathbb{K}\hbar), \\ &\mathscr{S}(\hbar,\mathbb{K}\rho) - \eth\mathscr{S}(\hbar,\rho)\mathscr{S}(\rho,\mathbb{K}\rho)\}. \end{aligned}$$

Corollary 4. *Suppose* $\mathbb{K} : \mathscr{U} \to \mathscr{U}$ *is a mapping on a nonempty subset of a complete suprametric space* $(\mathscr{U},\mathscr{S})$ *satisfying the requirements of* $(\mathscr{F}_{R_\tau})_{F_\mathscr{P}}$*-contraction, then mapping* $\mathbb{K}$ *concedes a fixed-point.*

Proof. If we take $\Phi = \Psi = \mathscr{U}$, the remaining proof is the same as of Theorem 2. □

Example 5. *Let* $\mathscr{U} = \{11,12,13,14,15,16\}$ *and distance function is defined as*

$$\mathscr{S}(\hbar,\rho) = \begin{cases} 0, & \text{if } \hbar = \rho, \\ |\hbar-\rho|^2 & \text{otherwise.} \end{cases}$$

By simple calculation, we obtain $\eth = 2$*, and* $(\mathscr{U},\mathscr{S})$ *is a suprametric space. Define* $\mathbb{K} : \mathscr{U} \to \mathscr{U}$ *such that*

$$\mathbb{K}\hbar = \begin{cases} 11 & \text{for } \hbar = \{11,12\} \\ 12 & \text{when } \hbar = 16. \end{cases}$$

The pair of mappings $(\mathscr{B},\mathbb{K})$ *satisfies* $(\mathscr{F}_{R_\tau})_{F_\mathscr{P}}$*-contraction* 4.2 *for all* $\hbar,\rho \in \mathscr{U}$. *Through fundamental calculations, we can conclude that*

$$\tau + \mathscr{F}(\mathscr{S}(\mathbb{K}12,\mathbb{K}16)) \leq \mathscr{F}(\mathscr{A}(12,16)),$$

$$\begin{aligned} \tau + \mathscr{F}(1) &\leq \mathscr{F}(\max\{16, 25, -775, -32\}), \\ \tau + \mathscr{F}(1) &\leq \mathscr{F}(25) \end{aligned}$$

for $\tau = 3.20$ *and* $\mathscr{F}(t) = \ln t$. *All the criteria of Corollary* 4 *are met. Consequently,* 11 *is a fixed-point of the mapping* $\mathbb{K}$.

5. Application to fractional BVP

This section is devoted to studying a boundary value problem associated with a nonlinear fractional differential equation of order $\varpi \in (2,3]$. It takes the form

$$\begin{cases} \mathbb{D}^{\varpi}\Lambda(p) + \Theta(p, \Lambda(p)) = 0, & p \in [0,1], \\ \Lambda(0) = \Lambda'(0) = 0, & \Lambda(1) = \Omega \int_0^1 \Lambda(r)\, dr, \end{cases} \tag{5.1}$$

where $\Omega > 0$, $\Omega \neq \varpi$, $\Theta : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given function and $\mathbb{D}^{\varpi}$ represents the Riemann-Liouville fractional derivative such that

$$\mathbb{D}^{\varpi}\Theta(p) = \begin{cases} \frac{1}{\Gamma(n-\varpi)} \frac{d^n}{dp^n} \left(\int_0^p \frac{\Theta(r)}{(p-r)^{\varpi+1-n}} dr \right) & \text{if } n-1 < \varpi < n \\ \frac{d^n}{dp^n}\Theta(p) & \text{if } n = \varpi. \end{cases}$$

The incorporation of integral boundary conditions renders this type of problem significant for modeling various phenomena, including blood flow, groundwater flow, changes in population, and heat-induced deformations. Although positive outcomes may be emphasized in certain related physical contexts, a thorough analysis requires investigating all possible solutions. Therefore, this research focuses on the existence and uniqueness of solutions of any nature for this problem.

In their earlier work, Cabada and Hamdi [39] investigated problem 5.1 for $\Omega \in (0, \varpi)$, assuming that Θ is continuous. Their focus was on identifying positive solutions to this problem. As an initial step, they derived an explicit representation of the Green's function associated with the boundary value problem, which is formulated as follows:

$$\begin{cases} \mathbb{D}^{\varpi}\Lambda(p) + q(p) = 0, & p \in [0,1], \\ \Lambda(0) = \Lambda'(0) = 0, & \Lambda(1) = \Omega \int_0^1 \Lambda(r)\, dr, \end{cases} \tag{5.2}$$

where q is any continuous function. The authors then identified several properties of this Green's function. By utilizing an integral operator that centers on this Green's function, they formulated conditions for the existence and uniqueness of positive solutions to the problem within the defined parameter interval $\Omega \in (0, \varpi)$. Additionally, they demonstrated that the linear problem 5.2 possesses a unique solution $\Lambda \in \mathbb{C}[0,1]$, expressed as follows:

$$\Lambda(p) = \int_0^1 \Xi(p,r) q(r) dr \tag{5.3}$$

where $\Xi(p,r)$ is the Green's function defined by

$$\Xi(p,r) = \begin{cases} \frac{p^{\varpi-1}(1-r)^{\varpi-1}(\varpi-\Omega+\Omega r) - (\varpi-\Omega)(p-r)^{\varpi-1}}{(\varpi-\Omega)\Gamma(\varpi)}, & \text{if} \quad 0 \leq r \leq p \leq 1 \\ \frac{p^{\varpi-1}(1-r)^{\varpi-1}(\varpi-\Omega+\Omega r)}{(\varpi-\Omega)\Gamma(\varpi)} blue, & \text{if} \quad 0 \leq p \leq r \leq 1 \end{cases} \tag{5.4}$$

where $\Omega \neq \varpi$ and $2 < \varpi \leq 3$. Thorough examination of the properties of the Green's function (5.4) is detailed in the work of Cabada and Hamdi [39]. In order to formulate our theorem regarding existence and uniqueness, we proceed to establish an L^2 norm estimate for the Green's function Ξ, which is an essential aspect of our analysis.

Lemma 1. *For $\Omega > 0$ along with $\Omega \neq \varpi$ and $2 < \varpi \leq 3$ holds. As a result, for every $p, r \in (0,1)$, the Green's function $\Xi(p,r) \in L^2(0,1)$ characterized by*

$$\int_0^1 |\Xi(p,r)|^2 dr < \frac{1}{(\Gamma(\varpi))^2}\left[\frac{4}{3} + \frac{\Omega}{3|\Omega-\varpi|} + \frac{\Omega^2}{30(\Omega-\varpi)^2}\right].$$

Proof. According to the definition of the Green's function (5.4), for $\Omega > 0$ along with $\Omega \neq \varpi$ and $2 < \varpi \leq 3$, then we have:
For a period of $0 < p \leq r < 1$, we arrive at

$$\begin{aligned} |\Xi(p,r)| &\leq \frac{p^{\varpi-1}(1-r)^{\varpi-1}(|\varpi-\Omega|+\Omega r)}{|\varpi-\Omega|\Gamma(\varpi)} \\ &\leq \frac{1}{\Gamma(\varpi)}\left(1+\frac{\Omega r}{|\varpi-\Omega|}\right)(1-r)^{\varpi-1}. \end{aligned} \tag{5.5}$$

In exchange for $0 < r \leq p < 1$, we conclude

$$\begin{aligned} |\Xi(p,r)| &\leq \frac{p^{\varpi-1}(1-r)^{\varpi-1}(2|\varpi-\Omega|+\Omega r)}{|\varpi-\Omega|\Gamma(\varpi)} \\ &\leq \frac{1}{\Gamma(\varpi)}\left(2+\frac{\Omega r}{|\varpi-\Omega|}\right)(1-r)^{\varpi-1}. \end{aligned} \tag{5.6}$$

As a result, for all p and r in the interval $(0,1)$,

$$|\Xi(p,r)|^2 \leq \frac{1}{(\Gamma(\varpi))^2}\left(4+\frac{4\Omega r}{|\varpi-\Omega|}+\frac{\Omega^2 r^2}{(\varpi-\Omega)^2}\right)(1-r)^{2\xi-2}. \tag{5.7}$$

From this, it can be concluded that

$$\begin{aligned} \int_0^1 |\Xi(p,r)|^2 dr &\leq \frac{4}{(\Gamma(\varpi))^2}\int_0^1 (1-r)^{2\xi-2}dr + \frac{4\Omega}{|\varpi-\Omega|(\Gamma(\varpi))^2}\int_0^1 r(1-r)^{2\xi-2}dr \\ &+ \frac{\Omega^2 r^2}{(\varpi-\Omega)^2(\Gamma(\varpi))^2}\int_0^1 r^2(1-r)^{2\xi-2}dr \\ &\leq \frac{1}{(\Gamma(\varpi))^2}\left[\frac{4}{2\xi-1}+\frac{2\Omega}{|\varpi-\Omega|\varpi(2\xi-1)}+\frac{\Omega^2}{(\varpi-\Omega)^2(2\xi-1)\varpi(2\xi+1)}\right] \\ &\leq \frac{1}{(\Gamma(\varpi))^2}\left[\frac{4}{3}+\frac{\Omega}{3|\varpi-\Omega|}+\frac{\Omega^2}{30(\varpi-\Omega)^2}\right]. \end{aligned}$$

□

Next, we present a mapping $\mathbb{K}:\mathbb{C}[0,1]\to\mathbb{C}[0,1]$ in such a way that

$$\mathbb{K}\Lambda(p)=\int_0^1 \Xi(p,r)\Theta(r,\Lambda(r))dr, \quad p\in[0,1]. \tag{5.8}$$

It is clear that if both functions $\Xi(p,r)$ and $\Theta(r,\Lambda(r))$ are integrable, then the expression on the right side of the integral equation remains continuous over the interval $[0,1]$. As a result, the mapping $\mathbb{K}$ operates within the space $\mathbb{C}[0,1]$, effectively transforming it into itself. According to Eq (5.3), a solution to problem 5.1 can be identified with a fixed-point of the mapping $\mathbb{K}$. Let us define the function $\mathscr{S}:\mathbb{C}[0,1]\times\mathbb{C}[0,1]\to[0,\infty)$ be presented as

$$\mathscr{S}(\Lambda,v)=\sup_{p\in[0,1]}|\Lambda(p)-v(p)|^2. \tag{5.9}$$

The function $\mathscr{S}$ defines a complete suprametric space on $\mathbb{C}[0,1]$ for $\eth\geq 2$. Based on this framework, we proceed to present an existence and uniqueness theorem that addresses the solution to problem 5.1.

Theorem 3. *Assume that* $2<\varpi\leq 3$ *along with*

$$\frac{1}{(\Gamma(\varpi))^2}\left[\frac{4}{3}+\frac{\Omega}{3|\Omega-\varpi|}+\frac{\Omega^2}{30(\Omega-\varpi)^2}\right]<\frac{1}{\exp(\tau)}, \tag{5.10}$$

where $\tau\in(0,+\infty)$ *is valid for any* $\Omega>0,\Omega\neq\varpi$*. Taking any* $\Lambda,v\in\mathbb{C}[0,1]$*, the function* $\Theta(r,\Lambda(r))$ *is integrable on* $[0,1]$ *along with the inequality*

$$|\Theta(r,\Lambda(r))-\Theta(r,v(r))|^2\leq|\Lambda(r)-v(r)|^2, \quad r\in[0,1].$$

As a result, the mapping $\mathbb{K}$ *introduced in Eq (5.8) possesses a unique fixed point. Therefore, the solution to problem 5.1, represented by* Λ^**, is unique within the set of continuous functions on the interval* $[0,1]$.

Proof. By applying the Cauchy-Schwarz inequality alongside the definition of the mapping $\mathbb{K}$ from Eq (5.8), we deduce the following:

$$\begin{aligned}
|\mathbb{K}\Lambda(p)-\mathbb{K}v(p)|^2 &= \left|\int_0^1 \Xi(p,r)\Theta(r,\Lambda(r))dr-\int_0^1 \Xi(p,r)\Theta(r,v(r))dr\right|^2\\
&\leq \left(\int_0^1 \Xi(p,r)dr\right)^2|\Theta(r,\Lambda(r))-\Theta(r,v(r))|^2\\
&\leq \frac{1}{(\Gamma(\varpi))^2}\left[\frac{4}{3}+\frac{\Omega}{3|\Omega-\varpi|}+\frac{\Omega^2}{30(\Omega-\varpi)^2}\right]|\Theta(r,\Lambda(r))-\Theta(r,v(r))|^2\\
&\leq \frac{1}{\exp(\tau)}|\Lambda(r)-v(r)|^2.
\end{aligned}$$

Taking the supremum over $[0,1]$ together with the definition of the suprametric 5.9, we obtain

$$\mathscr{S}(\mathbb{K}\Lambda,\mathbb{K}v)\leq\frac{1}{\exp(\tau)}\mathscr{S}(\Lambda,v),$$

which can be written as

$$\exp(\tau)\mathscr{S}(\mathbb{K}\Lambda, \mathbb{K}v) \leq \mathscr{S}(\Lambda, v).$$

By taking $\mathscr{F}(t) = \ln t$, we conclude

$$\tau + \mathscr{F}(\mathscr{S}(\mathbb{K}\Lambda, \mathbb{K}v)) \leq \mathscr{F}(\mathscr{S}(\Lambda, v)).$$

The mapping $\mathbb{K}$, as defined in Eq (5.8), is demonstrated to fulfill the conditions stipulated by Corollary 3. Consequently, it possesses a unique fixed-point, thereby implying the existence and uniqueness of a solution to the boundary value problem 5.1 within the function space $\mathbb{C}[0,1]$. □

6. Conclusions

In conclusion, this manuscript provides a comprehensive exploration of the interrelations between coincidence points, best proximity points, and fixed-point results using the framework of $(\mathscr{F}_\tau)_{C_\mathscr{P}}$-proximal contractions and rational-type $(\mathscr{F}_{R_\tau})_{C_\mathscr{P}}$-proximal contractions within suprametric spaces. By bridging theoretical concepts with practical examples, we have demonstrated the real-world applicability of these abstract ideas. Additionally, we have extended our investigation to boundary value problems (BVPs) associated with nonlinear fractional differential equations of order $2 < \varpi \leq 3$.

Through the transformation of these BVPs into integral equations, we derived the necessary conditions for the existence and uniqueness of fixed-points under $(\mathscr{F}_\tau)_{F_\mathscr{P}}$-contractions. A vital component of this study involved the approximation of the Green's function, which proved instrumental in confirming the existence and uniqueness of solutions to the problem. The results presented here not only enhance existing literature but also offer robust methodologies for tackling complex mathematical challenges, particularly in the realm of fractional differential equations. This work lays the foundation for future research in fixed-point theory and fractional calculus and their applications to nonlinear dynamic systems.

Open questions. In this study, we concentrated on the range $2 < \varpi \leq 3$, which coincides with the features of the Green's function and the analytical techniques applied. Extending the conclusions to other ranges, such as $0 < \varpi \leq 2$ or $\varpi > 3$, would require re-evaluating these features and applying fixed-point theorems for different fractional orders. We believe this is an attractive path for future research. Furthermore, this research is now restricted to the Riemann-Liouville fractional derivative and integral boundary conditions. Exploring alternate fractional derivatives, such as Caputo or Atangana-Baleanu, or boundary conditions, such as periodic or multi-point, may broaden the applicability of our findings. Additionally, reducing the assumptions on the fractional differential equation, or extending the methodology to higher-order fractional equations and coupled systems, may give new perspectives. For further insights into fractional calculus and contraction mappings, it would be interesting to check which conditions allow an approach to coupled systems, studying the works [35, 36].

Author contributions

All the authors contribute equally to this manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflict of interest.

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