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Research article

Soliton dynamics in the $(2 + 1)$-dimensional Nizhnik-Novikov-Veselov system via the Riccati modified extended simple equation method

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Abstract: The current study employs a transformation-based analytical technique, namely Riccati modified extended simple equation method (RMESEM) to construct and examine soliton phenomena in a prominent (2 + 1)-dimensional mathematical model namely Nizhnik-Novikov-Veselov system (NNVS), which has potential applications in exponentially localized structure interactions. The suggested RMESEM uses a variable transformation to turn the desired NNVS into a nonlinear ordinary differential equation (NODE). The resulting NODE is then assumed to have a closed-form solution, converting it into an algebraic system of equations. When the resulting algebraic system is dealt with RMESEM's strategy using Maple, a range of dark and bright soliton solutions in the form of rational, exponential, periodic, hyperbolic and rational-hyperbolic functions are revealed. Some 3D, contour and 2D graphs are plotted for visual representations of these soliton solutions that demonstrate their versatility. The findings deepen our understanding of the NNVS's dynamics, shedding light on its behavior and potential uses.

Keywords: Nizhnik-Novikov-Veselov system; nonlinear partial differential equations; Riccati modified extended simple equation method; dark and bright solitons; shallow water waves; Incompressible fluid
Mathematics Subject Classification: 34G20, 35A20, 35A22, 35R11

1. Introduction

Nonlinear partial differential equations (NPDEs) are widely used in various disciplines, including mathematical engineering and physics [1–5]. The explicit solutions of NPDEs are a key component of nonlinear scientific research. Several effective techniques have been developed to analytically solve

NPDEs, including the Bäcklund transformation method [6], the inverse scattering transform [7], the Darboux transformation [8], the (G'/G)-expansion method [9, 10], the exp-function and Jacobi elliptic function method [11–13], the Riccati mapping method [14], the Hirota bilinear method [15], the unified method and its generalized form [16, 17], the sech-function method [18], extended direct algebraic method (EDAM) [19]. It is important to recognize that while these methods greatly enhance our understanding of soliton dynamics and help us relate them to the frameworks that explain phenomena, they may also have drawbacks and shortcomings (such as the seven common errors) [20, 21]. Furthermore, many of these techniques are based on the Riccati equation [22] which are helpful for analyzing soliton phenomena in nonlinear models, considering that the Riccati equation has solitary solutions [23].

The study of nonlinear differential equations and their solutions has been a cornerstone of mathematical physics, with diverse applications across disciplines. For example, Kai and Yin explored Gaussian traveling waves in Schrodinger equations and soliton molecules in Sharma-Tasso-Olver-Burgers equations, respectively, highlighting intricate wave dynamics [24, 25]. He and Kai [26] further analyzed wave structures and chaotic behaviors in Kudryashov's equation, while Xie et al. [27] examined fractional damping in Duffing systems.

Soliton solutions have garnered a lot of attention recently considering that they have applications in a variety of nonlinear settings, such as nonlinear optics, shallow-water waves, Bose-Einstein condensates and plasma [28–33]. For example, analyses in detail the soliton solutions for several well-known NPDEs, including the Sine-Gordon equation, nonlinear Schrödinger equation, and Korteweg-de Vries (KdV) equations in [34]. Freeman's work [35] shows soliton solutions for three significant nonlinear evolution equations namely nonlinear Schrödinger equation, Kadomtsev-Petviashvili equation and Davey-Stewartson equation using Wronskian determinants which also highlights challenges in the Davey-Stewartson setting including phase variables associated with soliton count, requiring distinct Bäcklund transformations and demonstrates the intricacy of the soliton solutions of these equations. Javeed et al. [36] claim that the exponential function technique, which is based on a series of exponential functions, may be used to precisely solve Burger's equation, KdV equation and Zakharov-Kuznetsov (ZK) equation. By using the dressing approach, Wang et al. were able to correctly extract the three-component coupled Hirota hierarchy in [37]. Tian et al. [38] presented an open study concerning symmetries and the multipliers of conservation law, as well as an efficient and straightforward method for studying the symmetry-preserving discretization for certain types of generalized higher order models. Furthermore, Li et al. [39] provided the N-soliton solutions for the Cauchy problem of the generic n-component nonlinear Schrödinger equations. Additionally, a hypothesis on the fundamental principle of nonlinear wave transmission was put forth. Furthermore, Li et al. have performed some intriguing work using the steepest descent approach to derive the solutions of the Wadati-Konno-Ichikawa equation and the complicated short pulse equation [40]. In addition to proving the soliton resolution conjecture and the asymptotic stability of these equations' solutions, they also resolved the long-time asymptotic behavior of their solutions.

In this study, we aim for investigating and assessing the soliton solutions for (2 + 1)-dimensional NNVS. The mathematical representation of NNVM is articulated as [41]:

$$\begin{aligned} &s_t + \varrho s_{yyy} + \rho s_{xxx} + \varsigma s_y + \sigma s_x - 3\varrho(sz)_y - 3\rho(sr)_x = 0, \\ &r_y - s_x = 0, \quad z_x - s_y = 0, \end{aligned} \tag{1.1}$$

where $s = s(x, y, t)$, $r = r(x, y, t)$ & $z = z(x, y, t)$ and are the components of the dimensionless velocity [42], ρ, ϱ, σ & ς are constant coefficients. NNVS is the only isotropic Lax expansion of the KdV equation currently [43]. The NNVS is a mathematical model for nonlinear dynamics and theoretical physics. It is mostly employed to characterize particular kinds of integrable systems, such as integrable hierarchies and soliton theory. Since its proposal by Nizhnik in the 1980s [44], the model has been the subject of much mathematical physics research. It has applications in fields like fluid dynamics and plasma physics and is helpful in comprehending the behavior of various physical systems. In the realm of incompressible fluids, the NNVS is crucial for phenomena such as long internal waves, shallow-water waves and acoustic waves. A mechanical perturbation in an incompressible fluid that travels as a pressure variation without triggering significant changes in the fluid's density is known as an acoustic wave. Numerous scholars have examined the NNVS using a variety of techniques. For example, Boiti et al. tackled NNVS using inverse scattering transformation [45]. Osman et al. addressed this model using several analytical techniques [46].

In view of given literature, the primary objective of this work is to employ an efficient transformation-based method, namely RMESEM, for generating new families of soliton solutions for NNVS, including bright and dark soliton solutions. In the context of the NNVS, the recommended RMESEM for producing soliton solutions is incredibly robust because it is straightforward and effective algebraic analytical approach, which does not require complex numerical processes or linearization. Dependability and accuracy are ensured by the method's direct calculations, and its capacity to generate various families of soliton solutions provides important information about the fundamental characteristics of the NNVS model. Because of its simplicity and wide solution space, the RMESEM is also a powerful tool for studying the complex behavior of acoustic waves in the realm of incompressible fluids. Its potential applications cover several fields of shallow-water waves, protracted internal waves, and related disciplines.

This paper's remaining sections are organized as follows: We outline the operational mechanism of the RMESEM in Section 2. We construct soliton solutions for the NNVS using the RMESEM in Section 3. The dynamics of obtained soliton solutions are graphically represented and discussed in Section 4, An overview and a summary of our investigation's findings are provided in the conclusion section while last section gives an appendix.

2. The operational procedure of RMESEM

In this section, we outline the RMESEM's working mechanism. Consider the following NPDE:

$$E(s, r, z, s_t, r_x, z_y, s_x r_t, z_x s_y \ldots) = 0, \tag{2.1}$$

where $s = s(t, x, y)$, $r = r(t, x, y)$ & $z = z(t, x, y)$. To solve Eq (2.1), we follow the subsequent procedure:

First, a variable transformation is performed as:

$$\begin{aligned} s(t, x, y) &= S(\upsilon), \\ r(t, x, y) &= R(\upsilon), \\ z(t, x, y) &= Z(\upsilon), \\ \upsilon &= \kappa(x + y - \omega t). \end{aligned} \tag{2.2}$$

Equation (2.1) is transformed into the subsequent NODE by this transformation:

$$G(S, S', R, R', Z, Z', SR', Z'S', RS', \dots) = 0, \tag{2.3}$$

where the derivatives of S, R, and Z with respect to υ are represented by primes. Sometimes, we integrate Eq (2.3) to meet homogeneous balance criterion.

Next, we suppose the following series form solution for NODE in (2.3):

$$S(\upsilon) = R(\upsilon) = Z(\upsilon) = \sum_{j=0}^{\eta} p_j \left(\frac{\psi'(\upsilon)}{\psi(\upsilon)}\right)^j + \sum_{j=0}^{\eta-1} q_j \left(\frac{\psi'(\upsilon)}{\psi(\upsilon)}\right)^j \cdot \left(\frac{1}{\psi(\upsilon)}\right). \tag{2.4}$$

In this equation, the parameters $p_j(j = 0, ..., \eta)$ and $q_j(j = 0, ..., \eta - 1)$ represent the unknown constants that need to be discovered later, while $\psi(\upsilon)$ represents the solution to the resulting Riccati equation:

$$\psi'(\upsilon) = A + B\psi(\upsilon) + C(\psi(\upsilon))^2, \tag{2.5}$$

where A, B and C are constants.

Following that, the highest-order derivative and the greatest nonlinear term in Eq (2.3) are balanced homogenously to get the positive integer η required in Eq (2.4).

Then, when (2.4) is inserted into (2.3) or in the equation that results from the integration of (2.3), all the terms with the same powers of $\psi(\upsilon)$ are brought together. By using this procedure, an equation in terms of $\psi(\upsilon)$ is produced. An algebraic system of equations encoding the parameters $p_j(j = 0, ..., \eta)$ and $q_j(j = 0, ..., \eta - 1)$, along with additional associated parameters, are produced by setting the coefficients in this equation to zero.

An analytical evaluation of the resultant system of nonlinear algebraic equations is performed using Maple.

Subsequently, to construct new plethora of soliton solutions for (2.1), given in the corresponding Table 1, we compute and replace the values of unknown parameters with $\psi(\upsilon)$ ((2.5)'s solution) in (2.4).

3. Application of RMESEM

In this section, we aim to address NNVS given in Eq (1.1) to construct new families of soliton solution for it using RMESEM. We start with the transformation defined in Eq (2.2) which converts Eq (1.1) into the following NODE's system:

$$\begin{aligned}
&(-\omega + \varsigma + \sigma)S' + \kappa^2(\varrho + \rho)S''' - 3[\rho(SR)' + \varrho(SZ)'] = 0,\\
&\kappa S' = \kappa R',\\
&\kappa S' = \kappa Z'.
\end{aligned} \tag{3.1}$$

Integrating all of the equations in (3.1) with respect to υ while keeping the integration constant zero results in:

$$\begin{aligned}
&(-\omega + \varsigma + \sigma)S + \kappa^2(\varrho + \rho)S'' - 3[\rho(SR) + \varrho(SZ)] = 0,\\
&S = R,\\
&S = Z.
\end{aligned} \tag{3.2}$$

The following single NODE is obtained by placing the second and third parts of Eq (3.2) in the first part:

$$(-\omega+\varsigma+\sigma)S+\kappa^2(\varrho+\rho)U''-3(\rho+\varrho)S^2=0. \tag{3.3}$$

The principle of homogeneous balance between the nonlinear term S^2 and the highest order derivative S'' yields $\eta+2=2\eta$, which suggests $\eta=2$. The following series form solution for Eq (3.3) is obtained by replacing $\eta=2$ in Eq (2.4):

$$S(\upsilon)=\sum_{j=0}^{2}p_j\left(\frac{\psi'(\upsilon)}{\psi(\upsilon)}\right)^j+\sum_{j=0}^{1}q_j\left(\frac{\psi'(\upsilon)}{\psi(\upsilon)}\right)^j\cdot\left(\frac{1}{\psi(\upsilon)}\right). \tag{3.4}$$

We obtain an expression in $\psi(\upsilon)$ by inserting Eq (3.4) in Eq (3.3) and collecting each term with the same powers of $\psi(\upsilon)$. The following algebraic system of nonlinear equations is produced by setting all of the coefficients to zero:

$$-3\rho\,{p_2}^2C^4+6\kappa^2\varrho\,p_2C^4+6\kappa^2\rho\,p_2C^4-3\varrho\,{p_2}^2C^4=0,$$

$$\begin{aligned}&-6\varrho\,p_1C^3p_2+2\kappa^2\varrho\,p_1C^3-6\rho\,p_1C^3p_2-12\rho\,{p_2}^2BC^3\\&+14\kappa^2\varrho\,p_2BC^3-12\varrho\,{p_2}^2BC^3+2\kappa^2\rho\,p_1C^3+14\kappa^2\rho\,p_2BC^3=0,\end{aligned}$$

$$\begin{aligned}&-6\rho\,p_2C^3q_1+\varsigma\,p_2C^2-\omega\,p_2C^2+8\kappa^2\varrho\,p_2AC^3-18\varrho\,p_1Bp_2C^2+10\kappa^2\varrho\,p_2B^2C^2-3\rho\,{p_1}^2C^2\\&-12\varrho\,{p_2}^2AC^3-18\varrho\,{p_2}^2B^2C^2-6\varrho\,p_2C^3q_1-6\rho\,p_0p_2C^2-18\rho\,{p_2}^2B^2C^2-6\varrho\,p_0p_2C^2\\&+8\kappa^2\rho\,p_2AC^3+10\kappa^2\rho\,p_2B^2C^2+3\kappa^2\rho\,p_1BC^2-18\rho\,p_1Bp_2C^2+\sigma\,p_2C^2+3\kappa^2\varrho\,p_1BC^2\\&-12\rho\,{p_2}^2AC^3-3\varrho\,{p_1}^2C^2=0,\end{aligned}$$

$$\begin{aligned}&10\kappa^2\varrho\,p_2ABC^2+10\kappa^2\rho\,p_2ABC^2+2\sigma\,p_2BC+2\varsigma\,p_2BC-2\omega\,p_2BC-6\rho\,p_0p_1C\\&-6\rho\,{p_1}^2BC-12\rho\,{p_2}^2B^3C-6\rho\,p_2C^2q_0-6\varrho\,p_0p_1C-6\varrho\,{p_1}^2BC-6\varrho\,p_1C^2q_1\\&-12\varrho\,{p_2}^2B^3C-6\varrho\,p_2C^2q_0+\varsigma\,p_1C-\omega\,p_1C-18\rho\,p_1B^2p_2C+2\kappa^2\rho\,p_2B^3C\\&+2\kappa^2\varrho\,p_1AC^2+\kappa^2\varrho\,p_1B^2C+2\kappa^2\varrho\,p_2B^3C-12\rho\,p_0p_2BC-18\rho\,p_1Ap_2C^2\\&-36\rho\,{p_2}^2ABC^2-18\rho\,p_2BC^2q_1-12\varrho\,p_0p_2BC-18\varrho\,p_1Ap_2C^2\\&-36\varrho\,{p_2}^2ABC^2-18\varrho\,p_2BC^2q_1+2\kappa^2\rho\,p_1AC^2+\kappa^2\rho\,p_1B^2C\\&-18\varrho\,p_1B^2p_2C+\sigma\,p_1C-6\rho\,p_1C^2q_1=0,\end{aligned}$$

$$\begin{aligned}&-2\omega\,p_2AC-6\rho\,p_0p_1B+2\varsigma\,p_2AC+4\kappa^2\rho\,p_2B^2AC-6\rho\,p_0p_2B^2+2\kappa^2\varrho\,p_1BCA\\&+4\kappa^2\varrho\,p_2B^2AC-36\rho\,{p_2}^2AB^2C+\kappa^2\varrho\,q_1B^2C-12\rho\,p_0p_2AC-12\rho\,p_1Bq_1C+\kappa^2\rho\,q_0BC\\&+2\kappa^2\rho\,q_1AC^2+\kappa^2\rho\,q_1B^2C-3\varrho\,{q_1}^2C^2-3\rho\,{p_1}^2B^2+\sigma\,q_1C-3\varrho\,{p_2}^2B^4-3\rho\,{p_2}^2B^4\\&-\omega\,p_2B^2-\omega\,q_1C+\varsigma\,p_2B^2-\omega\,p_1B-6\rho\,p_0q_1C+\sigma\,p_2B^2-3\varrho\,{p_1}^2B^2+\varsigma\,q_1C\\&-3\rho\,{q_1}^2C^2+\varsigma\,p_1B+\sigma\,p_1B-6\rho\,{p_1}^2AC-6\rho\,p_1B^3p_2-6\rho\,p_1Cq_0-18\rho\,{p_2}^2A^2C^2\\&-6\varrho\,p_0p_1B-6\varrho\,p_0p_2B^2-6\varrho\,p_0q_1C-6\varrho\,p_1B^3p_2-6\varrho\,p_1Cq_0+4\kappa^2\,p_2A^2C^2(\rho+\varrho)\\&+\kappa^2\varrho\,q_0BC+2\kappa^2\varrho\,q_1AC^2+2\sigma\,p_2AC-18\varrho\,{p_2}^2A^2C^2-18\rho\,p_2AC^2q_1-18\rho\,p_2B^2q_1C\\&-12\rho\,p_2BCq_0-12\varrho\,p_0p_2AC-12\varrho\,p_1Bq_1C-36\varrho\,{p_2}^2AB^2C-18\varrho\,p_2AC^2q_1\\&-18\varrho\,p_2B^2q_1C-12\varrho\,p_2BCq_0-36\rho\,p_1Ap_2BC-36\varrho\,p_1Ap_2BC+2\kappa^2\rho\,p_1BCA+\sigma\,p_0\\&+\varsigma\,p_0-\omega\,p_0-3\rho\,{p_0}^2-3\varrho\,{p_0}^2-6\varrho\,{p_1}^2AC=0,\end{aligned}$$

$$2\,\sigma\,p_2AB + 8\,\kappa^2\rho\,q_1BCA - 6\,\varrho\,q_0q_1C - 6\,\rho\,q_0q_1C - 6\,\rho\,{q_1}^2BC - 6\,\varrho\,p_0p_1A - 6\,\varrho\,p_0q_1B$$
$$- 6\,\varrho\,{p_1}^2AB - 6\,\varrho\,p_1Bq_0 - 6\,\varrho\,p_1B^2q_1 - 12\,\varrho\,{p_2}^2AB^3 - 6\,\varrho\,p_2B^2q_0 - 6\,\varrho\,p_2B^3q_1 + \varsigma\,p_1A + \sigma\,p_1A$$
$$- 6\,\rho\,p_0q_0 - 6\,\varrho\,p_0q_0 - \omega\,p_1A - \omega\,q_1B + \varsigma\,q_1B + \sigma\,q_1B - 6\,\varrho\,{q_1}^2BC - \omega\,q_0 + \sigma\,q_0 + \varsigma\,q_0$$
$$+ 10\,\kappa^2\varrho\,p_2A^2BC + 8\,\kappa^2\varrho\,q_1BCA - 36\,\rho\,p_2ABq_1C - 36\,\varrho\,p_2ABq_1C + 2\,\kappa^2\rho\,p_2B^3A + 2\,\kappa^2\rho\,q_0AC$$
$$+ \kappa^2\varrho\,p_1B^2A - 12\,\rho\,p_1Aq_1C + 2\,\kappa^2\varrho\,p_2B^3A + 2\,\kappa^2\varrho\,q_0AC - 12\,\rho\,p_0p_2AB - 18\,\rho\,p_1A^2p_2C$$
$$- 12\,\varrho\,p_0p_2AB - 18\,\varrho\,p_1A^2p_2C - 36\,\rho\,{p_2}^2A^2BC - 18\,\varrho\,p_1Ap_2B^2 - 12\,\varrho\,p_1Aq_1C - 36\,\varrho\,{p_2}^2A^2BC$$
$$- 12\,\varrho\,p_2ACq_0 + 2\,\kappa^2\rho\,p_1CA^2 + \kappa^2\rho\,p_1B^2A + 10\,\kappa^2\rho\,p_2A^2BC - 18\,\rho\,p_1Ap_2B^2 + 2\,\kappa^2\varrho\,p_1CA^2$$
$$+ 2\,\varsigma\,p_2AB - 2\,\omega\,p_2AB + \kappa^2\rho\,q_0B^2 + \kappa^2\rho\,q_1B^3 + \kappa^2\varrho\,q_0B^2 + \kappa^2\varrho\,q_1B^3 - 6\,\rho\,p_0p_1A - 6\,\rho\,p_0q_1B$$
$$- 6\,\rho\,{p_1}^2AB - 6\,\rho\,p_1Bq_0 - 6\,\rho\,p_1B^2q_1 - 12\,\rho\,{p_2}^2AB^3 - 6\,\rho\,p_2B^2q_0 - 6\,\rho\,p_2B^3q_1 - 12\,\rho\,p_2ACq_0 = 0,$$

$$- 6\,\rho\,p_0p_2A^2 + 3\,\kappa^2\rho\,p_1A^2B + 10\,\kappa^2\rho\,p_2B^2A^2 - 6\,\rho\,p_0q_1A - 6\,\rho\,p_1Aq_0 - 12\,\rho\,p_1Aq_1B - 18\,\rho\,p_2A^2q_1C$$
$$+ 8\,\kappa^2\rho\,p_2A^3C + 3\,\kappa^2\rho\,q_0AB + 8\,\kappa^2\rho\,q_1CA^2 + 7\,\kappa^2\rho\,q_1B^2A + 3\,\kappa^2\varrho\,p_1A^2B + \sigma\,p_2A^2 + \sigma\,q_1A$$
$$+ \varsigma\,p_2A^2 + \varsigma\,q_1A - \omega\,p_2A^2 - \omega\,q_1A - 3\,\varrho\,{p_1}^2A^2 - 3\,\rho\,{p_1}^2A^2 - 3\,\varrho\,{q_1}^2B^2 - 3\,\rho\,{q_1}^2B^2 - 12\,\rho\,{p_2}^2A^3C$$
$$+ 10\,\kappa^2\varrho\,p_2B^2A^2 + 8\,\kappa^2\varrho\,p_2A^3C - 12\,\varrho\,p_2ABq_0 - 18\,\varrho\,p_2AB^2q_1 - 18\,\rho\,{p_2}^2A^2B^2 - 6\,\rho\,q_0q_1B$$
$$- 6\,\varrho\,p_0p_2A^2 - 6\,\varrho\,p_0q_1A - 6\,\varrho\,p_1Aq_0 - 12\,\varrho\,{p_2}^2A^3C - 18\,\varrho\,{p_2}^2A^2B^2 - 6\,\varrho\,q_0q_1B - 6\,\varrho\,{q_1}^2AC$$
$$+ 3\,\kappa^2\varrho\,q_0AB + 8\,\kappa^2\varrho\,q_1CA^2 + 7\,\kappa^2\varrho\,q_1B^2A - 18\,\rho\,p_1A^2p_2B - 12\,\rho\,p_2ABq_0 - 18\,\rho\,p_2AB^2q_1$$
$$- 18\,\varrho\,p_1A^2p_2B - 12\,\varrho\,p_1Aq_1B - 18\,\varrho\,p_2A^2q_1C - 3\,\rho\,{q_0}^2 - 3\,\varrho\,{q_0}^2 - 6\,\rho\,{q_1}^2AC = 0,$$

$$14\,\kappa^2\varrho\,p_2A^3B - 12\,\varrho\,{p_2}^2A^3B + 12\,\kappa^2\rho\,q_1A^2B + 2\,\kappa^2\varrho\,p_1A^3 + 2\,\kappa^2\rho\,q_0A^2 - 6\,\rho\,p_1A^2q_1$$
$$+ 12\,\kappa^2\varrho\,q_1A^2B - 6\,\varrho\,p_2A^2q_0 - 6\,\rho\,p_1A^3p_2 - 18\,\rho\,p_2A^2q_1B + 14\,\kappa^2\rho\,p_2A^3B + 2\,\kappa^2\rho\,p_1A^3$$
$$- 18\,\varrho\,p_2A^2q_1B - 6\,\varrho\,q_0q_1A - 6\,\rho\,q_0q_1A - 6\,\varrho\,p_1A^3p_2 - 6\,\rho\,p_2A^2q_0 - 6\,\varrho\,{q_1}^2AB + 2\,\kappa^2\varrho\,q_0A^2$$
$$- 12\,\rho\,{p_2}^2A^3B - 6\,\varrho\,p_1A^2q_1 - 6\,\rho\,{q_1}^2AB = 0,$$

and

$$6\,\kappa^2\rho\,q_1A^3 + 6\,\kappa^2\varrho\,p_2A^4 + 6\,\kappa^2\varrho\,q_1A^3 - 6\,\varrho\,p_2A^3q_1 - 3\,\varrho\,{q_1}^2A^2$$
$$- 3\,\rho\,{q_1}^2A^2 - 3\,\rho\,{p_2}^2A^4 - 6\,\rho\,p_2A^3q_1 + 6\,\kappa^2\rho\,p_2A^4 - 3\,\varrho\,{p_2}^2A^4 = 0.$$

The ensuing five cases of solutions are provided by using Maple to solve this system:

Case 1.

$$\begin{aligned} &p_0 = 0, p_1 = 0, p_2 = 0, q_0 = 0, q_1 = 2\,A\kappa^2, \omega = \omega, \kappa = \kappa, \rho = \rho, \\ &\varrho = \varrho, \sigma = \sigma, \varsigma = -\sigma + \omega - \kappa^2\chi\;(\varrho + \rho)\,. \end{aligned} \tag{3.5}$$

Case 2.

$$\begin{aligned} &p_0 = \frac{\kappa^2\chi}{3}, p_1 = 0, p_2 = 0, q_0 = 0, q_1 = 2\,A\kappa^2, \omega = \omega, \\ &\kappa = \kappa, \rho = \rho, \varrho = \varrho, \sigma = \sigma, \varsigma = -\sigma + \omega + \kappa^2\chi\;(\varrho + \rho)\,. \end{aligned} \tag{3.6}$$

Case 3.

$$\begin{aligned} &p_0 = 0, p_1 = -2\,\kappa^2B, p_2 = 2\,\kappa^2, q_0 = 0, q_1 = -2\,A\kappa^2, \omega = \omega, \\ &\kappa = \kappa, \rho = \rho, \varrho = \varrho, \sigma = \sigma, \varsigma = -\sigma + \omega - \kappa^2\chi\;(\varrho + \rho)\,. \end{aligned} \tag{3.7}$$

Case 4.

$$\begin{aligned} &p_0 = \frac{1}{3}\kappa^2\chi, p_1 = -2\kappa^2 B, p_2 = 2\kappa^2, q_0 = 0, q_1 = -2A\kappa^2, \\ &\omega = \omega, \kappa = \kappa, \rho = \rho, \varrho = \varrho, \sigma = \sigma, \varsigma = -\sigma + \omega + \kappa^2\chi\,(\varrho + \rho)\,. \end{aligned} \tag{3.8}$$

Case 5.

$$\begin{aligned} &p_0 = p_0, p_1 = p_1, p_2 = p_2, q_0 = q_0, q_1 = q_1, \omega = \omega, \\ &\kappa = \kappa, \rho = \rho, \varrho = -\rho, \sigma = \sigma, \varsigma = \omega - \sigma. \end{aligned} \tag{3.9}$$

The subsequent families of soliton solutions result from assuming Case 1 and using Eqs (2.2) and (3.4) with the corresponding solutions of (2.5) given in Table 1.

Family 1.1. Given $\chi < 0, C \neq 0$,

$$s_{1,1}(t,x,y) = \frac{A\kappa^2\chi\left(1 + \left(\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)\left(-\frac{1}{2}\frac{B}{C} + \frac{1}{2}\frac{\sqrt{-\chi}\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)}{C}\right)}{\left(B - \sqrt{-\chi}\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)}, \tag{3.10}$$

$$s_{1,2}(t,x,y) = \frac{A\kappa^2\chi\left(1 + \left(\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)\left(-\frac{1}{2}\frac{B}{C} - \frac{1}{2}\frac{\sqrt{-\chi}\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)}{C}\right)}{\left(B + \sqrt{-\chi}\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)}, \tag{3.11}$$

$$s_{1,3}(t,x,y) = \frac{2A\kappa^2\chi\left(1 + \sin\left(\sqrt{-\chi}\upsilon\right)\right)\left(-\frac{1}{2}\frac{B}{C} + \frac{1}{2}\frac{\sqrt{-\chi}\left(\tan\left(\sqrt{-\chi}\upsilon\right) + \sec\left(\sqrt{-\chi}\upsilon\right)\right)}{C}\right)}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)\left(B\cos\left(\sqrt{-\chi}\upsilon\right) - \sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right) - \sqrt{-\chi}\right)}, \tag{3.12}$$

and

$$s_{1,4}(t,x,y) = \frac{-2A\kappa^2\chi\left(\sin\left(\sqrt{-\chi}\upsilon\right) - 1\right)\left(-\frac{1}{2}\frac{B}{C} + \frac{1}{2}\frac{\sqrt{-\chi}\left(\tan\left(\sqrt{-\chi}\upsilon\right) - \sec\left(\sqrt{-\chi}\upsilon\right)\right)}{C}\right)}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)\left(B\cos\left(\sqrt{-\chi}\upsilon\right) - \sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right) + \sqrt{-\chi}\right)}. \tag{3.13}$$

Family 1.2. Given $\chi > 0, C \neq 0$,

$$s_{1,5}(t,x,y) = \frac{-A\kappa^2\chi\left(-1 + \left(\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2\right)\left(-\frac{1}{2}\frac{B}{C} - \frac{1}{2}\frac{\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)}{C}\right)}{\left(B + \sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)}, \tag{3.14}$$

$$s_{1,6}(t,x,y) = \frac{-2A\kappa^2\chi\left(-1 + i\sinh\left(\sqrt{\chi}\upsilon\right)\right)\left(-\frac{1}{2}\frac{B}{C} - \frac{1}{2}\frac{\sqrt{\chi}\left(\tanh\left(\sqrt{\chi}\upsilon\right) + i\,\mathrm{sech}\left(\sqrt{\chi}\upsilon\right)\right)}{C}\right)}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)\left(B\cosh\left(\sqrt{\chi}\upsilon\right) + \sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right) + i\sqrt{\chi}\right)}, \tag{3.15}$$

$$s_{1,7}(t,x,y) = \frac{2A\kappa^2\chi\left(1 + i\sinh\left(\sqrt{\chi}\upsilon\right)\right)\left(-\frac{1}{2}\frac{B}{C} - \frac{1}{2}\frac{\sqrt{\chi}\left(\tanh\left(\sqrt{\chi}\upsilon\right) - i\,\mathrm{sech}\left(\sqrt{\chi}\upsilon\right)\right)}{C}\right)}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)\left(B\cosh\left(\sqrt{\chi}\upsilon\right) + \sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right) - i\sqrt{\chi}\right)}, \tag{3.16}$$

and

$$s_{1,8}(t,x,y) = \frac{\frac{1}{2} A\kappa^2\chi \left(2\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2 - 1\right)\left(-\frac{1}{2}\frac{B}{C} - \frac{1}{4}\frac{\sqrt{\chi}\left(\tanh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right) - \coth\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)}{C}\right)}{\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)\left(\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)\left(2B\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right) - \sqrt{\chi}\right)}. \tag{3.17}$$

Family 1.3. Given $\chi = 0, B \neq 0$,

$$s_{1,9}(t,x,y) = 8\,\frac{A^2\kappa^2}{B^2\upsilon^2}. \tag{3.18}$$

Family 1.4. Given $B = n$, $A = ln(l \neq 0)$ and $C = 0$,

$$s_{1,10}(t,x,y) = 2\,l\kappa n^2 \mathrm{e}^{n\upsilon}. \tag{3.19}$$

The subsequent families of soliton solutions result from assuming Case 2 and using Eqs (2.2) and (3.4) with the corresponding solutions of (2.5) given in Table 1.

Family 2.1. Given $\chi < 0, C \neq 0$,

$$s_{2,1}(t,x,y) = -1/6\,\frac{\kappa^2\chi\left(-2C\left(\cos\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2 + 3A\right)}{C\left(\cos\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2}, \tag{3.20}$$

$$s_{2,2}(t,x,y) = 1/6\,\frac{\kappa^2\chi\left(-2C + 2C\left(\cos\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2 + 3A\right)}{C\left(-1 + \left(\cos\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)}, \tag{3.21}$$

$$s_{2,3}(t,x,y) = -\frac{1}{3}\,\frac{\kappa^2\chi\left(-\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)^2 C + 3A + 3A\sin\left(\sqrt{-\chi}\upsilon\right)\right)}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)^2 C}, \tag{3.22}$$

and

$$s_{2,4}(t,x,y) = \frac{1}{3}\,\frac{\kappa^2\chi\left(\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)^2 C + 3A\sin\left(\sqrt{-\chi}\upsilon\right) - 3A\right)}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)^2 C}. \tag{3.23}$$

Family 2.2. Given $\chi > 0, C \neq 0$,

$$s_{2,5}(t,x,y) = -1/6\,\frac{\kappa^2\chi\left(-2C\left(\cosh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2 + 3A\right)}{C\left(\cosh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2}, \tag{3.24}$$

$$s_{2,6}(t,x,y) = -1/6\,\frac{\kappa^2\chi\left(-2C\left(\cosh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2 + 3A\right)}{C\left(\cosh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2}, \tag{3.25}$$

$$s_{2,7}(t,x,y) = -\frac{1}{3}\,\frac{\kappa^2\chi\left(-\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)^2 C + 3\,A + 3\,iA\sinh\left(\sqrt{\chi}\upsilon\right)\right)}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)^2 C}, \tag{3.26}$$

and

$$s_{2,8}(t,x,y) = -\frac{1}{2}4\,\frac{\left(-8\,C\left(\cosh\left(\frac{1}{4}\,\sqrt{\chi}\upsilon\right)\right)^4 + 8\,C\left(\cosh\left(\frac{1}{4}\,\sqrt{\chi}\upsilon\right)\right)^2 + 6\,A\left(\cosh\left(\frac{1}{4}\,\sqrt{\chi}\upsilon\right)\right)^2 - 3\,A\right)\kappa^2\chi}{C\left(\left(\cosh\left(\frac{1}{4}\,\sqrt{\chi}\upsilon\right)\right)^2 - 1\right)\left(\cosh\left(\frac{1}{4}\,\sqrt{\chi}\upsilon\right)\right)^2}. \tag{3.27}$$

Family 2.3. Given $\chi = 0, B \neq 0$,

$$s_{2,9}(t,x,y) = \frac{1}{3}\,\kappa^2\chi + 8\,\frac{A^2\kappa^2}{B^2\upsilon^2}. \tag{3.28}$$

Family 2.4. Given $B = n$, $A = ln(l \neq 0)$ and $C = 0$,

$$s_{2,10}(t,x,y) = \frac{1}{3}\,\kappa^2\chi + 2l\,A\kappa n^2 \mathrm{e}^{n\upsilon}. \tag{3.29}$$

The subsequent families of soliton solutions result from assuming Case 3 and using Eqs (2.2) and (3.4) with the corresponding solutions of (2.5) given in Table 1.

Family 3.1. Given $\chi < 0, C \neq 0$,

$$\begin{aligned} s_{3,1}(t,x,y) = &- \frac{\kappa^2 B\chi\left(1 + \left(\tan\left(\frac{1}{2}\,\sqrt{-\chi}\upsilon\right)\right)^2\right)}{B - \sqrt{-\chi}\tan\left(\frac{1}{2}\,\sqrt{-\chi}\upsilon\right)} + \frac{1}{2}\,\frac{\kappa^2\chi^2\left(1 + \left(\tan\left(\frac{1}{2}\,\sqrt{-\chi}\upsilon\right)\right)^2\right)^2}{\left(B - \sqrt{-\chi}\tan\left(\frac{1}{2}\,\sqrt{-\chi}\upsilon\right)\right)^2} \\ &- \frac{A\kappa^2\chi\left(1 + \left(\tan\left(\frac{1}{2}\,\sqrt{-\chi}\upsilon\right)\right)^2\right)\left(-\frac{1}{2}\,\frac{B}{C} + \frac{1}{2}\,\frac{\sqrt{-\chi}\tan\left(\frac{1}{2}\,\sqrt{-\chi}\upsilon\right)}{C}\right)}{\left(B - \sqrt{-\chi}\tan\left(\frac{1}{2}\,\sqrt{-\chi}\upsilon\right)\right)}, \end{aligned} \tag{3.30}$$

$$\begin{aligned} s_{3,2}(t,x,y) = &- \frac{\kappa^2 B\chi\left(1 + \left(\cot\left(\frac{1}{2}\,\sqrt{-\chi}\upsilon\right)\right)^2\right)}{B + \sqrt{-\chi}\cot\left(\frac{1}{2}\,\sqrt{-\chi}\upsilon\right)} + \frac{1}{2}\,\frac{\kappa^2\chi^2\left(1 + \left(\cot\left(\frac{1}{2}\,\sqrt{-\chi}\upsilon\right)\right)^2\right)^2}{\left(B + \sqrt{-\chi}\cot\left(\frac{1}{2}\,\sqrt{-\chi}\upsilon\right)\right)^2} \\ &- \frac{A\kappa^2\chi\left(1 + \left(\cot\left(\frac{1}{2}\,\sqrt{-\chi}\upsilon\right)\right)^2\right)\left(-\frac{1}{2}\,\frac{B}{C} - \frac{1}{2}\,\frac{\sqrt{-\chi}\cot\left(\frac{1}{2}\,\sqrt{-\chi}\upsilon\right)}{C}\right)}{\left(B + \sqrt{-\chi}\cot\left(\frac{1}{2}\,\sqrt{-\chi}\upsilon\right)\right)}, \end{aligned} \tag{3.31}$$

$$\begin{aligned} s_{3,3}(t,x,y) = &- 2\,\frac{\kappa^2 B\chi\left(1 + \sin\left(\sqrt{-\chi}\upsilon\right)\right)}{\cos\left(\sqrt{-\chi}\upsilon\right)\left(B\cos\left(\sqrt{-\chi}\upsilon\right) - \sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right) - \sqrt{-\chi}\right)} \\ &+ 2\,\frac{\kappa^2\chi^2\left(1 + \sin\left(\sqrt{-\chi}\upsilon\right)\right)^2}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)^2\left(B\cos\left(\sqrt{-\chi}\upsilon\right) - \sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right) - \sqrt{-\chi}\right)^2} \\ &- \frac{2\,A\kappa^2\chi\left(1 + \sin\left(\sqrt{-\chi}\upsilon\right)\right)\left(-\frac{1}{2}\,\frac{B}{C} + \frac{1}{2}\,\frac{\sqrt{-\chi}\left(\tan\left(\sqrt{-\chi}\upsilon\right) + \sec\left(\sqrt{-\chi}\upsilon\right)\right)}{C}\right)}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)\left(B\cos\left(\sqrt{-\chi}\upsilon\right) - \sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right) - \sqrt{-\chi}\right)}, \end{aligned} \tag{3.32}$$

and

$$s_{3,4}(t,x,y) = 2\,\frac{\kappa^2 B\chi\left(\sin\left(\sqrt{-\chi}\upsilon\right)-1\right)}{\cos\left(\sqrt{-\chi}\upsilon\right)\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)+\sqrt{-\chi}\right)} + 2\,\frac{\kappa^2\chi^2\left(\sin\left(\sqrt{-\chi}\upsilon\right)-1\right)^2}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)^2\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)+\sqrt{-\chi}\right)^2} + \frac{2A\kappa^2\chi\left(\sin\left(\sqrt{-\chi}\upsilon\right)-1\right)\left(-\frac{1}{2}\frac{B}{C}+\frac{1}{2}\frac{\sqrt{-\chi}\left(\tan\left(\sqrt{-\chi}\upsilon\right)-\sec\left(\sqrt{-\chi}\upsilon\right)\right)}{C}\right)}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)+\sqrt{-\chi}\right)}. \tag{3.33}$$

Family 3.2. Given $\chi > 0, C \neq 0$,

$$s_{3,5}(t,x,y) = \frac{\kappa^2 B\chi\left(-1+\left(\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2\right)}{B+\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)} + \frac{1}{2}\,\frac{\kappa^2\chi^2\left(-1+\left(\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2\right)^2}{\left(B+\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2} + \frac{A\kappa^2\chi\left(-1+\left(\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2\right)\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)}{C}\right)}{\left(B+\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)}, \tag{3.34}$$

$$s_{3,6}(t,x,y) = 2\,\frac{\kappa^2 B\chi\left(-1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)}{\cosh\left(\sqrt{\chi}\upsilon\right)\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)+i\sqrt{\chi}\right)} + 2\,\frac{\kappa^2\chi^2\left(-1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)^2}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)^2\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)+i\sqrt{\chi}\right)^2} + \frac{2A\kappa^2\chi\left(-1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\chi}\left(\tanh\left(\sqrt{\chi}\upsilon\right)+i\,\mathrm{sech}\left(\sqrt{\chi}\upsilon\right)\right)}{C}\right)}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)+i\sqrt{\chi}\right)}, \tag{3.35}$$

$$s_{3,7}(t,x,y) = -2\,\frac{\kappa^2 B\chi\left(1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)}{\cosh\left(\sqrt{\chi}\upsilon\right)\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)-i\sqrt{\chi}\right)} + 2\,\frac{\kappa^2\chi^2\left(1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)^2}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)^2\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)-i\sqrt{\chi}\right)^2} - \frac{2A\kappa^2\chi\left(1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\chi}\left(\tanh\left(\sqrt{\chi}\upsilon\right)-i\,\mathrm{sech}\left(\sqrt{\chi}\upsilon\right)\right)}{C}\right)}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)-i\sqrt{\chi}\right)}, \tag{3.36}$$

and

$$s_{3,8}(t,x,y) = -\frac{1}{2}\frac{\kappa^2 B\chi\left(2\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2-1\right)}{\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\left(2B\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)-\sqrt{\chi}\right)}$$
$$+\frac{1}{8}\frac{\kappa^2\chi^2\left(2\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2-1\right)^2}{\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2\left(\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2\left(2B\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)-\sqrt{\chi}\right)^2} \tag{3.37}$$
$$-\frac{\frac{1}{2}A\kappa^2\chi\left(2\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2-1\right)\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{4}\frac{\sqrt{\chi}\left(\tanh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)-\coth\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)}{C}\right)}{\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)\left(\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)\left(2B\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)-\sqrt{\chi}\right)}.$$

Family 3.3. Given $\chi = 0, B \neq 0$,

$$s_{3,9}(t,x,y) = 4\frac{\kappa^2 B}{\upsilon\,(B\upsilon+2)} + 8\frac{\kappa^2}{\upsilon^2\,(B\upsilon+2)^2} - 8\frac{A^2\kappa^2}{B^2\upsilon^2}. \tag{3.38}$$

Family 3.4. Given $\chi = 0$, in case when $B = C = 0$,

$$s_{3,10}(t,x,y) = 2\frac{\kappa^2}{\upsilon^2} - 2A^2\kappa^2. \tag{3.39}$$

Family 3.5. Given $\chi = 0$, in case when $B = A = 0$,

$$s_{3,11}(t,x,y) = 2\frac{\kappa^2}{\upsilon^2}. \tag{3.40}$$

Family 3.6. Given $B = n$, $A = ln(l \neq 0)$ and $C = 0$,

$$s_{3,12}(t,x,y) = -2\frac{\kappa^2 n^2 e^{n\upsilon}}{e^{n\upsilon}-l} + 2\frac{\kappa^2 n^2\,(e^{n\upsilon})^2}{(e^{n\upsilon}-l)^2} - 2\,l\kappa n^2 n e^{n\upsilon}. \tag{3.41}$$

Family 3.7. Given $B = n$, $C = ln(l \neq 0)$ and $A = 0$,

$$s_{3,13}(t,x,y) = -2\frac{\kappa^2 n\left(-se^{s\upsilon} + se^{s\upsilon+n\upsilon}l - e^{\upsilon(s+n)}ln\right)e^{-s\upsilon}}{-1+le^{n\upsilon}}$$
$$+2\frac{\kappa^2\left(-se^{s\upsilon} + se^{s\upsilon+n\upsilon}l - e^{\upsilon(s+n)}ln\right)^2(e^{-s\upsilon})^2}{(-1+le^{n\upsilon})^2}. \tag{3.42}$$

Family 3.8. Given $A = 0$, $C \neq 0$ and $B \neq 0$,

$$s_{3,14}(t,x,y) = -2\frac{\kappa^2 B^2\,(\sinh(B\upsilon)-\cosh(B\upsilon))}{-\cosh(B\upsilon)+\sinh(B\upsilon)-k_2} + 2\frac{\kappa^2 B^2\,(\sinh(B\upsilon)-\cosh(B\upsilon))^2}{(-\cosh(B\upsilon)+\sinh(B\upsilon)-k_2)^2}, \tag{3.43}$$

and

$$s_{3,15}(t,x,y) = -2\,\frac{\kappa^2 B^2 k_2}{\cosh(B\upsilon)+\sinh(B\upsilon)+k_2} + 2\,\frac{\kappa^2 B^2 {k_2}^2}{\left(\cosh(B\upsilon)+\sinh(B\upsilon)+k_2\right)^2}. \tag{3.44}$$

The subsequent families of soliton solutions result from assuming Case 4 and using Eqs (2.2) and (3.4) with the corresponding solutions of (2.5) given in Table 1.

Family 4.1. Given $\chi < 0, C \neq 0$,

$$\begin{aligned} s_{4,1}(t,x,y) = & \frac{1}{3}\kappa^2\chi - \frac{\kappa^2 B\chi\left(1+\left(\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)}{B-\sqrt{-\chi}\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)} + \frac{1}{2}\,\frac{\kappa^2\chi^2\left(1+\left(\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)^2}{\left(B-\sqrt{-\chi}\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2} \\ & - \frac{A\kappa^2\chi\left(1+\left(\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)\left(-\frac{1}{2}\frac{B}{C}+\frac{1}{2}\frac{\sqrt{-\chi}\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)}{C}\right)}{\left(B-\sqrt{-\chi}\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)}, \end{aligned} \tag{3.45}$$

$$\begin{aligned} s_{4,2}(t,x,y) = & \frac{1}{3}\kappa^2\chi - \frac{\kappa^2 B\chi\left(1+\left(\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)}{B+\sqrt{-\chi}\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)} + \frac{1}{2}\,\frac{\kappa^2\chi^2\left(1+\left(\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)^2}{\left(B+\sqrt{-\chi}\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2} \\ & - \frac{A\kappa^2\chi\left(1+\left(\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{-\chi}\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)}{C}\right)}{\left(B+\sqrt{-\chi}\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)}, \end{aligned} \tag{3.46}$$

$$\begin{aligned} s_{4,3}(t,x,y) = & \frac{1}{3}\kappa^2\chi - 2\,\frac{\kappa^2 B\chi\left(1+\sin\left(\sqrt{-\chi}\upsilon\right)\right)}{\cos\left(\sqrt{-\chi}\upsilon\right)\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\right)} \\ & + 2\,\frac{\kappa^2\chi^2\left(1+\sin\left(\sqrt{-\chi}\upsilon\right)\right)^2}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)^2\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\right)^2} \\ & - \frac{2A\kappa^2\chi\left(1+\sin\left(\sqrt{-\chi}\upsilon\right)\right)\left(-\frac{1}{2}\frac{B}{C}+\frac{1}{2}\frac{\sqrt{-\chi}\left(\tan\left(\sqrt{-\chi}\upsilon\right)+\sec\left(\sqrt{-\chi}\upsilon\right)\right)}{C}\right)}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\right)}, \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} s_{4,4}(t,x,y) = & \frac{1}{3}\kappa^2\chi + 2\,\frac{\kappa^2 B\chi\left(\sin\left(\sqrt{-\chi}\upsilon\right)-1\right)}{\cos\left(\sqrt{-\chi}\upsilon\right)\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)+\sqrt{-\chi}\right)} \\ & + 2\,\frac{\kappa^2\chi^2\left(\sin\left(\sqrt{-\chi}\upsilon\right)-1\right)^2}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)^2\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)+\sqrt{-\chi}\right)^2} \\ & + \frac{2A\kappa^2\chi\left(\sin\left(\sqrt{-\chi}\upsilon\right)-1\right)\left(-\frac{1}{2}\frac{B}{C}+\frac{1}{2}\frac{\sqrt{-\chi}\left(\tan\left(\sqrt{-\chi}\upsilon\right)-\sec\left(\sqrt{-\chi}\upsilon\right)\right)}{C}\right)}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)+\sqrt{-\chi}\right)}. \end{aligned} \tag{3.48}$$

Family 4.2. Given $\chi > 0, C \neq 0$,

$$s_{4,5}(t,x,y) = \frac{1}{3}\kappa^2\chi + \frac{\kappa^2 B\chi\left(-1+\left(\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2\right)}{B+\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)} + \frac{1}{2}\frac{\kappa^2\chi^2\left(-1+\left(\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2\right)^2}{\left(B+\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2}$$
$$+\frac{A\kappa^2\chi\left(-1+\left(\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2\right)\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)}{C}\right)}{\left(B+\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)}, \tag{3.49}$$

$$s_{4,6}(t,x,y) = \frac{1}{3}\kappa^2\chi + 2\frac{\kappa^2 B\chi\left(-1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)}{\cosh\left(\sqrt{\chi}\upsilon\right)\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)+i\sqrt{\chi}\right)}$$
$$+2\frac{\kappa^2\chi^2\left(-1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)^2}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)^2\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)+i\sqrt{\chi}\right)^2}$$
$$+\frac{2A\kappa^2\chi\left(-1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\chi}\left(\tanh\left(\sqrt{\chi}\upsilon\right)+i\,\mathrm{sech}\left(\sqrt{\chi}\upsilon\right)\right)}{C}\right)}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)+i\sqrt{\chi}\right)}, \tag{3.50}$$

$$s_{4,7}(t,x,y) = \frac{1}{3}\kappa^2\chi - 2\frac{\kappa^2 B\chi\left(1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)}{\cosh\left(\sqrt{\chi}\upsilon\right)\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)-i\sqrt{\chi}\right)}$$
$$+2\frac{\kappa^2\chi^2\left(1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)^2}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)^2\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)-i\sqrt{\chi}\right)^2}$$
$$-\frac{2A\kappa^2\chi\left(1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\chi}\left(\tanh\left(\sqrt{\chi}\upsilon\right)-i\,\mathrm{sech}\left(\sqrt{\chi}\upsilon\right)\right)}{C}\right)}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)-i\sqrt{\chi}\right)}, \tag{3.51}$$

and

$$s_{4,8}(t,x,y) = \frac{1}{3}\kappa^2\chi - \frac{1}{2}\frac{\kappa^2 B\chi\left(2\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2-1\right)}{\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\left(2B\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)-\sqrt{\chi}\right)}$$
$$+\frac{1}{8}\frac{\kappa^2\chi^2\left(2\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2-1\right)^2}{\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2\left(\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2\left(2B\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)-\sqrt{\chi}\right)^2} \tag{3.52}$$
$$-\frac{\frac{1}{2}A\kappa^2\chi\left(2\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2-1\right)\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{4}\frac{\sqrt{\chi}\left(\tanh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)-\coth\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)}{C}\right)}{\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)\left(\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)\left(2B\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)-\sqrt{\chi}\right)}.$$

Family 4.3. Given $\chi = 0, B \neq 0$,

$$s_{4,9}(t,x,y) = 4\frac{\kappa^2 B}{\upsilon\,(B\upsilon+2)} + 8\frac{\kappa^2}{\upsilon^2\,(B\upsilon+2)^2} - 8\frac{A^2\kappa^2}{B^2\upsilon^2}. \tag{3.53}$$

Family 4.4. Given $\chi = 0$, in case when $B = C = 0$,

$$s_{4,10}(t,x,y) = 2\,\frac{\kappa^2}{\upsilon^2} - 2\,A^2\kappa^2. \tag{3.54}$$

Family 4.5. Given $\chi = 0$, in case when $B = A = 0$,

$$s_{4,11}(t,x,y) = 2\,\frac{\kappa^2}{\upsilon^2}. \tag{3.55}$$

Family 4.6. Given $B = n$, $A = ln(l \neq 0)$ and $C = 0$,

$$\begin{aligned} s_{4,12}(t,x,y) =& \frac{1}{3}\,\kappa^2 n - 2\,\frac{\kappa^2 n^2 e^{n\upsilon}}{e^{n\upsilon} - l} \\ &+ 2\,\frac{\kappa^2 n^2 \left(e^{n\upsilon}\right)^2}{\left(e^{n\upsilon} - l\right)^2} - 2\,l\kappa n^2 n e^{n\upsilon}. \end{aligned} \tag{3.56}$$

Family 4.7. Given $B = n$, $C = ln(l \neq 0)$ and $A = 0$,

$$\begin{aligned} s_{4,13}(t,x,y) =& \frac{1}{3}\,\kappa^2 n - 2\,\frac{\kappa^2 n\left(-s e^{s\upsilon} + s e^{s\upsilon + n\upsilon} l - e^{\upsilon(s+n)} ln\right) e^{-s\upsilon}}{-1 + l e^{n\upsilon}} \\ &+ 2\,\frac{\kappa^2\left(-s e^{s\upsilon} + s e^{s\upsilon + n\upsilon} l - e^{\upsilon(s+n)} ln\right)^2 \left(e^{-s\upsilon}\right)^2}{\left(-1 + l e^{n\upsilon}\right)^2}. \end{aligned} \tag{3.57}$$

Family 4.8. Given $A = 0$, $C \neq 0$ and $B \neq 0$,

$$\begin{aligned} s_{4,14}(t,x,y) =& \frac{1}{3}\,\kappa^2 B - 2\,\frac{\kappa^2 B^2 \left(\sinh(B\upsilon) - \cosh(B\upsilon)\right)}{-\cosh(B\upsilon) + \sinh(B\upsilon) - k_2} \\ &+ 2\,\frac{\kappa^2 B^2 \left(\sinh(B\upsilon) - \cosh(B\upsilon)\right)^2}{\left(-\cosh(B\upsilon) + \sinh(B\upsilon) - k_2\right)^2}, \end{aligned} \tag{3.58}$$

and

$$\begin{aligned} s_{4,15}(t,x,y) =& \frac{1}{3}\,\kappa^2 B - 2\,\frac{\kappa^2 B^2 k_2}{\cosh(B\upsilon) + \sinh(B\upsilon) + k_2} \\ &+ 2\,\frac{\kappa^2 B^2 {k_2}^2}{\left(\cosh(B\upsilon) + \sinh(B\upsilon) + k_2\right)^2}. \end{aligned} \tag{3.59}$$

The subsequent families of soliton solutions result from assuming Case 5 and using Eqs (2.2) and (3.4) with the corresponding solutions of (2.5) given in Table 1.

Table 1. Solutions $\psi(\upsilon)$ of Riccati equation in (2.5) and the formation of $\left(\frac{\psi'(\upsilon)}{\psi(\upsilon)}\right)$. Where $k_1, k_2 \in$ R, $\chi = B^2 - 4CA$ and $\zeta = \cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)$.

S. No.	Family	Constraint(s)	$\psi(\upsilon)$	$\left(\frac{\psi'(\upsilon)}{\psi(\upsilon)}\right)$
1	Trigonometric Solutions	$\chi < 0, \quad C \neq 0$	$-\frac{B}{2C} + \frac{\sqrt{-\chi}\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)}{2C}$,	$-\frac{\chi\left(1+\left(\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)}{2(-B+\sqrt{-\chi}\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right))}$,
			$-\frac{B}{2C} - \frac{\sqrt{-\chi}\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)}{2C}$,	$\frac{\left(1+\left(\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)\chi}{2(B+\sqrt{-\chi}\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right))}$,
			$-\frac{B}{2C} + \frac{\sqrt{-\chi}\left(\tan\left(\sqrt{-\chi}\upsilon\right)+\left(\sec\left(\sqrt{-\chi}\upsilon\right)\right)\right)}{2C}$,	$\frac{-\chi\left(1+\sin\left(\sqrt{-\chi}\upsilon\right)\right)\sec\left(\sqrt{-\chi}\upsilon\right)}{-B\cos\left(\sqrt{-\chi}\upsilon\right)+\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)+\sqrt{-\chi}}$,
			$-\frac{B}{2C} + \frac{\sqrt{-\chi}\left(\tan\left(\sqrt{-\chi}\upsilon\right)-\left(\sec\left(\sqrt{-\chi}\upsilon\right)\right)\right)}{2C}$.	$\frac{\chi\left(\sin\left(\sqrt{-\chi}\upsilon\right)-1\right)\sec\left(\sqrt{-\chi}\upsilon\right)}{-B\cos\left(\sqrt{-\chi}\upsilon\right)+\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}}$.
2	Hyperbolic Solutions	$\chi > 0, \quad C \neq 0$	$-\frac{B}{2C} - \frac{\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)}{2C}$,	$-\frac{\left(-1+\left(\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2\right)\chi}{2(B+\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right))}$,
			$-\frac{B}{2C} - \frac{\sqrt{\chi}\left(\tanh\left(\sqrt{\chi}\upsilon\right)+i\left(\operatorname{sech}\left(\sqrt{\chi}\upsilon\right)\right)\right)}{2C}$,	$\frac{-\chi\left(-1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)\operatorname{sech}\left(\sqrt{\chi}\upsilon\right)}{B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)+i\sqrt{\chi}}$,
			$-\frac{B}{2C} - \frac{\sqrt{\chi}\left(\tanh\left(\sqrt{\chi}\upsilon\right)-i\left(\operatorname{sech}\left(\sqrt{\chi}\upsilon\right)\right)\right)}{2C}$,	$\frac{-\chi\left(1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)\operatorname{sech}\left(\sqrt{\chi}\upsilon\right)}{-B\cosh\left(\sqrt{\chi}\upsilon\right)-\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)+i\sqrt{\chi}}$,
			$-\frac{B}{2C} - \frac{\sqrt{\chi}\left(\coth\left(\sqrt{\chi}\upsilon\right)+\left(\operatorname{csch}\left(\sqrt{\chi}\upsilon\right)\right)\right)}{2C}$.	$-\frac{\chi\left(2\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2-1\right)}{4\zeta\left(-2B\zeta+\sqrt{\chi}\right)}$.
3	Rational Solutions	$\chi = 0$	$\frac{-2A(B\upsilon+2)}{B^2\upsilon}$,	$\frac{-2}{\upsilon(B\upsilon+2)}$,
		$\chi = 0, \& B = C = 0$	υA,	$\frac{1}{\upsilon}$,
		$\chi = 0, \& B = A = 0$	$-\frac{1}{\upsilon C}$.	$-\frac{1}{\upsilon}$.
4	Exponential Solutions	$C = 0, \& B = n, A = ln$	$\varsigma^{n\upsilon} - l$,	$\frac{n e^{n\upsilon}}{e^{n\upsilon}-l}$,
		$A = 0, \& B = n, C = ln$	$\frac{\varsigma^{n\upsilon}}{1-l\varsigma^{n\upsilon}}$.	$-\frac{n}{-1+le^{n\upsilon}}$.
5	Rational-Hyperbolic Solutions	$A = 0, \& B \neq 0, C \neq 0$	$-\frac{Bk_1}{C(\cosh(B\upsilon)-\sinh(B\upsilon)+k_2)}$,	$\frac{B(\sinh(B\upsilon)-\cosh(B\upsilon))}{-\cosh(B\upsilon)+\sinh(B\upsilon)-k_2}$,
			$-\frac{B(\cosh(B\upsilon)+\sinh(B\upsilon))}{C(\cosh(B\upsilon)+\sinh(B\upsilon)+k_2)}$.	$\frac{Bk_2}{\cosh(B\upsilon)+\sinh(B\upsilon)+k_2}$.

Family 5.1. Given $\chi < 0, C \neq 0$,

$$s_{5,1}(t,x,y) = p_0 + \frac{1}{2}\,\frac{p_1\chi\left(1+\left(\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)}{B-\sqrt{-\chi}\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)} + \frac{1}{4}\,\frac{p_2\chi^2\left(1+\left(\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)^2}{\left(B-\sqrt{-\chi}\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2}$$
$$+ q_0\left(-\frac{1}{2}\,\frac{B}{C} + \frac{1}{2}\,\frac{\sqrt{-\chi}\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)}{C}\right)$$
$$+ \frac{\frac{1}{2}q_1\chi\left(1+\left(\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)\left(-\frac{1}{2}\frac{B}{C}+\frac{1}{2}\frac{\sqrt{-\chi}\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)}{C}\right)}{\left(B-\sqrt{-\chi}\tan\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)}, \tag{3.60}$$

$$s_{5,2}(t,x,y) = p_0 + \frac{1}{2}\,\frac{p_1\chi\left(1+\left(\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)}{B+\sqrt{-\chi}\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)} + \frac{1}{4}\,\frac{p_2\chi^2\left(1+\left(\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)^2}{\left(B+\sqrt{-\chi}\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2}$$
$$+ q_0\left(-\frac{1}{2}\,\frac{B}{C} - \frac{1}{2}\,\frac{\sqrt{-\chi}\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)}{C}\right)$$
$$+ \frac{\frac{1}{2}q_1\chi\left(1+\left(\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)^2\right)\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{-\chi}\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)}{C}\right)}{\left(B+\sqrt{-\chi}\cot\left(\frac{1}{2}\sqrt{-\chi}\upsilon\right)\right)}, \tag{3.61}$$

$$s_{5,3}(t,x,y) = p_0 + \frac{p_1\chi\left(1+\sin\left(\sqrt{-\chi}\upsilon\right)\right)}{\cos\left(\sqrt{-\chi}\upsilon\right)\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\right)}$$
$$+ \frac{p_2\chi^2\left(1+\sin\left(\sqrt{-\chi}\upsilon\right)\right)^2}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)^2\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\right)^2}$$
$$+ q_0\left(-\frac{1}{2}\,\frac{B}{C} + \frac{1}{2}\,\frac{\sqrt{-\chi}\left(\tan\left(\sqrt{-\chi}\upsilon\right)+\sec\left(\sqrt{-\chi}\upsilon\right)\right)}{C}\right)$$
$$+ \frac{q_1\chi\left(1+\sin\left(\sqrt{-\chi}\upsilon\right)\right)\left(-\frac{1}{2}\frac{B}{C}+\frac{1}{2}\frac{\sqrt{-\chi}\left(\tan\left(\sqrt{-\chi}\upsilon\right)+\sec\left(\sqrt{-\chi}\upsilon\right)\right)}{C}\right)}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\right)}, \tag{3.62}$$

and

$$s_{5,4}(t,x,y) = p_0 - \frac{p_1\chi\left(\sin\left(\sqrt{-\chi}\upsilon\right)-1\right)}{\cos\left(\sqrt{-\chi}\upsilon\right)\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)+\sqrt{-\chi}\right)}$$
$$+ \frac{p_2\chi^2\left(\sin\left(\sqrt{-\chi}\upsilon\right)-1\right)^2}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)^2\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)+\sqrt{-\chi}\right)^2}$$
$$+ q_0\left(-\frac{1}{2}\,\frac{B}{C} + \frac{1}{2}\,\frac{\sqrt{-\chi}\left(\tan\left(\sqrt{-\chi}\upsilon\right)-\sec\left(\sqrt{-\chi}\upsilon\right)\right)}{C}\right)$$

$$-\frac{q_1\chi\left(\sin\left(\sqrt{-\chi}\upsilon\right)-1\right)\left(-\frac{1}{2}\frac{B}{C}+\frac{1}{2}\frac{\sqrt{-\chi}\left(\tan\left(\sqrt{-\chi}\upsilon\right)-\sec\left(\sqrt{-\chi}\upsilon\right)\right)}{C}\right)}{\left(\cos\left(\sqrt{-\chi}\upsilon\right)\right)\left(B\cos\left(\sqrt{-\chi}\upsilon\right)-\sqrt{-\chi}\sin\left(\sqrt{-\chi}\upsilon\right)+\sqrt{-\chi}\right)}. \quad (3.63)$$

Family 5.2. Given $\chi > 0, C \neq 0$,

$$\begin{aligned}
s_{5,5}(t,x,y) =&p_0-\frac{1}{2}\frac{p_1\chi\left(-1+\left(\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2\right)}{B+\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)}+\frac{1}{4}\frac{p_2\chi^2\left(-1+\left(\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2\right)^2}{\left(B+\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2}\\
&+q_0\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)}{C}\right)\\
&-\frac{\frac{1}{2}q_1\chi\left(-1+\left(\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)^2\right)\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)}{C}\right)}{\left(B+\sqrt{\chi}\tanh\left(\frac{1}{2}\sqrt{\chi}\upsilon\right)\right)},
\end{aligned} \quad (3.64)$$

$$\begin{aligned}
s_{5,6}(t,x,y) =&p_0-\frac{p_1\chi\left(-1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)}{\cosh\left(\sqrt{\chi}\upsilon\right)\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)+i\sqrt{\chi}\right)}\\
&+\frac{p_2\chi^2\left(-1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)^2}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)^2\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)+i\sqrt{\chi}\right)^2}\\
&+q_0\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\chi}\left(\tanh\left(\sqrt{\chi}\upsilon\right)+i\operatorname{sech}\left(\sqrt{\chi}\upsilon\right)\right)}{C}\right)\\
&-\frac{q_1\chi\left(-1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\chi}\left(\tanh\left(\sqrt{\chi}\upsilon\right)+i\operatorname{sech}\left(\sqrt{\chi}\upsilon\right)\right)}{C}\right)}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)+i\sqrt{\chi}\right)},
\end{aligned} \quad (3.65)$$

$$\begin{aligned}
s_{5,7}(t,x,y) =&p_0+\frac{p_1\chi\left(1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)}{\cosh\left(\sqrt{\chi}\upsilon\right)\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)-i\sqrt{\chi}\right)}\\
&+\frac{p_2\chi^2\left(1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)^2}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)^2\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)-i\sqrt{\chi}\right)^2}\\
&+q_0\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\chi}\left(\tanh\left(\sqrt{\chi}\upsilon\right)-i\operatorname{sech}\left(\sqrt{\chi}\upsilon\right)\right)}{C}\right)\\
&+\frac{q_1\chi\left(1+i\sinh\left(\sqrt{\chi}\upsilon\right)\right)\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{2}\frac{\sqrt{\chi}\left(\tanh\left(\sqrt{\chi}\upsilon\right)-i\operatorname{sech}\left(\sqrt{\chi}\upsilon\right)\right)}{C}\right)}{\left(\cosh\left(\sqrt{\chi}\upsilon\right)\right)\left(B\cosh\left(\sqrt{\chi}\upsilon\right)+\sqrt{\chi}\sinh\left(\sqrt{\chi}\upsilon\right)-i\sqrt{\chi}\right)},
\end{aligned} \quad (3.66)$$

and

$$s_{5,8}(t,x,y) = p_0 + \frac{1}{4}\frac{p_1\chi\left(2\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2-1\right)}{\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\left(2B\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)-\sqrt{\chi}\right)}$$
$$+\frac{1}{16}\frac{p_2\chi^2\left(2\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2-1\right)^2}{\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2\left(\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2\left(2B\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)-\sqrt{\chi}\right)^2}$$
$$+q_0\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{4}\frac{\sqrt{\chi}\left(\tanh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)-\coth\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)}{C}\right)$$
$$+\frac{\frac{1}{4}q_1\chi\left(2\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)^2-1\right)\left(-\frac{1}{2}\frac{B}{C}-\frac{1}{4}\frac{\sqrt{\chi}\left(\tanh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)-\coth\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)}{C}\right)}{\left(\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)\left(\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\right)\left(2B\cosh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)\sinh\left(\frac{1}{4}\sqrt{\chi}\upsilon\right)-\sqrt{\chi}\right)}. \tag{3.67}$$

Family 5.3. Given $\chi = 0, B \neq 0$,

$$s_{5,9}(t,x,y) = p_0 - 2\frac{p_1}{\upsilon(B\upsilon+2)} + 4\frac{p_2}{\upsilon^2(B\upsilon+2)^2} - 2\frac{q_0A(B\upsilon+2)}{B^2\upsilon} + 4\frac{q_1A}{B^2\upsilon^2}. \tag{3.68}$$

Family 5.4. Given $\chi = 0$, in case when $B = C = 0$,

$$s_{5,10}(t,x,y) = p_0 + \frac{p_1}{\upsilon} + \frac{p_2}{\upsilon^2} + q_0A\upsilon + q_1A. \tag{3.69}$$

Family 5.5. Given $\chi = 0$, in case when $B = A = 0$,

$$s_{5,11}(t,x,y) = p_0 - \frac{p_1}{\upsilon} + \frac{p_2}{\upsilon^2} - \frac{q_0}{C\upsilon} + \frac{q_1}{\upsilon^2 C}. \tag{3.70}$$

Family 5.6. Given $B = n$, $A = ln(l \neq 0)$ and $C = 0$,

$$s_{5,12}(t,x,y) = p_0 + \frac{p_1 n e^{n\upsilon}}{e^{n\upsilon}-l} + \frac{p_2 n^2 (e^{n\upsilon})^2}{(e^{n\upsilon}-l)^2} + q_0(e^{n\upsilon}-l) + q_1 n e^{n\upsilon}. \tag{3.71}$$

Family 5.7. Given $B = n$, $C = ln(l \neq 0)$ and $A = 0$,

$$s_{5,13}(t,x,y) = p_0 + \frac{p_1\left(-se^{s\upsilon}+se^{s\upsilon+n\upsilon}l-e^{\upsilon(s+n)}ln\right)e^{-s\upsilon}}{-1+le^{n\upsilon}}$$
$$+\frac{p_2\left(-se^{s\upsilon}+se^{s\upsilon+n\upsilon}l-e^{\upsilon(s+n)}ln\right)^2(e^{-s\upsilon})^2}{(-1+le^{n\upsilon})^2}$$
$$+\frac{q_0e^{s\upsilon}}{1-le^{n\upsilon}}+\frac{q_1\left(-se^{s\upsilon}+se^{s\upsilon+n\upsilon}l-e^{\upsilon(s+n)}ln\right)e^{-s\upsilon}e^{s\upsilon}}{(-1+le^{n\upsilon})(1-le^{n\upsilon})}. \tag{3.72}$$

Family 5.8. Given $A = 0$, $C \neq 0$ and $B \neq 0$,

$$\begin{aligned} s_{5,14}(t,x,y) =& p_0 + \frac{p_1 B\left(\sinh(B\upsilon) - \cosh(B\upsilon)\right)}{-\cosh(B\upsilon) + \sinh(B\upsilon) - k_2} + \frac{p_2 B^2\left(\sinh(B\upsilon) - \cosh(B\upsilon)\right)^2}{\left(-\cosh(B\upsilon) + \sinh(B\upsilon) - k_2\right)^2} \\ & - \frac{q_0 k_1 B}{C\left(\cosh(B\upsilon) - \sinh(B\upsilon) + k_2\right)} + \frac{q_1 B^2\left(\sinh(B\upsilon) - \cosh(B\upsilon)\right) k_1}{C\left(\cosh(B\upsilon) - \sinh(B\upsilon) + k_2\right)^2}, \end{aligned} \tag{3.73}$$

and

$$\begin{aligned} s_{5,15}(t,x,y) =& p_0 + \frac{p_1 B k_2}{\cosh(B\upsilon) + \sinh(B\upsilon) + k_2} + \frac{p_2 B^2 {k_2}^2}{\left(\cosh(B\upsilon) + \sinh(B\upsilon) + k_2\right)^2} \\ & - \frac{q_0 B\left(\cosh(B\upsilon) + \sinh(B\upsilon)\right)}{C\left(\cosh(B\upsilon) + \sinh(B\upsilon) + k_2\right)} - \frac{q_1 B^2 k_2\left(\cosh(B\upsilon) + \sinh(B\upsilon)\right)}{C\left(\cosh(B\upsilon) + \sinh(B\upsilon) + k_2\right)^2}. \end{aligned} \tag{3.74}$$

In above all solutions,

$$\upsilon = \kappa(x + y - \omega t).$$

4. Discussion and graphs

The several kink solitons found in the system under study are shown graphically in this section. Using the RMESEM, we extract and display a variety of wave patterns, most notably dark and bright kink solitons in 2D, contour, and 3D forms. Different parameter choices provide informative and intelligible graphics. Notably, the work's findings are novel, and to the best of our knowledge, no prior literature has reported the use of these mathematical methodologies to the $(2 + 1)$-dimensional NNVS. Our present study is peculiar in that we have discovered novel families of soliton solutions, such as periodic, exponential, rational, hyperbolic, and rational-hyperbolic families. The ability to infer solutions obtained by other analytical techniques, like the tan-method, EDAM, F-expansion method, (G'/G)-expansion method, and numerous others, is another unique advantage of our analytical methodology.

Axiom 4.1. The following transpires after the setup of $p_1 = p_2 = q_1 = 0$ in Eq (3.4):

$$S(\upsilon) = \frac{q_0}{\psi(\upsilon)}. \tag{4.1}$$

This displays the closed form solution for the F-expansion, EDAM, and tan-function methods. Thus, attaining $p_1 = p_2 = q_1 = 0$, our findings may also provide the solutions generated by the F-expansion technique, EDAM, and tan-function method.

Axiom 4.2. Similarly, after setting up $q_0 = q_1 = 0$ in Eq (3.4) the same solution structure emerges:

$$S(\upsilon) = \sum_{j=0}^{2} p_j \left(\frac{\psi'(\upsilon)}{\psi(\upsilon)}\right)^j. \tag{4.2}$$

This is the series-form solution that is produced by using the (G'/G)-expansion method in conjunction with the Riccati equation.

Moreover, dark and bright kink solitons are two types of the found kink soliton wave phenomena that are essential to the study of the NNVS and associated mathematical frameworks. The presence of only these two types of kink solitons in NNVS solutions can be attributed to physical characteristics since in exponentially localized interactions, dark and bright kinks likely represent the fundamental building blocks of localized waves, governing energy transport and system stability, symmetry constraints because the inner parameter-dependent symmetry criteria of the KP model, from which NNVS derives, might limit soliton solutions to transitions between different background states and integrability properties as NNVS being derived as an isotropic Lax expansion of the KdV equation inherently restricts its soliton families to stable, localized structures like dark and bright kinks. Each of these two types of kink soliton has unique characteristics and implications in the context of nonlinear wave processes. Abrupt field transitions are a common feature of many physical phenomena, including kink waves. Bright solitons are localized wave packets that exhibit a sudden amplitude change that mimics a kink in the waveform character. Typically, they signify a sudden change from a situation with a smaller amplitude to one with a greater amplitude. Bright kink solitons can help us understand how systems move from one phase to another by modeling phase transitions in a range of physical contexts. On the other hand, dark solitons show a dip in their wave pattern due to a regional decrease in their amplitude. They are solutions to nonlinear wave equations that maintain their shape while traveling at constant speeds. Dark solitons are essential for understanding stability in nonlinear systems. Because of their resistance to disruptions, they are ideal for use in communication technologies where signal dependability is crucial. In physical systems, such as Bose-Einstein condensation or certain fluid dynamics scenarios, dark solitons can be employed to represent regions of low density in a medium. Lastly, dark-bright solitons are composite wave patterns composed of a bright and a dark soliton. They often appear in systems when the interplay of two different wave modes produces a stable configuration. Dark-bright solitons are a balance between attracting and repulsive forces in nonlinear media. Understanding the coexistence and interaction of different wave types is crucial in fields like nonlinear optics and plasma physics, and their study helps with this. In conclusion, understanding nonlinear wave processes requires an understanding of all types of kink soliton discovered. Particularly in fields like stability and wave propagation, their interactions and other unique properties provide valuable insights into a variety of physics applications.

Moreover, Figure 1 is drawn for dark multiple bell-shaped kink soliton solution $s_{1,5}(t, x, y)$ given in Eq (3.14) with $A = 1, B = 5, C = 4, \kappa = 1, \omega = 3, t = 0$. Figure 2 is drawn for bright kink soliton solution $s_{1,9}(t, x, y)$ given in Eq (3.18) with $A = 4, B = 8, C = 4, \kappa = 0.5e - 3, \omega = 0.05, t = 1$. Figure 3 is drawn for bright kink soliton solution $s_{2,2}(t, x, y)$ given in Eq (3.21) with $A = 1, B = 1, C = 1, \kappa = 2, \omega = 0.0015, t = 5$. Figure 4 is drawn for dark kink soliton solution $s_{2,8}(t, x, y)$ given in Eq (3.27) with $A = 8, B = 10, C = 2, \kappa = 0.002, \omega = 0.0075, t = 10$. Figure 5 is drawn for dark multiple bell-shaped kink soliton solution $s_{3,5}(t, x, y)$ given in Eq (3.34) with $A = 4, B = 10, C = 4, \kappa = 0.0065, \omega = 0.0115, t = 20$. Figure 6 is drawn for dark bell-shaped kink soliton solution $s_{3,14}(t, x, y)$ given in Eq (3.43) with $A = 0, B = 5, C = 2, \kappa = 0.085, \omega = 0.0215, t = 50, k_2 = 1$. Figure 7 is drawn for bright kink soliton solution $s_{4,1}(t, x, y)$ given in Eq (3.45) with $A = 1, B = 3, C = 5, \kappa = 0.055, \omega = 0.045, t = 100$. Figure 8 is drawn for bright kink soliton solution $s_{4,13}(t, x, y)$ given in Eq (3.57) with $s = 3, l = 2, A = 0, B = s, C = sl, \kappa = 0.0009, \omega = 0.0005, t = 500$. Figure 9 is drawn for dark-bright or lump-like kink soliton solution $s_{5,8}(t, x, y)$ given in Eq (3.67) with $A = 12, B = 13, C = 3, \kappa = 0.001, \omega = 0.0445, t = 10, p_0 = 0, p_1 = 1, p_2 = 1, q_0 = 2, q_1 = 3$.

Figure 10 is drawn for bright kink soliton solution $s_{5,11}(t,x,y)$ given in Eq (3.70) with $A = 0, B = 0, C = 5, \kappa = 0.054, \omega = 0.00035, t = 5, p_0 = 1, p_1 = 2, p_2 = 2, q_0 = 3, q_1 = 5$. In a nutshell, bright kinks can represent areas of elevated density, energy, pressure; these are frequently linked to localized amplification or detrimental interference. These solitons are essential for simulating phenomenon like rogue or shock waves because they represent regions where wave interactions intensify local intensities. Dark kink solitons are frequently used to represent areas of decreased densities or energies in physical systems, such as fluid flow or wave propagation, in the setting of exponentially localized structural interactions in NNVS. These solitons indicate dispersive or dissipative areas, which are helpful in comprehending how energy dissipates in fluids or plasmas.

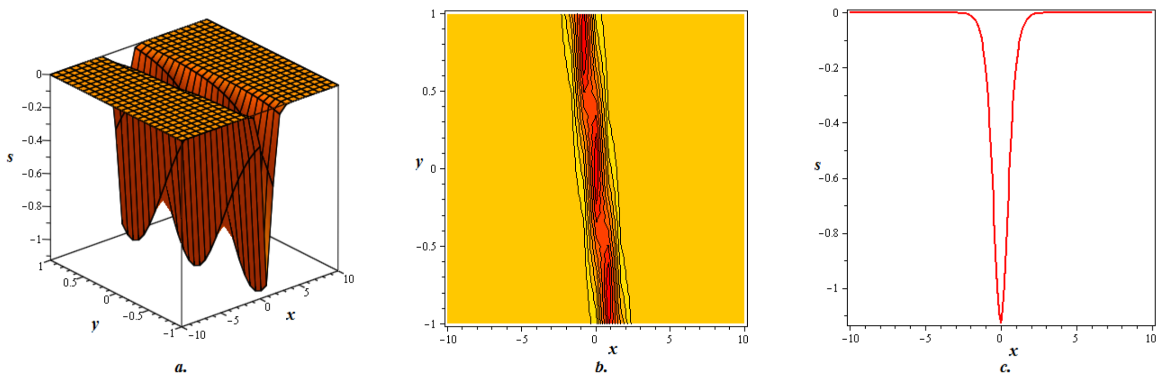

Figure 1. The a. 3D, b. contour and c. 2D ($y = 0$) plots of kink soliton solution $s_{1,5}(t,x,y)$ in Eq (3.14) are illustrated with $A = 1, B = 5, C = 4, \kappa = 1, \omega = 3, t = 0$. Overall, this profile shows dark multiple bell-shaped kink soliton.

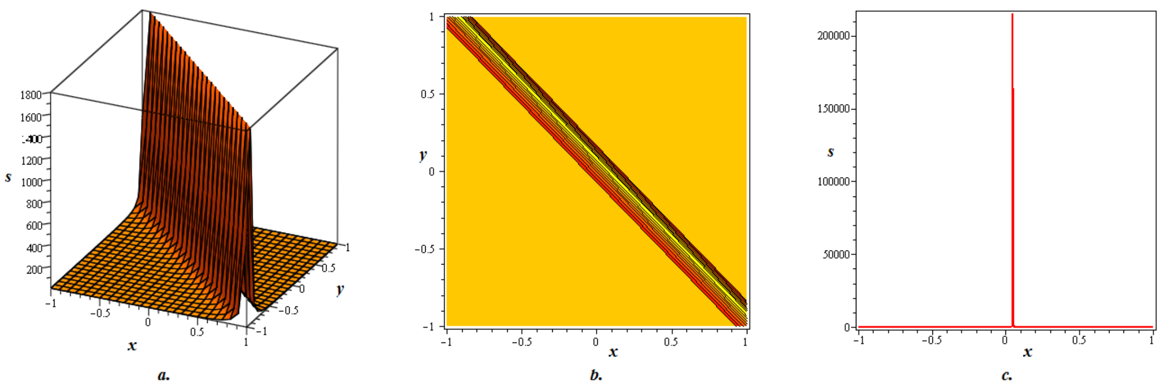

Figure 2. The a. 3D, b. contour and c. 2D ($y = 0$) plots of kink soliton solution $s_{1,9}(t,x,y)$ in Eq (3.18) are illustrated with $A = 4, B = 8, C = 4, \kappa = 0.5e-3, \omega = 0.05, t = 1$. Overall, this profile shows bright kink soliton.

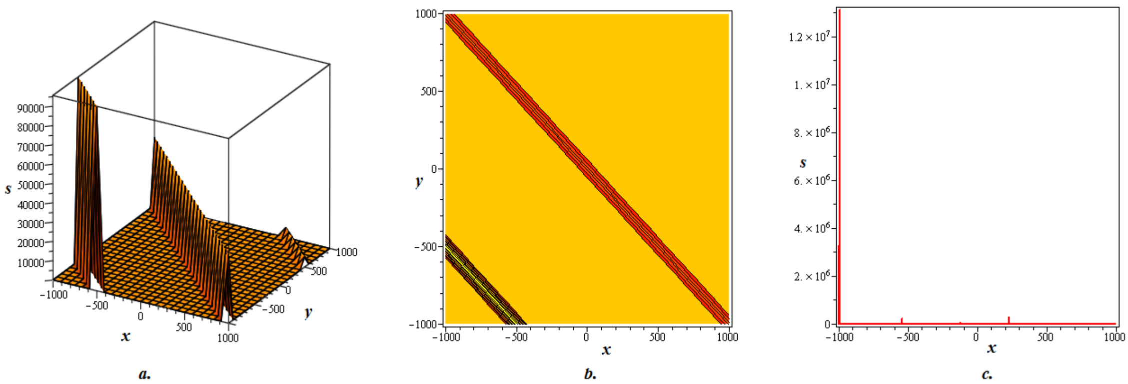

Figure 3. The a. 3D, b. contour and c. 2D ($y = 1000$) plots of kink soliton solution $s_{2,2}(t, x, y)$ in Eq (3.21) are illustrated with $A = 1, B = 1, C = 1, \kappa = 2, \omega = 0.0015, t = 5$. Overall, this profile shows bright kink soliton.

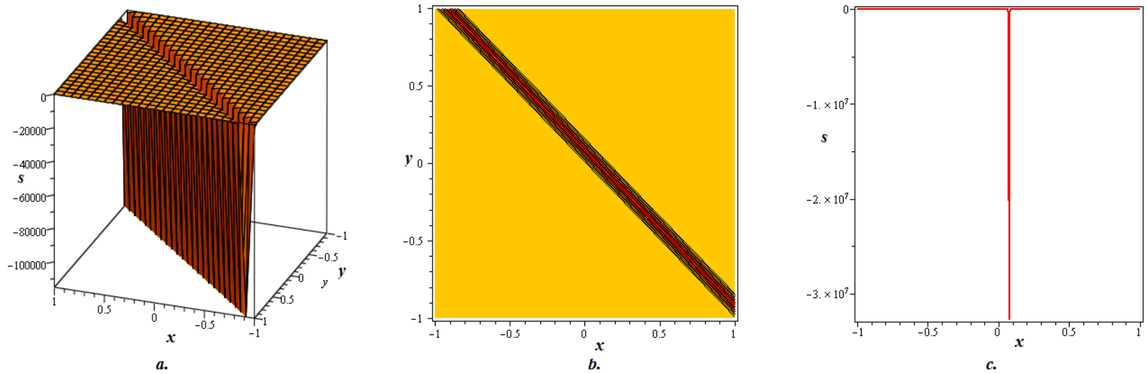

Figure 4. The a. 3D, b. contour and c. 2D ($y = 0$) plots of kink soliton solution $s_{2,8}(t, x, y)$ in Eq (3.27) are illustrated with $A = 8, B = 10, C = 2, \kappa = 0.002, \omega = 0.0075, t = 10$. Overall, this profile shows dark kink soliton solution.

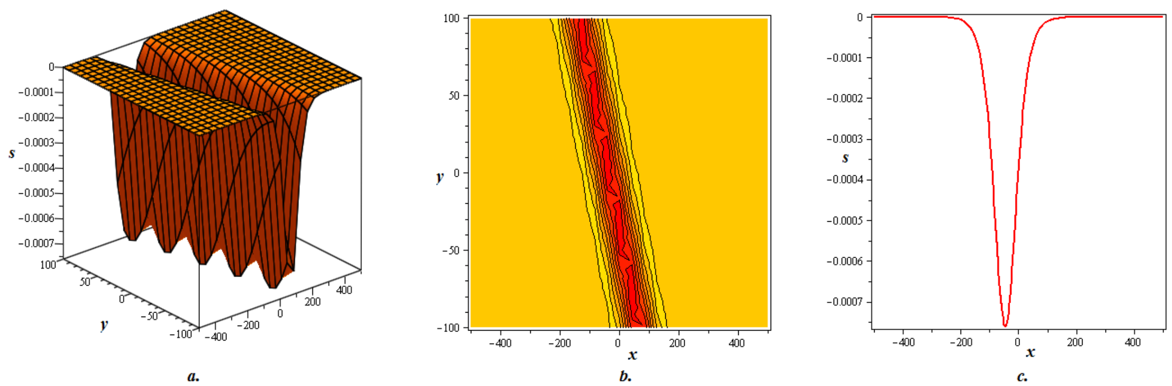

Figure 5. The a. 3D, b. contour and c. 2D ($y = 10$) plots of kink soliton solution $s_{3,5}(t, x, y)$ in Eq (3.34) are illustrated with $A = 4, B = 10, C = 4, \kappa = 0.0065, \omega = 0.0115, t = 20$. Overall, this profile shows dark multiple bell-shaped kink soliton.

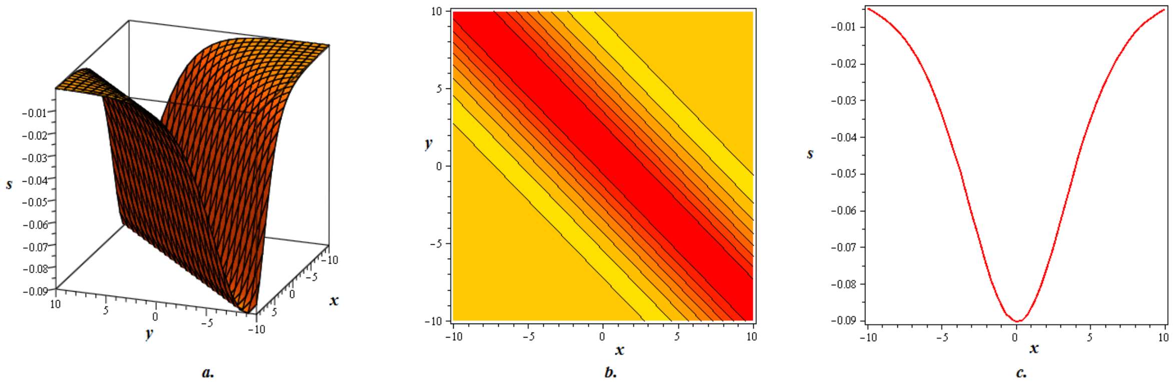

Figure 6. The a. 3D, b. contour and c. 2D ($y = 1$) plots of kink soliton solution $s_{3,14}(t, x, y)$ in Eq (3.43) are illustrated with $A = 0, B = 5, C = 2, \kappa = 0.085, \omega = 0.0215, t = 50, k_2 = 1$. Overall, this profile shows dark bell-shaped kink soliton.

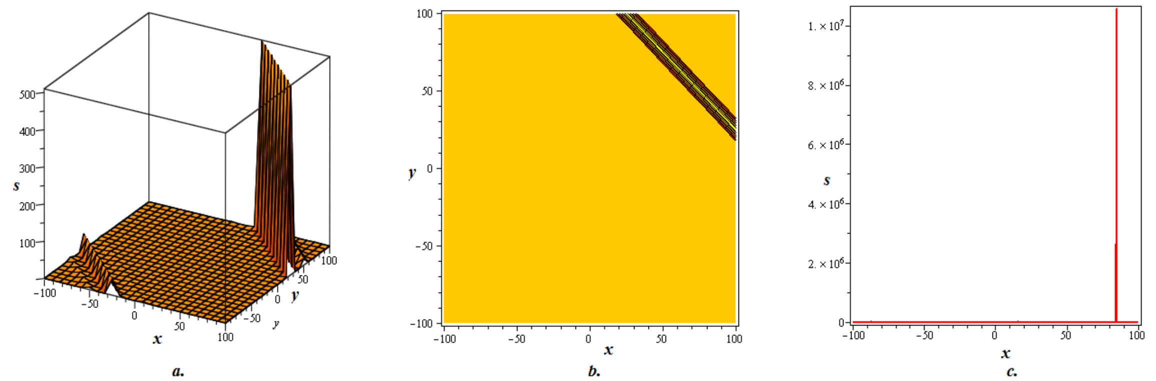

Figure 7. The a. 3D, b. contour and c. 2D ($y = 100$) plots of kink soliton solution $s_{4,1}(t, x, y)$ in Eq (3.45) are illustrated with $A = 1, B = 3, C = 5, \kappa = 0.055, \omega = 0.045, t = 100$. Overall, this profile shows bright kink soliton.

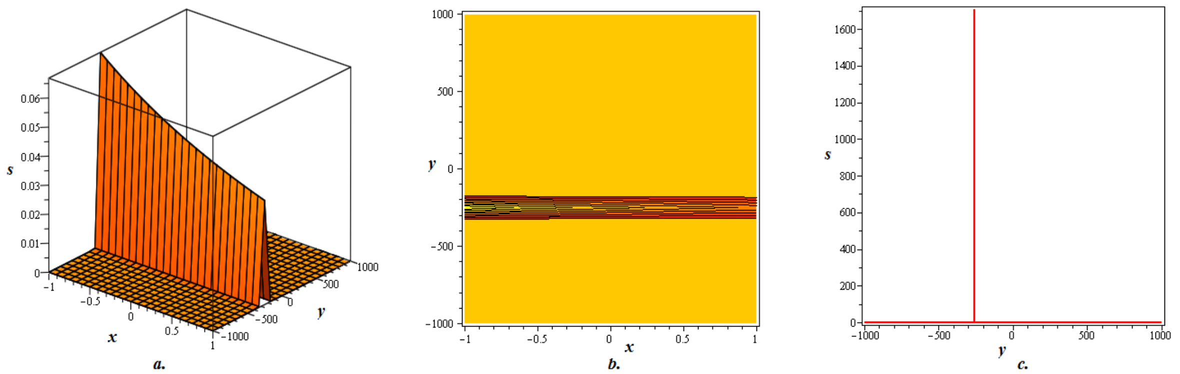

Figure 8. The a. 3D, b. contour and c. 2D ($x = 1$) plots of kink soliton solution $s_{4,13}(t, x, y)$ in Eq (3.57) are illustrated with $s = 3, l = 2, A = 0, B = s, C = sl, \kappa = 0.0009, \omega = 0.0005, t = 500$. Overall, this profile shows bright kink soliton.

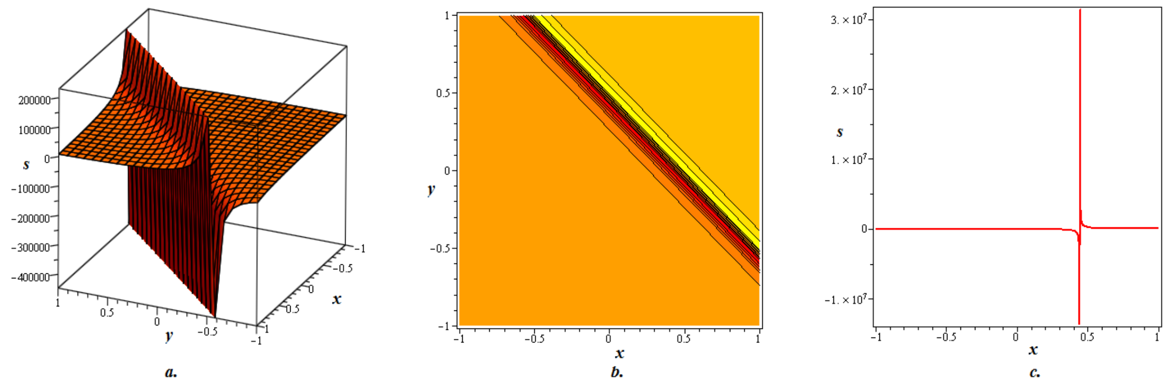

Figure 9. The a. 3D, b. contour and c. 2D ($y = 0$) plots of kink soliton solution $s_{5,8}(t, x, y)$ in Eq (3.67) are illustrated with $A = 12, B = 13, C = 3, \kappa = 0.001, \omega = 0.0445, t = 10, p_0 = 0, p_1 = 1, p_2 = 1, q_0 = 2, q_1 = 3$. Overall, this profile shows dark-bright or lump-like kink soliton.

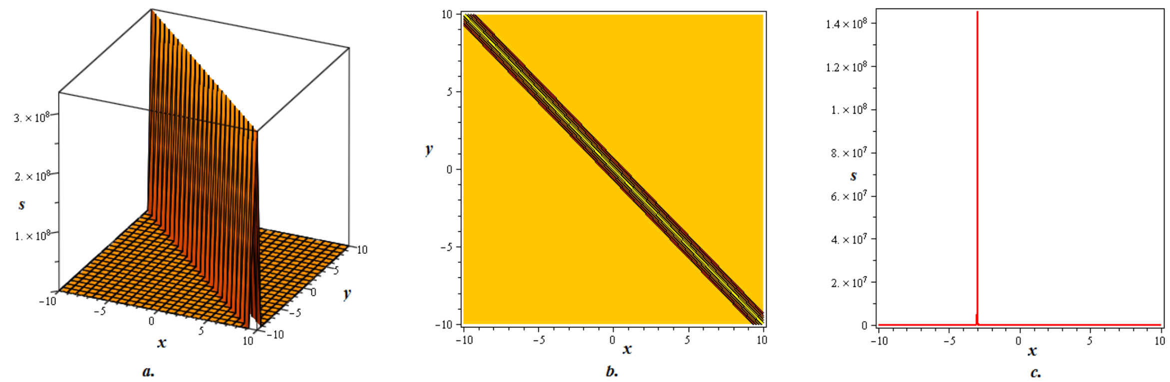

Figure 10. The a. 3D, b. contour and c. 2D ($y = 3$) plots of kink soliton solution $s_{5,11}(t, x, y)$ in Eq (3.70) are illustrated with $A = 0, B = 0, C = 5, \kappa = 0.054, \omega = 0.00035, t = 5, p_0 = 1, p_1 = 2, p_2 = 2, q_0 = 3, q_1 = 5$. Overall, this profile shows bright kink soliton.

5. Conclusions

In this section, our study of the (2 + 1)-dimensional NNVS produced significant outcomes. Using the RMESEM, we successfully established a broad range of kink soliton solutions for the aimed model which prominently reveal dark and bright kink soliton phenomena. These findings are essential to comprehending the underlying behavior of the NNVS. Some 3D, contour and 2D graphs are plotted for visual representations of these soliton solutions that demonstrate their versatility. With implications for long interior waves, shallow-water waves, and beyond, these findings also advance our knowledge of the dynamics of acoustic waves in incompressible fluids. The importance of the NNVS in theoretical physics and nonlinear dynamics is highlighted in this work, along with its usefulness and the potential for further development in related mathematical models and physical systems. Furthermore, while the RMESEM has greatly advanced our understanding of soliton dynamics and their relationship to the model under study, it is important to acknowledge the limitations of this methodology, particularly, this method fails when dealing with NPDEs with nonlinear terms and highest order derivatives that are not homogeneously balanced. Notwithstanding this limitation, the present investigation demonstrates that the methodology employed in this work is extremely straightforward and efficacious for nonlinear problems in a variety of natural science domains.

Appendix

Since we know from Eq (3.2) that:

$$S(\upsilon) = R(\upsilon) = Z(\upsilon). \tag{5.1}$$

As a result, as demonstrated in ((3.10)–(3.74)), the solutions of the system given in Eq (1.1) that correspond to the functions $r(t, x, y)$ and $z(t, x, y)$ are the same as those for $s(t, x, y)$. Therefore, in order to prevent duplication, we do not restate these solutions in the Section 3.

Author's contributions

Naveed Iqbal: Conceptualization, Methodology, Investigation, Writing-review & editing; Meshari Alesemi: Software, Formal analysis, Resources, Writing-review & editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflict of interests regarding the publication of this paper.

References

1. M. Aldandani, A. A. Altherwi, M. M. Abushaega, Propagation patterns of dromion and other solitons in nonlinear Phi-Four (ϕ^4) equation, *AIMS Math.*, **9** (2024), 19786–19811. https://doi.org/10.3934/math.2024966
2. R. Qahiti, N. M. A. Alsafri, H. Zogan, A. A. Faqihi, Kink soliton solution of integrable Kairat-X equation via two integration algorithms, *AIMS Math.*, **9** (2024), 30153–30173. https://doi.org/10.3934/math.20241456
3. M. Y. Almusawa, H. Almusawa, Exploring the diversity of kink solitons in (3 + 1)-dimensional Wazwaz-Benjamin-Bona-Mahony equation, *Mathematics*, **12** (2024), 3340. https://doi.org/10.3390/math12213340

4. M. N. Alshehri, S. Althobaiti, A. Althobaiti, R. I. Nuruddeen, H. S. Sambo, A. F. Aljohani, Solitonic analysis of the newly introduced three-dimensional nonlinear dynamical equations in fluid mediums, *Mathematics*, **12** (2024), 3205. https://doi.org/10.3390/math12203205
5. H. Khan, R. Shah, J. F. Gómez-Aguilar, D. Baleanu, P. Kumam, Travelling waves solution for fractional-order biological population model, *Math. Model. Nat. Phenom.*, **16** (2021), 32. https://doi.org/10.1051/mmnp/2021016
6. G. F. Yu, H. W. Tam, A vector asymmetrical NNV equation: soliton solutions, bilinear Bäcklund transformation and Lax pair, *J. Math. Anal. Appl.*, **344** (2008), 593–600. https://doi.org/10.1016/j.jmaa.2008.02.057
7. M. A. Ablowitz, P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, Cambridge University Press, 1991. https://doi.org/10.1017/CBO9780511623998
8. V. Matveev, M. A. Salle, *Darboux transformations and solitons*, Vol. 17, Berlin: Springer, 1991.
9. B. Q. Li, Y. L. Ma, Rich soliton structures for the Kraenkel-Manna-Merle (KMM) system in ferromagnetic materials, *J. Supercond. Nov. Magn.*, **31** (2018), 1773–1778. https://doi.org/10.1007/s10948-017-4406-9
10. B. Li, Y. Ma, The non-traveling wave solutions and novel fractal soliton for the $(2+1)$-dimensional Broer-Kaup equations with variable coefficients, *Commun. Nonlinear Sci. Numer. Simul.*, **16** (2011), 144–149. https://doi.org/10.1016/j.cnsns.2010.02.011
11. Y. L. Ma, B. Q. Li, Y. Y. Fu, A series of the solutions for the Heisenberg ferromagnetic spin chain equation, *Math. Methods Appl. Sci.*, **41** (2018), 3316–3322. https://doi.org/10.1002/mma.4818
12. Y. Ma, B. Li, C. Wang, A series of abundant exact travelling wave solutions for a modified generalized Vakhnenko equation using auxiliary equation method, *Appl. Math. Comput.*, **211** (2009), 102–107. https://doi.org/10.1016/j.amc.2009.01.036
13. B. Q. Li, Y. L. Ma, Periodic solutions and solitons to two complex short pulse (CSP) equations in optical fiber, *Optik*, **144** (2017), 149–155. https://doi.org/10.1016/j.ijleo.2017.06.114
14. M. Zhang, Y. L. Ma, B. Q. Li, Novel loop-like solitons for the generalized Vakhnenko equation, *Chinese Phys. B*, **22** (2013), 030511. https://doi.org10.1088/1674-1056/22/3/030511
15. B. Q. Li, Y. L. Ma, Multiple-lump waves for a (3 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation arising from incompressible fluid, *Comput. Math. Appl.*, **76** (2018), 204–214. https://doi.org/10.1016/j.camwa.2018.04.015
16. M. S. Osman, Nonlinear interaction of solitary waves described by multi-rational wave solutions of the (2 + 1)-dimensional Kadomtsev-Petviashvili equation with variable coefficientss, *Nonlinear Dyn.*, **87** (2017), 1209–1216. https://doi.org/10.1007/s11071-016-3110-9
17. H. I. Abdel-Gawad, Towards a unified method for exact solutions of evolution equations. An application to reaction diffusion equations with finite memory transport, *J. Stat. Phys.*, **147** (2012), 506–518. https://doi.org/10.1007/s10955-012-0467-0
18. C. F. Wei, New solitary wave solutions for the fractional Jaulent-Miodek hierarchy model, *Fractals*, **31** (2023), 2350060. https://doi.org/10.1142/S0218348X23500603
19. S. M. Mirhosseini-Alizamini, H. Rezazadeh, K. Srinivasa, A. Bekir, New closed form solutions of the new coupled Konno-Oono equation using the new extended direct algebraic method, *Pramana*, **94** (2020), 52. https://doi.org/10.1007/s12043-020-1921-1

20. N. A. Kudryashov, Seven common errors in finding exact solutions of nonlinear differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, **14** (2009), 3507–3529. https://doi.org/10.1016/j.cnsns.2009.01.023
21. Z. Navickas, M. Ragulskis, Comments on “A new algorithm for automatic computation of solitary wave solutions to nonlinear partial differential equations based on the Exp-function method”, *Appl. Math. Comput.*, **243** (2014), 419–425. https://doi.org/10.1016/j.amc.2014.06.029
22. A. O. Antonova, N. A. Kudryashov, Generalization of the simplest equation method for nonlinear non-autonomous differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, **19** (2014), 4037–4041. https://doi.org/10.1016/j.cnsns.2014.03.035
23. Z. Navickas, R. Marcinkevicius, I. Telksniene, T. Telksnys, M. Ragulskis, Structural stability of the hepatitis C model with the proliferation of infected and uninfected hepatocytes, *Math. Comput. Model. Dyn. Syst.*, **30** (2024), 51–72. https://doi.org/10.1080/13873954.2024.2304808
24. Y. Kai, Z. Yin, On the Gaussian traveling wave solution to a special kind of Schrödinger equation with logarithmic nonlinearity, *Mod. Phys. Lett. B*, **36** (2021), 2150543. https://doi.org/10.1142/S0217984921505436
25. Y. Kai, Z. Yin, Linear structure and soliton molecules of Sharma-Tasso-Olver-Burgers equation, *Phys. Lett. A*, **452** (2022), 128430. https://doi.org/10.1016/j.physleta.2022.128430
26. Y. He, Y. Kai, Wave structures, modulation instability analysis and chaotic behaviors to Kudryashov’s equation with third-order dispersion, *Nonlinear Dyn.*, **112** (2024), 10355–10371. https://doi.org/10.1007/s11071-024-09635-3
27. J. Xie, Z. Xie, H. Xu, Z. Li, W. Shi, J. Ren, et al., Resonance and attraction domain analysis of asymmetric duffing systems with fractional damping in two degrees of freedom, *Chaos Soliton. Fract.*, **187** (2024), 115440. https://doi.org/10.1016/j.chaos.2024.115440
28. M. S. Iqbal, A. R. Seadawy, M. Z. Baber, Demonstration of unique problems from Soliton solutions to nonlinear Selkov-Schnakenberg system, *Chaos Soliton. Fract.*, **162** (2022), 112485. https://doi.org/10.1016/j.chaos.2022.112485
29. M. S. Alber, R. Camassa, D. D. Holm, J. E. Marsden, On the link between umbilic geodesics and soliton solutions of nonlinear PDEs, *Proc. R. Soc. London. Ser. A: Math. Phys. Sci.*, **450** (1995), 677–692. https://doi.org/10.1098/rspa.1995.0107
30. W. Hereman, A. Nuseir, Symbolic methods to construct exact solutions of nonlinear partial differential equations, *Math. Comput. Simul.*, **43** (1997), 13–27. https://doi.org/10.1016/S0378-4754(96)00053-5
31. Z. Zhao, Y. Chen, B. Han, Lump soliton, mixed lump stripe and periodic lump solutions of a (2 + 1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation, *Mod. Phys. Lett. B*, **31** (2017), 1750157. https://doi.org/10.1142/S0217984917501573
32. W. X. Ma, Lump solutions to the Kadomtsev-Petviashvili equation, *Phys. Lett. A*, **379** (2015), 1975–1978. https://doi.org/10.1016/j.physleta.2015.06.061
33. J. Y. Yang, W. X. Ma, Lump solutions to the BKP equation by symbolic computation, *Int. J. Mod. Phys. B*, **30** (2016), 1640028. https://doi.org/10.1142/S0217979216400282
34. M. A. Helal, Soliton solution of some nonlinear partial differential equations and its applications in fluid mechanics, *Chaos Soliton. Fract.*, **13** (2002), 1917–1929. https://doi.org/10.1016/S0960-0779(01)00189-8

35. N. C. Freeman, Soliton solutions of non-linear evolution equations, *IMA J. Appl. Math.*, **32** (1984), 125–145. https://doi.org/10.1093/imamat/32.1-3.125
36. S. Javeed, K. Saleem Alimgeer, S. Nawaz, A. Waheed, M. Suleman, D. Baleanu, et al., Soliton solutions of mathematical physics models using the exponential function technique, *Symmetry*, **12** (2020), 176. https://doi.org/10.3390/sym12010176
37. Z. Y. Wang, S. F. Tian, J. Cheng, The ∂^{-}-dressing method and soliton solutions for the three-component coupled Hirota equations, *J. Math. Phys.*, **62** (2021), 093510. https://doi.org/10.1063/5.0046806
38. S. F. Tian, M. J. Xu, T. T. Zhang, A symmetry-preserving difference scheme and analytical solutions of a generalized higher-order beam equation, *Proc. R. Soc. A*, **477** (2021), 20210455. https://doi.org/10.1098/rspa.2021.0455
39. Y. Li, S. F. Tian, J. J. Yang, Riemann-Hilbert problem and interactions of solitons in the n-component nonlinear Schrödinger equations, *Stud. Appl. Math.*, **148** (2022), 577–605. https://doi.org/10.1111/sapm.12450
40. Z. Q. Li, S. F. Tian, J. J. Yang, On the soliton resolution and the asymptotic stability of N-soliton solution for the Wadati-Konno-Ichikawa equation with finite density initial data in space-time solitonic regions, *Adv. Math.*, **409** (2022), 108639. https://doi.org/10.1016/j.aim.2022.108639
41. T. Aktürk, Ç. Kubal, Analysis of wave solutions of $(2+1)$-dimensional Nizhnik-Novikov-Veselov equation, *Ordu Univ. J. Sci. Tecnol.*, **11** (2021), 13–24.
42. P. G. Estévez, S. Leble, A wave equation in $2+1$: Painlevé analysis and solutions, *Inverse Probl.*, **11** (1995), 925. https://doi.org10.1088/0266-5611/11/4/018
43. Y. Ren, H. Zhang, New generalized hyperbolic functions and auto-Bácklund transformation to find new exact solutions of the $(2+1)$-dimensional NNV equation, *Phys. Lett. A*, **357** (2006), 438–448. https://doi.org/10.1016/j.physleta.2006.04.082
44. L. P. Nizhnik, Integration of multidimensional nonlinear equations by the method of the inverse problem, *Doklady Akademii Nauk*, Russian Academy of Sciences, **254** (1980), 332–335.
45. M. B. Hossen, H. O. Roshid, M. Z. Ali , Multi-soliton, breathers, lumps and interaction solution to the (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov equation, *Heliyon*, **5** (2019), e02548. https://doi.org/10.1016/j.heliyon.2019.e02548
46. M. S. Osman, H. I. Abdel-Gawad, Multi-wave solutions of the $(2+1)$-dimensional Nizhnik-Novikov-Veselov equations with variable coefficients, *Eur. Phys. J. Plus*, **130** (2015), 215. https://doi.org/10.1140/epjp/i2015-15215-1
47. H. Yasmin, N. H. Aljahdaly, A. M. Saeed, R. Shah, Probing families of optical soliton solutions in fractional perturbed Radhakrishnan-Kundu-Lakshmanan model with improved versions of extended direct algebraic method, *Fractal Fract.*, **7** (2023), 512. https://doi.org/10.3390/fractalfract7070512

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