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Research article

The analysis of the traveling wave solutions of the Hirota-Ramani equation via the modified Kudryashov method

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Abstract: In this paper, the modified Kudryashov method is utilized to construct the exact traveling solutions to the Hirota-Ramani equation. The Hirota-Ramani equation holds significant importance as a fundamental model in the examination of nonlinear and integrable systems. It offers valuable theoretical insights and practical applications across multiple domains of physics and applied mathematics. The modified Kudryashov method was utilized to acquire the novel solutions of the Hirota-Ramani equation. Consequently, numerous analytical exact solutions have been derived, including rational, trigonometric, and hyperbolic function solutions. This method is potent, effective, and serves as an option for developing new solutions to many sorts of fractional differential equations utilized in mathematical physics.

Keywords: Kudryashov method; the Hirota-Ramani equation; traveling wave solution
Mathematics Subject Classification: 35A24, 35A99, 35C07

1. Introduction

Nonlinear evolution equations (NLEEs) have been extensively studied in several fields of mathematical and physical sciences, including physics, biology, and chemistry. The analytical solutions of these equations are crucial, since many mathematical and physical models are characterized by nonlinear partial differential equations (PDEs). Within the realm of possible solutions to NLEEs, specific solutions exist that rely solely on a singular combination of variables such as solitons. A soliton, in the fields of mathematical and physical sciences, refers to a solitary wave, wave packet, or pulse that sustains its shape and travels at a consistent velocity due to self-reinforcement. Solitons arise from the simultaneous nullification of nonlinear events inside the environment. Solitons emerge as the solutions of a broad category of weakly nonlinear dispersive partial differential equations (PDEs) that describe physical systems [1].

An analytical method for determining the precise solutions of nonlinear differential equations, particularly those pertaining to integrable systems and soliton theory is the modified Kudryashov method (MKM). This method provides novelty and advantages over more conventional approaches for solving nonlinear PDEs when applied to the Hirota-Ramani equation (HRE). The ability to apply the MKM to non-integrable equations as well as integrable ones (such as many soliton equations) is one of its main innovations. The HRE belongs to the class of integrable systems; nonetheless, the method is flexible and powerful, since it may be used to equations without a rich integrable structure [2].

Finding rational solutions to nonlinear equations, that is, solutions written as ratios of polynomials, is a specialty of the MKM. Rational solutions, in the framework of the HRE, might offer fresh perspectives on the system's behavior that may be difficult to gain by other techniques, such the inverse scattering transform or Hirota's bilinear method. Particularly for multi-soliton solutions, traditional techniques for solving nonlinear PDEs, such as the inverse scattering transform, can be computationally and technically taxing. By converting the original nonlinear equation into a manageable, easier to solve polynomial form, the MKM streamlines the solution procedure. Due to its simplification, the approach is quite effective in obtaining precise solutions [3,4].

The capacity to produce a broad range of solutions, such as singular solutions (solutions that display singularities, like blow-ups) and soliton solutions (localized, non-dispersive waves) is another innovation of the MKM. Like many soliton equations, the HRE can have both soliton-like and more complicated solution structures; this diversity can be captured via the MKM [4].

The computationally efficient Kudryashov approach consists of reducing the original equation to a reduced form in which an ansatz or a good estimate of the solution's form can be used. In contrast, several conventional techniques for determining soliton solutions could be difficult to automate and call for certain integrability requirements [4].

A large class of nonlinear PDEs can be solved using the generalizable method. The HRE is used not only to extend and modify it to other kinds of equations in mathematical physics. Because of its versatility, it can be used to investigate novel exact solutions in more general domains such as fluid dynamics, nonlinear optics, and plasma physics. The HRE is used in fluid mechanics to model the behavior of shallow water waves or internal waves [5–7]. The HRE is used in nonlinear optics, especially in modeling optical solitons and nonlinear light propagation [6,8,9]. The HRE is used in plasma physics, particularly to model the dynamics of nonlinear waves and plasma solitons [6].

While Hirota's bilinear technique is more specialized for multi-soliton integrable systems, MKM has a wider range of applications [10]. Compared with the tanh-coth technique, MKM is more flexible and offers a wider range of solution types than merely hyperbolic solutions [10]. Compared with the Exp-function approach, MKM offers more solution types and is not restricted to exponential forms [11].

While HPM is better suited for approximate solutions and perturbative analysis, MKM offers accurate solutions [12]. While Lie group analysis offers deeper insights into the structure of the equation through symmetries, MKM is more useful for discovering accurate solutions immediately [8].

When looking for precise, closed-form solutions, the MKM stands out for its adaptability and simplicity in solving both integrable and nonintegrable equations [3,4]. While it may not reveal as much about the soliton structure or symmetry features of the equation as some other methods, it is very good at discovering rational and single solutions [3,4].

The HRE is a nonlinear PDE that is primarily studied in the field of soliton theory and integrable systems. In the framework of integrable equations that admit exact solutions, it was found by R. Hirota and A. Ramani that solitons are stable, locally distributed wave packets that hold their shape during propagation [6].

Wave propagation phenomena are connected to the HRE. In physics, soliton equations are used to characterize different kinds of waves that propagate over space without dispersing, including waves in water, light in optical fibers, or sound in specific media. Soliton solutions in mathematics arise from a delicate equilibrium between dispersion, which refers to the phenomenon where different wave frequencies propagate at varying velocities, and nonlinearity, which indicates that the amplitude of a wave influences its speed or shape. Like other integrable models, the HRE captures a specific nonlinear interaction that maintains the wave across time. Usually, unique mathematical methods like the inverse scattering transform, Bäcklund transformations, or Hirota's bilinear method are used to find its solutions. These instruments aid in the precise determination of multi-soliton solutions, which depict the interplay of multiple solitons. As a member of the integrable PDEs, the HRE is used in many different domains. Nonlinear Schrödinger-type equations, which are connected to the HRE, explain how light pulses can generate non-dispersive, persistent solitons in optical fibers. For long-distance communication networks, these solitons are essential. The HRE can represent internal waves in stratified fluids or shallow water waves, much like the Korteweg-de Vries (KdV) equation. Waveforms that are steady and move without releasing energy are known as soliton appearances. In plasma environments, where charged particles interact to propagate waves, solitonic structures arise. Nonlinear wave equations are useful for modeling phenomena like Langmuir waves and ion-acoustic waves. In the field of integrable systems research, the HRE aids in the comprehension of the intricate mathematical structures that provide precise solutions. Solitons of such systems show how nonlinearity and dispersion can coexist in a stable way; they are frequently characterized by an unlimited number of conserved quantities [13–17].

In this paper, we analyse the HRE [13]

$$u_t - u_{xxt} + ru_x(1 - u_t) = 0, \tag{1}$$

where $r \neq 0$ and $u(x, t)$ is the amplitude of related to the wave mode; $u(x, t)$ represents wave amplitude, particle density, or another field quantity. The term of u_t is essential in describing the time-dependent behavior of the wave. It indicates that the system evolves dynamically rather than being static. The term of $-u_{xxt}$ reflects the influence of dispersion on the temporal evolution of the wave. In optics, it could describe group velocity dispersion, while in fluid dynamics, it represents the dispersive nature of shallow water waves or ion-acoustic waves in plasma. The nonlinear term $ru_x(1 - u_t)$ combines the spatial derivative of the wave u_x with a term that is dependent on u_t. The parameter r controls the strength of this nonlinearity. This term represents the interaction between the wave's spatial gradient (u_x) and its temporal evolution $(1 - u_t)$. The term ru_x reflects the influence of the spatial gradient on the system, often linked to nonlinear steepening or amplification of waves. The term of $1 - u_t$ acts as a modulation factor, dynamically adjusting the strength of the nonlinearity based on the wave's temporal behavior.

The exp-function approach was used to solve the HRE, and several traveling soliton solutions were obtained [18]. The inverse scattering method can be used to completely integrate this equation. Eq (1) was analyzed in [19], resulting in the discovery of novel solutions. In [20], the $\left(\frac{G'}{G}\right)$- expansion approach was introduced as a means to generate precise traveling solutions for the HRE. The HRE finds extensive use in several fields of physics, including plasma physics, fluid physics, and quantum field theory. Furthermore, it elucidates a diverse range of wave phenomena in both plasma and solid states [20]. The Lie symmetry method has been utilized to analyze the HRE [20]. There are distinct methods to solve PDEs in the literature [21–26].

In this study, the MKM is applied for the first time to obtain exact solutions of the HRE. The work introduces an innovative approach by developing a generalized ansatz tailored to the complex structure of the equation, enabling the derivation of novel solutions that have not been previously reported in the literature. Furthermore, the solutions obtained encompass a broader parameter range, enhancing the applicability of the equation in physical models.

The subsequent sections of the paper are structured in the following manner: The core concept of the MKM is introduced in Section 2. The traveling wave solutions of the HRE are shown in Section 3. The conclusion is presented in Section 4.

2. Modified Kudryashov method

We consider the general PDE of the following formula [27–30]:

$$G(u, u_x, u_t, u_{xx}, u_{tt}, \dots) = 0, \tag{2}$$

where G represents the polynomial and derivatives of $u = u(x,t)$, which comprises the nonlinear terms and the highest-order derivatives.

First step

We analyze the traveling wave solutions of Eq (2) with the the traveling wave transform

$$u(x,t) = u(\xi), \xi = cx + kt, \tag{3}$$

where c is the frequency of the wave and k is the height of the wave. If we substitute the derivative terms obtained from the wave transform in the PDE (2), the general form of the following nonlinear ordinary differential equation is obtained:

$$H(u, u', u'', \dots) = 0. \tag{4}$$

Differentiation of u with respect to ξ in Eq (4).

Second step

We propose that the exact solutions of Eq (4) can be expressed in the following manner:

$$u(\xi) = \frac{\sum_{i=0}^{N} a_i \Psi^i(\xi)}{\sum_{j=0}^{M} b_j \Psi^j(\xi)}, \tag{5}$$

where $a_i \ (i = 0,1,\dots,N), b_j \ (j = 0,1,\dots,M), \Psi = \frac{1}{1 \pm e^{\xi}}$ and the function Ψ is the solution of the equation

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\xi} = \Psi^2(\xi) - \Psi(\xi). \tag{6}$$

Third step

According to the method, we suppose that the solution of Eq (4) can be represented in the form

$$u(\xi) = \frac{a_0 + a_1\Psi + a_2\Psi^2 + \cdots + a_N\Psi^N + \cdots}{b_0 + b_1\Psi + b_2\Psi^2 + \cdots + b_M\Psi^M + \cdots}. \tag{7}$$

To determine the values M and N in (7), which represent the pole orders for the general solution of (4), we employ the classical Kudryashov approach by balancing the highest-order nonlinear terms in (4), thereby deriving formulas for M and N. Thus, we can obtain certain values of M and N.

Fourth step

Substituting (5) into (4) produces a polynomial $R(\Psi)$ in terms of Ψ. By setting the coefficients of $R(\Psi)$ to zero, we obtain a system of algebraic equations. By resolving this system, we can describe the variable coefficients $a_0, a_1, a_2, \ldots, a_N, b_0, b_1, b_2, \ldots,$ and b_M. Thus, we can ascertain the exact solutions to (4).

The derived solutions may rely on rational, trigonometric, and hyperbolic functions.

3. Application

Using the method that was first presented by Hirota and Ramani, the goal of this part is to derive analytic wave solutions to the HRE in Eq (1).

The HRE is a nonlinear dispersive equation and plays a critical role in understanding the dynamics of nonlinear wave interactions in various physical systems. This equation is particularly useful for investigating moving frames and preserving essential conservation laws within discrete systems.

For solving this nonlinear equation, it is first reduced it to a nonlinear ordinary differential equation utilizing the following wave transform:

$$u(x,t) = u(\xi), \xi = kx + ct, \tag{8}$$

where c is the frequency of the wave and k is the height of the wave.

Substituting the necessary derivative terms are obtained by wave transform (8) in Eq (1) yields

$$(c + rk)u' - kc^2u''' - rck(u')^2 = 0. \tag{9}$$

Substituting (5) into (9) and balancing the highest-order nonlinear term in terms of $(u')^2$and u''' in (9), we obtain

$$N - M + 2 = 2N - 2M \Rightarrow N = M + 2. \tag{10}$$

For $M = 1,$ we have $N = 3.$ In this case, the solution function of this equation examined in Eq (5) is as follows:

$$u(\xi) = \frac{\sum_{i=0}^{3} A_i\Psi^i\,(\xi)}{\sum_{j=0}^{1} B_j\Psi^j\,(\xi)} = \frac{A_0 + A_1\Psi + A_2\Psi^2 + A_3\Psi^3}{B_0 + B_1\Psi}, \tag{11}$$

where A_i $(i = 0,1,2,3)$, B_j $(j = 0,1)$ are constants.

Equation (11) is used to generate the derivative terms in Eq (9). The algebraic equations which consists of the coefficients of Ψ are established when each of these scenarios is realized and expressed in its appropriate form. After this algebraic system of equations is solved, the coefficients in Eq (11) can be found. It is assumed that these coefficients are substituted in the solution function. The ability of the solution functions to produce the mathematical model is verified. Lastly, graphs simulating the behavior of the discovered solution functions with suitable parameters are produced.

Case 1:

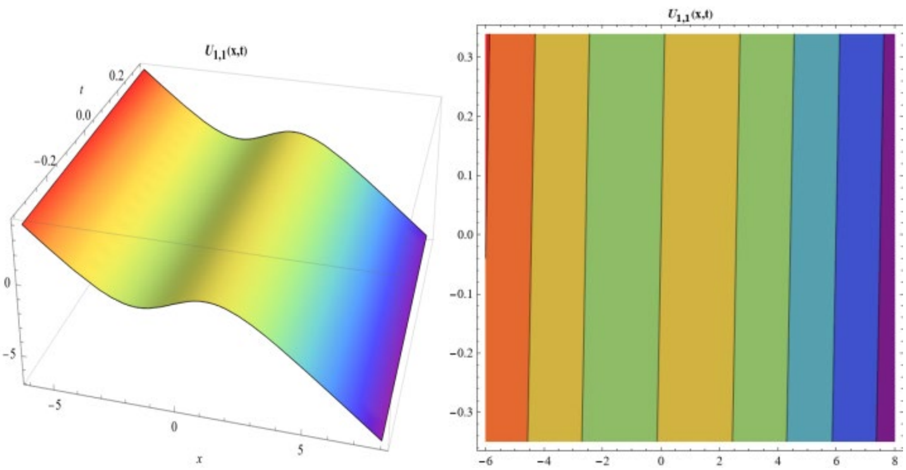
$$A_0 \to -\frac{kB_0}{r}, A_1 \to \frac{A_3}{6} + \frac{6kB_0}{r}, A_2 \to -A_3 - \frac{6kB_0}{r}, B_1 = -\frac{rA_3}{6k}, c \to -\frac{kr}{1+k^2}.$$

According to the coefficients found, the moving solution functions of the mathematical expression are presented below.

$$u_{1,1}(x,t) = \left(\frac{k\left(ct + kx - 6\text{arcTanh}\left[\text{Tanh}\left[\frac{1}{2}(ct+kx)\right]\right] + 6\text{Tanh}\left[\frac{1}{2}(ct+kx)\right]\right)}{2r} \right), \quad (12)$$

$$u_{1,2}(x,t) = \left(-\frac{k\left(ct + kx - 3\text{Coth}\left[\frac{1}{2}(ct+kx)\right]\right)}{r} \right). \quad (13)$$

Since the solutions found for $u_{1,1}$ and $u_{1,2}$ are the hyperbolic, trigonometric and rational functions, they have the feature of being periodic functions. The periodicity signifies that the system has a repetitive nature, which allows for predictions of future behavior based on previous patterns.



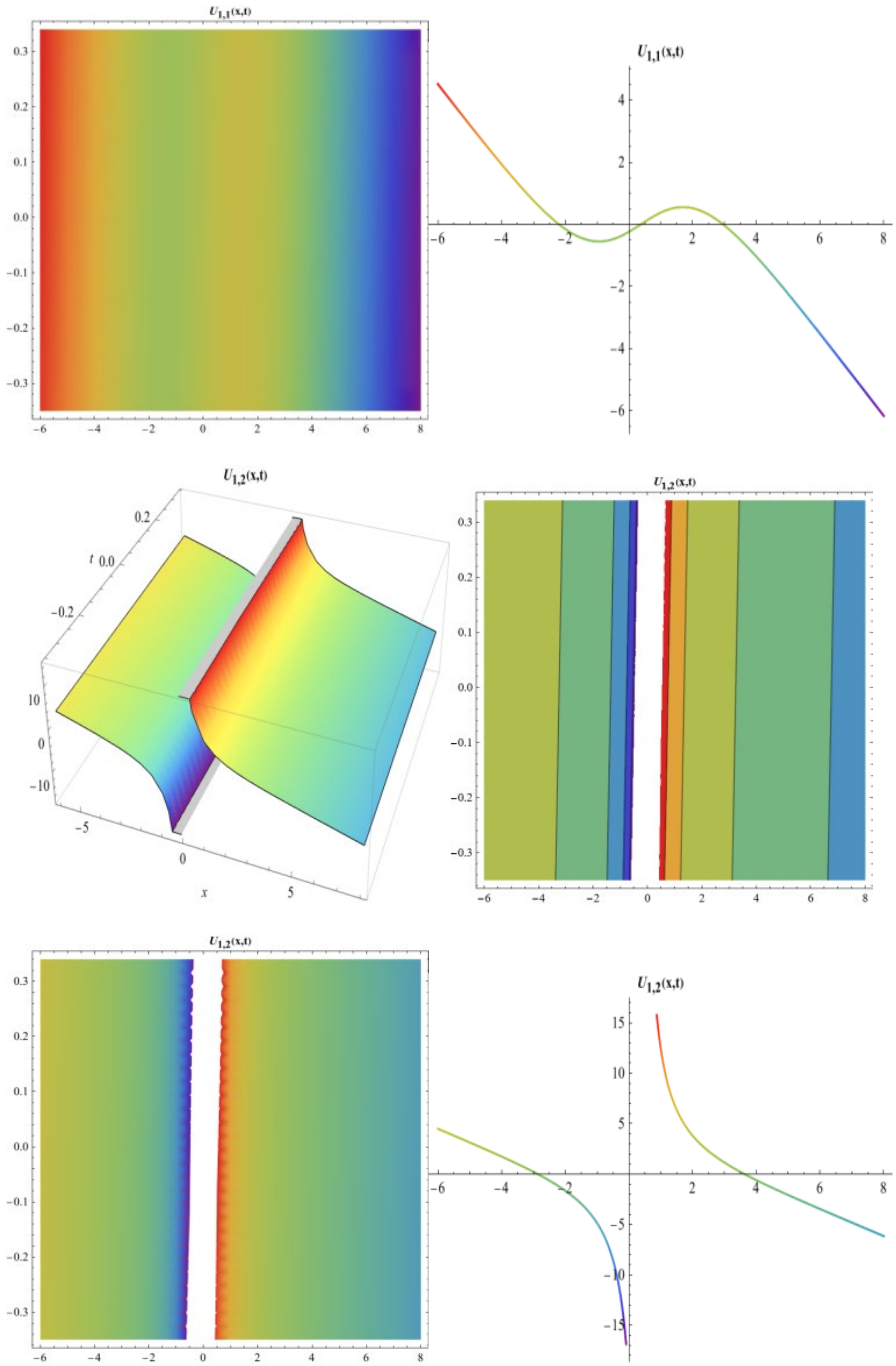


Figure 1. The 2D, 3D, density, contour graphs of Eqs (12) and (13) for $k = -1, B_0 = 0.24, r = 0.75, A_3 = 3, A_0 = 0.32, A_1 = -1.42, A_2 = -1.08, B_1 = 0.375, c = 0.375,$ and $t = 1$.

The solutions to (12) and (13) include the parameters c, k, r, A_0, A_1, A_2, A_3, B_0, and B_1. For the values of the frequency of the wave $c = 0.375$ and the parameters $k = -1, r = 0.75, A_0 = 0.32, A_1 = -1.42$, $A_2 = -1.08, B_0 = 0.24$, and $B_1 = 0.375$, the kink shape and singular kink solutions were acquired. The kink shape soliton in Figure 1 for $u_{1,1}(x,t)$ descends from left to right. In Figure 1, the three-dimensional (3D), density, and contour graphs for $u_{1,1}(x,t)$ and $u_{1,2}(x,t)$ have been demonstrated within the interval $-6 \leq x \leq 8, -0.3 \leq t \leq 0.3$. From the behavior of the solution $u_{1,2}(x,t)$ in Figure 1, it is seen that it is a singular kink soliton. The two-dimensional (2D) graph in Figure 1 shows the solutions $u_{1,1}(x,t)$ and $u_{1,2}(x,t)$ for $t = 1$.

Case 2:

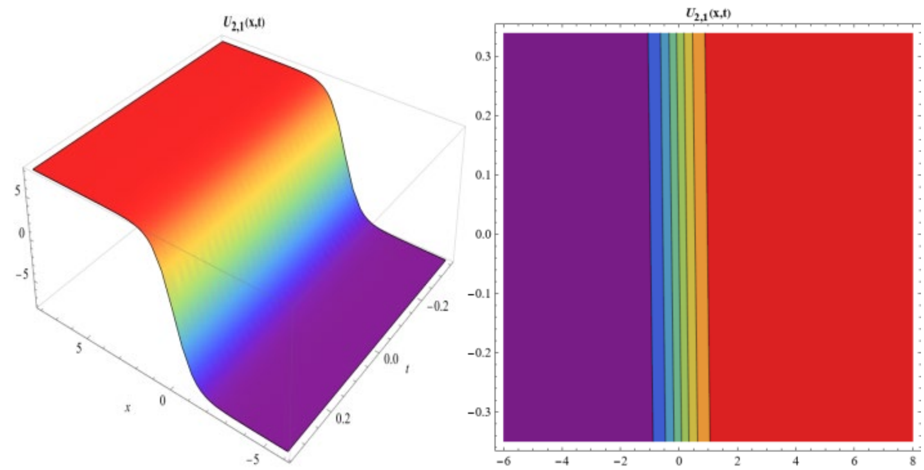
$$A_0 \to 0, A_1 \to \frac{6kB_0}{r}, \qquad A_2 \to -A_3 - \frac{6kB_0}{r}, B_1 = -\frac{rA_3}{6k}, \ c \to \frac{kr}{-1+k^2}.$$

According to the coefficients found, the moving solution functions of the mathematical expression are presented below.

$$u_{2,1}(x,t) = \left(\frac{3k\text{Tanh}\left[\frac{1}{2}(ct+kx)\right]}{r}\right), \tag{14}$$

$$u_{2,2}(x,t) = \left(\frac{3k\text{Coth}\left[\frac{1}{2}(ct+kx)\right]}{r}\right). \tag{15}$$

The solutions of $u_{2,1}$ and $u_{2,2}$ have the property of being periodic functions, since they are the hyperbolic, trigonometric, and rational functions. The system's periodicity indicates its repeating character, allowing for the prediction of future behavior based on previous trends.



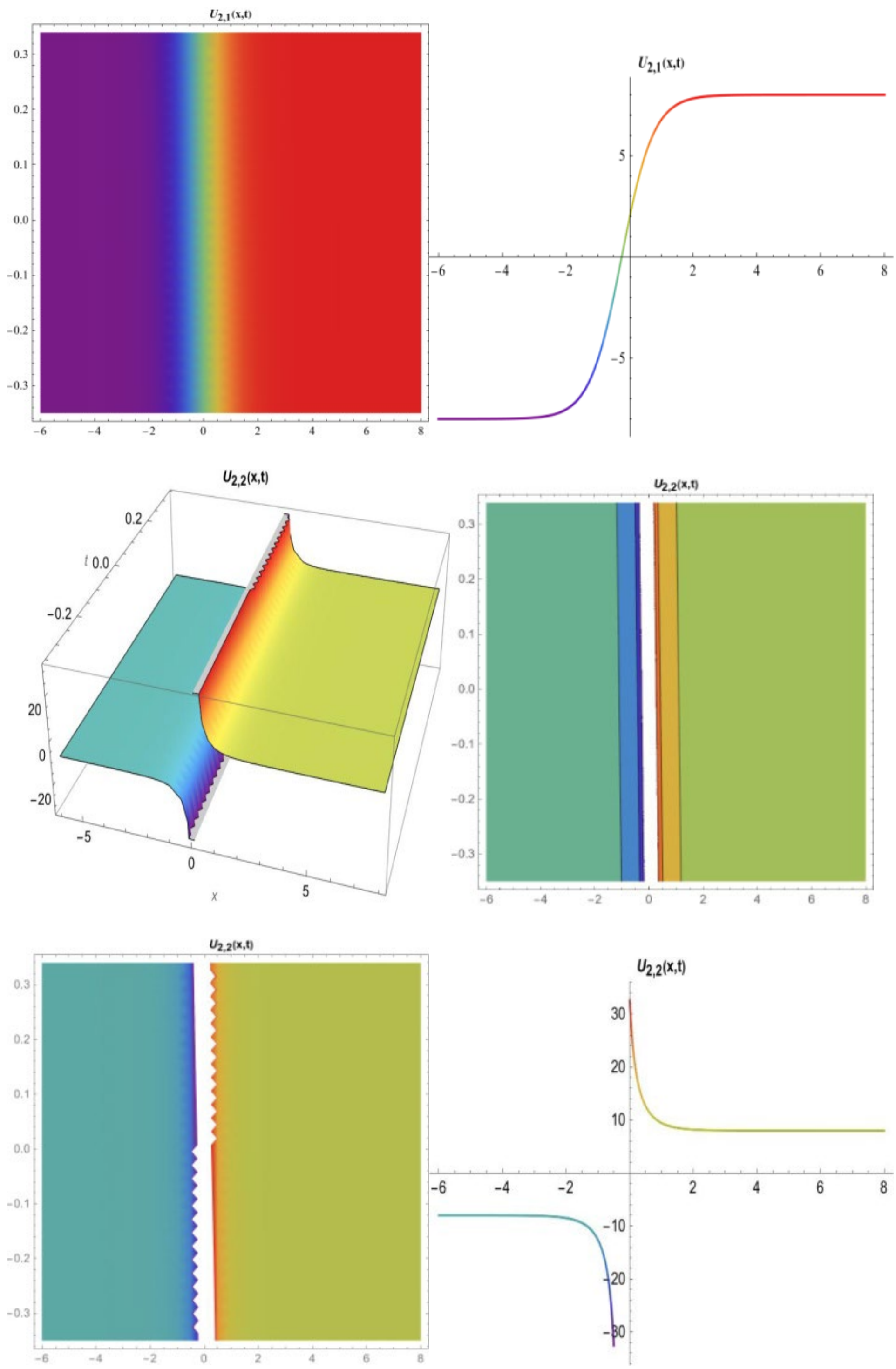


Figure 2. The 2D, 3D, density, contour graphs of Eqs (14) and (15) for $k = -2, B_0 = 0.24, r = 0.75, A_3 = 3, A_0 = 0, A_1 = -3.84, A_2 = 0.84, B_1 = 0.1875, c = -0.5$, and $t = 1$.

Similarly, the solutions to (14) and (15) include the parameters c, k, r, A_0, A_1, A_2, A_3, B_0, and B_1. For the values of the frequency of the wave $c = -0.5$ and the parameters $k = -2, r = 0.75, A_0 = 0$, $A_1 = -3.84, A_2 = 0.84, A_3 = 3, B_0 = 0.24$, and $B_1 = 0.1875$, the kink shape and singular kink solutions were obtained. The kink soliton in Figure 2 for $u_{2,1}(x,t)$ descends from left to right. In Figure 2, the three-dimensional (3D), density, contour graphs for $u_{2,1}(x,t)$ and $u_{2,2}(x,t)$ have been indicated within the interval $-6 \leq x \leq 8, -0.3 \leq t \leq 0.3$. From the behavior of the solution $u_{2,2}(x,t)$ in Figure 2, it is seen that it is a singular kink soliton. The two-dimensional (2D) graph in Figure 2 shows the solutions $u_{2,1}(x,t)$ and $u_{2,2}(x,t)$ for $t = 1$.

Case 3:

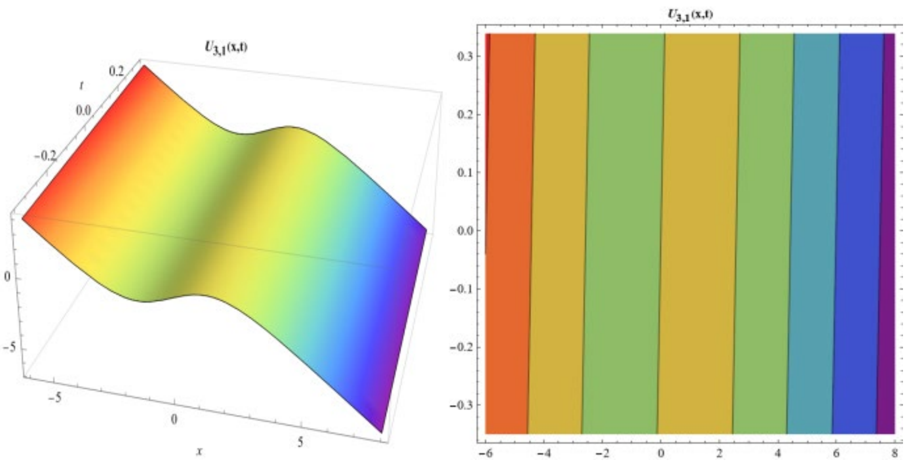
$$A_0 \to 0, A_1 \to \frac{A_3}{6}, A_2 \to -A_3, B_0 = 0, B_1 = \frac{rA_3}{6}, c \to \frac{r}{2}, k = -1.$$

According to the coefficients found, the moving solution functions of the mathematical expression are presented below.

$$u_{3,1}(x,t) = \left(\frac{ct + kx - 3\mathrm{Tanh}\left[\frac{1}{2}(ct + kx)\right]}{r}\right), \tag{16}$$

$$u_{3,2}(x,t) = \left(\frac{ct + kx - 3\mathrm{Coth}\left[\frac{1}{2}(ct + kx)\right]}{r}\right). \tag{17}$$

Since the solutions of $u_{3,1}$ and $u_{3,2}$ are hyperbolic, trigonometric, and rational, they have the characteristic of being periodic functions. Because of the system's periodicity, which denotes its repeating nature, future behavior can be predicted using previous behaviors.



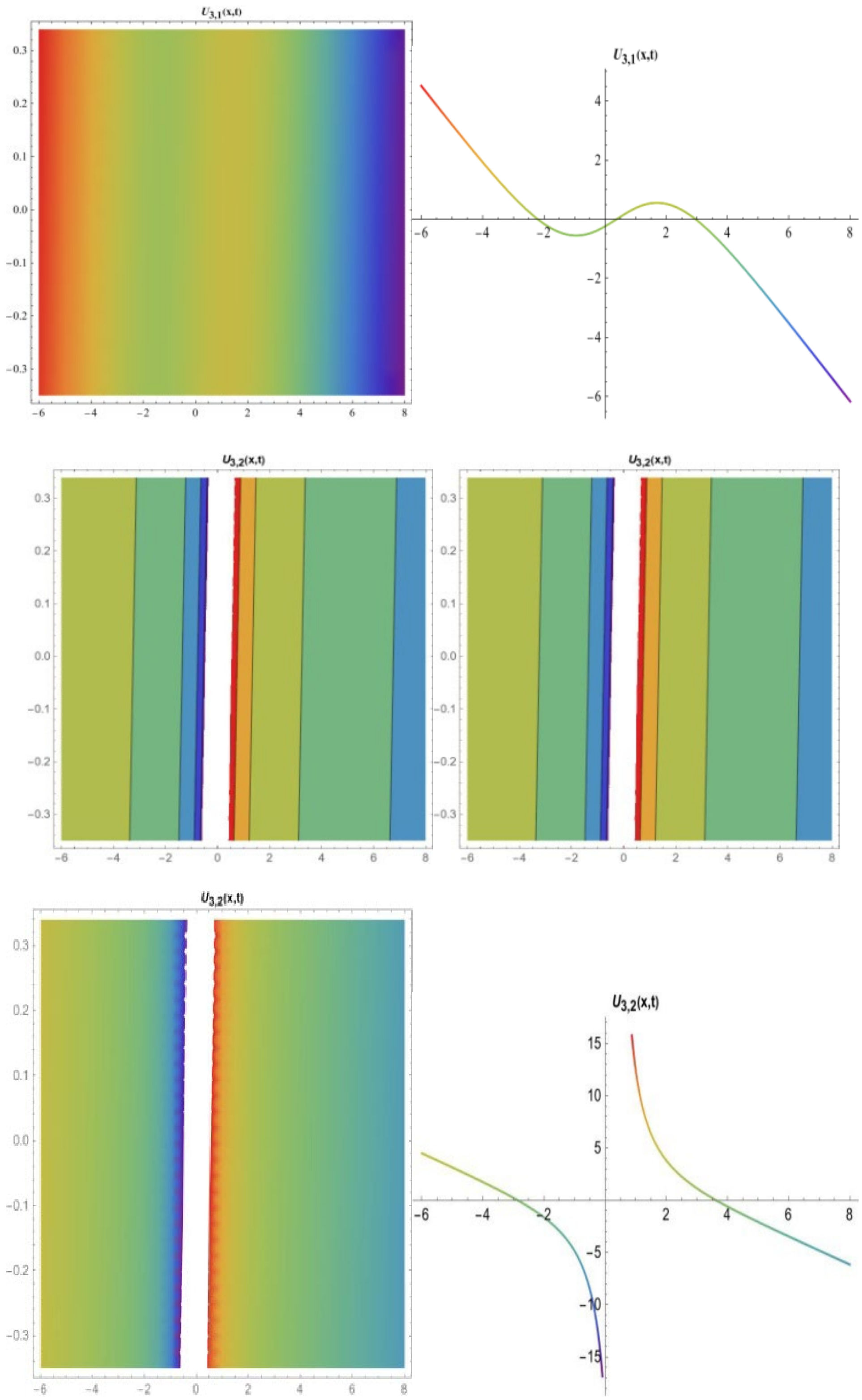


Figure 3. The 2D, 3D, density, and contour graphs of Eqs (16) and (17) for $k = -1, B_0 = 0, r = 0.75, A_3 = 3, A_0 = 0, A_1 = 0.5, A_2 = -3, B_1 = 0.375, c = 0.375,$ and $t = 1$.

Moreover, the solutions of (16) and (17) include the parameters c, k, r, A_0, A_1, A_2, A_3, B_0, and B_1. For the values of the frequency of the wave $c = 0.375$ and the parameters $k = -1, r =$

$0.75, A_0 = 0,\ A_1 = 0.5, A_2 = -3, A_3 = 3, B_0 = 0$, and $B_1 = 0.375$, it produces the smooth kink soliton and the singular kink solution, respectively. Figure 3 for $u_{3,1}(x,t)$ indicates the smooth kink soliton. From the behavior of the solution $u_{3,2}(x,t)$ in Figure 3, it is observed that it is a singular kink soliton. In Figure 3, the 3D, density and contour graphs for $u_{3,1}(x,t)$ and $u_{3,2}(x,t)$ have been demonstrated within the interval $-6 \leq x \leq 8, -0.3 \leq t \leq 0.3$. The 2D graph in Figure 3 indicates the solutions $u_{3,1}(x,t)$ and $u_{3,2}(x,t)$ for $t = 1$.

The exact solutions of the HRE derived using the modified Kudryashov method in three separate cases provide deep insights into the behavior of nonlinear wave phenomena in various physical systems. The HRE, an important tool in nonlinear evolution equations, governs the dynamics of complex wave interactions, which often involve the interaction of dispersion and nonlinearity. Using the modified Kudryashov method, an effective and systematic approach to obtain exact solutions, this study reveals the delicate balance between these competing effects. The obtained solutions are interpreted as representing stable, localized wave structures such as solitons that maintain their shape during propagation and rogue waves that exhibit extreme amplitudes and are of particular interest in nonlinear optics, fluid dynamics, and plasma physics. These wave structures are crucial for understanding wave propagation in nonlinear media as they reveal the conditions necessary for their formation, stability, and continuity. This study advances our understanding of nonlinear wave dynamics and provides a solid foundation for further research in various scientific fields such as plasma physics, nonlinear optics, and fluid mechanics.

Therefore, the values of the parameters acquired via Mathematica 13.1 in Case 1, Case 2 and Case 3 provide acceptable innovative solutions.

4. Conclusions

In this study, modified Kudryashov method has efficiently been applied to obtain the exact solutions to the nonlinear HRE. We have derived novel rational, trigonometric, and hyperbolic function solutions for the HRE, differing from earlier published solutions [9,14–17,19,20]. It has improved diverse sorts of exact solitary wave solutions, namely kink, singular kink and smooth kink solitons, which are the types of solitons obtained via the modified Kudryashov method. The solutions are determined and discussed in relation to hyperbolic and trigonometric functions with important applications in fluid dynamics, nonlinear optics, etc. In actuality, solitary solutions capture a wide range of essential characteristics, including weak shock waves in plasmas, the propagation of waves in an elastic tube filled with liquid, the distribution of electromagnetic pulses in nonlinear optical fibers, and the stability of the Stokes wave in water. By giving the indefinite parameters concrete values, 3D and 2D graphs have been presented to ensure the accuracy of the results. The primary benefit of this method is that it uses an applied technique that is straightforward and useful for analyzing nonlinear evolution equations, compatible with symbolic computation, and yields. This approach is easy to implement and understand. Furthermore, it has a high degree of ability to construct the analytical solutions that are crucial for interpreting the nonlinear events connected to various scientific and technological areas. In the future, the following types of studies can be done: Exploring more complex solutions of the HRE, extending the method to higher-dimensional systems, performing stability analysis, comparisons with numerical simulations, studying generalized or coupled systems, applying the equation to physical models, and further developing analytical techniques. These efforts may deepen the understanding of nonlinear wave phenomena and open up new applications in various fields of physics and engineering.

Author contributions

Aslı Alkan: Conceptualization, funding, writing--review and editing, project administration, validation; Mehmet Kayalar: Conceptualization, data curation, writing--review and editing; Hasan Bulut: Visualization, resources, formal analysis, validation, investigation. All authors have read and agreed to the published version of the manuscript.

Use of Generative AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

There is no conflict of interest among the authors.

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