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Research article

Fixed point results for inward and outward enriched Kannan mappings

Yahya Almalki[1], **Muhammad Usman Ali**[2,*] **and Monairah Alansari**[3]

[1] Department of Mathematics, College of Sciences, King Khalid University, Abha 61413, Saudi Arabia

[2] Department of Mathematics, COMSATS University Islamabad, Islamabad Campus, Islamabad, Pakistan

[3] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

* **Correspondence:** Email: muh_usman_ali@yahoo.com, usman.ali@comsats.edu.pk.

Abstract: This article presents the concepts of inward and outward enriched Kannan mappings, as well as inward and outward enriched Bianchini mappings. Through appropriate results based on certain conditions, the article also examines the existence of fixed points for such mappings. Some examples are also developed to support the presented notions and results.

Keywords: enriched Kannan mapping; enriched Bianchini mapping; fixed points
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1. Introduction and preliminaries

Let (M, d) be a metric space. A Banach contraction mapping $G : M \to M$ satisfies the following inequality:

$$d(Gp, Gq) \leq kd(p, q) \ \forall p, q \in M,$$

where $k \in [0, 1)$.

Two key characteristics of the Banach contraction mapping [1, 2] are as follows:
1) It is continuous.
2) If (M, d) is complete, it contains a unique fixed point in M.

A Kannan contraction mapping $G : M \to M$ satisfies the following inequality:

$$d(Gp, Gq) \leq k[d(p, Gp) + d(q, Gq)] \ \forall p, q \in M,$$

where $k \in [0, 1/2)$.

Two important characteristics of the Kannan contraction mapping [3] are as follows:
1) It may be discontinuous.
2) If (M, d) is complete, it contains a unique fixed point in M.

Both of the above mappings are also Picard operators on complete metric spaces (M, d), that is, Fix(G)={p} and $G^n p_0 \to p$ as $n \to \infty$, for any $p_0 \in M$. These mappings are considered one of the most effective tools in nonlinear analysis due to their wide domain of applicability.

The literature on metric fixed-point theory is rich and contains many interesting results. Within the last decades, we have seen concepts like α-admissibility and α-ψ-contraction mapping [4]; ϕ-nearly contraction mapping and ϕ-nearly nonexpansive mapping [5]; asymptotically T-regular mapping in modular G-metric spaces [6]; relation theoretic contractions [7]; product-operated metric spaces [8]; F-contraction mapping [9]; interpolative Kannan mapping [10]; enriched Kannan mapping [11]; MR-Kannan mapping [12]; etc.

Berinde and Păcurar [11] used the technique of enriching the contractive type mappings through the Krasnoselskii averaging process and introduced the notions of enriched Kannan mapping and enriched Bianchini mapping. They also proved the existence of a fixed point for these notions with the help of Kransnoselskii iteration. Berinde and Păcurar also presented the notions of enriched Banach contraction [13] and enriched ϕ-contraction in convex metric spaces [14]. The directions provided by Berinde and Păcurar inspired the other researchers. As a result, we have seen enriched ϕ-contraction using normed linear spaces [15]; enriched multivalued contraction [16]; enriched Kannan-type semigroup mapping [17]; MR-Kannan-type interpolative contractions [18]; enriched ρ-contraction and enriched ρ-Kannan mapping in modular function spaces [19]; enriched Suzuki mappings in Hadamard spaces [20]; Wardowski-type enriched contractive mappings [21]; enriched generalized Bianchini mapping [22]; enriched Kannan-type mappings in convex metric spaces [23]; etc.

The literature review provides a lot of applications of enriched type contraction mappings in other topics of mathematics, like split feasibility problems [11, 22], variational inequality problems [11, 22], linear systems of equations [22], and integral equations [23]. Numerous scientific and technical domains have recognized possible uses of the above-listed topics. For example, these problems are helpful for phase retrieval, compressed sensing, and image reconstruction in signal processing. They are employed in data analysis, optimization, and machine learning within the discipline of computer science. They are also useful in equilibrium problems, game theory, and finance in the context of economics. In addition to the above, these topics have a lot of other uses in engineering and transportation problems.

The notions of enriched Kannan mapping and enriched Bianchini mapping, along with related fixed-point results, are mentioned below.

Definition 1.1. *[11]* Let $(M, \|\cdot\|)$ be a normed linear space. A mapping $G : M \to M$ is said to be an enriched Kannan mapping if there exist $l \in [0, \infty)$ and $k \in [0, 1/2)$ such that

$$\|l(p-q) + Gp - Gq\| \le k(\|p - Gp\| + \|q - Gq\|) \; \forall p, q \in M.$$

Theorem 1.1. *[11]* Let $(M, \|\cdot\|)$ be a Banach space, and let $G : M \to M$ be an enriched Kannan mapping. Then, G contains a unique fixed point in M.

Definition 1.2. *[11]* Let $(M, \|\cdot\|)$ be a normed linear space. A mapping $G : M \to M$ is said to be an enriched Bianchini mapping if there exist $l \in [0, \infty)$ and $k \in [0, 1)$ such that

$$\|l(p-q) + Gp - Gq\| \le k \max\{\|p - Gp\|, \|q - Gq\|\} \; \forall p, q \in M.$$

Theorem 1.2. *[11]* Let $(M, \|\cdot\|)$ be a Banach space, and let $G : M \to M$ be an enriched Bianchini mapping. Then, G contains a unique fixed point in M.

In the next section, we have presented a few generalizations of the above-mentioned enriched Kannan mapping and enriched Bianchini mapping, along with corresponding fixed-point results. The generalizations are obtained in two different ways. One generalization is based on an inward modification of the ideas of Berinde and Păcurar [11], while the other generalization is based on an outward modification.

Throughout the article, we denote the set of all natural numbers by $\mathbb{N}$ and the set of all real numbers by $\mathbb{R}$.

2. Main results

In this section, we have generalized the notions of Berinde and Păcurar [11] as inward and outward enriched mappings by using the methodology of Samet et al. [4] and have studied the existence of fixed points for these new notions.

The following definition presents the concept of an inward enriched Kannan mapping.

Definition 2.1. Let $(M, \|\cdot\|)$ be a normed linear space. A mapping $G : M \to M$ is called an inward enriched Kannan mapping if, for each $p, q \in M$, we have

$$\|\theta(p,q)(p-q) + Gp - Gq\| \leq k(\|p - Gp\| + \|q - Gq\|), \tag{2.1}$$

where $\theta : M \times M \to \mathbb{R}$ is a mapping and $k \in [0, 1/2)$.

The above definition can also be expressed in the form of metric spaces, as given below.

Let (M, d) be a metric space. A mapping $G : M \to M$ is called an inward enriched Kannan mapping if, for each $p, q \in M$, we have

$$d(\theta(p,q)p + Gp, \theta(p,q)q + Gq) \leq k(d(p, Gp) + d(q, Gq)), \tag{2.2}$$

where $\theta : M \times M \to \mathbb{R}$ is a mapping and $k \in [0, 1/2)$.

We now provide an example to support the concept of inward enriched Kannan mapping.

Example 2.1. Consider $M = \{1, 2, 3\}$ with a usual metric defined on it. Define $G : M \to M$ by

$$Gp = \begin{cases} 1, & p = 1, \\ 2, & p = 3, \\ 3, & p = 2, \end{cases}$$

and $\theta : M \times M \to \mathbb{R}$ by

$$\theta(p,q) = \begin{cases} -2, & p, q \in \{1, 2\} \text{ with } p \neq q, \\ -1/2, & p, q \in \{1, 3\} \text{ with } p \neq q, \\ 1, & p, q \in \{2, 3\} \text{ with } p \neq q, \\ 2, & p = q. \end{cases}$$

The readers can easily verify that the definition of inward enriched Kannan mapping is satisfied in this example. But the mapping mentioned above is neither a Kannan mapping nor an enriched Kannan mapping. To see this, consider $p = 2$, $q = 3$, and $p = 1$, $q = 3$, respectively.

The subsequent outcome guarantees the existence of a fixed point for an inward enriched Kannan mapping.

Theorem 2.1. Let $(M, \|\cdot\|)$ be a Banach space, and let $G : M \to M$ be an inward enriched Kannan mapping. Let $\eta \in (0, 1)$ be such that the following conditions hold:

(i) There exists $p_0 \in M$ with $\theta(p_0, (1-\eta)p_0 + \eta Gp_0) = l$;

(ii) For each $p, q \in M$ with $\theta(p, q) = l$, we get $\theta((1-\eta)p + \eta Gp, (1-\eta)q + \eta Gq) = l$;

(iii) For each sequence $\{p_n\}$ in M with $\theta(p_n, p_{n+1}) = l\ \forall n \in \mathbb{N}$ and $p_n \to p$ as $n \to \infty$, we have $\theta(p_n, p) = l\ \forall n \in \mathbb{N}$;

where $l = \frac{1}{\eta} - 1$. Then, G contains a fixed point in M. Moreover, if $\theta(p, q) > -1\ \forall p, q \in M$, then G contains a unique fixed point in M.

Proof. By the condition (i), we say that there exists $p_0 \in M$ with $\theta(p_0, (1-\eta)p_0 + \eta Gp_0) = l$. Define $p_1 = (1-\eta)p_0 + \eta Gp_0$. Then, we can write $\theta(p_0, p_1) = l$. By (2.1), we get

$$\|\theta(p_0, p_1)(p_0 - p_1) + Gp_0 - Gp_1\| \le k(\|p_0 - Gp_0\| + \|p_1 - Gp_1\|).$$

That is,

$$\|l(p_0 - p_1) + Gp_0 - Gp_1\| \le k(\|p_0 - Gp_0\| + \|p_1 - Gp_1\|).$$

Since $l = \frac{1}{\eta} - 1$, the above inequality is equivalent to

$$\left\|\left(\frac{1}{\eta} - 1\right)(p_0 - p_1) + Gp_0 - Gp_1\right\| \le k(\|p_0 - Gp_0\| + \|p_1 - Gp_1\|).$$

Simplifying the above inequality, we obtain

$$\left\|G_\eta p_0 - G_\eta p_1\right\| \le k(\|p_0 - G_\eta p_0\| + \|p_1 - G_\eta p_1\|),$$

where G_η is an averaged map, that is, $G_\eta p = (1-\eta)p + \eta Gp\ \forall p \in M$. By condition (ii) and $\theta(p_0, p_1) = l$, we get $\theta((1-\eta)p_0 + \eta Gp_0, (1-\eta)p_1 + \eta Gp_1) = l$. Again, define $p_2 = (1-\eta)p_1 + \eta Gp_1$. Then, we have $\theta(p_1, p_2) = l$. Again, by (2.1), we get

$$\|\theta(p_1, p_2)(p_1 - p_2) + Gp_1 - Gp_2\| \le k(\|p_1 - Gp_1\| + \|p_2 - Gp_2\|).$$

That is,

$$\left\|\left(\frac{1}{\eta} - 1\right)(p_1 - p_2) + Gp_1 - Gp_2\right\| \le k(\|p_1 - Gp_1\| + \|p_2 - Gp_2\|).$$

This implies

$$\left\|G_\eta p_1 - G_\eta p_2\right\| \le k(\|p_1 - G_\eta p_1\| + \|p_2 - G_\eta p_2\|).$$

Continuing the above procedure, we can obtain a sequence $\{p_n\}$ in M with the following properties:

- $p_n = G_\eta p_{n-1}\ \forall n \in \mathbb{N}$;
- $\theta(p_{n-1}, p_n) = l\ \forall n \in \mathbb{N}$;
- The following inequality is valid for each $n \in \mathbb{N}$:

$$\left\|G_\eta p_{n-1} - G_\eta p_n\right\| \leq k(\|p_{n-1} - G_\eta p_{n-1}\| + \|p_n - G_\eta p_n\|). \tag{2.3}$$

We can rewrite (2.3) as

$$\|p_n - p_{n+1}\| \leq k(\|p_{n-1} - p_n\| + \|p_n - p_{n+1}\|)\ \forall n \in \mathbb{N}.$$

This inequality further shows that

$$\|p_n - p_{n+1}\| \leq \frac{k}{1-k}\|p_{n-1} - p_n\|\ \forall n \in \mathbb{N}. \tag{2.4}$$

Using (2.4) and induction, we obtain

$$\|p_n - p_{n+1}\| \leq \left(\frac{k}{1-k}\right)^n \|p_0 - p_1\|\ \forall n \in \mathbb{N}. \tag{2.5}$$

Next, we will show that $\{p_n\}$ is a Cauchy in M by using the triangle property of normed space and (2.5).

$$\|p_n - p_m\| \leq \sum_{j=n}^{m-1} \left\|p_j - p_{j+1}\right\| \leq \sum_{j=n}^{m-1} \left(\frac{k}{1-k}\right)^j \|p_0 - p_1\|\ \forall n < m. \tag{2.6}$$

The fact $\frac{k}{1-k} < 1$ provides the convergence of $\sum_{j=1}^{\infty} \left(\frac{k}{1-k}\right)^j$. Thus, from (2.6), we get $\lim_{n,m\to\infty} \|p_n - p_m\| = 0$. That is, $\{p_n\}$ is a Cauchy in a Banach space M. Thus, we get a point p_* in M such that $p_n \to p_*$ as $n \to \infty$. From condition (iii) and the fact that $\theta(p_{n-1}, p_n) = l\ \forall n \in \mathbb{N}$, we say that $\theta(p_{n-1}, p_*) = l\ \forall n \in \mathbb{N}$. By (2.1), we obtain

$$\|\theta(p_{n-1}, p_*)(p_{n-1} - p_*) + Gp_{n-1} - Gp_*\| \leq k(\|p_{n-1} - Gp_{n-1}\| + \|p_* - Gp_*\|)\ \forall n \in \mathbb{N}.$$

That is,

$$\left\|\left(\frac{1}{\eta} - 1\right)(p_{n-1} - p_*) + Gp_{n-1} - Gp_*\right\| \leq k(\|p_{n-1} - Gp_{n-1}\| + \|p_* - Gp_*\|)\ \forall n \in \mathbb{N}.$$

This yields

$$\left\|G_\eta p_{n-1} - G_\eta p_*\right\| \leq k(\|p_{n-1} - G_\eta p_{n-1}\| + \|p_* - G_\eta p_*\|)\ \forall n \in \mathbb{N}.$$

We can also express the above inequality as

$$\left\|p_n - G_\eta p_*\right\| \leq k(\|p_{n-1} - p_n\| + \|p_* - G_\eta p_*\|)\ \forall n \in \mathbb{N}. \tag{2.7}$$

By using the triangle property of the norm and (2.7), we obtain

$$\|p_* - G_\eta p_*\| \leq \|p_* - p_n\| + \|p_n - G_\eta p_*\|$$

$$\leq \quad \|p_* - p_n\| + k(\|p_{n-1} - p_n\| + \|p_* - G_\eta p_*\|) \ \forall n \in \mathbb{N}.$$

As $n \to \infty$, the above inequality reduces to

$$\|p_* - G_\eta p_*\| \leq k\|p_* - G_\eta p_*\|.$$

This inequality only exists when $\|p_* - G_\eta p_*\| = 0$, that is, $p_* = G_\eta p_*$. Hence, $p_* = (1-\eta)p_* + \eta G p_*$, that is, $p_* = Gp_*$.

We now discuss the uniqueness of a fixed point of G. Consider we have two fixed points, p_* and q_*, then by hypothesis, we have $\theta(p_*, q_*) > -1$, that is $\theta(p_*, q_*) + 1 > 0$. Using (2.1), we obtain

$$\|\theta(p_*, q_*)(p_* - q_*) + Gp_* - Gq_*\| = \|(\theta(p_*, q_*) + 1)(p_* - q_*)\| \leq k(\|p_* - Gp_*\| + \|q_* - Gq_*\|) = 0.$$

This implies $p_* = q_*$. Hence, the fixed point of G is unique. □

Remark 2.1. The uniqueness of a fixed point can also be obtained in the preceding theorem by substituting $\theta(p, q) > -1 \ \forall p, q \in M$ with $p \in Gp$ and $q \in Gq$ for the condition $\theta(p, q) > -1 \ \forall p, q \in M$.

Remark 2.2. Theorem 2.1 reduces to Theorem 1.1 with $l > 0$ by defining $\theta(p, q) = l \ \forall p, q \in M$, where $l > 0$ is a fixed constant.

The following illustration clarifies the requirements of the aforementioned theorem.

Example 2.2. Consider $M = \mathbb{R}$ with $\|p\| = |p| \ \forall p \in M$. Define $G : M \to M$ and $\theta : M \times M \to \mathbb{R}$ by

$$Gp = \begin{cases} \frac{-2p}{3}, & p > 0, \\ 0, & p \leq 0, \end{cases}$$

and

$$\theta(p, q) = \begin{cases} \frac{1}{4}, & p, q \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

First, we discuss the existence of (2.1) by the following cases:
Case 1. For $p, q \leq 0$, we have $Gp = 0 = Gq$. Then, we obtain

$$\begin{aligned} \|\theta(p, q)(p - q) + Gp - Gq\| &= |(1/4)(p - q)| \\ &\leq (1/4)(|p| + |q|) \\ &= (1/4)(\|p - Gp\| + \|q - Gq\|) \\ &\leq \frac{2.25}{5}\Big(\|p - Gp\| + \|q - Gq\|\Big). \end{aligned}$$

Case 2. For $p, q > 0$, we have $Gp = \frac{-2p}{3}$ and $Gq = \frac{-2q}{3}$. Then, we obtain

$$\begin{aligned} \|\theta(p, q)(p - q) + Gp - Gq\| &= \left|0(p - q) + \frac{-2p}{3} - \frac{-2q}{3}\right| \\ &= \frac{2}{3}(|p - q|) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{2.25}{5}\left(\left|\frac{5p}{3}\right| + \left|\frac{5q}{3}\right|\right)\\
&= \frac{2.25}{5}\Big(\|p - Gp\| + \|q - Gq\|\Big).
\end{aligned}
$$

Case 3. For $p \ge 0$ and $q < 0$, we have $Gp = \frac{-2p}{3}$ and $Gq = 0$. Then, we obtain

$$
\begin{aligned}
\|\theta(p,q)(p-q) + Gp - Gq\| &= \left|0(p-q) + \frac{-2p}{3} - 0\right|\\
&= \frac{2}{3}\Big(|p|\Big)\\
&\le \frac{2.25}{5}\left(\left|\frac{5p}{3}\right| + |q|\right)\\
&= \frac{2.25}{5}\Big(\|p - Gp\| + \|q - Gq\|\Big).
\end{aligned}
$$

By considering the above cases, we see that (2.1) holds with $k = \frac{2.25}{5}$. Next, we discuss the existence of Axioms (i)–(iii). By given θ, we assume $l = \frac{1}{4}$; this implies $\eta = \frac{4}{5}$. For Axiom (i), take $p_0 = -1$, then $(1-\eta)p_0 + \eta Gp_0 = \left(1 - \frac{4}{5}\right)(-1) + \frac{4}{5}(0) < 0$, thus $\theta(p_0, (1-\eta)p_0 + \eta Gp_0) = \frac{1}{4}$ with $\eta = \frac{4}{5}$. If $p, q \le 0$, then $(1-\eta)p + \eta Gp = (1-\eta)p \le 0$ and $(1-\eta)q + \eta Gq = (1-\eta)q \le 0$. Hence, Axiom (ii) holds; that is, $\theta(p,q) = \frac{1}{4}$ implies $\theta((1-\eta)p + \eta Gp, (1-\eta)q + \eta Gq) = \frac{1}{4}$ with $\eta = \frac{4}{5}$. Also, for each sequence $\{p_n\}$ in M with $\theta(p_n, p_{n+1}) = \frac{1}{4}$ $\forall n \in \mathbb{N}$ and $p_n \to p$ as $n \to \infty$, we have $\theta(p_n, p) = \frac{1}{4}$ $\forall n \in \mathbb{N}$. Hence, all conditions of Theorem 2.1 are valid in this example, and G has a fixed point.

The following definition presents the concept of an outward enriched Kannan mapping.

Definition 2.2. Let $(M, \|\cdot\|)$ be a normed linear space. A mapping $G : M \to M$ is called an outward enriched Kannan mapping if there exist $l \in [0, \infty)$ and $k \in [0, 1/2)$ such that

$$
\theta(p,q)\|l(p-q) + Gp - Gq\| \le k(\|p - Gp\| + \|q - Gq\|)\ \forall p, q \in M, \tag{2.8}
$$

where $\theta : M \times M \to [0, \infty)$ is a mapping.

The above definition can also be expressed in terms of metric spaces, as follows:

Let (M, d) be a metric space. A mapping $G : M \to M$ is called an outward enriched Kannan mapping if there exist $l \in [0, \infty)$ and $k \in [0, 1/2)$ such that

$$
\theta(p,q)d(lp + Gp, lq + Gq) \le k(d(p, Gp) + d(q, Gq))\ \forall p, q \in M, \tag{2.9}
$$

where $\theta : M \times M \to [0, \infty)$ is a mapping.

To strengthen the idea of outward enriched Kannan mapping, we have developed the following example:

Example 2.3. Consider $M = [0, \infty)$ with a usual metric defined on it. Define $G : M \to M$ by

$$
Gp = \begin{cases} 1 - p, & p \in [0, 1], \\ p, & p \ge 1, \end{cases}
$$

and $\theta : M \times M \to [0, \infty)$ by

$$\theta(p, q) = \begin{cases} 1, & p, q \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that the definition of outward enriched Kannan mapping holds for this example with $l = 1-2k$ for each $k \in [0, 1/2)$. But the mapping mentioned above is not an enriched Kannan mapping; for instance, consider $p, q \geq 1$.

Outward enriched Kannan mappings also possess fixed points under certain conditions, as stated in the following theorem.

Theorem 2.2. Let $(M, \|\cdot\|)$ be a Banach space, and let $G : M \to M$ be an outward enriched Kannan mapping. Let the following conditions also hold:

(i) There exists $p_0 \in M$ such that $\theta(p_0, (1-\eta)p_0 + \eta Gp_0) = 1$;

(ii) For each $p, q \in M$ with $\theta(p, q) = 1$, we have $\theta((1-\eta)p + \eta Gp, (1-\eta)q + \eta Gq) = 1$;

(iii) For each sequence $\{p_n\}$ in M with $\theta(p_n, p_{n+1}) = 1$ $\forall n \in \mathbb{N}$ and $p_n \to p$ as $n \to \infty$, we have $\theta(p_n, p) = 1$ $\forall n \in \mathbb{N}$.

Where $\eta = \frac{1}{l+1}$ and l is the constant that appears in (2.8). Then, G contains a fixed point in M. Moreover, if $\theta(p, q) > 0$ $\forall p, q \in M$, then G contains a unique fixed point in M.

Proof. First, we discuss the proof for an outward enriched Kannan mapping with $l > 0$. The condition (i) of the theorem implies that $p_0 \in M$ with $\theta(p_0, (1-\eta)p_0 + \eta Gp_0) = 1$ with $\eta = \frac{1}{l+1}$. Set $p_1 = (1-\eta)p_0 + \eta Gp_0$. Then, we can write $\theta(p_0, p_1) = 1$. By (2.8), we obtain

$$\theta(p_0, p_1)\|l(p_0 - p_1) + Gp_0 - Gp_1\| \leq k(\|p_0 - Gp_0\| + \|p_1 - Gp_1\|).$$

Since $\theta(p_0, p_1) = 1$ and $l = \frac{1}{\eta} - 1 > 0$, the above inequality is equivalent to

$$\left\|\left(\frac{1}{\eta} - 1\right)(p_0 - p_1) + Gp_0 - Gp_1\right\| \leq k(\|p_0 - Gp_0\| + \|p_1 - Gp_1\|).$$

This implies

$$\left\|G_\eta p_0 - G_\eta p_1\right\| \leq k(\|p_0 - G_\eta p_0\| + \|p_1 - G_\eta p_1\|),$$

where G_η is an averaged map with $\eta = \frac{1}{l+1}$. As $\theta(p_0, p_1) = 1$ then condition (ii) provides $\theta((1-\eta)p_0 + \eta Gp_0, (1-\eta)p_1 + \eta Gp_1) = 1$ with $\eta = \frac{1}{l+1}$. By setting $p_2 = (1-\eta)p_1 + \eta Gp_1$, we write $\theta(p_1, p_2) = 1$. Again, by (2.8), we obtain

$$\theta(p_1, p_2)\|l(p_1 - p_2) + Gp_1 - Gp_2\| \leq k(\|p_1 - Gp_1\| + \|p_2 - Gp_2\|).$$

After performing a few simplifications, we obtain

$$\left\|G_\eta p_1 - G_\eta p_2\right\| \leq k(\|p_1 - G_\eta p_1\| + \|p_2 - G_\eta p_2\|).$$

Working with this procedure, we can obtain a sequence $\{p_n\}$ in M with the following properties:

- $p_n = G_\eta p_{n-1}$ $\forall n \in \mathbb{N}$ with $\eta = \frac{1}{l+1}$;
- $\theta(p_{n-1}, p_n) = 1$ $\forall n \in \mathbb{N}$;
- For an averaged map G_η with $\eta = \frac{1}{l+1}$, we obtain

$$\left\|G_\eta p_{n-1} - G_\eta p_n\right\| \leq k(\|p_{n-1} - G_\eta p_{n-1}\| + \|p_n - G_\eta p_n\|)\ \forall n \in \mathbb{N}. \tag{2.10}$$

We can rewrite (2.10) as

$$\|p_n - p_{n+1}\| \leq k(\|p_{n-1} - p_n\| + \|p_n - p_{n+1}\|)\ \forall n \in \mathbb{N}.$$

By collecting the same terms on one side of the inequality, we obtain

$$\|p_n - p_{n+1}\| \leq \frac{k}{1-k}\|p_{n-1} - p_n\|\ \forall n \in \mathbb{N}. \tag{2.11}$$

Using (2.11) and induction, we obtain

$$\|p_n - p_{n+1}\| \leq \left(\frac{k}{1-k}\right)^n \|p_0 - p_1\|\ \forall n \in \mathbb{N}. \tag{2.12}$$

From (2.12), one can see that

$$\|p_n - p_m\| \leq \sum_{j=n}^{m-1} \left\|p_j - p_{j+1}\right\| \leq \sum_{j=n}^{m-1} \left(\frac{k}{1-k}\right)^j \|p_0 - p_1\|\ \forall n < m. \tag{2.13}$$

Thus, $\lim_{n,m\to\infty} \|p_n - p_m\| = 0$. That is, $\{p_n\}$ is Cauchy in a Banach space M. Hence, a point p_* exists in M such that $p_n \to p_*$ as $n \to \infty$. Since $\theta(p_{n-1}, p_n) = 1$ $\forall n \in \mathbb{N}$, then by the condition (iii), we obtain $\theta(p_{n-1}, p_*) = 1$ $\forall n \in \mathbb{N}$. By (2.8), we obtain

$$\theta(p_{n-1}, p_*)\,\|l(p_{n-1} - p_*) + Gp_{n-1} - Gp_*\| \leq k(\|p_{n-1} - Gp_{n-1}\| + \|p_* - Gp_*\|)\ \forall n \in \mathbb{N}.$$

That is,

$$\left\|\left(\frac{1}{\eta} - 1\right)(p_{n-1} - p_*) + Gp_{n-1} - Gp_*\right\| \leq k(\|p_{n-1} - Gp_{n-1}\| + \|p_* - Gp_*\|)\ \forall n \in \mathbb{N}.$$

By simplifying the above inequality, we obtain

$$\left\|G_\eta p_{n-1} - G_\eta p_*\right\| \leq k(\|p_{n-1} - G_\eta p_{n-1}\| + \|p_* - G_\eta p_*\|)\ \forall n \in \mathbb{N}.$$

Equivalently, we say

$$\left\|p_n - G_\eta p_*\right\| \leq k(\|p_{n-1} - p_n\| + \|p_* - G_\eta p_*\|)\ \forall n \in \mathbb{N}. \tag{2.14}$$

The limiting case of (2.14), as $n \to \infty$, is provided below:

$$\|p_* - G_\eta p_*\| \leq k\|p_* - G_\eta p_*\|\ \forall n \in \mathbb{N},$$

which only holds when $\|p_* - G_\eta p_*\| = 0$, that is, $p_* = G_\eta p_*$. Hence, $p_* = (1-\eta)p_* + \eta G p_*$, that is, $p_* = Gp_*$.

We next talk about the case where $l = 0$. The theorem's formulation gives us the following conditions for $l = 0$.

- For each $p, q \in M$, we have

$$\theta(p,q)\|Gp - Gq\| \le k(\|p - Gp\| + \|q - Gq\|) \; \forall p, q \in M;$$

- There exists $p_0 \in M$ such that $\theta(p_0, Gp_0) = 1$;
- For each $p, q \in M$ with $\theta(p, q) = 1$, we get $\theta(Gp, Gq) = 1$;
- For each sequence $\{p_n\}$ in M with $\theta(p_n, p_{n+1}) = 1 \; \forall n \in \mathbb{N}$ and $p_n \to p$ as $n \to \infty$, we have $\theta(p_n, p) = 1 \; \forall n \in \mathbb{N}$.

It is guaranteed that there is a fixed point of G under these circumstances. Since $G_\eta = G_1 = G$ and $p_n = Gp_{n-1} \; \forall n \in \mathbb{N}$.

We now discuss the uniqueness of a fixed point of G. Let there exist two fixed points, p_* and q_*, then by hypothesis, we have $\theta(p_*, q_*) > 0$. Using (2.8), we obtain

$$\begin{aligned} \theta(p_*, q_*)\|l(p_* - q_*) + Gp_* - Gq_*\| &= \theta(p_*, q_*)\|(l+1)(p_* - q_*)\| \\ &\le k(\|p_* - Gp_*\| + \|q_* - Gq_*\|) = 0. \end{aligned}$$

This implies $p_* = q_*$. Hence, G has a unique fixed point. □

Remark 2.3. In the aforementioned theorem, the uniqueness of a fixed point can also be established by substituting $\theta(p, q) > 0 \; \forall p, q \in M$ with $p \in Gp$ and $q \in Gq$ for the condition $\theta(p, q) > 0 \; \forall p, q \in M$.

Remark 2.4. Theorem 2.2 reduces to Theorem 1.1 by defining $\theta(p, q) = 1 \; \forall p, q \in M$.

The following example supports and explains the above theorem.

Example 2.4. Consider $M = \mathbb{R}$ with $\|p\| = |p| \; \forall p \in M$. Define $G : M \to M$ by

$$Gp = \begin{cases} 1 - p, & p \in [0, 1], \\ p, & \text{otherwise}, \end{cases}$$

and $\theta : M \times M \to [0, \infty)$ by

$$\theta(p, q) = \begin{cases} 1, & p, q \in [0, 1], \\ 0, & \text{otherwise}. \end{cases}$$

The definition of outward enriched Kannan mapping holds for this example with $l = \frac{1}{2}$ and $k = \frac{1}{4}$. For instance, consider the following cases:
Case 1. For $p, q \in [0, 1]$, we have $Gp = 1 - p$ and $Gq = 1 - q$. Then, we obtain

$$\begin{aligned} \theta(p,q)\|l(p - q) + Gp - Gq\| &= 1\left(\left|\frac{1}{2}(p - q) + (1 - p) - (1 - q)\right|\right) \\ &= \left|\frac{1}{2}(p - q) - (p - q)\right| \\ &= \frac{1}{2}(|p - q|) \\ &= \frac{1}{4}(|2p - 2q|) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}(|(2p-1)-(2q-1)|) \\
&\leq \frac{1}{4}(|2p-1|+|2q-1|) \\
&= k(\|p-Gp\|+\|q-Gq\|).
\end{aligned}
$$

Case 2. For $p \notin [0,1]$ and $q \in M$. Then, we obtain

$$\theta(p,q)\|l(p-q)+Gp-Gq\| = 0 \leq k(\|p-Gp\|+\|q-Gq\|).$$

Next, we discuss the existence of Axioms (i)–(iii) with $\eta = \frac{1}{l+1} = \frac{1}{\frac{1}{2}+1} = \frac{2}{3}$, since the definition of outward enriched Kannan mapping holds with $l = \frac{1}{2}$. For Axiom (i), consider $p_0 = 0$, then $(1-\eta)p_0 + \eta Gp_0 = \left(1-\frac{2}{3}\right)(0) + \frac{2}{3}(1) \in [0,1]$, thus $\theta(p_0,(1-\eta)p_0+\eta Gp_0) = 1$. If $p,q \in [0,1]$, then $(1-\eta)p+\eta Gp = (1-\frac{2}{3})p+\frac{2}{3}(1-p) \leq p+(1-p) = 1$ and $(1-\eta)q+\eta Gq = (1-\frac{2}{3})q+\frac{2}{3}(1-q) \leq q+(1-q) = 1$. Hence, Axiom (ii) holds; that is, $\theta(p,q) = 1$ implies $\theta((1-\eta)p+\eta Gp,(1-\eta)q+\eta Gq) = 1$ with $\eta = \frac{2}{3}$. Also, for each sequence $\{p_n\}$ in M with $\theta(p_n,p_{n+1}) = 1$ $\forall n \in \mathbb{N}$ and $p_n \to p$ as $n \to \infty$, we have $\theta(p_n,p) = 1$ $\forall n \in \mathbb{N}$. Hence, all conditions of Theorem 2.2 are valid in this example, and G has a fixed point.

Remark 2.5. In the above example, by considering $p = 2$ and $q = 3$, we conclude the following:

- Theorem 2.1 is not applicable for the given θ.
- Theorem 1.1 is not applicable.
- Kannan fixed-point theorem [3] is not applicable.

We now provide generalized notions of enriched Bianchini mappings and discuss the existence of fixed points for these notions.

Definition 2.3. Let $(M,\|\cdot\|)$ be a normed linear space. A mapping $G : M \to M$ is called an inward enriched Bianchini mapping if, for each $p,q \in M$, we have

$$\|\theta(p,q)(p-q)+Gp-Gq\| \leq k\max\{\|p-Gp\|,\|q-Gq\|\} \tag{2.15}$$

where $\theta : M \times M \to \mathbb{R}$ is a mapping and $k \in [0,1)$.

The existence of a fixed point for the aforementioned idea is given by the following theorem.

Theorem 2.3. Let $(M,\|\cdot\|)$ be a Banach space, and let $G : M \to M$ be an inward enriched Bianchini mapping. Let $\eta \in (0,1)$ be such that the following conditions hold:

(i) There exists $p_0 \in M$ with $\theta(p_0,(1-\eta)p_0+\eta Gp_0) = l$;

(ii) For each $p,q \in M$ with $\theta(p,q) = l$, we have $\theta((1-\eta)p+\eta Gp,(1-\eta)q+\eta Gq) = l$;

(iii) For each sequence $\{p_n\}$ in M with $\theta(p_n,p_{n+1}) = l$ $\forall n \in \mathbb{N}$ and $p_n \to p$ as $n \to \infty$, we have $\theta(p_n,p) = l$ $\forall n \in \mathbb{N}$;

where $l = \frac{1}{\eta} - 1$. Then, G contains a fixed point in M. Moreover, if $\theta(p,q) > -1$ $\forall p,q \in M$, then G contains a unique fixed point in M.

Proof. From the condition (i), we obtain $p_0 \in M$ with $\theta(p_0, (1-\eta)p_0 + \eta G p_0) = l$. By letting $p_1 = (1-\eta)p_0 + \eta G p_0$, we can write $\theta(p_0, p_1) = l$. By (2.15), we obtain

$$\|\theta(p_0, p_1)(p_0 - p_1) + Gp_0 - Gp_1\| \leq k \max\{\|p_0 - Gp_0\|, \|p_1 - Gp_1\|\}.$$

This inequality is equivalent to the following.

$$\left\|\left(\frac{1}{\eta} - 1\right)(p_0 - p_1) + Gp_0 - Gp_1\right\| \leq k \max\{\|p_0 - Gp_0\|, \|p_1 - Gp_1\|\}.$$

Simplifying the above inequality, we obtain

$$\left\|G_\eta p_0 - G_\eta p_1\right\| \leq k \max\{\|p_0 - G_\eta p_0\|, \|p_1 - G_\eta p_1\|\}.$$

As $\theta(p_0, p_1) = l$, by (ii), we get $\theta((1-\eta)p_0 + \eta G p_0, (1-\eta)p_1 + \eta G p_1) = l$. We can also rewrite it as $\theta(p_1, p_2) = l$ by setting $p_2 = (1-\eta)p_1 + \eta G p_1$. Again, by (2.15), we obtain

$$\|\theta(p_1, p_2)(p_1 - p_2) + Gp_1 - Gp_2\| \leq k \max\{\|p_1 - Gp_1\|, \|p_2 - Gp_2\|\}.$$

That is,

$$\left\|\left(\frac{1}{\eta} - 1\right)(p_1 - p_2) + Gp_1 - Gp_2\right\| \leq k \max\{\|p_1 - Gp_1\|, \|p_2 - Gp_2\|\}.$$

This implies

$$\left\|G_\eta p_1 - G_\eta p_2\right\| \leq k \max\{\|p_1 - G_\eta p_1\|, \|p_2 - G_\eta p_2\|\}.$$

Continuing the same pattern, we get a sequence $\{p_n\}$ in M with the following properties:

- $p_n = G_\eta p_{n-1}\ \forall n \in \mathbb{N}$;
- $\theta(p_{n-1}, p_n) = l\ \forall n \in \mathbb{N}$;
- The following inequality is valid for each $n \in \mathbb{N}$:

$$\left\|G_\eta p_{n-1} - G_\eta p_n\right\| \leq k \max\{\|p_{n-1} - G_\eta p_{n-1}\|, \|p_n - G_\eta p_n\|\}. \tag{2.16}$$

We can rewrite (2.16) as

$$\|p_n - p_{n+1}\| \leq k \max\{\|p_{n-1} - p_n\|, \|p_n - p_{n+1}\|\}\ \forall n \in \mathbb{N}.$$

In order to proceed with the proof, we assume that the above inequality yields the following:

$$\|p_n - p_{n+1}\| \leq k\|p_{n-1} - p_n\|\ \forall n \in \mathbb{N}. \tag{2.17}$$

Otherwise, we obtain a fixed point of G_η and hence of G. Using (2.17) and induction, we obtain

$$\|p_n - p_{n+1}\| \leq k^n\|p_0 - p_1\|\ \forall n \in \mathbb{N}. \tag{2.18}$$

Following the proof of Theorem 2.1, we say that (2.18) guarantees that $\{p_n\}$ is Cauchy in a Banach space M. Hence, $\{p_n\}$ converges to some p_* in M, that is, $p_n \to p_*$ as $n \to \infty$. From the condition (iii) and the fact that $\theta(p_{n-1}, p_n) = l \; \forall n \in \mathbb{N}$, we say that $\theta(p_{n-1}, p_*) = l \; \forall n \in \mathbb{N}$. By (2.15), we obtain

$$\|\theta(p_{n-1}, p_*)(p_{n-1} - p_*) + Gp_{n-1} - Gp_*\| \leq k \max\{\|p_{n-1} - Gp_{n-1}\|, \|p_* - Gp_*\|\} \; \forall n \in \mathbb{N}.$$

That is,

$$\left\|\left(\frac{1}{\eta} - 1\right)(p_{n-1} - p_*) + Gp_{n-1} - Gp_*\right\| \leq k \max\{\|p_{n-1} - Gp_{n-1}\|, \|p_* - Gp_*\|\} \; \forall n \in \mathbb{N}.$$

This yields

$$\left\|G_\eta p_{n-1} - G_\eta p_*\right\| \leq k \max\{\|p_{n-1} - G_\eta p_{n-1}\|, \|p_* - G_\eta p_*\|\} \; \forall n \in \mathbb{N}.$$

That is,

$$\left\|p_n - G_\eta p_*\right\| \leq k \max\{\|p_{n-1} - p_n\|, \|p_* - G_\eta p_*\|\} \; \forall n \in \mathbb{N}. \tag{2.19}$$

Letting $n \to \infty$ in (2.19), we obtain

$$\|p_* - G_\eta p_*\| \leq k\|p_* - G_\eta p_*\| \; \forall n \in \mathbb{N}.$$

This implies $\|p_* - G_\eta p_*\| = 0$, that is, $p_* = G_\eta p_*$. Hence, $p_* = (1-\eta)p_* + \eta G p_*$ and we obtain $p_* = Gp_*$.

Next, we discuss the uniqueness of a fixed point of G. Suppose that there are two fixed points, p_* and q_*, then by hypothesis, we have $\theta(p_*, q_*) > -1$, that is, $\theta(p_*, q_*) + 1 > 0$. Using (2.15), we obtain

$$\|\theta(p_*, q_*)(p_* - q_*) + Gp_* - Gq_*\| = \|(\theta(p_*, q_*) + 1)(p_* - q_*)\| \leq k \max\{\|p_* - Gp_*\|, \|q_* - Gq_*\|\} = 0.$$

The above inequality implies that $p_* = q_*$. Hence, G contains a unique fixed point. □

Remark 2.6. Theorem 2.3 reduces to Theorem 1.2 with $l > 0$ by letting $\theta(p, q) = l \; \forall p, q \in M$, where $l > 0$ is a fixed constant.

To support the above theorem, we provide the following example.

Example 2.5. Consider $M = \mathbb{R}$ with $\|p\| = |p| \; \forall p \in M$. Define $G : M \to M$ and $\theta : M \times M \to \mathbb{R}$ by

$$Gp = \begin{cases} -p, & p > 0, \\ 0, & p \leq 0, \end{cases}$$

and

$$\theta(p, q) = \begin{cases} \frac{1}{3}, & p, q \leq 0, \\ 0, & \text{otherwise}. \end{cases}$$

To discuss the existence of (2.15), we consider the following cases.
Case 1. For $p, q \leq 0$, we have $Gp = 0 = Gq$. Then, we obtain

$$\|\theta(p, q)(p - q) + Gp - Gq\| = |(1/3)(p - q)|$$

$$
\begin{aligned}
&\leq (1/3)(|p| + |q|) \\
&\leq \frac{2}{3}\Big(\max\{\|p - Gp\|, \|q - Gq\|\}\Big).
\end{aligned}
$$

Case 2. For $p, q > 0$, we have $Gp = -p$ and $Gq = -q$. Then, we obtain

$$
\begin{aligned}
\|\theta(p,q)(p-q) + Gp - Gq\| &= |0(p-q) + (-p) - (-q)| \\
&= |p - q| \\
&< \max\{|p|, |q|\} \\
&< \frac{2}{3}\Big(\max\{\|p - Gp\|, \|q - Gq\|\}\Big).
\end{aligned}
$$

Case 3. For $p \leq 0$ and $q > 0$, we have $Gp = 0$ and $Gq = -q$. Then, we obtain

$$
\begin{aligned}
\|\theta(p,q)(p-q) + Gp - Gq\| &= |0(p-q) + 0 - (-q)| \\
&= |q| \\
&< \frac{2}{3}\Big(\max\{|p|, |2q|\}\Big) \\
&= \frac{2}{3}\Big(\max\{\|p - Gp\|, \|q - Gq\|\}\Big).
\end{aligned}
$$

The above cases show that (2.15) holds with $k = \frac{2}{3}$. We now discuss the existence of Axioms (i)–(iii). By given θ, we assume $l = \frac{1}{3}$; this implies $\eta = \frac{3}{4}$. For Axiom (i), consider $p_0 = -2$, then $(1-\eta)p_0 + \eta Gp_0 = \left(1 - \frac{3}{4}\right)(-2) + \frac{3}{4}(0) < 0$, thus $\theta(p_0, (1-\eta)p_0 + \eta Gp_0) = \frac{1}{3}$ with $\eta = \frac{3}{4}$. If $p, q \leq 0$, then $(1-\eta)p + \eta Gp = (1-\eta)p < 0$ and $(1-\eta)q + \eta Gq = (1-\eta)q < 0$. Hence, Axiom (ii) holds; that is, $\theta(p,q) = \frac{1}{3}$ implies $\theta((1-\eta)p + \eta Gp, (1-\eta)q + \eta Gq) = \frac{1}{3}$ with $\eta = \frac{3}{4}$. Also, for each sequence $\{p_n\}$ in M with $\theta(p_n, p_{n+1}) = \frac{1}{3}$ $\forall n \in \mathbb{N}$ and $p_n \to p$ as $n \to \infty$, we have $\theta(p_n, p) = \frac{1}{3}$ $\forall n \in \mathbb{N}$. Hence, all conditions of Theorem 2.3 are valid in this example, and G has a fixed point.

Remark 2.7. For $p = 1$ and $q = 0$, we can calculate $\|\theta(p,q)(p-q)+G(p)-G(q)\| = |0(1-0)+(-1)-0| = 1$, $\|p - Gp\| = 2$, and $\|q - Gq\| = 0$. Hence, it is trivial to mention that the above-defined example is not an inward enriched Kannan mapping, and Theorem 2.1 is not applicable in this example.

The following definition provides the concept of outward enriched Bianchini mapping.

Definition 2.4. Let $(M, \|\cdot\|)$ be a normed linear space. A mapping $G : M \to M$ is called an outward enriched Bianchini mapping if there exist $l \in [0, \infty)$ and $k \in [0, 1)$ such that

$$
\theta(p,q)\|l(p-q) + Gp - Gq\| \leq k \max\{\|p - Gp\|, \|q - Gq\|\} \ \forall p, q \in M, \tag{2.20}
$$

where $\theta : M \times M \to [0, \infty)$ is a mapping.

Theorem 2.4. Let $(M, \|\cdot\|)$ be a Banach space, and let $G : M \to M$ be an outward enriched Bianchini mapping. Let the following conditions also hold:

(i) There exists $p_0 \in M$ such that $\theta(p_0, (1-\eta)p_0 + \eta Gp_0) = 1$;

(ii) For each $p, q \in M$ with $\theta(p,q) = 1$, we have $\theta((1-\eta)p + \eta Gp, (1-\eta)q + \eta Gq) = 1$;

(iii) For each sequence $\{p_n\}$ in M with $\theta(p_n, p_{n+1}) = 1\ \forall n \in \mathbb{N}$ and $p_n \to p$ as $n \to \infty$, we have $\theta(p_n, p) = 1\ \forall n \in \mathbb{N}$.

Where $\eta = \frac{1}{l+1}$ and l is the constant that appears in (2.20). Then, G contains a fixed point in M. Moreover, if $\theta(p, q) > 0\ \forall p, q \in M$, then G contains a unique fixed point in M.

Proof. First, assume that G is an outward enriched Bianchini mapping with a value of l greater than zero. The condition (i) ensures $p_0 \in M$ such that $\theta(p_0, (1-\eta)p_0 + \eta G p_0) = 1$ with $\eta = \frac{1}{l+1}$. Assume that $p_1 = (1-\eta)p_0 + \eta G p_0$. Then, we say $\theta(p_0, p_1) = 1$. Using (2.20), we obtain

$$\theta(p_0, p_1)\|l(p_0 - p_1) + Gp_0 - Gp_1\| \leq k \max\{\|p_0 - Gp_0\|, \|p_1 - Gp_1\|\}. \tag{2.21}$$

As $\theta(p_0, p_1) = 1$ and $l = \frac{1}{\eta} - 1 > 0$, then (2.21) provides

$$\left\| \left(\frac{1}{\eta} - 1\right)(p_0 - p_1) + Gp_0 - Gp_1 \right\| \leq k \max\{\|p_0 - Gp_0\|, \|p_1 - Gp_1\|\}.$$

After performing some calculations, it implies that

$$\left\| G_\eta p_0 - G_\eta p_1 \right\| \leq k \max\{\|p_0 - G_\eta p_0\|, \|p_1 - G_\eta p_1\|\},$$

where G_η is an averaged map with $\eta = \frac{1}{l+1}$. As $\theta(p_0, p_1) = 1$, then the condition (ii) implies $\theta((1-\eta)p_0 + \eta G p_0, (1-\eta)p_1 + \eta G p_1) = 1$ with $\eta = \frac{1}{l+1}$. By considering $p_2 = (1-\eta)p_1 + \eta G p_1$, we say that $\theta(p_1, p_2) = 1$. Again, by (2.20), we obtain

$$\theta(p_1, p_2)\|l(p_1 - p_2) + Gp_1 - Gp_2\| \leq k \max\{\|p_1 - Gp_1\|, \|p_2 - Gp_2\|\}.$$

After performing a few simplifications, we obtain

$$\left\| G_\eta p_1 - G_\eta p_2 \right\| \leq k \max\{\|p_1 - G_\eta p_1\|, \|p_2 - G_\eta p_2\|\}.$$

This procedure generates a sequence $\{p_n\}$ in M with the following properties:

- $p_n = G_\eta p_{n-1}\ \forall n \in \mathbb{N}$ with $\eta = \frac{1}{l+1}$;
- $\theta(p_{n-1}, p_n) = 1\ \forall n \in \mathbb{N}$;
- For an averaged map G_η with $\eta = \frac{1}{l+1}$, we obtain

$$\left\| G_\eta p_{n-1} - G_\eta p_n \right\| \leq k \max\{\|p_{n-1} - G_\eta p_{n-1}\|, \|p_n - G_\eta p_n\|\}\ \forall n \in \mathbb{N}. \tag{2.22}$$

We can restate (2.22) as

$$\|p_n - p_{n+1}\| \leq k \max\{\|p_{n-1} - p_n\|, \|p_n - p_{n+1}\|\}\ \forall n \in \mathbb{N}. \tag{2.23}$$

If we assume that $\max\{\|p_{n-1} - p_n\|, \|p_n - p_{n+1}\|\} = \|p_n - p_{n+1}\|$ for some n, then we obtain a fixed point of G_η and G. To continue the proof, we assume it is not possible, then by (2.23), we obtain

$$\|p_n - p_{n+1}\| \leq k\|p_{n-1} - p_n\|\ \forall n \in \mathbb{N}. \tag{2.24}$$

Using (2.24) and induction, we obtain

$$\|p_n - p_{n+1}\| \leq k^n \|p_0 - p_1\| \ \forall n \in \mathbb{N}. \tag{2.25}$$

From (2.25), one can see that

$$\|p_n - p_m\| \leq \sum_{j=n}^{m-1} \left\| p_j - p_{j+1} \right\| \leq \sum_{j=n}^{m-1} k^j \|p_0 - p_1\| \ \forall n < m. \tag{2.26}$$

Thus, $\lim_{n,m\to\infty} \|p_n - p_m\| = 0$. That is, $\{p_n\}$ is Cauchy in a Banach space M. Hence, a point p_* exists in M such that $p_n \to p_*$ as $n \to \infty$. Since $\theta(p_{n-1}, p_n) = 1 \ \forall n \in \mathbb{N}$, then by following the condition (iii), we obtain $\theta(p_{n-1}, p_*) = 1 \ \forall n \in \mathbb{N}$. By (2.20), we obtain

$$\theta(p_{n-1}, p_*) \|l(p_{n-1} - p_*) + Gp_{n-1} - Gp_*\| \leq k \max\{\|p_{n-1} - Gp_{n-1}\|, \|p_* - Gp_*\|\} \ \forall n \in \mathbb{N}.$$

That is,

$$\left\| \left(\frac{1}{\eta} - 1 \right) (p_{n-1} - p_*) + Gp_{n-1} - Gp_* \right\| \leq k \max\{\|p_{n-1} - Gp_{n-1}\|, \|p_* - Gp_*\|\} \ \forall n \in \mathbb{N}.$$

By simplifying the above inequality, we obtain

$$\left\| G_\eta p_{n-1} - G_\eta p_* \right\| \leq k \max\{\|p_{n-1} - G_\eta p_{n-1}\|, \|p_* - G_\eta p_*\|\} \ \forall n \in \mathbb{N}.$$

Equivalently, we say

$$\left\| p_n - G_\eta p_* \right\| \leq k \max\{\|p_{n-1} - p_n\|, \|p_* - G_\eta p_*\|\} \ \forall n \in \mathbb{N}. \tag{2.27}$$

The limiting case of (2.27), as $n \to \infty$, is provided below.

$$\|p_* - G_\eta p_*\| \leq k \|p_* - G_\eta p_*\|,$$

which only holds when $\|p_* - G_\eta p_*\| = 0$, that is, $p_* = G_\eta p_*$. Hence, $p_* = (1 - \eta)p_* + \eta G p_*$, that is, $p_* = Gp_*$.

Next, we discuss the situation in which $l = 0$. For $l = 0$, we obtain the following conditions from the statement of the theorem:

- For each $p, q \in M$, we have

$$\theta(p, q)\|Gp - Gq\| \leq k \max\{\|p - Gp\|, \|q - Gq\|\} \ \forall p, q \in M.$$

- There exists $p_0 \in M$ such that $\theta(p_0, Gp_0) = 1$.
- For each $p, q \in M$ with $\theta(p, q) = 1$, we have $\theta(Gp, Gq) = 1$.
- For each sequence $\{p_n\}$ in M with $\theta(p_n, p_{n+1}) = 1 \ \forall n \in \mathbb{N}$ and $p_n \to p$ as $n \to \infty$, we have $\theta(p_n, p) = 1 \ \forall n \in \mathbb{N}$.

Under these conditions, the existence of a fixed point of G is guaranteed. Since $G_\eta = G_1 = G$ and $p_n = Gp_{n-1}$ $\forall n \in \mathbb{N}$.

To discuss the uniqueness of a fixed point of G, assume that G has two fixed points, p_* and q_*, then by hypothesis, we obtain $\theta(p_*, q_*) > 0$. By using (2.20), we obtain

$$\begin{aligned}\theta(p_*, q_*)\|l(p_* - q_*) + Gp_* - Gq_*\| &= \theta(p_*, q_*)\|(l+1)(p_* - q_*)\| \\ &\leq k\{\|p_* - Gp_*\|, \|q_* - Gq_*\|\} = 0.\end{aligned}$$

The above inequality implies $p_* = q_*$. Hence, G contains a unique fixed point. □

Remark 2.8. Theorem 2.4 reduces to Theorem 1.2 by assuming $\theta(p, q) = 1$ $\forall p, q \in M$.

The following example supports the above result.

Example 2.6. Consider $M = \mathbb{R}$ with $\|p\| = |p|$ $\forall p \in M$. Define $G : M \to M$ and $\theta : M \times M \to [0, \infty)$ by

$$Gp = \begin{cases} p, & p > 0, \\ 0, & p \leq 0, \end{cases}$$

and

$$\theta(p, q) = \begin{cases} 1, & p, q \leq 0, \\ 0, & \text{otherwise}. \end{cases}$$

The existence of (2.20) with $l = \frac{1}{3}$ and $k = \frac{2}{3}$ will be discussed by the following cases.
Case 1. For $p, q \leq 0$, we have $Gp = 0 = Gq$. Then, we obtain

$$\begin{aligned}\theta(p, q)\|l(p - q) + Gp - Gq\| &= |(1/3)(p - q)| \\ &\leq (1/3)(|p| + |q|) \\ &\leq \frac{2}{3}\Big(\max\{\|p - Gp\|, \|q - Gq\|\}\Big).\end{aligned}$$

Case 2. For $p, q > 0$, we have $Gp = p$ and $Gq = q$. Then, we obtain

$$\begin{aligned}\theta(p, q)\|l(p - q) + Gp - Gq\| &= 0\,|(1/3)(p - q) + p - q| \\ &= 0 \\ &= \frac{2}{3}\Big(\max\{\|p - Gp\|, \|q - Gq\|\}\Big).\end{aligned}$$

Case 3. For $p \leq 0$ and $q > 0$, we have $Gp = 0$ and $Gq = q$. Then, we obtain

$$\begin{aligned}\theta(p, q)\|l(p - q) + Gp - Gq\| &= 0\,|(1/3)(p - q) + 0 - q| \\ &= 0 \\ &\leq \frac{2}{3}\Big(\max\{\|p - Gp\|, \|q - Gq\|\}\Big).\end{aligned}$$

Hence, (2.20) holds with $k = \frac{2}{3}$ and $l = \frac{1}{3}$. As $l = \frac{1}{3}$, this provides $\eta = \frac{3}{4}$, since $\eta = \frac{1}{l+1}$. For Axiom (i), consider $p_0 = -1$, then $(1 - \eta)p_0 + \eta Gp_0 = \left(1 - \frac{3}{4}\right)(-1) + \frac{3}{4}(0) < 0$, thus $\theta(p_0, (1 - \eta)p_0 + \eta Gp_0) = 1$ with $\eta = \frac{3}{4}$. If $p, q \leq 0$, then $(1 - \eta)p + \eta Gp = (1 - \eta)p \leq 0$ and $(1 - \eta)q + \eta Gq = (1 - \eta)q \leq 0$. Hence, Axiom (ii) holds; that is, $\theta(p, q) = 1$ implies $\theta((1 - \eta)p + \eta Gp, (1 - \eta)q + \eta Gq) = 1$ with $\eta = \frac{3}{4}$. Also, for each sequence $\{p_n\}$ in M with $\theta(p_n, p_{n+1}) = 1$ $\forall n \in \mathbb{N}$ and $p_n \to p$ as $n \to \infty$, we have $\theta(p_n, p) = 1$ $\forall n \in \mathbb{N}$. Hence, all conditions of Theorem 2.4 are valid in this example, and G has a fixed point.

Remark 2.9. For $p = 2$ and $q = 0$, we see that $\|\theta(p,q)(p-q) + G(p) - G(q)\| = |0(2-0) + (2) - 0| = 2$, $\|p - Gp\| = 0$, and $\|q - Gq\| = 0$. Hence, the above-defined example is not an inward enriched Bianchini mapping, and Theorem 2.3 is not applicable in this example.

Remark 2.10. The following algorithms are useful for obtaining a fixed point of G and can be extracted from the proofs of the above-mentioned theorems.

- For each $p_0 \in M$ with $\theta(p_0, G_\eta p_0) = \frac{1}{\eta} - 1$, where $\eta \in (0,1)$, a sequence $\{p_n\}$ defined by $p_n = G_\eta p_{n-1}$ $\forall n \in \mathbb{N}$ converges to a fixed point of G, provided that either Theorem 2.1 or Theorem 2.3 holds.
- For each $p_0 \in M$ with $\theta(p_0, G_{\frac{1}{l+1}} p_0) = 1$, $l \geq 0$, a sequence $\{p_n\}$ defined by $p_n = G_{\frac{1}{l+1}} p_{n-1}$ $\forall n \in \mathbb{N}$ converges to a fixed point of G, provided that either Theorem 2.2 or Theorem 2.4 holds.

Open problem: We invite the researchers to extend the other existing enriched contraction type mappings by using the techniques presented in this article.

3. Conclusions

This article presents the notions of inward and outward enriched Kannan mappings, as well as inward and outward enriched Bianchini mappings. These notions are the generalized forms of the notions given by Berinde and Păcurar, known as enriched Kannan mapping and enriched Bianchini mapping. The generality of the stated notions is also supported by examples. The existence of fixed points for the aforementioned notions in Banach spaces is also studied.

Author contributions

Y. A.: Validation; formal analysis; investigation; original draft preparation; review and editing.

M. U. A.: Conceptualization; formal analysis; investigation; project administration; original draft preparation; review and editing.

M. A.: Formal analysis; investigation; original draft preparation; review and editing.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

Authors declare that they have no competing interests.

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