Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 3197–3206.
DOI:10.3934/math.2025148
Received: 23 December 2024
Revised: 14 February 2025
Accepted: 14 February 2025
Published: 20 February 2025

Research article

Some identities connecting Stirling numbers, central factorial numbers and higher-order Bernoulli polynomials

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Abstract: By utilizing the generating function of higher-order Bernoulli polynomials, we uncover novel relationships that intertwine higher-order Bernoulli polynomials, higher-order Bernoulli numbers, Stirling numbers of the second kind, and central factorial numbers of the second kind. Leveraging these interconnections, we successfully rederive the identities formulated by Qi and Taylor, specifically those pertaining to Stirling numbers of the second kind and central factorial numbers of the second kind. Additionally, we derive series expansions for both positive integer and real powers of the sinc and sinhc functions.

Keywords: Bernoulli polynomial; Bernoulli number; series expansion; central factorial numbers of the second kind; Stirling numbers of the second kind
Mathematics Subject Classification: 05A19, 11B73, 11B83

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}_0$, the Stirling numbers of the second kind, denoted as $S(n, k)$, (as referenced in [5]), are defined by the equation:

$$x^n = \sum_{k=0}^{n} S(n, k)(x)_k, \tag{1.1}$$

where the falling factorial $(x)_k$ is given by $x(x-1)\cdots(x-k+1)$ for $k \geq 1$ and $(x)_0 = 1$. Additionally, the Stirling numbers of the second kind $S(n, k)$ can be generated by the following series expansion:

$$\frac{(e^z - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k)\frac{z^n}{n!}. \tag{1.2}$$

As an analogue of the Stirling numbers of the second kind, the central factorial numbers of the second kind, denoted as $T(n,k)$ (as referenced in [2, 3, 12]), are defined by the equation:

$$x^n = \sum_{k=0}^{n} T(n,k) x^{[k]}, \tag{1.3}$$

where the central factorial $x^{[k]}$ is given by $x(x+\frac{k}{2}-1)(x+\frac{k}{2}-2)\cdots(x+\frac{k}{2}-k+1)$ for $k \geq 1$ and $x^{[0]} = 1$ (as introduced and studied by Steffensen [17]). The central factorial numbers of the second kind $T(n,k)$ can be generated by the following series expansion:

$$\frac{1}{k!}\left(2\sinh\frac{z}{2}\right)^k = \sum_{n=k}^{\infty} T(n,k)\frac{z^n}{n!}. \tag{1.4}$$

An explicit expression was provided by Riordan in [16], as follows:

$$T(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\frac{k}{2}-j\right)^n. \tag{1.5}$$

By comparing (1.4) with (1.2), we obtain a formula that relates $S(n,k)$ and $T(n,k)$:

$$T(n,k) = \sum_{j=k}^{n}(-1)^{k-j}\binom{n}{j}\left(\frac{k}{2}\right)^{n-j} S(j,k). \tag{1.6}$$

In the recent paper by Qi and Taylor [15], with the assistance of the Faà di Bruno formula [5], a pivotal tool in combinatorial analysis, several series expansions for any positive integer powers of the sinc and hyperbolic sinc functions were derived in terms of the central factorial numbers and the Stirling numbers of the second kind, respectively. The expansions are as follows:

$$\operatorname{sinc}^l z = \sum_{k=0}^{\infty}(-1)^k\frac{T(2k+l,l)}{\binom{2k+l}{l}}\frac{(2z)^{2k}}{(2k)!}, \tag{1.7}$$

$$\operatorname{sinc}^l z = \sum_{k=0}^{\infty}(-1)^k\left[\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}\left(\frac{2}{l}\right)^j\frac{S(j+l,l)}{\binom{j+l}{l}}\right]\frac{(2z)^{2k}}{(2k)!}, \tag{1.8}$$

$$\operatorname{sinhc}^l z = \sum_{k=0}^{\infty}\frac{T(2k+l,l)}{\binom{2k+l}{l}}\frac{(2z)^{2k}}{(2k)!}, \tag{1.9}$$

where $l \in \mathbb{N}_0$, and the functions are defined as:

$$\operatorname{sinc} z = \begin{cases} \frac{\sin z}{z}, & z \neq 0, \\ 1, & z = 0, \end{cases}$$

and

$$\operatorname{sinhc} z = \begin{cases} \frac{\sinh z}{z}, & z \neq 0, \\ 1, & z = 0. \end{cases}$$

Furthermore, for any $r \in \mathbb{R}$, they derived several series expansions for real powers of the sinc and sinhc functions:

$$\operatorname{sinc}^r z = 1 + \sum_{q=1}^{\infty} (-1)^q \left[\sum_{k=1}^{2q} \frac{\langle -r \rangle_k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{T(2q+j, j)}{\binom{2q+j}{j}} \right] \frac{(2z)^{2q}}{(2q)!}, \tag{1.10}$$

$$\operatorname{sinc}^r z = 1 + \sum_{q=1}^{\infty} (-1)^q \left[\sum_{k=1}^{2q} \frac{\langle -r \rangle_k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \times \sum_{m=0}^{2q} (-1)^m \binom{2q}{m} \left(\frac{j}{2} \right)^m \frac{S(2q+j-m, j)}{\binom{2q+j-m}{j}} \right] \frac{(2z)^{2q}}{(2q)!}, \tag{1.11}$$

$$\operatorname{sinhc}^r z = 1 + \sum_{q=1}^{\infty} \left[\sum_{k=1}^{2q} \frac{\langle -r \rangle_k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{T(2q+j, j)}{\binom{2q+j}{j}} \right] \frac{(2z)^{2q}}{(2q)!}, \tag{1.12}$$

$$\operatorname{sinhc}^r z = 1 + \sum_{q=1}^{\infty} \left[\sum_{k=1}^{2q} \frac{\langle -r \rangle_k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \times \sum_{m=0}^{2q} (-1)^m \binom{2q}{m} \left(\frac{j}{2} \right)^m \frac{S(2q+j-m, j)}{\binom{2q+j-m}{j}} \right] \frac{(2z)^{2q}}{(2q)!}, \tag{1.13}$$

where $\langle r \rangle_k = r(r+1) \cdots (r+k-1)$ for $k \geq 1$ and $\langle r \rangle_0 = 1$. Utilizing these series expansions, they also presented several identities for the central factorial numbers of the second kind and the Stirling numbers of the second kind. Furthermore, Qi [14] explored connections among central factorial numbers, the Stirling numbers, and specific matrix inverses, and derived several closed-form formulas and inequalities.

In this paper, starting from the generating function of higher-order Bernoulli polynomials, we establish novel connections among higher-order Bernoulli polynomials, higher-order Bernoulli numbers, Stirling numbers of the second kind, and central factorial numbers of the second kind. By leveraging these connections, we rederive several identities attributed to Qi and Taylor, which pertain to Stirling numbers of the second kind and central factorial numbers of the second kind. Furthermore, we derive novel series expansions for both positive integer powers and real powers of the sinc and sinhc functions.

2. Identities linking Stirling numbers, central factorial numbers, and higher-order Bernoulli polynomials

The higher-order Bernoulli polynomials $B_k^{(\alpha)}(x)$ [6, 8, 11, 18], also known as Nörlund polynomials [10, 13], are typically defined through their generating functions as follows:

$$\left(\frac{z}{e^z - 1} \right)^{\alpha} e^{xz} = \sum_{k=0}^{\infty} B_k^{(\alpha)}(x) \frac{z^k}{k!}. \tag{2.1}$$

Specifically, $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$ is referred to as the higher-order Bernoulli numbers [1, 7]. When $\alpha = 1$, these reduce to the classical Bernoulli numbers B_n (see [4, 9, 19]). In this section, we aim to elucidate the intricate relationships among higher-order Bernoulli numbers, higher-order Bernoulli polynomials, Stirling numbers of the second kind, and central factorial numbers of the second kind.

The following identity presents a relationship between higher-order Bernoulli numbers and Stirling numbers of the second kind.

Lemma 2.1. *[18, Example 7.1] For $l, n \in \mathbb{N}_0$,*

$$B_n^{(-l)} = \frac{S(n+l, l)}{\binom{n+l}{l}}. \tag{2.2}$$

Proof. By setting $\alpha = -l$ and $x = 0$ in (2.1), we obtain the following equation:

$$\left(\frac{z}{e^z - 1}\right)^{-l} = \sum_{k=0}^{\infty} B_k^{(-l)} \frac{z^k}{k!}.$$

Utilizing (1.2), we can rewrite the left-hand side as:

$$\left(\frac{z}{e^z - 1}\right)^{-l} = \left(\frac{e^z - 1}{z}\right)^{l} = l! z^{-l} \sum_{n=l}^{\infty} S(n, l) \frac{z^n}{n!}.$$

By substituting n by $n + l$, we obtain

$$\left(\frac{z}{e^z - 1}\right)^{-l} = l! \sum_{n=0}^{\infty} S(n+l, l) \frac{z^n}{(n+l)!}.$$

Therefore,

$$\sum_{k=0}^{\infty} B_k^{(-l)} \frac{z^k}{k!} = l! \sum_{n=0}^{\infty} S(n+l, l) \frac{z^n}{(n+l)!}.$$

By comparing the coefficients of $z^n/n!$ on both sides, we derive Eq (2.2). □

Below is a relationship presented between higher-order Bernoulli polynomials and central factorial numbers.

Lemma 2.2. *For $l, n \in \mathbb{N}_0$,*

$$B_n^{(-l)}\left(-\frac{l}{2}\right) = \frac{T(n+l, l)}{\binom{n+l}{l}}. \tag{2.3}$$

Proof. By setting $\alpha = -l$ and $x = -l/2$ in (2.1), we derive the following equation:

$$\left(\frac{z}{e^z - 1}\right)^{-l} e^{-\frac{l}{2}z} = \sum_{k=0}^{\infty} B_k^{(-l)}\left(-\frac{l}{2}\right) \frac{z^k}{k!}.$$

Utilizing (1.4), we can rewrite the left-hand side as:

$$\left(\frac{z}{e^z - 1}\right)^{-l} e^{-\frac{l}{2}z} = \left(\frac{e^{\frac{z}{2}} - e^{-\frac{z}{2}}}{z}\right)^{l} = l! z^{-l} \sum_{n=l}^{\infty} T(n, l) \frac{z^n}{n!}.$$

By substituting n with $n + l$, we obtain

$$\left(\frac{z}{e^z - 1}\right)^{-l} e^{-\frac{l}{2}z} = l! \sum_{n=0}^{\infty} T(n+l, l) \frac{z^n}{(n+l)!}.$$

Therefore,

$$\sum_{k=0}^{\infty} B_k^{(-l)}\left(-\frac{l}{2}\right)\frac{z^k}{k!} = l!\sum_{n=0}^{\infty} T(n+l,l)\frac{z^n}{(n+l)!}.$$

By comparing the coefficients of $z^n/n!$ on both sides, we obtain Eq (2.3). □

Lemma 2.3. *For $r \in \mathbb{R}$ and $n \in \mathbb{N}_0$,*

$$B_{2n+1}^{(r)}\left(\frac{r}{2}\right) = 0. \tag{2.4}$$

Proof. Since the function $\left(\frac{z}{e^z-1}\right)^r e^{\frac{r}{2}z} = \left(\frac{z}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}\right)^r$ is even, we can directly obtain the desired result by applying Eq (2.1). □

Using the property of $B_n^{(-l)}(-\frac{l}{2})$ for $l \in \mathbb{N}_0$, we obtain the following corollaries.

Corollary 2.1. *[15, Theorem 3] For $l, n \in \mathbb{N}_0$,*

$$T(2n+1+l,l) = 0. \tag{2.5}$$

Proof. From (2.4), we know that $B_{2n+1}^{(-l)}\left(-\frac{l}{2}\right) = 0$ for $l \in \mathbb{N}_0$. Combining this with (2.3), we can derive the desired result. □

Corollary 2.2. *[15, Theorem 1] For $l \in \mathbb{N}_0$ and $z \in \mathbb{C}$,*

$$\operatorname{sinc}^l z = \sum_{k=0}^{\infty}(-1)^k \frac{T(2k+l,l)}{\binom{2k+l}{l}}\frac{(2z)^{2k}}{(2k)!}. \tag{2.6}$$

Proof. We start with the expression:

$$\operatorname{sinc}^l z = \left(\frac{e^{iz}-e^{-iz}}{2iz}\right)^l = \left(\frac{2iz}{e^{2iz}-1}\right)^{-l} e^{-ilz}.$$

According to (2.1) and (2.4), we have

$$\operatorname{sinc}^l z = \sum_{k=0}^{\infty} B_k^{(-l)}\left(-\frac{l}{2}\right)\frac{(2iz)^k}{k!} = \sum_{k=0}^{\infty}(-1)^k B_{2k}^{(-l)}\left(-\frac{l}{2}\right)\frac{(2z)^{2k}}{(2k)!}. \tag{2.7}$$

Substituting (2.3) into (2.7), we obtain the desired result. □

Corollary 2.3. *For $l \in \mathbb{N}_0$ and $z \in \mathbb{C}$,*

$$\operatorname{sinhc}^l z = \sum_{k=0}^{\infty} \frac{T(2k+l,l)}{\binom{2k+l}{l}}\frac{(2z)^{2k}}{(2k)!}. \tag{2.8}$$

Proof. We start with the expression:

$$\operatorname{sinhc}^l z = \left(\frac{e^z - e^{-z}}{2z}\right)^l = \left(\frac{2z}{e^{2z}-1}\right)^{-l} e^{-lz}.$$

According to (2.1) and (2.4), we have

$$\operatorname{sinhc}^l z = \sum_{k=0}^{\infty} B_k^{(-l)}\left(-\frac{l}{2}\right)\frac{(2z)^k}{k!} = \sum_{k=0}^{\infty} B_{2k}^{(-l)}\left(-\frac{l}{2}\right)\frac{(2z)^{2k}}{(2k)!}. \tag{2.9}$$

Substituting (2.3) into (2.9), we obtain the desired result. □

Based on the properties of higher-order Bernoulli polynomials, we can derive a formula for $B_n^{(-l)}\left(-\frac{l}{2}\right)$ in terms of Stirling numbers of the second kind, as well as a formula for $B_n^{(-l)}$ in terms of central factorial numbers of the second kind.

Theorem 2.1. *For $l, n \in \mathbb{N}_0$,*

$$B_n^{(-l)}\left(-\frac{l}{2}\right) = \sum_{k=0}^{n}\binom{n}{k}\left(-\frac{l}{2}\right)^k \frac{S(n-k+l, l)}{\binom{n-k+l}{l}}, \tag{2.10}$$

$$B_n^{(-l)} = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{l}{2}\right)^k \frac{T(n-k+l, l)}{\binom{n-k+l}{l}}. \tag{2.11}$$

Proof. By (2.1), we have

$$B_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n}{k} x^k B_{n-k}^{(\alpha)}.$$

Choosing $\alpha = -l$ and $x = -l/2$ yields

$$B_n^{(-l)}\left(-\frac{l}{2}\right) = \sum_{k=0}^{n}\binom{n}{k} B_{n-k}^{(-l)} \cdot \left(-\frac{l}{2}\right)^k. \tag{2.12}$$

Substituting (2.2) into (2.12), we obtain (2.10). From (2.12), we derive

$$\left(-\frac{l}{2}\right)^{-n} B_n^{(-l)}\left(-\frac{l}{2}\right) = \sum_{k=0}^{n}\binom{n}{k}\left(-\frac{l}{2}\right)^{-k} B_k^{(-l)},$$

which can be rearranged to

$$\left(-\frac{l}{2}\right)^{-n} B_n^{(-l)} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\left(-\frac{l}{2}\right)^{-k} B_k^{(-l)}\left(-\frac{l}{2}\right).$$

Simplifying this expression results in

$$B_n^{(-l)} = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{l}{2}\right)^k B_{n-k}^{(-l)}\left(-\frac{l}{2}\right). \tag{2.13}$$

Substituting (2.3) into (2.13), we obtain (2.11). □

Corollary 2.4. *For $l, n \in \mathbb{N}_0$,*

$$\frac{T(n+l,l)}{\binom{n+l}{l}} = \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{l}{2}\right)^k \frac{S(n-k+l,l)}{\binom{n-k+l}{l}}, \tag{2.14}$$

$$\frac{S(n+l,l)}{\binom{n+l}{l}} = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{l}{2}\right)^k \frac{T(n-k+l,l)}{\binom{n-k+l}{l}}. \tag{2.15}$$

Proof. By comparing (2.3) with (2.10), we derive (2.14). Similarly, by comparing (2.2) with (2.11), we obtain (2.15). □

For (2.14) and its alternative proof, one should also refer to Corollary 6 in [15].

Corollary 2.5. *[15, Theorem 4] For $l, n \in \mathbb{N}_0$,*

$$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \left(\frac{2}{l}\right)^k \frac{S(k+l,l)}{\binom{k+l}{l}} = 0. \tag{2.16}$$

Proof. By using (2.4) and (2.10) and replacing k by $2n+1-k$, we arrive at (2.16). □

Furthermore, we consider the case of the r-th order Bernoulli polynomial with $r \in \mathbb{R}$.

Lemma 2.4. *For $r \in \mathbb{R}, n \in \mathbb{N}_0$,*

$$B_{2n}^{(-r)}\left(-\frac{r}{2}\right) = \sum_{j=0}^{2n} \binom{r}{j} \binom{2n-r}{2n-j} B_{2n}^{(-j)}\left(-\frac{j}{2}\right). \tag{2.17}$$

Proof. From (2.1), we have

$$B_{2n}^{(-r)}\left(-\frac{r}{2}\right) = \left[\frac{z^{2n}}{(2n)!}\right] \left(\frac{z}{e^{\frac{z}{2}} - e^{-\frac{z}{2}}}\right)^{-r} = \left[\frac{z^{2n}}{(2n)!}\right] \sum_{k=0}^{2n} \binom{r}{k} \left(\frac{e^{\frac{z}{2}} - e^{-\frac{z}{2}}}{z} - 1\right)^k.$$

Since

$$\sum_{k=0}^{2n} \binom{r}{k} \left(\frac{e^{\frac{z}{2}} - e^{-\frac{z}{2}}}{z} - 1\right)^k = \sum_{k=0}^{2n} \binom{r}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \left(\frac{e^{\frac{z}{2}} - e^{-\frac{z}{2}}}{z}\right)^j = \sum_{k=0}^{2n} \binom{r}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{q=0}^{\infty} B_{2q}^{(-j)}\left(-\frac{j}{2}\right) \frac{z^{2q}}{(2q)!},$$

we obtain

$$B_{2n}^{(-r)}\left(-\frac{r}{2}\right) = \sum_{k=0}^{2n} \binom{r}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} B_{2n}^{(-j)}\left(-\frac{j}{2}\right) = \sum_{j=0}^{2n} (-1)^j B_{2n}^{(-j)}\left(-\frac{j}{2}\right) \sum_{k=j}^{2n} (-1)^k \binom{r}{k} \binom{k}{j}.$$

Simple calculations yields

$$\begin{aligned} \sum_{k=j}^{2n} (-1)^k \binom{r}{k} \binom{k}{j} &= \sum_{k=j}^{2n} \binom{-r+k-1}{k} \binom{k}{j} = \binom{-r+j-1}{j} \sum_{k=j}^{2n} \binom{-r+k-1}{k-j} \\ &= \binom{-r+j-1}{j} \binom{2n-r}{2n-j} = (-1)^j \binom{r}{j} \binom{2n-r}{2n-j}. \end{aligned} \tag{2.18}$$

Therefore, we derive the desired result. □

Utilizing Lemma 2.4, we can derive explicit expressions for $B_{2n}^{(-r)}\left(-\frac{r}{2}\right)$ in terms of central factorial numbers of the second kind and Stirling numbers of the second kind.

Theorem 2.2. *For $r \in \mathbb{R}, n \in \mathbb{N}_0$,*

$$B_{2n}^{(-r)}\left(-\frac{r}{2}\right) = \sum_{j=0}^{2n} \binom{r}{j}\binom{2n-r}{2n-j}\frac{T(2n+j,j)}{\binom{2n+j}{j}}, \tag{2.19}$$

$$B_{2n}^{(-r)}\left(-\frac{r}{2}\right) = \sum_{j=0}^{2n} \binom{r}{j}\binom{2n-r}{2n-j}\sum_{k=0}^{2n}\binom{2n}{k}\left(-\frac{j}{2}\right)^k \frac{S(2n-k+j,j)}{\binom{2n-k+j}{j}}. \tag{2.20}$$

Proof. By substituting (2.3) and (2.10) into (2.17), we obtain (2.19) and (2.20), respectively. □

Corollary 2.6. *Let $r \in \mathbb{R}$. We have*

$$\operatorname{sinc}^r z = \sum_{q=0}^{\infty}(-1)^q \frac{(2z)^{2q}}{(2q)!}\sum_{j=0}^{2q}\binom{r}{j}\binom{2q-r}{2q-j}\frac{T(2q+j,j)}{\binom{2q+j}{j}}, \tag{2.21}$$

$$\operatorname{sinc}^r z = \sum_{q=0}^{\infty}(-1)^q \frac{(2z)^{2q}}{(2q)!}\sum_{j=0}^{2q}\binom{r}{j}\binom{2q-r}{2q-j}\sum_{k=0}^{2q}\binom{2q}{k}\left(-\frac{j}{2}\right)^k \frac{S(2q-k+j,j)}{\binom{2q-k+j}{j}}. \tag{2.22}$$

Proof. According to (2.1), we have

$$\operatorname{sinc}^r z = \left(\frac{2iz}{e^{2iz}-1}\right)^{-r} e^{-irz} = \sum_{q=0}^{\infty}(-1)^q B_{2q}^{(-r)}\left(-\frac{r}{2}\right)\frac{(2z)^{2q}}{(2q)!}. \tag{2.23}$$

Substituting (2.19) into (2.23), we immediately derive (2.21). Similarly, substituting (2.20) into (2.23), we arrive at (2.22). □

Corollary 2.7. *Let $r \in \mathbb{R}$. We have*

$$\operatorname{sinhc}^r z = \sum_{q=0}^{\infty}\frac{(2z)^{2q}}{(2q)!}\sum_{j=0}^{2q}\binom{r}{j}\binom{2q-r}{2q-j}\frac{T(2q+j,j)}{\binom{2q+j}{j}}, \tag{2.24}$$

$$\operatorname{sinhc}^r z = \sum_{q=0}^{\infty}\frac{(2z)^{2q}}{(2q)!}\sum_{j=0}^{2q}\binom{r}{j}\binom{2q-r}{2q-j}\sum_{k=0}^{2q}\binom{2q}{k}\left(-\frac{j}{2}\right)^k \frac{S(2q-k+j,j)}{\binom{2q-k+j}{j}}. \tag{2.25}$$

Proof. According to (2.1), we have

$$\operatorname{sinhc}^r z = \left(\frac{2z}{e^{2z}-1}\right)^{-r} e^{-rz} = \sum_{q=0}^{\infty} B_{2q}^{(-r)}\left(-\frac{r}{2}\right)\frac{(2z)^{2q}}{(2q)!}. \tag{2.26}$$

Substituting (2.19) into (2.26), we immediately derive (2.24). Similarly, substituting (2.20) into (2.26), we arrive at (2.25). □

Remark 2.1. *In fact, starting from* (2.18)*, we obtain the following equality:*

$$\sum_{j=0}^{2q}\binom{r}{j}\binom{2q-r}{2q-j}\frac{T(2q+j,j)}{\binom{2q+j}{j}}=\sum_{k=0}^{2q}\binom{r}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{T(2q+j,j)}{\binom{2q+j}{j}},$$

which demonstrates that (1.10) *and* (1.11) *are equivalent to our results, namely,* (2.21) *and* (2.22)*. Compared to* (1.10) *and* (1.11)*, ours appear to be more concise. Similarly,* (1.12) *is equivalent to* (2.24)*, and* (1.13) *is equivalent to* (2.25)*.*

3. Conclusions

In conclusion, our study has uncovered novel connections among higher-order Bernoulli polynomials, higher-order Bernoulli numbers, Stirling numbers of the second kind, and the central factorial numbers of the second kind, through the utilization of the generating function of higher-order Bernoulli polynomials. By leveraging these connections, we have validated the identities proposed by Qi and Taylor concerning Stirling numbers of the second kind and the central factorial numbers of the second kind. Additionally, we have derived series expansions for the sinc and sinhc functions, encompassing both their positive integer and real power terms.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The work was supported by the Zhejiang Provincial Natural Science Foundation of China (Grant No. LTGY23H240002).

Conflict of interest

The author declares that he has no competing interests.

References

1. A. Bayad, Y. Simsek, On generating functions for parametrically generalized polynomials involving combinatorial, Bernoulli and Euler polynomials and numbers, *Symmetry*, **14** (2022), 654. https://doi.org/10.3390/sym14040654
2. P. Butzer, M. Schmidt, E. Stark, L. Vogt, Central factorial numbers, their main properties and some applications, *Numer. Func. Anal. Opt.*, **10** (1989), 419–488. https://doi.org/10.1080/01630568908816313
3. P. Butzer, M. Schmidt, Central factorial numbers and their role in finite difference calculus and approximation, *Approximat. Theor.*, **58** (1990), 127–150.

4. W. Chu, C. Y. Wang, Convolution formulae for Bernoulli numbers, *Integr. Transf. Spec. F.*, **21** (2010), 437–457. https://doi.org/10.1080/10652460903360861
5. L. Comtet, *Advanced combinatorics: the art of finite and infinite expansions*, Dordrecht: Springer, 1974. https://doi.org/10.1007/978-94-010-2196-8
6. M. Dağlı, Closed formulas and determinantal expressions for higher-order Bernoulli and Euler polynomials in terms of Stirling numbers, *RACSAM*, **115** (2021), 32. https://doi.org/10.1007/s13398-020-00970-9
7. F. Howard, Congruences and recurrences for Bernoulli numbers of higher order, *Fibonacci Quart.*, **32** (1994), 316–328. https://doi.org/10.1080/00150517.1994.12429204
8. N. Kilar, Y. Simsek, H. M. Srivastava, Recurrence relations, associated formulas, and combinatorial sums for some parametrically generalized polynomials arising from an analysis of the Laplace transform and generating functions, *Ramanujan J.*, **61** (2023), 731–756. https://doi.org/10.1007/s11139-022-00679-w
9. N. Li, W. Chu, Explicit formulae for Bernoulli numbers, *AIMS Mathematics*, **9** (2024), 28170–28194. https://doi.org/10.3934/math.20241366
10. G. Liu, H. M. Srivastava, Explicit formulas of the Nörlund polynomials $B_n^{(x)}$ and $b_n^{(x)}$, *Comput. Math. Appl.*, **51** (2006), 1377–1384. https://doi.org/10.1016/j.camwa.2006.02.003
11. Y. Luke, *The special functions and their approximations*, New York: Acdemic Press, 1969.
12. M. Merca, Connections between central factorial numbers and Bernoulli polynomials, *Period Math. Hung.*, **73** (2016), 259–264. https://doi.org/10.1007/s10998-016-0140-5
13. N. Nörlund, *Vorlesungen über differenzenrechnung*, Berlin: Springer-Verlag, 1924. https://doi.org/10.1007/978-3-642-50824-0
14. F. Qi, Series and connections among central factorial numbers, Stirling numbers, inverse of Vandermonde matrix, and normalized remainders of Maclaurin series expansions, *Mathematics*, **13** (2025), 223. https://doi.org/10.3390/math13020223
15. F. Qi, P. Taylor, Series expansions for powers of sinc function and closed-form expressions for specific partial Bell polynomials, *Appl. Anal. Discr. Math.*, **18** (2024), 92–115. https://doi.org/10.2298/AADM230902020Q
16. J. Riordan, *Combinatorial identities*, New York: Wiley, 1968.
17. J. Steffensen, *Interpolation*, Baltimore: Williams & Wilkins, 1927.
18. W. Wang, Generalized higher order Bernoulli number pairs and generalized Stirling number paris, *J. Math. Anal. Appl.*, **364** (2010), 255–274. https://doi.org/10.1016/j.jmaa.2009.10.023
19. X. Wang, W. Chu, Reciprocal relations for Bernoulli and Euler numbers/polynomials, *Integr. Transf. Spec. F.*, **29** (2018), 831–841. https://doi.org/10.1080/10652469.2018.1501047