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Research article

The anti-periodic solutions of incommensurate fractional-order Cohen-Grossberg neural network with inertia

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Abstract: A class of incommensurate fractional-order Cohen-Grossberg neural networks with inertia was investigated in this paper. First, the sufficient conditions for the boundedness of the solutions of the system were derived using the properties of fractional-order calculus. Second, by constructing a sequence of solutions in the system and using the Ascoli-Arzela theorem, the sufficient conditions for the existence of an anti-period solution and the global asymptotical stability of the system were deduced. Finally, the correctness of theoretical reasoning results was verified by a numerical simulation.

Keywords: incommensurate; fractional-order; Cohen-Grossberg neural networks; anti-periodic solution; global asymptotical stability
Mathematics Subject Classification: 34K20, 92B20

1. Introduction

Fractional-order calculus is an operator that was derived from the generalization of integral-order calculus to non-integral calculus. It was first mentioned in the letters between Leibniz and L'Hospital in 1695. In the following 300 years, owing to the lack of accurate physical meaning and application background, the research about fractional-order calculus remained only on pure theory [1, 2]. After Mandelbrot first proposed in 1982, there were extensive examples of fractional-order in nature and many scientific and technological fields, and the self-similarity between integers and fractional-order parts was demonstrated [3]. The fractional-order calculus, as the dynamic basis of fractal geometry and fractal dimension, has become a hot research subject.

Compared with the traditional integer-order model, a fractional-order model provides an effective tool to describe the inherent memory and genetic characteristics of real materials and processes [4, 5]. Fractional-order dynamic systems have been employed in electromagnetic waves, electrolyte polarization, viscoelastic systems, economy, biology, system control, medicine, and so on [6, 7]. In real world processes, especially complex processes, they are more likely to be fractional-order systems.

The early research model of the integer-order neural networks contained only the first derivative of the system state. In 1987, Babcock and Westervelt added inductance to the circuit of the analog neural network, where the inertial term is the second derivative term of the system, and the obvious chaotic and bifurcation behaviors have been obtained [8]. Therefore, an integral-order inertial neural network was proposed. From a biological perspective, the increase of inertial terms has a strong biological background, such as the critical behavior of squid axons in the small-signal range, which can be viewed as caused by inductance or capacitance [9]; the membrane of some specialized neurons, for example, hair cells in the vertebrate cochlea, electroreceptors in certain fish (e.g., electric fish and cartilaginous fishes), retinas in lower vertebrates, and so on, can be achieved by adding inductors to analog circuitry [10]; even some non-neuronal cells, such as the Purkinje fibers of the heart muscle and individual skeletal muscle fibers, exhibit electrical properties, just as their membranes containing inductors [11].

Compared with neural networks without inertia, the inertial neural networks have more complex dynamic characteristics. In applications, the addition of inertia is helpful to the memory search of disorders. Therefore, the research of inertial neural networks has attracted more and more attention from domestic and foreign scholars in recent years [12, 13]. For example, the condition of global asymptotic stability and the condition of global robust stability of the integer-order inertial neural networks are obtained by Linear matrix inequality in reference [12], and the periodicity and synchronization of the integer-order inertial neural networks are studied by employing the matrix measure method and Alan formula [14, 15].

The Cohen-Grossberg neural network is one of the most representative neural networks because it contains many well-known neural networks as its special cases, such as the Hopfield Neural Network and cellular neural network. Cohen-Grossberg neural networks can describe many models from neurobiology, population biology, and evolution. The introduction of the inertia term in the networked system can stimulate the neural network to produce complex dynamic behaviors, such as periodic oscillation, bifurcation, dissipation, and synchronization. When the integer-order systems cannot be used to describe the problems, using a fractional-order system is often more concise and efficient to fit the actual situation. Fractional-order systems have great potential to surpass integer-order systems and have shown wider application value. Therefore, a growing number of scholars began to study fractional-order calculus and obtained many results in different fields. The problems of the numerical solution, chaos, stability control, and synchronization of fractional-order differential equation are hot topics in nonlinear research [16–18].

The model of a fractional-order inertial neural network is a differential equation with two different fractional-order derivative terms, whose dynamic behavior is more complex and can be used widely. Therefore, it is of great significance and value to study the dynamic characteristics of fractional-order inertial neural networks. Moreover, the research of fractional-order inertial neural networks mainly encompass two kinds: One is the fractional-order of two different fractional-order derivatives which are multiples; for example, in [14, 19–21], the global asymptotic stability, Mittag-Leffler stability, and periodic stability of fractional-order inertial neural networks, and Cohen-Grossberg neural networks have been studied. Another kind of inertial neural network has two different fractional-order derivatives that are not multiplies, but only the results of asymptotic stability or synchronous stability have been studied, as shown in [22, 23].

The existence and stability of anti-periodic solutions of the systems have not been studied. In practical application, the dynamic behavior of nonlinear differential systems is not only periodic, but also anti-periodic, so the existence and stability of anti-periodic solutions have attracted much attention. Let $x(t)$ be the solution of the system, and there exits a constant $\omega > 0$ that satisfies $x(t+\omega) = -x(t)$, then $x(t)$ is said to be an anti-periodic solution of the system. In real world problems, anti-periodic phenomena are widely observed in fields such as biology, physics, and engineering, such as, anti-periodic oscillation, vibration, and anti-periodic solutions of impulse equations. Therefore, it is practical to study the anti-periodic solutions. There are many results for anti-periodic solutions for fractional-order Hopfield neural networks and neural networks, such as those in [24–27]. For incommensurate fractional-order Cohen-Grossberg neural networks with inertia, the relative study has not been explored yet, so in this paper, we study the existence and the global asymptotical stability of anti-periodic solutions for a class of incommensurate fractional-order inertial neural networks that are characterized by two different fractional-order derivatives that are not multiples.

This is a new subject that is worthy of study, it will provide a new theoretical basis for the practical application of dynamic performance.

The major ideas and innovations of this paper are listed below:

1) The incommensurate fractional-order Cohen-Grossberg neural network with inertia is proposed for the first time. The system proposed, including the general fractional-order Cohen-Grossberg neural networks and the fractional-order Cohen-Grossberg neural networks with inertia as the special cases, extends the scope of the research.

2) By constructing the sequence of solutions in the system and the Ascoli-Arzela theorem, the boundedness of the solution and the existence of the anti-periodic solution is derived. The research methods are innovative.

3) By constructing a Lyapunov function, the sufficient conditions for the global asymptotical stability of the anti-periodic solution of the system are derived. The ways adopted are innovative.

4) The results are new, and they provide a theoretical basis for the study of the stability of the system.

In this paper, the incommensurate fractional-order Cohen-Grossberg neural networks with inertia are considered, which are described as

$$\begin{aligned} D_t^{\alpha}(x_i(t)) &= -\gamma_i D_t^{\beta}(x_i(t)) - \alpha_i(x_i(t))[h_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \\ &\quad - \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) - I_i(t)], \qquad (t>0), \end{aligned} \tag{1.1}$$

for all $i = 1, 2, \cdots, n$, where D_t^{α} and D_t^{β} are the Riemann-Liouville fractional derivative with order $\alpha, \beta (0 < \beta \leq 1, \beta \leq \alpha \leq \beta + 1)$; $x_i(t)$ is the state variable of the ith neuron at time t; and $\alpha_i(\cdot) > 0$, $h_i(\cdot)$ are the abstract amplification function and the behavior function of the ith neuron; $\gamma_i > 0$ is the damping coefficient; a_{ij} and b_{ij} are the connection weights; $f_j(\cdot)$ is the activation function of the jth neuron; $\tau_{ij} > 0$ are time delays; $I_i(t)$ is the external input of the ith neuron at time t.

The initial conditions of system (1.1) are:

$$\begin{cases} x_i(t) = \chi_i(t), \\ D_t^{\beta}(x_i(t)) = \psi_i(t), \end{cases} \quad -\tau \le t \le 0, \;\; \tau = \max_{0\le i,j\le n}\{\tau_{ij}\}, \;\; i = 1, 2, \cdots, n, \tag{1.2}$$

where $\chi_i(t)$ and $\psi_i(t)$ are continuous and bounded.

Remark:

(1) If $\alpha = \beta$, system (1.1) is the general fractional-order Cohen-Grossberg neural network:

$$D_t^{\alpha}(x_i(t)) = -\alpha_i(x_i(t))[h_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) - I_i(t)], \qquad (t > 0).$$

(2) If $\alpha = 2\beta$, system (1.1) is the fractional-order Cohen-Grossberg neural network with inertia of commensurate orders:

$$D_t^{2\beta}(x_i(t)) = -\gamma_i D_t^{\beta}(x_i(t)) - \alpha_i(x_i(t))[h_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) - I_i(t)], \qquad (t > 0),$$

for the two kinds of neural networks, they have some results, such as [16] and [17].

2. Preliminaries

Definition 2.1 [2] The fractional-order Riemann-Liouville integral of function $f(t)$ with order q is defined as

$${}_{t_0}D_t^{-q} f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\upsilon)^{q-1} f(\upsilon) d\upsilon, \;\; t \ge t_0 \ge 0,$$

where $\Gamma(q) = \int_0^{+\infty} t^{q-1} e^{-t} dt$, $q > 0$.

Definition 2.2 [2] The Riemann-Liouville fractional-order derivative with order q is defined as

$${}_{t_0}^{RL}D_t^{q} f(t) = D_{t_0}^{n}(D_t^{q-n} f(t)) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(r)}{(t-r)^{q-n+1}} dr,$$

where $q > 0$ is a real number and $n - 1 \le q \le n$, $t \ge t_0 \ge 0$.

Let $X = \{x | x = (x_1, x_2, \cdots, x_n)^T, x_i \in R\}$; thus, it is easy to see that X is a Banach space with norm $\|x\| = [\sum_{i=1}^{n} |x_i(t)|^p]^{\frac{1}{p}}, p \ge 1$.

Definition 2.3 Let $\mu(t) = (\mu_1(t), \mu_2(t), \cdots, \mu_n(t))^T$ be a solution of system (1.1), and $\nu(t) = (\nu_1(t), \nu_2(t), \cdots, \nu_n(t))^T$ be the solution of system (1.1) under initial conditions $\nu_i(t) = \overline{\chi}_i(t)$, $D_t^d(\nu_i(t)) = \overline{\psi}_i(t)$ for $-\tau \le t \le 0$. If

$$\lim_{t\to\infty} |\mu(t) - \nu(t)| = 0,$$

then the solution $\nu(t)$ of system (1.1) is globally asymptotically stable.

Definition 2.4 [28] If $h(x)$ is continuous in $\mathbb{R}$, and $h(t+\omega) = -h(t)$ for $t \in \mathbb{R}$, then $h(x)$ is called an anti-period function, where $\omega > 0$.

Lemma 2.1 [23] If $u(t) \in \mathbb{R}$ is continuous and derivable in $[0, \delta]$, and $0 < q < 1, n - 1 < p < n$, then

(1) $D_t^p D_t^q u(t) = D_t^{p+q} u(t)$.

(2) $D_t^{-p}D_t^q u(t) = D_t^{-p+q}u(t)$.
(3) $D_t^{-q}C = \frac{C}{\Gamma(q+1)}t^q$.

Lemma 2.2 [23] If $\chi(t)$ is derivable, and $\chi'(t)$ is continuous, then

$$\frac{1}{2}D_t^q\chi^2(t) \leq \chi(t)D_t^q\chi(t), \quad 0 < q \leq 1.$$

Lemma 2.3 [14] If $u(t)$ is continuous in $[0, +\infty)$, and there is $m_1 > 0, m_2 > 0$, which satisfies $u(t) \leq -m_1 D_{0,t}^{-q}u(t) + m_2$, then for $t \geq 0$

$$u(t) \leq m_2 E_q(-m_1 t^q),$$

where $E_q(\cdot)$ is a Mittag-Leffler function with parameter q.

Lemma 2.4(The Ascoli-Arzela Theorem) X is a topological space, which is separable, $F : \{f : X \to \mathbb{R}^n\}$ is a family of functions. If F is equi-continous everywhere in X, for any $x \in X$, $\{f(x) : f \in F\}$ is a bounded subset in $\mathbb{R}^n$, then each function sequence has a subsequence that is uniformly convergent in any compact subset of X.

The research is established on the assumptions for $i, j = 1, 2, \cdots, n$:

$\mathbf{H_1}$: $\alpha_i(\cdot)$ is bounded, and has bounded derivative. That is, there is $\underline{\alpha}_i \geq 0, \overline{\alpha}_i > 0, A_i > 0$, which satisfies

$$0 \leq \underline{\alpha}_i \leq \alpha_i(x_i(t)) \leq \overline{\alpha}_i, \ |\alpha_i'(\cdot)| \leq A_i.$$

$\mathbf{H_2}$: $f_j(\cdot)$ satisfies Lipschitz conditions with constant l_j, and is bounded with $\overline{f}_j > 0$. That is,

$$|f_j(s) - f_j(v)| \leq l_j|s - v|, \ |f_j(\cdot)| \leq \overline{f}_j,$$

for all $s, v \in \mathbb{R}$.

$\mathbf{H_3}$: $\vartheta_i(x_i)$ has a bounded derivative of x_i, where $\vartheta_i(x_i) = \alpha_i(x_i)h_i(x_i)$. That is, there is $\underline{\vartheta}_i > 0, \overline{\vartheta}_i > 0$, which satisfies

$$0 \leq \underline{\vartheta}_i \leq \vartheta_i'(x_i) \leq \overline{\vartheta}_i.$$

$\mathbf{H_4}$: $I_i(t)$ is bounded. That is,

$$|I_i(t)| \leq I_i, \quad I_i > 0.$$

$\mathbf{H_5}$: $\alpha_i(x_i)I_i(t+\omega) = -\alpha_i(-x_i)I_i(t)$, $\alpha_i(x_i)h_i(x_i) = -\alpha_i(-x_i)h_i(-x_i)$, $\alpha_i(x_i)f_i(x_i) = -\alpha_i(-x_i)f_i(-x_i)$, where $\omega > 0$.

Let $\xi_i(t) = D_t^\beta x_i(t) + \eta_i x_i(t), \eta_i > 0$, from (1.1) one has

$$\begin{cases} D_t^\beta x_i(t) = \xi_i(t) - \eta_i x_i(t), \\ D_t^{\alpha-\beta}[\xi_i(t) - \eta_i x_i(t)] = -\gamma_i\xi_i(t) + \gamma_i\eta_i x_i(t) \\ \qquad -\alpha_i(x_i(t))[h_i(x_i(t)) - \sum\limits_{j=1}^n a_{ij}f_j(x_j(t)) \\ \qquad - \sum\limits_{j=1}^n b_{ij}f_j(x_j(t-\tau_{ij})) - I_i(t)], \end{cases} \tag{2.1}$$

3. Major results

Theorem 3.1 If Assumptions $H_1 - H_4$ hold, and $x_i(t)$ is the solution of system (1.1), then $x_i(t)$ and $D_t^{\beta} x_i(t)$ $(i = 1, 2, ..., n)$ are bounded on $(0, T]$, where T is a positive finite real number.
Proof. From Assumption H_3,

$$\alpha_i(x_i(t))h_i(x_i(t)) - \alpha_i(0)h_i(0) = \vartheta_i'(\theta)x_i(t),$$

where θ is between 0 and $x_i(t)$. From (1.1), one has

$$\begin{aligned} D_t^{\alpha} x_i(t) &= -\gamma_i D_t^{\beta} x_i(t) - \alpha_i(x_i(t))h_i(x_i(t)) + \alpha_i(0)h_i(0) - \alpha_i(0)h_i(0) \\ &\quad +\alpha_i(x_i(t))[\sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) + I_i(t)] \\ &= -\gamma_i D_t^{\beta} x_i(t) - \vartheta_i'(\theta)x_i(t) - \alpha_i(0)h_i(0) \\ &\quad +\alpha_i(x_i(t))[\sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) + I_i(t)], \end{aligned}$$

and

$$\begin{aligned} D_t^{\alpha}|x_i(t)| &\leq -\gamma_i D_t^{\beta}|x_i(t)| - \vartheta_i'(\theta)|x_i(t)| + |\alpha_i(0)h_i(0)| \\ &\quad +\overline{\alpha}_i[\sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + I_i] \\ &\leq -\gamma_i D_t^{\beta}|x_i(t)| - \underline{\vartheta}_i|x_i(t)| + |\alpha_i(0)h_i(0)| \\ &\quad +\overline{\alpha}_i[\sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + I_i], \end{aligned}$$

if $0 < t \leq T < +\infty$, from Lemma 2.1 and the formula above, one has

$$\begin{aligned} D_t^{\beta} D_t^{\alpha-\beta}|x_i(t)| = D_t^{\alpha}|x_i(t)| &\leq -\gamma_i D_t^{\beta}|x_i(t)| - \underline{\vartheta}_i|x_i(t)| + |\alpha_i(0)h_i(0)| \\ &\quad +\overline{\alpha}_i[\sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + I_i], \end{aligned}$$

and

$$\begin{aligned} D_t^{\alpha-\beta}|x_i(t)| &\leq -\gamma_i|x_i(t)| - \underline{\vartheta}_i D_t^{-\beta}|x_i(t)| \\ &\quad +D_t^{-\beta}[\overline{\alpha}_i \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + \overline{\alpha}_i I_i + |\alpha_i(0)h_i(0)|] \\ &\leq -\gamma_i|x_i(t)| + \frac{1}{\Gamma(\beta+1)}[\overline{\alpha}_i \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + \overline{\alpha}_i I_i + |\alpha_i(0)h_i(0)|]T^{\beta}, \end{aligned}$$

$$|x_i(t)| \leq -\gamma_i D_t^{-\alpha+\beta}|x_i(t)| + \frac{1}{\Gamma(\beta+1)}\frac{1}{\Gamma(\alpha-\beta+1)}[\overline{\alpha}_i \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j$$

$$+\overline{\alpha}_i I_i + |\alpha_i(0)h_i(0)|]T^{\alpha}.$$

From Lemma 2.3,

$$|x_i(t)| \leq A_i E_{\alpha-\beta}(-\gamma_i t^{\alpha-\beta}),$$

where

$$A_i = \frac{1}{\Gamma(\beta+1)}\frac{1}{\Gamma(\alpha-\beta+1)}[\overline{\alpha}_i \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + \overline{\alpha}_i I_i + |\alpha_i(0)h_i(0)|]T^{\alpha},$$

hence, $x_i(t)$ is bounded in $(0, T]$.

On the other hand, if $\xi_i(t) = D_t^{\beta} x_i(t) + \eta_i x_i(t), \eta_i > 0$, then system (1.1) can be transformed to (2.1), from the second formula in (2.1),

$$\begin{aligned}
D_t^{\alpha-\beta}(\xi_i(t) - \eta_i x_i(t)) &= -\gamma_i(\xi_i(t) - \eta_i x_i(t)) - \alpha_i(x_i(t))[h_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \\
&\quad - \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) - I_i(t)] \\
&= -\gamma_i(\xi_i(t) - \eta_i x_i(t)) - \alpha_i(x_i(t))h_i(x_i(t)) - \alpha_i(0)h_i(0) + \alpha_i(0)h_i(0) \\
&\quad + \alpha_i(x_i(t))[\sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) + I_i(t)] \\
&= -\gamma_i(\xi_i(t) - \eta_i x_i(t)) - \vartheta_i^{'}(\theta)x_i(t) + \alpha_i(0)h_i(0) \\
&\quad + \alpha_i(x_i(t))[\sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) + I_i(t)].
\end{aligned} \tag{3.1}$$

According to the boundedness of $x_i(t)$, there is $M_i > 0$, such that $|x_i(t)| \leq M_i$, and from (3.1), one has

$$\begin{aligned}
D_t^{\alpha-\beta}|\xi_i(t) - \eta_i x_i(t)| &\leq -\gamma_i|\xi_i(t) - \eta_i x_i(t)| + |\vartheta_i^{'}(\theta)||x_i(t)| + |\alpha_i(0)h_i(0)| \\
&\quad + \overline{\alpha}_i \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + \overline{\alpha}_i I_i,
\end{aligned}$$

if $0 < t \leq T < +\infty$, from the formula above, one has

$$\begin{aligned}
|\xi_i(t) - \eta_i x_i(t)| &\leq -\gamma_i D_t^{-\alpha+\beta}|\xi_i(t) - \eta_i x_i(t)| + D_t^{-\alpha+\beta}[\overline{\vartheta}_i M_i + |\alpha_i(0)h_i(0)| \\
&\quad + \overline{\alpha}_i \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + \overline{\alpha}_i I_i] \\
&\leq -\gamma_i D_t^{-\alpha+\beta}|\xi_i(t) - \eta_i x_i(t)| + \frac{1}{\Gamma(\alpha-\beta+1)}[\overline{\vartheta}_i M_i + |\alpha_i(0)h_i(0)| \\
&\quad + \overline{\alpha}_i \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + \overline{\alpha}_i I_i]T^{\alpha-\beta} \\
&= -\gamma_i D_t^{-\alpha+\beta}|\xi_i(t) - \eta_i x_i(t)| + \frac{B_i}{\Gamma(\alpha-\beta+1)}T^{\alpha-\beta},
\end{aligned} \tag{3.2}$$

where

$$B_i = \frac{1}{\Gamma(\alpha-\beta+1)}[\overline{\alpha}_i \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + \overline{\alpha}_i I_i + \overline{\vartheta}_i M_i + |\alpha_i(0)h_i(0)|].$$

From Lemma 2.3

$$|\xi_i(t) - \eta_i x_i(t)| \le \frac{B_i}{\Gamma(\alpha-\beta+1)} T^{\alpha-\beta} E_{\alpha-\beta}(-\gamma_i t^{\alpha-\beta}),$$

from the first formula in (2.1), $D_t^{\beta} x_i(t) = \xi_i(t) - \eta_i x_i(t)$, one can see that $D_t^{\beta} x_i(t)$ is bounded in $[0, T]$.

Theorem 3.2 If Assumptions H_1–H_5 are satisfied in system (1.1), then system (1.1) has at least one anti-period solution.

Proof. For any $s \in \mathbb{N}$, if ω satisfies Assumption H_5, then from (1.1) one has

$$\begin{cases} D_t^{\beta}[(-1)^{s+1} x_i(t+(s+1)\omega)] = (-1)^{s+1}[-\eta_i x_i(t+(s+1)\omega) + \xi_i(t+(s+1)\omega)], \\ D_t^{\alpha-\beta}[(-1)^{s+1}\xi_i(t+(s+1)\omega)] = (-1)^{s+1}\{-\gamma_i \xi_i(t+(s+1)\omega) + \gamma_i\eta_i x_i(t+(s+1)\omega) \\ \qquad -\alpha_i(x_i(t+(s+1)\omega))[h_i(x_i(t+(s+1)\omega)) \\ \qquad -\sum\limits_{j=1}^{n} a_{ij} f_j(x_j(t+(s+1)\omega)) - \sum\limits_{j=1}^{n} b_{ij} f_j(x_j(t+(s+1)\omega - \tau_{ij})) \\ \qquad +I_i(t+(s+1)\omega)] + \eta_i D_t^{\alpha-\beta}[x_i(t+(s+1)\omega)]\}. \end{cases} \tag{3.3}$$

From the Assumptions in this theorem,

$$\begin{cases} D_t^{\beta}[(-1)^{s+1} x_i(t+(s+1)\omega)] = -\eta_i(-1)^{s+1} x_i(t+(s+1)\omega) + (-1)^{s+1}\xi_i(t+(s+1)\omega), \\ D_t^{\alpha-\beta}[(-1)^{s+1}\xi_i(t+(s+1)\omega)] = -\gamma_i(-1)^{s+1}\xi_i(t+(s+1)\omega) + \gamma_i\eta_i(-1)^{s+1} x_i(t+(s+1)\omega) \\ \qquad -\alpha_i((-1)^{s+1} x_i(t+(s+1)\omega))[h_i((-1)^{s+1} x_i(t+(s+1)\omega)) \\ \qquad -\sum\limits_{j=1}^{n} a_{ij}[f_j((-1)^{s+1} x_j(t+(s+1)\omega)) - \sum\limits_{j=1}^{n} b_{ij}[f_j((-1)^{s+1} x_j(t+(s+1)\omega - \tau_{ij})) \\ \qquad +I_i(t)] + \eta_i D_t^{\alpha-\beta}((-1)^{s+1} x_i(t+(s+1)\omega)), \end{cases} \tag{3.4}$$

which means $(-1)^{s+1} x_i(t+(s+1)\omega)$ and $(-1)^{s+1}\xi_i(t+(s+1)\omega)$ are the solutions of system (2.1). As the boundedness of $x_i(t)$ and $D_t^{\beta} x_i(t)$ is proved in Theorem 3.1, one can deduce from $\xi_i(t) = D_t^{\beta} x_i(t) + \eta_i x_i(t)$ that $(-1)^{s+1} x_i(t+(s+1)\omega)$ and $(-1)^{s+1}\xi_i(t+(s+1)\omega)$ are bounded and differentiable; thus, the functional sequences $\{(-1)^{s+1} x_i(t+(s+1)\omega)\}$ and $\{(-1)^{s+1}\xi_i(t+(s+1)\omega)\}$ are equicontinuous and uniformly bounded. According to Ascoli-Arzela Theorem, a subsequence $\{s\omega\}_{s\in N}$ is chosen so that $\{(-1)^s x_i(t+s\omega)\}_{s\in N}$, $\{(-1)^s \xi_i(t+s\omega)\}_{s\in N}$ uniformly converge to functions $x_i^*(t)$ and $\xi_i^*(t)$, respectively, which are continuous on arbitrary compact sets. Thus, one has

$$\lim_{s\to+\infty}(-1)^s x_i(t+s\omega) = x_i^*(t),$$

$$\lim_{s\to+\infty}(-1)^s \xi_i(t+s\omega) = \xi_i^*(t), i = 1, 2, \cdots, n.$$

Then it is going to prove that $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^T$ and $\xi^*(t) = (\xi_1^*(t), \xi_2^*(t), \cdots, \xi_n^*(t))^T$ are the anti-period solutions of system (2.1).

As

$$x^*(t+\omega) = \lim_{s\to+\infty}(-1)^s x(t+\omega+s\omega) = -\lim_{s\to+\infty}(-1)^{s+1} x(t+(s+1)\omega) = -x^*(t),$$

$$\xi^*(t+\omega) = \lim_{s\to+\infty}(-1)^s\xi(t+\omega+s\omega) = -\lim_{s\to+\infty}(-1)^{s+1}\xi(t+(s+1)\omega) = -\xi^*(t),$$

one can see that $x^*(t)$ and $\xi^*(t)$ are anti-period functions.

From (3.3), one has

$$\begin{cases} D_t^{\beta}[(-1)^s x_i(t+s\omega)] = -\eta_i(-1)^s x_i(t+s\omega) + (-1)^s\xi_i(t+s\omega), \\ D_t^{\alpha-\beta}[(-1)^s\xi_i(t+s\omega)] = -\gamma_i(-1)^s\xi_i(t+s\omega) + \gamma_i\eta_i(-1)^s x_i(t+s\omega) \\ \qquad -\alpha_i(-1)^s(x_i(t+s\omega))[h_i((-1)^s x_i(t+s\omega)) - \sum_{j=1}^{n} a_{ij}f_j((-1)^s x_j(t+s\omega)) \\ \qquad - \sum_{j=1}^{n} b_{ij}f_j((-1)^s x_j(t+s\omega-\tau_{ij})) - I_i(t)] + \eta_i D_t^{\alpha-\beta}(-1)^s x_i(t+s\omega), \end{cases} \tag{3.5}$$

since $f_j(\cdot)$ is continuous, $(-1)^s x_i(t+s\omega)$ and $(-1)^s\xi_i(t+s\omega)$ uniformly converge to continuous functions $x_i^*(t)$ and $\xi_i^*(t)$ respectively. Thus, from (3.5), one has

$$\lim_{s\to+\infty} D_t^{\beta}(-1)^s x_i(t+s\omega) = D_t^{\beta}x_i^*(t),$$

$$\lim_{s\to+\infty} D_t^{\alpha-\beta}(-1)^s\xi_i(t+s\omega) = D_t^{\alpha-\beta}\xi_i^*(t),$$

and

$$\begin{cases} D_t^{\beta}x_i^*(t) = -\eta_i x_i^*(t) + \xi_i^*(t), \\ D_t^{\alpha-\beta}\xi_i^*(t) = -\gamma_i\xi_i^*(t) + \gamma_i\eta_i x_i^*(t) - \alpha_i(x_i^*(t))[h_i(x_i^*(t)) - \sum_{j=1}^{n} a_{ij}f_j(x_j^*(t)) \\ \qquad - \sum_{j=1}^{n} b_{ij}f_j(x_j^*(t-\tau_{ij})) - I_i(t)] + \eta_i D_t^{\alpha-\beta}x_i^*(t), \end{cases} \tag{3.6}$$

which means $x^*(t)$, $\xi^*(t)$ are the solutions of system (2.1). Thus, $x^*(t)$ is the anti-period solution of system (1.1).

Theorem 3.3 If Assumptions H_1 to H_5 are satisfied, and $1 - 2\gamma_i\eta_i - \overline{\vartheta}_i > 0$,

$$c_i = \eta_i + \gamma_i\eta_i^2 - \eta_i\overline{\vartheta}_i - \frac{1}{2}(1 - 2\gamma_i\eta_i - \underline{\vartheta}_i) - \sum_{j=1}^{n}[(\frac{1}{2}+\eta_i)|a_{ji}| + \frac{1}{2}\eta_i^2|b_{ji}|]l_i\overline{\alpha}_j$$

$$-(\frac{1}{2}+\eta_i)\overline{\vartheta}_i[\sum_{j=1}^{n}(|a_{ij}|+|b_{ij}|)\overline{f}_j + I_i] > 0,$$

$$d_i = \gamma_i - \frac{1}{2}\sum_{j=1}^{n}(|a_{ji}|+|b_{ji}|)l_i\overline{\alpha}_i - \frac{1}{2}\overline{\vartheta}_i[\sum_{j=1}^{n}(|a_{ij}|+|b_{ij}|)\overline{f}_j + I_i] - \frac{1}{2}(1 - 2\gamma_i\eta_i - \underline{\vartheta}_i) > 0,$$

for $i = 1, 2, \cdots n$, then the anti-period solution of system (1.1) is global asymptotically stable on $[0, T]$, where T is a finite real number.

Proof. From Theorem 3.2, there is at least one anti-period solution of system (1.1). Suppose that $\overline{x}(t) = (\overline{x}_1(t), \overline{x}_2(t), ..., \overline{x}_n(t))^T$ is the anti-period solution of system (1.1), and $x(t)$ is the solution of system (1.1).

Let $z_i(t) = x_i(t)-\overline{x}_i(t)$, $\nu_i(t) = \xi_i(t)-\overline{\xi}_i(t)$, where $\xi_i(t) = D_t^{\beta}\xi_i(t)+\eta_i\xi_i(t)$, $\overline{\xi}_i(t) = D_t^{\beta}\overline{x}_i(t)+\eta_i\overline{x}_i(t)$,$\eta_i > 0$, from (2.1), one has

$$\begin{cases} D_t^{\beta}z_i(t) = -\eta_i z_i(t) + \nu_i(t) \\ D_t^{\alpha-\beta}[\nu_i(t) - \eta_i z_i(t)] = -\gamma_i[\nu_i(t) - \eta_i z_i(t)] + \alpha_i(\overline{x}_i(t))h_i(\overline{x}_i(t)) - \alpha_i(x_i(t))h_i(x_i(t)) \\ \qquad +\alpha_i(x_i(t))[\sum_{j=1}^{n} a_{ij}(f_j(x_j(t)) - f_j(\overline{x}_j(t))) + \sum_{j=1}^{n} b_{ij}(f_j(x_j(t-\tau_{ij})) - f_j(\overline{x}_j(t-\tau_{ij})))] \\ \qquad +[\alpha_i(x_i(t)) - \alpha_i(\overline{x}_i(t))][\sum_{j=1}^{n} a_{ij}f_j(\overline{x}_j(t)) + \sum_{j=1}^{n} b_{ij}f_j(\overline{x}_j(t-\tau_{ij})) + I_i(t)]. \end{cases} \tag{3.7}$$

From Assumptions H_1 and H_3,

$$\begin{cases} \alpha_i(\overline{x}_i(t))h_i(\overline{x}_i(t)) - \alpha_i(x_i(t))h_i(x_i(t)) = -\vartheta_i'(\theta_i)z_i(t), \\ \alpha_i(x_i(t)) - \alpha_i(\overline{x}_i(t)) = \alpha_i'(\theta_i^*)z_i(t), \end{cases} \tag{3.8}$$

where θ_i and θ_i^* are between $x_i(t)$ and $\overline{x}_i(t)$.

Consider the Lyapunov function

$$W(t, z(t), \nu(t)) = \sum_{i=1}^{n}\{\frac{1}{2}D_t^{\beta-1}z_i^2(t) + \frac{1}{2}D_t^{\alpha-\beta-1}[\nu_i(t) - \eta_i z_i(t)]^2 + \sum_{j=1}^{n}|b_{ij}|l_j\int_{t-\tau_{ij}}^{t} z_j^2(s)ds\}, \tag{3.9}$$

where $z(t) = (z_1(t), z_2(t), ..., z_n(t))^T$, $\nu(t) = (\nu_1(t), \nu_2(t), ..., \nu_n(t))^T$, one has

$$\begin{aligned} W_t'(t, z(t), \nu(t)) &= \sum_{i=1}^{n}\{\frac{1}{2}D_t^{\beta}z_i^2(t) + \frac{1}{2}D_t^{\alpha-\beta}[\nu_i(t) - \eta_i z_i(t)]^2 + \sum_{j=1}^{n}|b_{ij}|l_j(z_j^2(t) - z_j^2(t-\tau_{ij}))\} \\ &\leq \sum_{i=1}^{n}\{z_i(t)D_t^{\beta}z_i(t) + [\nu_i(t) - \eta_i z_i(t)]D_t^{\alpha-\beta}[\nu_i(t) - \eta_i z_i(t)] \\ &\quad + \sum_{j=1}^{n}|b_{ij}|l_j(z_j^2(t) - z_j^2(t-\tau_{ij}))\} \\ &= \sum_{i=1}^{n}\{z_i(t)[-\eta_i z_i(t) + \nu_i(t)] + [\nu_i(t) - \eta_i z_i(t)][-\gamma_i(\nu_i(t) - \eta_i z_i(t)) \\ &\quad +\alpha_i(\overline{x}_i(t))h_i(\overline{x}_i(t)) - \alpha_i(x_i(t))h_i(x_i(t)) + \alpha_i(x_i(t))[\sum_{j=1}^{n} a_{ij}(f_j(x_j(t) - f_j(\overline{x}_j(t))) \\ &\quad + \sum_{j=1}^{n} b_{ij}(f_j(x_j(t-\tau_{ij}) - f_j(\overline{x}_j(t-\tau_{ij}))))] + (\nu_i(t) - \eta_i z_i(t))(\alpha_i(x_i(t)) - \alpha_i(\overline{x}_i(t))) \\ &\quad [\sum_{j=1}^{n} a_{ij}f_j(\overline{x}_j(t)) + \sum_{j=1}^{n} b_{ij}f_j(\overline{x}_j(t-\tau_{ij})) + I_i(t)] + |b_{ij}|l_j(z_j^2(t) - z_j^2(t-\tau_{ij}))\}. \end{aligned} \tag{3.10}$$

As

$$0 < 1 - 2\gamma_i\eta_i - \overline{\vartheta}_i \leq 1 - 2\gamma_i\eta_i - \vartheta_i'(\theta) \leq 1 - 2\gamma_i\eta_i - \underline{\vartheta}_i,$$

from Assumptions $H_1 \sim H_4$ and (3.8), (3.10) can be reduced to

$$
\begin{aligned}
W_t^{'}(t,z(t),\nu(t))) \leq & \sum_{i=1}^{n}\{-\eta_i(1+\gamma_i\eta_i)z_i^2(t)-\gamma_i\nu_i^2(t)+(1-2\gamma_i\eta_i)z_i(t)\nu_i(t)-\vartheta_i^{'}(\theta)z_i(t)\nu_i(t)\\
& +\eta_i\vartheta_i^{'}(\theta)z_i^2(t)+[\sum_{j=1}^{n}|a_{ji}|l_i\overline{\alpha}_j+\overline{\vartheta}_i\sum_{j=1}^{n}(|a_{ij}|+|b_{ij}|)\overline{f}_j+\overline{\vartheta}_iI_i](|\nu_i(t)|+\eta_i|z_i(t)|)|z_i(t)|\\
& +\overline{\alpha}_i\sum_{j=1}^{n}|b_{ij}|l_j(|\nu_i(t)|+\eta_i|z_i(t)|)|z_j(t-\tau_{ij})|+\sum_{j=1}^{n}|b_{ij}|l_j(z_j^2(t)-z_j^2(t-\tau_{ij}))\}\\
\leq & \sum_{i=1}^{n}\{-\{\eta_i+\gamma_i\eta_i^2-\eta_i\overline{\vartheta}_i-\frac{1}{2}(1-2\gamma_i\eta_i-\underline{\vartheta}_i)-\sum_{j=1}^{n}[(\frac{1}{2}+\eta_i)|a_{ji}|+\frac{1}{2}\eta_i^2|b_{ji}|]l_i\overline{\alpha}_j\\
& -(\frac{1}{2}+\eta_i)\overline{\vartheta}_i\sum_{j=1}^{n}(|a_{ij}|+|b_{ij}|)\overline{f}_j-(\frac{1}{2}+\eta_i)\overline{\vartheta}_iI_i\}z_i^2(t)-[\gamma_i-\frac{1}{2}\sum_{j=1}^{n}(|a_{ji}|+|b_{ji}|)]l_i\overline{\alpha}_i\\
& -\frac{1}{2}\overline{\vartheta}_i\sum_{j=1}^{n}(|a_{ij}|+|b_{ij}|)\overline{f}_j-\frac{1}{2}\overline{\vartheta}_iI_i-\frac{1}{2}(1-2\gamma_i\eta_i-\underline{\vartheta}_i)\nu_i^2(t)\}\\
& =\sum_{i=1}^{n}\{-c_iz_i^2(t)-d_i\nu_i^2(t)\}.\\
& <0.
\end{aligned}
$$

That if $W_t^{'}(t,z(t),\nu(t)))<0$, then

$$0\leq W(t,z(t),\nu(t)))\leq W(0,z(0),\nu(0))),$$

and from (3.9)

$$0\leq\frac{1}{2}D_t^{\beta-1}z_i^2(t)\leq W(0,z(0),\nu(0))),$$

which leads to

$$
\begin{aligned}
\frac{1}{2}D_t^{\beta-1}z_i^2(t) &= \frac{1}{2\Gamma(1-\beta)}\int_0^t(t-r)^{-\beta}z_i^2(r)dr\\
&= \frac{1}{2\Gamma(1-\beta)}[\int_0^{\frac{t}{2}}(t-r)^{-\beta}z_i^2(r)dr+\int_{\frac{t}{2}}^t(t-r)^{-\beta}z_i^2(r)dr]\\
&\leq W(0,z(0),\nu(0))).
\end{aligned}
$$

On the other hand, according to integral mean value theorem, there is $\xi\in(\frac{t}{2},t)$ which satisfies

$$D_t^{\beta-1}z_i^2(t)\leq\frac{1}{\Gamma(1-\beta)}\int_{\frac{t}{2}}^t(t-r)^{-\beta}z_i^2(r)dr=\frac{z_i^2(\xi)}{\Gamma(1-\beta)}\int_{\frac{t}{2}}^t(t-r)^{-\beta}dr$$

$$=\frac{z_i^2(\xi)}{\Gamma(2-\beta)}(\frac{t}{2})^{1-\beta}\leq W(0,z(0),\nu(0))),$$

one can see that $1-\beta > 0, \xi \in (\frac{t}{2}, t)$, so $\xi \to +\infty$ as $t \to +\infty$, which means

$$\lim_{t\to+\infty} z_i^2(t) = \lim_{\xi\to+\infty} z_i^2(\xi) = 0.$$

Thus, one has

$$\lim_{t\to+\infty} \|x(t) - \overline{x}(t)\| = \lim_{t\to+\infty} \sum_{i=1}^{n} |x_i(t) - \overline{x}_i(t)| = \lim_{t\to+\infty} \sum_{i=1}^{n} z_i(t) = 0,$$

from Definition 2.3, the anti-period solution of system (1.1) is global asymptotically stable.

4. A numerical example

Consider the incommensurate fractional-order Cohen-Grossberg neural networks with inertia as

$$D_t^{\alpha}(x_i(t)) = -\gamma_i D_t^{\beta}(x_i(t)) - \alpha_i(x_i(t))[h_i(x_i(t)) - \sum_{j=1}^{3} a_{ij} f_j(x_j(t))$$
$$- \sum_{j=1}^{3} b_{ij} f_j(x_j(t-\tau_{ij})) - I_i(t)], \qquad (t > 0) \qquad i = 1, 2, 3. \tag{4.1}$$

Example 4.1. The parameters in the system are

$$\alpha = 1.2,\ \beta = 0.5,\ \gamma_1 = 1,\ \gamma_2 = 0.5,\ \gamma_3 = 0.75,\ \eta_1 = \frac{1}{3},\ \eta_2 = \frac{1}{2},\ \eta_3 = \frac{5}{12},$$
$$a_{11} = 0.15,\ a_{12} = 0.10,\ a_{13} = 0.10,\ a_{21} = 0.12,\ a_{22} = 0.10,\ a_{23} = 0.15,$$
$$a_{31} = 0.10,\ a_{32} = 0.15,\ a_{33} = 0.12,\ b_{11} = 0.10,\ b_{12} = 0.15,\ b_{13} = 0.10,$$
$$b_{21} = 0.10,\ b_{22} = 0.10,\ b_{23} = 0.20,\ b_{31} = 0.12,\ b_{32} = 0.10,\ b_{33} = 0.15,$$

and the functions are

$$h_1(x_1) = 6x_1,\ \ h_2(x_2) = 2.6x_2,\ \ h_3(x_3) = 4x_3,\ \ I_i(t) = 0.01 sin2t,$$
$$f_j(x_j) = 0.1 sin(2x_j),\ \ i, j = 1, 2, 3.$$

Let

$$\alpha_1(x_1) = \frac{1}{40}(2 - \frac{1}{1+x_1^2}),\ \ \alpha_2(x_2) = \frac{3}{80}(2 + \frac{1}{1+x_2^2}).$$
$$\alpha_3(x_3) = \frac{1}{32}(3 + \frac{1}{1+x_3^2}),$$

Then one can see that

$$0 \le 0.025 \le \alpha_1(x_1) = \frac{1}{40}(2 - \frac{1}{1+x_1^2}) \le 0.05,$$
$$0 \le 0.075 \le \alpha_2(x_2) = \frac{3}{80}(2 + \frac{1}{1+x_2^2}) \le 0.1125,$$

$$0 \le 0.093 \le \alpha_3(x_3) = \frac{1}{32}(3 + \frac{1}{1+x_3^2}) \le 0.125,$$

$$|f_j(s) - f_j(v)| \le 0.1|s - v|, \ |f_j(\cdot)| \le 0.1,$$

for all $s, v \in \mathbb{R}$, $j = 1, 2, 3$.

$$\vartheta_1(x_1) = \alpha_1(x_1)h_1(x_1) = 0.15x_1(2 - \frac{1}{1+x_1^2}),$$

$$\vartheta_2(x_2) = \alpha_2(x_2)h_2(x_2) = \frac{7.8}{80}x_2(2 + \frac{1}{1+x_2^2}),$$

$$\vartheta_3(x_3) = \alpha_3(x_3)h_3(x_3) = \frac{1}{8}x_3(3 + \frac{1}{1+x_3^2}),$$

$$0 \le 0.15 \le \vartheta_1'(x_1) \le 0.319,$$

$$0 \le 0.1828 \le \vartheta_2'(x_2) \le 0.292,$$

$$0 \le 0.359 \le \vartheta_3'(x_3) \le 0.5,$$

$$|I_i(t)| \le 0.01,$$

$\alpha_i(x_i)I_i(t+\omega) = -\alpha_i(-x_i)I_i(t)$, $\alpha_i(x_i)h_i(x_i) = -\alpha_i(-x_i)h_i(-x_i)$, $\alpha_i(x_i)f_i(x_i) = -\alpha_i(-x_i)f_i(-x_i)$, where $\omega = \pi/2 > 0$. Thus, Assumptions H_1-H_5 are satisfied, and the constants in the Assumptions are

$$\underline{\alpha}_1 = 0.025, \ \overline{\alpha}_1 = 0.05, \ \underline{\alpha}_2 = 0.075, \ \overline{\alpha}_2 = 0.1125, \ \underline{\alpha}_3 = 0.093, \ \overline{\alpha}_3 = 0.125,$$

$$\underline{\vartheta}_1 = 0.15, \ \overline{\vartheta}_1 = 0.319, \ \underline{\vartheta}_2 = 0.1828, \ \overline{\vartheta}_2 = 0.292, \ \underline{\vartheta}_3 = 0.359, \ \overline{\vartheta}_3 = 0.5,$$

$$I_1 = I_2 = I_3 = 0.01, \ l_1 = l_2 = l_3 = 0.1, \ \overline{f}_1 = \overline{f}_2 = \overline{f}_3 = 0.1,$$

after calculation, one has:

$$1 - 2\gamma_1\eta_1 - \overline{\vartheta}_1 \approx 0.015 > 0, 1 - 2\gamma_2\eta_2 - \overline{\vartheta}_2 = 0.208 > 0, 1 - 2\gamma_3\eta_3 - \overline{\vartheta}_3 = 0.208 > 0,$$

$$c_i = \eta_i + \gamma_i\eta_i^2 - \eta_i\overline{\vartheta}_i - \frac{1}{2}(1 - 2\gamma_i\eta_i - \underline{\vartheta}_i) - \sum_{j=1}^{3}[(\frac{1}{2} + \eta_i)|a_{ji}| + \frac{1}{2}\eta_i^2|b_{ji}|]l_i\overline{\alpha}_j$$

$$-(\frac{1}{2} + \eta_i)\overline{\vartheta}_i[\sum_{j=1}^{3}(|a_{ij}| + |b_{ij}|)\overline{f}_j + I_i] > 0,$$

where $c_1 \approx 0.224, c_2 \approx 0.316 > 0, c_3 \approx 0.111 > 0$.

$$d_i = \gamma_i - \frac{1}{2}\sum_{j=1}^{3}(|a_{ji}| + |b_{ji}|)l_i\overline{\alpha}_i - \frac{1}{2}\overline{\vartheta}_i[\sum_{j=1}^{3}(|a_{ij}| + |b_{ij}|)\overline{f}_j + I_i] - \frac{1}{2}(1 - 2\gamma_i\eta_i - \underline{\vartheta}_i) > 0,$$

where $d_1 \approx 0.894 > 0$, $d_2 \approx 0.325 > 0$, $d_3 \approx 0.218 > 0$.

That is to say, the conditions in Theorem 3.3 are satisfied. It can be seen that system (4.1) has an anti-periodic solution with a period of $\pi/2$ and it is global asymptotically stable.

On the other hand, we get the states of the Example through numerical simulation, as shown in Figure 1. It can be seen from the figure that it is consistent with the theoretical result of Theorem 3.3.

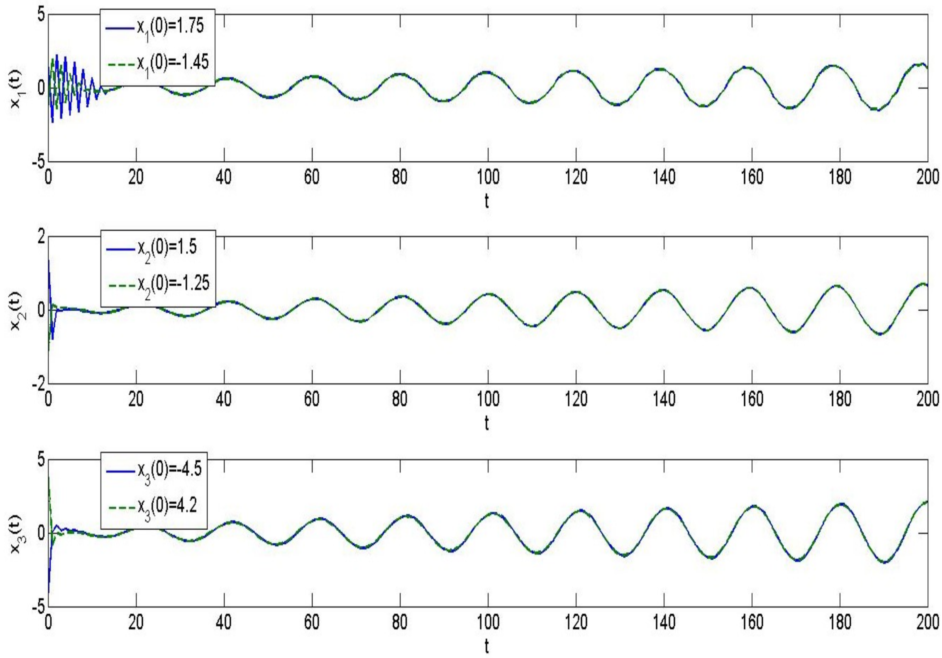

Figure 1. The trajectories of states $x_1(t), x_2(t), x_3(t)$ in system (4.1) in Example 4.1.

5. Conclusions

Stability is the basic condition to ensure the normal operation of the system. Moreover, it is of great significance both in control theory and in practical application in various fields. It is of great practical value to simulate complex biological neural networks with different derivatives α and β, especially when they are incommensurate.

In this paper, the incommensurate fractional-order Cohen-Grossberg neural networks with inertia is studied. By appropriate variable substitution, and using the Ascoli-Arzela Theorem, the sufficient conditions for the boundedness of the system are deduced as Theorem 3.1. The existence of anti-period solutions is investigated in Theorem 3.2. The global asymptotically stability of an anti-period solution is derived in Theorem 3.3. Furthermore, an example is simulated to verify the correctness of the results.

The results are new and verified by simulation, which provide a new basis for the theoretical study and practical application of the system. Similarly, the ideas and methods adopted can be further used to study the performance of other incommensurate fractional-order neural networks. The following two aspects can be further studied using the research ideas in this paper:

1) For the stability problems of fractional-order BAM neural networks with inertia and Cohen-Grossberg-BAM neural networks with inertia;

2) If we change the Riemann-Liouville fractional-order derivative to Caputos fractional-order derivative, we can get similar results.

The two aspects above are also the directions we are heading in the future.

Author contributions

These authors contributed equally to this work.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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