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Research article

Admissible interval-valued monotone comparative statics methods applied in games with strategic complements

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Abstract: In many theories and applications with uncertainty, using intervals to characterize uncertainty is simple and operable. It is crucial to choose a proper order for an interval method. In general, a total order is superior to a partial order for those applications in which we have to make a final decision. Motivated by this idea, we generalized interval-valued monotone comparative statics (MCS) with a partial order to interval-valued MCS with a total order, to be more precise, with an admissible order. The generalization was not trival. We obtained a necessary and sufficient condition for MCS by a series of new concepts such as an interval-valued quasi-super-modular function and an interval-valued single crossing property with an admissible order. The same condition was only sufficient in the existing literature. Furthermore, we illustrated the efficiency of the interval-valued MCS with the lexicographical orders and XY-order, which are well-known admissible orders. Finally, we applied our results in interval games with strategic complements to get the monotonity of the best response correspondence for player i.

Keywords: admissible orders; monotone comparative statics; interval-valued quasisupermodular function; interval-valued single crossing property; interval games with strategic complements
Mathematics Subject Classification: 90C, 91A, 91B

1. Introduction

The real world is full of uncertainty. How to characterize all kinds of uncertainties has been a hot topic for researchers. Generally speaking, stochastic and fuzzy approaches are used widely, but parameters describing uncertain information in stochastic methods are considered as random variables. They should have corresponding probability distribution functions, which is hard to obtain in practice. Similarly, parameters with uncertainty in fuzzy methods are always seen as fuzzy sets, whose fuzzy membership functions should be provided in advance. To some extent, the above embarrassment can be solved by using intervals to describe uncertainty. In fact, interval methods have been applied in all kinds of uncertain problems [1–4].

Comparing between intervals is essential in an interval method, so it is important to choose a proper order relation between intervals to represent a decision-maker's preference for interval profits. The order relation $\preceq_{LU}$ is widely used for intervals of real numbers, which inherits from the usual order between real numbers [1–5]. The order compares both the upper and lower bounds of two intervals simultaneously, called LU-order, which represents the decision-maker's preference for the alternative with the higher minimum profit and maximum profit. However, the pursuit for both optimal results is restricted in practice. Another drawback of LU-order is that it is only a partial order. In fact, using LU-order usually gets a group of choices rather than a unique result, which makes it harder for decision-makers to make decisions. It seems more realistic to use such an order in which the first indices of two intervals are compared, then the second will be done if their first indices are equal. Clearly, this is a total order such as the lexicographical orders and XY-order [6]. Meanwhile, using a total order can provide decision-makers a unique result to facilitate their decision-making process.

Considering the importance of total orders in applications, Bustince et al. proposed admissible orders by refining LU-order on $L([0,1])$, where $L([0,1])$ denotes closed subintervals of [0,1] [7]. The lexicographical orders and XY-order, as the most used examples of total orders, are both specific admissible orders. Afterward, the research on admissible orders was published successively [7–14]. It is worth to note that admissible orders have been extended and applied in all kinds of areas such as linear and nonlinear optimization [15,16], decision-making [17–19], classification [20–23], clustering [24] and image processing [25].

However, there are few applications in economics and game theories to use admissible orders to deal with uncertain problems as far as we know. This is one of the contributions of our paper to use admissible orders for monotone comparative statics (MCS). Comparative statics is a common notion in economics, which is to compare two equilibrium (statics) before and after changing economic situations. The usual methods to analyze comparative statics are to apply the implicit function theorem to first-order conditions or to exploit the identities of duality theory. However, these methods need some strong assumptions such as convexity, smoothness, strict second derivative conditions, and so on. All these assumptions couldn't be necessary for any meaningful comparative statics conclusions [26]. In 1978, Topkis proposed an important monotone theorem for maximizer sets by defining so-called super-modularity and increasing differences for real-valued functions [27]. MCS began to become a hot topic [26–34]. MCS is different from the above classical methods, which provides a clear direction for decision-makers to increase or decrease related parameters. Furthermore, MCS is consistent with strategic complements in games [35].

Let X be a lattice and T be a partially ordered set with partial orders $\leq_X, \leq_T$, respectively. A function $f : X \to \mathfrak{R}$ is called a super-modular function (SMF) if $f(x) + f(y) \leq f(x \wedge y) + f(x \vee y)$ for all x, y in X. A function $f : X \times T \to \mathfrak{R}$ satisfies increasing differences property (IDP) in (x, t) if $f(x_2, t_1) - f(x_1, t_1) \leq f(x_2, t_2) - f(x_1, t_2)$ for $x_1 \leq_X x_2$ and $t_1 \leq_T t_2$. In applications [35–39], super-modularity means the incremental returns, associated with increases in other decision variables, goes up with increasing in one decision variable. Similarly, IDP means the marginal returns can be raised by increasing a parameter. They both express the concept of complementarity to some extent, which is the key to codirectional incentives in economics, but they are cardinal and do not necessarily hold true for MCS [27].

The more general concepts of quasi-super-modularity and single crossing property were proposed by Milgrom and Shannon [26]. A function $f : X \to \Re$ is a quasi-super-modular function (QSMF) if $f(x \wedge y) \leq f(x) \Rightarrow f(y) \leq f(x \vee y)$ and $f(x \wedge y) < f(x) \Rightarrow f(y) < f(x \vee y)$ for all x, y in X. A function $f : X \times T \to \Re$ satisfies single crossing property (SCP) if $f(x_1, t_1) \leq f(x_2, t_1) \Rightarrow f(x_1, t_2) \leq f(x_2, t_2)$ and $f(x_1, t_1) < f(x_2, t_1) \Rightarrow f(x_1, t_2) < f(x_2, t_2)$ for $x_1 \leq_X x_2$ and $t_1 \leq_T t_2$. Clearly, quasi-super-modularity and SCP are weaker than super-modularity and IDP, respectively, but they are ordinal and sufficient and necessary for MCS [26]. Furthermore, only an order relation is needed in QSMF and SCP, compared to SMF and IDP.

The success of QSMF and SCP in a real-valued situation is undoubted, but uncertainty in economic environment is inevitable. So, Li and Luo generalized SMF, IDP, QSMF, and SCP to obtain a series of corresponding interval-valued concepts by using LU-order [5]. This generalization does not undermine the sufficient condition for MCS, but the necessity cannot be preserved. In our opinion, the reason is that LU-order is a veritable partial order, which is limited in the proof of necessity. In this paper, we generalize those real-valued concepts above by using an admissible order to preserve the sufficiency and necessity successfully for MCS under interval uncertainty.

Furthermore, we apply the conclusions in interval games with strategic complements. Classical games with strategic complements (GSC) are connected with MCS naturally based on the synchronization [35]. Considering uncertainty in games, an interval game emerged from bankruptcy situations by employing fuzzy sets [40]. Thereafter, interval games were studied by many researchers [41–46], but most of these topics discuss solvability or existence of Nash equilibria rather than monotonicity. In our paper, we pay more attention to monotonicity of the best response correspondence for player i in an interval GSC, which is another contribution that differs from previous research.

The remainder of the paper is organized as follows. In Section 2, we review some preliminaries, including lattice theory, admissible orders, and interval arithmetic. In Section 3, we generalize the interval-valued SMF, IDP, QSMF, and SCP with an admissible order and show their essential properties. Furthermore, we modify the results by using the specific admissible orders, i.e., the lexicographical orders and XY-order. In Section 4, we propose a monotone theorem, which is sufficient and necessary for interval-valued MCS, by using those generalized concepts in the previous section. We illustrate the superiority of admissible orders and apply the monotone theorem in an economic example. In Section 5, we apply interval-valued MCS in an interval GSC. Two different games are analyzed in this section, which are an interval-valued coordination game and an interval-valued Bertrand oligopoly model. Finally, we draw our conclusions in the last section.

2. Preliminaries

Let X be a nonempty set with a reflexive, antisymmetric, and transitive order $\leq_X$, and call it a partially ordered set or a poset. A strict order relation $<_X$ in a poset X is that $x <_X y$ if, and only if, $x \leq_X y$ and $x \neq y$. A poset X is total or linear, called a chain, if any two elements in X are comparable by $\leq_X$.

For any x and y in a poset X, $x \vee y$ is denoted the least upper bound of x and y, and $x \wedge y$ is denoted the greatest lower bound. A poset X is a lattice if $x \vee y$ and $x \wedge y$ exist in X for any x and y in X. We call a subset S of X a sub-lattice if S is closed under the two operations. Refer to [47] for more details about lattice theory.

In this paper, we study those maximization problems in which objective functions are interval-valued in $\mathfrak{R}$. We denote $I(\mathfrak{R}) = \{A = [\underline{a}, \overline{a}] \| \underline{a}, \overline{a} \in \mathfrak{R}, \underline{a} \leq \overline{a}\}$ as the set of all closed intervals in $\mathfrak{R}$. The intervals we discuss in this paper are in $I(\mathfrak{R})$.

For all $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ in $I(\mathfrak{R})$, if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$, then $A \leq_{LU} B$, which is a well-known LU-order. The following is the definition of an admissible order, extended from $L([0, 1])$ to $I(\mathfrak{R})$.

Definition 2.1. *The order $\leq_{adm}$ is called an admissible order if it is a total order first, and for all $A, B \in I(\mathfrak{R})$, $A \leq_{adm} B$ whenever $A \leq_{LU} B$.*

The definition is abstract, but Bustince et al. proposed a method to generate a concrete admissible order by two aggregation functions [7], which can be extended to $I(\mathfrak{R})$ naturally. Here, we focus on the so-called (α, β)-orders, a class of admissible orders generated by k_α mappings.

Definition 2.2. *Let $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$, and $\alpha, \beta \in [0, 1]$, $\alpha \neq \beta$. $A \leq_{(\alpha,\beta)} B$ if, and only if, $k_\alpha(A) < k_\alpha(B)$, or $k_\alpha(A) = k_\alpha(B)$ and $k_\beta(A) \leq k_\beta(B)$, where $k_\alpha(A) = (1 - \alpha)\underline{a} + \alpha\overline{a}$ and $k_\beta(A) = (1 - \beta)\underline{a} + \beta\overline{a}$. $A \prec_{(\alpha,\beta)} B$ if, and only if, $A \leq_{(\alpha,\beta)} B$ and $A \neq B$.*

In the literature on the structure of admissible orders, the researchers usually choose (α, β)-orders to study [48,49]. Specifically, the lexicographical order 1 on $I(\mathfrak{R})$ is an (α, β)-order when $\alpha = 0$ and $\beta = 1$. Similarly, the lexicographical order 2 is an (α, β)-order when $\alpha = 1$ and $\beta = 0$. XY-order is also an (α, β)-order, in which $\alpha = \frac{1}{2}$ and $\beta = 1$. For convenience, we give the definitions of the three widely used orders in their usual forms.

Definition 2.3. *Let $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$. $A \leq_{Lex1} B$ if, and only if, $\underline{a} < \underline{b}$, or $\underline{a} = \underline{b}$ and $\overline{a} \leq \overline{b}$. If $A \leq_{Lex1} B$ and $A \neq B$, then $A \prec_{Lex1} B$.*

Definition 2.4. *Let $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$. $A \leq_{Lex2} B$ if, and only if, $\overline{a} < \overline{b}$, or $\overline{a} = \overline{b}$ and $\underline{a} \leq \underline{b}$. If $A \leq_{Lex2} B$ and $A \neq B$, then $A \prec_{Lex2} B$.*

Definition 2.5. *Let $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$. $A \leq_{XY} B$ if, and only if, $\underline{a} + \overline{a} < \underline{b} + \overline{b}$, or $\underline{a} + \overline{a} = \underline{b} + \overline{b}$ and $\overline{a} - \underline{a} \leq \overline{b} - \underline{b}$. If $A \leq_{XY} B$ and $A \neq B$, then $A \prec_{XY} B$.*

Remark 2.1. *It has been proved that $A \leq_{XY} B$ if, and only if, $\underline{a} + \overline{a} < \underline{b} + \overline{b}$, or $\underline{a} + \overline{a} = \underline{b} + \overline{b}$ and $\overline{a} \leq \overline{b}$ in [7]. Considering an easier operation, we choose this way to compare two intervals rather than Definition 2.5 when we use XY-order.*

In this paper, we still need some operations between intervals, which are listed as follows.

Definition 2.6. *Let $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$. $A + B = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$. $kA = [k\underline{a}, k\overline{a}]$, where k is a nonnegative real number.* $A \ominus B = [\min(\underline{a} - \underline{b}, \overline{a} - \overline{b}), \max(\underline{a} - \underline{b}, \overline{a} - \overline{b})]$.

Remark 2.2. *The subtraction $\ominus$ should be denoted more exactly by $\ominus_{gH}$, which is called a generalized Hukuhara difference (gH-difference) between two intervals [50]. In fact, there are other interval subtractions such as Hukuhara difference [51] and g-difference [52]. So far, gH-difference has been widely used owing to its good performance in interval problems [53–57]. We use simplified mark $\ominus$ where there is no confusion.*

3. Interval-valued QSMF and SCP with an (α, β)-order

In this section, we generalize the real-valued concepts of SMF, IDP, QSMF, and SCP to interval-valued ones with an (α, β)-order, and investigate some properties to characterize them for applications.

Let X be a lattice and T be a poset. $f : X \to I(\Re)$ is an interval-valued function, which maps any $x \in X$ to a related closed interval $f(x) = [\underline{f}(x), \overline{f}(x)]$. Here, $\underline{f}(x)$ and $\overline{f}(x)$ are real-valued.

Definition 3.1. *An interval-valued function $f : X \to I(\Re)$ is called an interval-valued super-modular function with an (α, β)-order ($I_{(\alpha,\beta)}$SMF) if for all x, y in X,*

$$f(x) + f(y) \preceq_{(\alpha,\beta)} f(x \wedge y) + f(x \vee y). \tag{3.1}$$

The definition is a formal generalization of the classical SMF. Li and Luo did the same generalization using LU-order, called an interval SMF (ISMF) [5]. To distinguish and compare with our related definitions, we call ISMF in [5] I_{LU}SMF in our paper. In fact, an I_{LU}SMF must be an $\text{I}_{(\alpha,\beta)}$SMF because LU-order is refined in an admissible order, an (α, β)-order of course.

Example 3.1. *Let $f : X \to I(\Re)$ be an interval-valued function.*

1) If X is a chain, then f is an $I_{(\alpha,\beta)}$SMF.

2) Let $x = (x_1, x_2) \in X = \Re^2$ and $f(x) = [x_1x_2, x_1^2 + x_2^2]$. Then, f is an $I_{(\alpha,\beta)}$SMF since it is an I_{LU}SMF [5]. The graph of f is illustrated in Figure 3.1.

3) Let $x = (x_1, x_2) \in X = [-\frac{\pi}{4}, \frac{\pi}{4}] \times [-\frac{\pi}{4}, \frac{\pi}{4}]$ and $f(x) = [x_1x_2, cos x_1x_2]$. f is an $I_{(\alpha,\beta)}$SMF, where $\alpha = 0$ and $\beta = 1$. Note that it is a general $I_{(\alpha,\beta)}$SMF rather than an I_{LU}SMF. The graph of f is illustrated in Figure 3.2.

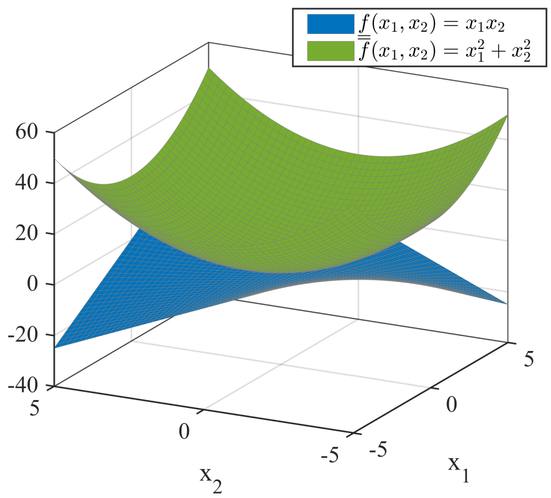

Figure 3.1. The graph of interval-valued function f in Example 3.1-2.

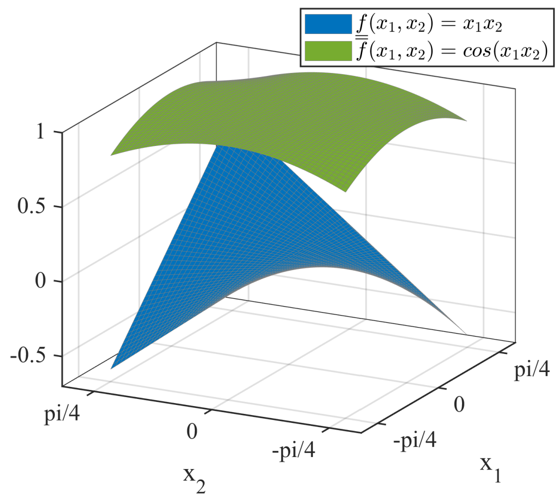

Figure 3.2. The graph of interval-valued function f in Example 3.1-3.

Definition 3.2. *An interval-valued function $f : X \times T \to I(\Re)$ satisfies interval-valued increasing differences property in (x, t) with an (α, β)-order ($I_{(\alpha,\beta)}$IDP) if for $x_1 \preceq_X x_2$ and $t_1 \preceq_T t_2$,*

$$f(x_2, t_1) \ominus f(x_1, t_1) \preceq_{(\alpha,\beta)} f(x_2, t_2) \ominus f(x_1, t_2). \tag{3.2}$$

Considering the relationship between LU-order and an (α,β)-order, we have the following theorem by using the related results in [5].

Theorem 3.1. *Let $f : X \times T \to I(\Re)$ be an interval-valued function. If $\underline{f}$ and $\overline{f}$ are super-modular in x, then f is an $I_{(\alpha,\beta)}$SMF. If $\underline{f}$ and $\overline{f}$ satisfy IDP in (x,t), then f satisfies $I_{(\alpha,\beta)}$IDP.*

However, if we compare two intervals A and B by an (α,β)-order, the key is the relationship between $k_\alpha(A)$ and $k_\alpha(B)$, or $k_\beta(A)$ and $k_\beta(B)$, not between the endpoints of two intervals. In fact, a general k_α mapping carries more information on uncertainty. For better understanding (α,β)-order, we give two lemmas about k_α as follows.

Lemma 3.1. *Let $\alpha \in [0,1]$ and $A = [\underline{a},\overline{a}] \in I(\Re)$. $k_\alpha : I(\Re) \to \Re$ is a real-valued function, where $k_\alpha(A) = (1-\alpha)\underline{a} + \alpha\overline{a}$. Then, $k_\alpha(A+B) = k_\alpha(A) + k_\alpha(B)$, and for $\lambda > 0$, $k_\alpha(\lambda A) = \lambda k_\alpha(A)$.*

The proof is clear. The lemma reveals the linearity of k_α to a great extent.

Lemma 3.2. *Let $f : X \times T \to I(\Re)$. Denote $k_\alpha \circ f$ the composition of k_α and f, which is a function from $X \times T$ into $\Re$. If $\underline{f}$ and $\overline{f}$ are super-modular in x, then $k_\alpha \circ f$ is super-modular in x; and if $\underline{f}$ and $\overline{f}$ satisfy IDP in (x,t), $k_\alpha \circ f$ satisfies IDP.*

Theorem 3.2. *Let $f : X \to I(\Re)$, and $\alpha,\beta \in [0,1], \alpha \neq \beta$. If $k_\alpha \circ f$ and $k_\beta \circ f$ are super-modular, then f is an $I_{(\alpha,\beta)}$SMF.*

Proof. For any $x, y \in X$, we need to prove $f(x) + f(y) \preceq_{(\alpha,\beta)} f(x \wedge y) + f(x \vee y)$, namely, $[\underline{f}(x) + \underline{f}(y), \overline{f}(x) + \overline{f}(y)] \preceq_{(\alpha,\beta)} [\underline{f}(x \wedge y) + \underline{f}(x \vee y), \overline{f}(x \wedge y) + \overline{f}(x \vee y)]$. That is, $(1-\alpha)(\underline{f}(x) + \underline{f}(y)) + \alpha(\overline{f}(x) + \overline{f}(y)) < (1-\alpha)(\underline{f}(x \wedge y) + \underline{f}(x \vee y)) + \alpha(\overline{f}(x \wedge y) + \overline{f}(x \vee y))$, or the relation is equal and $(1-\beta)(\underline{f}(x) + \underline{f}(y)) + \beta(\overline{f}(x) + \overline{f}(y)) \leq (1-\beta)(\underline{f}(x \wedge y) + \underline{f}(x \vee y)) + \beta(\overline{f}(x \wedge y) + \overline{f}(x \vee y))$. After rearranging, we need to prove that $k_\alpha \circ f(x) + k_\alpha \circ f(y) < k_\alpha \circ f(x \wedge y) + k_\alpha \circ f(x \vee y)$, or the relation is equal and $k_\beta \circ f(x) + k_\beta \circ f(y) \leq k_\beta \circ f(x \wedge y) + k_\beta \circ f(x \vee y)$. This is natural if $k_\alpha \circ f$ and $k_\beta \circ f$ are super-modular. The proof is finished. □

f is an I_{LU}SMF if, and only if, $\underline{f}$ and $\overline{f}$ are super-modular in [5]. However, it is not necessary for an $\mathrm{I}_{(\alpha,\beta)}$SMF that $k_\alpha \circ f$ and $k_\beta \circ f$ are super-modular, which will be shown in the following example.

Example 3.2. *Let $X = \{x, y, x \wedge y, x \vee y\}$, in which x and y are incomparable. $f : X \to I(\Re)$ is an interval-valued function, where $f(x) = [0,1]$, $f(y) = [-1,2]$, $f(x \wedge y) = [0,1]$, $f(x \vee y) = [0,1]$. If $\alpha = \frac{1}{3}, \beta = \frac{2}{3}$, $f(x) + f(y) \preceq_{(\alpha,\beta)} f(x \wedge y) + f(x \vee y)$ for any $x, y \in X$, that is, f is an $I_{(\alpha,\beta)}$SMF. However, $k_\beta \circ f(x) + k_\beta \circ f(y) > k_\beta \circ f(x \wedge y) + k_\beta \circ f(x \vee y)$. $k_\beta \circ f$ is not super-modular.*

Remark 3.1. *In Example 3.2, we also have $\overline{f}(x) + \overline{f}(y) > \overline{f}(x \wedge y) + \overline{f}(x \vee y)$. That is to say, $\overline{f}$ is not super-modular. Moreover, f is not an I_{LU}SMF, which is proposed in [5]. So, when the interval-valued SMF is needed, I_{LU}SMF becomes invalid, but our $I_{(\alpha,\beta)}$SMF can still be used. This is a discrete example, unlike the continuous example in Example 3.1-3.*

Actually, when we compare two intervals using an (α,β)-order, we always compare their k_α terms first. We don't need to compare their k_β terms unless the k_α terms are equal. From this point, we think that $k_\alpha \circ f$ is more important than $k_\beta \circ f$ in characterizing properties of an interval-valued function by using an (α,β)-order. Theorem 3.2 and Example 3.2 provide some clues for this idea. The following theorem is also a support, whose proof is obvious and omitted.

Theorem 3.3. *Let $f : X \to I(\Re)$, and $\alpha,\beta \in [0,1], \alpha \neq \beta$. If f is an $I_{(\alpha,\beta)}$SMF, $k_\alpha \circ f$ is super-modular.*

Note that $k_\alpha(A \ominus B) \neq k_\alpha(A) - k_\alpha(B)$ according to Definition 2.6, which leads to an unsatisfying result on $\mathrm{I}_{(\alpha,\beta)}$IDP. The fact is that f does not satisfy $\mathrm{I}_{(\alpha,\beta)}$IDP even if $k_\alpha \circ f$ and $k_\beta \circ f$ both satisfy IDP.

Example 3.3. *Let $X = \{x, y, x \wedge y, x \vee y\}$, in which x and y are incomparable, and $T = \{1,2\}$ with a natural order. $f : X \times T \to I(\Re)$, whose corresponding interval-values are listed in Table 3.1. For $\alpha = \frac{1}{4}, \beta \in [0,1]$ and $\beta \neq \alpha$, $k_\alpha \circ f$ and $k_\beta \circ f$ satisfy IDP. However, $f(x,1) \ominus f(x \wedge y, 1) \succ_{(\alpha,\beta)} f(x,2) \ominus f(x \wedge y, 2)$, that is, f does not satisfy $I_{(\alpha,\beta)}$IDP.*

Table 3.1. Corresponding interval-values of f in Example 3.3.

T \ X	x	y	$x \wedge y$	$x \vee y$
1	[0,2]	[0,2]	[0,1]	[0,2]
2	[0,1]	[0,1]	[-1,3]	[0,1]

Example 3.3 indicates that the subtraction $\ominus$ on $I(\Re)$ is different greatly from the natural subtraction on $\Re$, which is not a converse of the interval addition. In fact, other subtractions between intervals are like this one, except for H-difference, but H-difference does not always exist [51].

Fortunately, SCP is a better substitute for IDP, even if it is weaker. Actually, the weaker SCP and QSMF are just conditions to get a nondecreasing maximizer [26]. We generalize QSMF and SCP for an interval-valued function by an (α,β)-order as follows.

Definition 3.3. *A function $f : X \to I(\Re)$ is called an interval-valued quasi-super-modular function with an (α,β)-order ($I_{(\alpha,\beta)}$QSMF) if for any x, y in X,*

$$\begin{cases} f(x \wedge y) \preceq_{(\alpha,\beta)} f(x) \Rightarrow f(y) \preceq_{(\alpha,\beta)} f(x \vee y), \\ f(x \wedge y) \prec_{(\alpha,\beta)} f(x) \Rightarrow f(y) \prec_{(\alpha,\beta)} f(x \vee y). \end{cases} \tag{3.3}$$

Theorem 3.4. *Let $f : X \to I(\Re)$, and $\alpha,\beta \in [0,1], \alpha \neq \beta$. If $k_\alpha \circ f$ and $k_\beta \circ f$ are quasi-super-modular, f is an $I_{(\alpha,\beta)}$QSMF.*

The proof of the theorem is similar to the one of Theorem 3.2, but the converse is not true. In the following example, $k_\alpha \circ f$ is not quasisupermodular even if f is an $\mathrm{I}_{(\alpha,\beta)}$QSMF. The result shows the complexity under the weaker conditions, compared with Theorem 3.3.

Example 3.4. *Let $X = \{x, y, x \wedge y, x \vee y\}$, in which x and y are incomparable. $f : X \to I(\Re)$ is given by $f(x) = [1,3]$, $f(y) = [0,2]$, $f(x \wedge y) = [0,1]$, $f(x \vee y) = [0,4]$. If $\alpha = \frac{1}{2}, \beta = 1$, which is XY-order, f is an $I_{(\alpha,\beta)}$QSMF but $k_\alpha \circ f$ is not a QSMF. The fact is that $k_\alpha \circ f(x) = k_\alpha \circ f(x \vee y)$ when $k_\alpha \circ f(x \wedge y) < k_\alpha \circ f(y)$.*

Another point worth noting is that f may not necessarily be an $\mathrm{I}_{(\alpha,\beta)}$QSMF even if $\underline{f}$ and $\overline{f}$ are both quasi-super-modular. For example, we redefine f by $f(x) = [1,1.5]$, $f(y) = [0,4]$, $f(x \wedge y) = [0,2]$, $f(x \vee y) = [1,2]$ in Example 3.4. It can be verified that $\underline{f}$ and $\overline{f}$ are quasi-super-modular, but when $f(x \wedge y) \preceq_{(\alpha,\beta)} f(x)$, we have $f(y) \succ_{(\alpha,\beta)} f(x \vee y)$ for $\alpha = \frac{1}{2}$ and $\forall \beta \in [0,1], \alpha \neq \beta$. We realize again that it is a better choice to pay more attention to $k_\alpha \circ f$ and $k_\beta \circ f$.

Definition 3.4. *An interval-valued function $f : X \times T \to I(\Re)$ satisfies interval-valued single crossing property with an (α,β)-order ($I_{(\alpha,\beta)}SCP$) if for $x_1 \leq_X x_2$ and $t_1 \leq_T t_2$,*

$$\begin{cases} f(x_1,t_1) \preceq_{(\alpha,\beta)} f(x_2,t_1) \Rightarrow f(x_1,t_2) \preceq_{(\alpha,\beta)} f(x_2,t_2), \\ f(x_1,t_1) \prec_{(\alpha,\beta)} f(x_2,t_1) \Rightarrow f(x_1,t_2) \prec_{(\alpha,\beta)} f(x_2,t_2). \end{cases} \tag{3.4}$$

Similarly, we characterize a function satisfying $I_{(\alpha,\beta)}$SCP by $k_\alpha \circ f$ and $k_\beta \circ f$ in the following theorem, which is an answer to the trouble in $I_{(\alpha,\beta)}$IDP. Its proof is obvious and omitted.

Theorem 3.5. *Let $f : X \times T \to I(\Re)$, $\alpha, \beta \in [0,1]$ and $\alpha \neq \beta$. If $k_\alpha \circ f$ and $k_\beta \circ f$ satisfy SCP, f satisfies $I_{(\alpha,\beta)}SCP$.*

In a real-valued case, an SMF is a QSMF and a function satisfying IDP has SCP. Here, we can prove that an $I_{(\alpha,\beta)}$SMF is still an $I_{(\alpha,\beta)}$QSMF, but a function satisfying $I_{(\alpha,\beta)}$IDP may not satisfy $I_{(\alpha,\beta)}$SCP.

Example 3.5. *Let $X = \{x, y, x \wedge y, x \vee y\}$, in which x and y are incomparable, and $T = \{1, 2\}$ with a natural order. $f : X \times T \to I(\Re)$, whose corresponding interval-values are listed in Table 3.2. Then, f satisfies $I_{(\alpha,\beta)}IDP$ if $\alpha = \frac{3}{4}$. However, when $f(x \wedge y, 1) \preceq_{(\alpha,\beta)} f(x, 1)$, $f(x \wedge y, 2) \succ_{(\alpha,\beta)} f(x, 2)$, that is, the function does not satisfy $I_{(\alpha,\beta)}SCP$.*

Table 3.2. Corresponding interval-values of f in Example 3.5.

T \ X	x	y	$x \wedge y$	$x \vee y$
1	[0,1]	[0,1]	[0,1]	[0,3]
2	[0,1]	[-1,3]	[-1,3]	[0,5]

Examples 3.3 and 3.5 indicate that the generalization of IDP with a general (α,β)-order is not successful. However, when we use a specific (α,β)- order, the lexicographical order 1, the corresponding results can be modified.

We change $\preceq_{(\alpha,\beta)}$ to $\preceq_{Lex1}$ in corresponding definitions above to obtain I_{Lex1}SMF, I_{Lex1}IDP, I_{Lex1}QSMF, and I_{Lex1}SCP. In this case, $k_\alpha \circ f = \underline{f}$ and $k_\beta \circ f = \overline{f}$.

Theorem 3.6. *Let $f : X \times T \to I(\Re)$. If $\underline{f}$ and $\overline{f}$ satisfy IDP, then f satisfies $I_{Lex1}IDP$. Furthermore, if f satisfies $I_{Lex1}IDP$, then it satisfies $I_{Lex1}SCP$.*

Proof. The first conclusion is the specific result of Theorem 3.1 if $\alpha = 0$ and $\beta = 1$. We pay more attention to the second one, which is not true for a general (α,β)-order. Suppose that f has I_{Lex1}IDP, $x_1 \leq_X x_2$, $t_1 \leq_T t_2$, and $f(x_1,t_1) \preceq_{Lex1} f(x_2,t_1)$. Then, $0 \preceq_{Lex1} f(x_2,t_1) \ominus f(x_1,t_1)$; furthermore, $0 \preceq_{Lex1} f(x_2,t_2) \ominus f(x_1,t_2)$ by I_{Lex1}IDP. So, $f(x_1,t_2) \preceq_{Lex1} f(x_2,t_2)$. The strict inequality can be proved similarly. In conclusion, f satisfies I_{Lex1}SCP. □

When substituting the lexicographical order 2 for the lexicographical order 1, we can get the similar results. Hence, using the lexicographical orders in theories and applications seems to be a good choice, and indeed they are, but information on uncertainty carried at the two endpoints of an interval may be more stuffless than on some point in the interval.

Another (α,β)-order, i.e., XY-order, is a well-known alternative. So, we supply similar concepts with XY-order, i.e., I_{XY}SMF, I_{XY}IDP, I_{XY}QSMF, and I_{XY}SCP. Note that $k_\alpha \circ f = \frac{\underline{f}+\overline{f}}{2}$ and $k_\beta \circ f = \overline{f}$ in XY-order. For convenience, we denote $f_c = \frac{\underline{f}+\overline{f}}{2}$.

It is a pity that there are not similar results as in Theorem 3.6. We define the corresponding interval-values of f in Table 3.3, where f has the same domain as Example 3.5. We can verify f_c and $\overline{f}$ satisfy IDP. However, $f(x,1) \ominus f(x \wedge y, 1) >_{XY} f(x,2) \ominus f(x \wedge y, 2)$, that is, f does not satisfy I_{XY}IDP.

Table 3.3. Corresponding interval-values of redefined f.

T \ X	x	y	$x \wedge y$	$x \vee y$
1	[2,3]	[0,3]	[0,3]	[2,3]
2	[0.5,1.5]	[-1,1]	[-1,1]	[0.5,1.5]

4. Interval-valued MCS with an (α,β)-order

Li and Luo tried to generalize the sufficient and necessary condition for MCS in real-valued cases to interval-valued cases [5]. However, the generalization is partially successful, in which the necessity fails. In this section, we fill the gap by using $I_{(\alpha,\beta)}$QSMF and $I_{(\alpha,\beta)}$SCP.

Here, we need a set order to compare the sets of maximizers and constraints.

Definition 4.1. *Let X be a lattice with the given order $\leq_X$. For S_1, S_2 of power set $P(X)$, $S_1 \leq_S S_2$ if for each $x \in S_1$ and $y \in S_2$, $x \wedge y \in S_1$ and $x \vee y \in S_2$.*

The order is called a strong set order, introduced by Veinott [58]. Note that $S \leq_S S$ if, and only if, S is a sub-lattice of X. A set-valued function $A : T \to P(X)$ is monotone nondecreasing if $A(t_1) \leq_S A(t_2)$ whenever $t_1 \leq_T t_2$, where T is a poset.

Theorem 4.1. *Let $f : X \times T \to I(\Re)$, $S \subseteq X$ and $\alpha, \beta \in [0,1], \alpha \neq \beta$. Then,* $\arg\max_{x\in S} f(x,t)$ *is monotone nondecreasing in (t,S) if, and only if, f is an $I_{(\alpha,\beta)}$QSMF in x and satisfies $I_{(\alpha,\beta)}$SCP in (x,t).*

Proof. Denote $A(t,S) = \arg\max_{x\in S} f(x,t)$ for convenience. We prove the sufficiency first. Suppose $S_1 \leq_S S_2$ and $t_1 \leq_T t_2$, then we will show $A(t_1,S_1) \leq_S A(t_2,S_2)$, that is, $\forall x \in A(t_1,S_1)$ and $\forall y \in A(t_2,S_2)$, $x \wedge y \in A(t_1,S_1)$, and $x \vee y \in A(t_2,S_2)$. Since $x \in A(t_1,S_1)$ and $S_1 \leq_S S_2$, then $f(x \wedge y, t_1) \preceq_{(\alpha,\beta)} f(x,t_1)$. Clearly, $f(y,t_1) \preceq_{(\alpha,\beta)} f(x \vee y, t_1)$ because f is an $I_{(\alpha,\beta)}$QSMF in x. Furthermore, $f(y,t_2) \preceq_{(\alpha,\beta)} f(x \vee y, t_2)$ because f satisfies $I_{(\alpha,\beta)}$SCP in (x,t). Hence, $x \vee y \in A(t_2,S_2)$. Similarly, we have $x \wedge y \in A(t_1,S_1)$.

Next, we solve the necessity of $I_{(\alpha,\beta)}$QSMF. For any x, y in X, let $S_1 = \{x, x \wedge y\}$ and $S_2 = \{y, x \vee y\}$, then $S_1 \leq_S S_2$. If $f(x \wedge y, t) \preceq_{(\alpha,\beta)} f(x,t)$, then $x \in A(t,S_1)$. Considering $A(t,S_1) \leq_S A(t,S_2)$, $x \vee y \in A(t,S_2)$, that is, $f(y,t) \preceq_{(\alpha,\beta)} f(x \vee y, t)$. Furthermore, if $f(x \wedge y, t) \prec_{(\alpha,\beta)} f(x,t)$, then $A(t,S_1) = \{x\}$. Clearly, $A(t,S_2) = \{x \vee y\}$, that is, $f(y,t) \prec_{(\alpha,\beta)} f(x \vee y, t)$.

Finally, we prove that $I_{(\alpha,\beta)}$SCP is also necessary. Let $x \leq_X y, t_1 \leq_T t_2$ and $S = \{x,y\}$. $f(x,t_1) \preceq_{(\alpha,\beta)} f(y,t_1)$ implies $y \in A(t_1,S)$. Since $A(t_1,S) \leq_S A(t_2,S)$, then $y \in A(t_2,S)$, that is, $f(x,t_2) \preceq_{(\alpha,\beta)} f(y,t_2)$. The implication in a strict situation holds in a similar way. □

We have two corollaries in the following. One is a simplified result above and the other is about the structure of a maximizer, which are proved by Theorem 4.1 easily and omitted.

Corollary 4.1. *Let $f : X \to I(\Re)$, $S \subseteq X$, and $\alpha, \beta \in [0,1], \alpha \neq \beta$. Then,* $\arg\max_{x\in S} f(x)$ *is monotone nondecreasing in S if, and only if, f is an $I_{(\alpha,\beta)}$QSMF.*

Corollary 4.2. *Let $f : X \to I(\Re)$, S be a sub-lattice of X ,$\alpha, \beta \in [0,1]$ and $\alpha \neq \beta$. If f is an $I_{(\alpha,\beta)}$QSMF,* $\arg\max_{x\in S} f(x)$ *is a sub-lattice of S.*

Sidering the results in Theorems 3.4 and 3.5, we have a new conclusion as follows. It is sufficient and easy to verify for a monotone nondecreasing maximizer.

Theorem 4.2. *Let $f : X \times T \to I(\Re)$, $S \subseteq X$, and $\alpha, \beta \in [0,1], \alpha \neq \beta$. If $k_\alpha \circ f$ and $k_\beta \circ f$ are QSMF and satisfy SCP, then* $\arg\max_{x\in S} f(x,t)$ *is monotone nondecreasing in (t, S).*

The conclusions above are true, of course, when we use the lexicographical orders or XY-order in the related statements. In the following example, we show that when you choose an order in uncertain situations, it shows your preferences on uncertainty, which may bring different results.

Example 4.1. *Let $X = \{x, y, x \wedge y, x \vee y\}$, in which x and y are incomparable, and $T = \{1,2\}$ with a natural order. $f : X \times T \to I(\Re)$, whose corresponding interval-values is given in Table 4.1.*

Table 4.1. Corresponding interval-values of f in Example 4.1.

T \ X	x	y	$x \wedge y$	$x \vee y$
1	[-1,2]	[-1,2]	[-1.5,3]	[0,1]
2	[0,3]	[0,3]	[-0.5,4]	[1,2]

We can verify that $\underline{f}$ and $\overline{f}$ are SMF, then f_c is SMF based on Lemma 3.2. Considering an SMF is a QSMF, $\underline{f}$, $\overline{f}$, and f_c are all QSMF. Similarly, $\underline{f}$ and $\overline{f}$ satisfy IDP, then f_c satisfies IDP, that is, they have SCP.

Based on the theorems in [5] and Theorem 4.2, $\arg\max_{x\in X} f(x,t)$ *is monotone nondecreasing in t when we use LU-order, the lexicographical orders, and XY-order, respectively.*

Actually, $\arg\max_{x\in X} f(x,1) = \{x, y, x \wedge y, x \vee y\}$ *and* $\arg\max_{x\in X} f(x,2) = \{x, y, x \wedge y, x \vee y\}$ *by LU-order, which is meaningless for making decisions. Such a result is not unexpected. Due to partial incomparability of LU-order, using it for analysis may not provide effective assistance to decision-makers. In other words, the interval-valued MCS in [5] is restrictive.*

However, when we use the lexicographical order 1, $\arg\max_{x\in X} f(x,1) = \{x \vee y\}$ *and* $\arg\max_{x\in X} f(x,2) = \{x \vee y\}$. $\arg\max_{x\in X} f(x,1) = \{x \wedge y\}$ *and* $\arg\max_{x\in X} f(x,2) = \{x, y\}$ *by the lexicographical order 2.* $\arg\max_{x\in X} f(x,1) = \{x \wedge y\}$ *and* $\arg\max_{x\in X} f(x,2) = \{x \wedge y\}$ *by XY-order. Though the results are different, they are clear and valuable to decision-makers.*

Furthermore, we take an example in economics to show how to use MCS to adjust economic behavior.

Example 4.2. *Consider the effects of a short run increase in the market size on the monopoly price. Let the number of customers be N. The objective of the firm is to maximize the profit function $f(q, N) = Nq(P(q) + p) - (C(Nq) + cq)$ by choosing the quantity $q \in K$ sold to each customer, where $P(q) + p$ is inverse demand, $C(Nq) + cq$ are costs, and $p, c \geq 0$.*

Note that the two function P and C are interval-valued here given the uncertainty of rewards and costs in most economical situations. Denote $P(q) = [\underline{P}(q), \overline{P}(q)]$ *and* $C(q) = [\underline{C}(q), \overline{C}(q)]$ *for* $q \in K$*. Then,* $f = [\underline{f}(q, N), \overline{f}(q, N)]$ *is interval-valued, which is compared by using the lexicographical order 1.*

Clearly, $\overline{f}$ *is an* I_{Lex1}*QSMF in q since K is a chain. Next, we show f has* I_{Lex1}*SCP in* (q, N)*. f can be rewritten by a family of functions* $U(q, y, N) = Ny - C(Nq)$ *and* $y = h(q, \alpha_1, \alpha_2) = qP(q) + \alpha_1 q + \alpha_2$*, where* $\alpha_1 = p - c$ *and* α_2 *is arbitrary. Then,* $\underline{f}(q, N)$ *and* $\overline{f}(q, N)$ *satisfy SCP in* (q, N) *if* $\underline{C}(q)$ *and* $\overline{C}(q)$ *are both concave by using Theorem 11 in [26].*

Consequently, if $\underline{C}(q)$ *and* $\overline{C}(q)$ *of the cost function C are concave,* $\arg\max_{q\in K} f(q, N)$*, denoted by* $q^*(N)$*, is nondecreasing in N for all choices of K and P, based on Theorem 4.2. Similarly, if* $\underline{C}(q)$ *and* $\overline{C}(q)$ *are convex,* $q^*(N)$ *is nonincreasing in N for all choices of K and P.*

It is worth noting that the above result has nothing to do with the properties of the inverse demand function P. In other words, it is enough for the firm to adjust its costs only if it wants a positive cycle.

5. Interval GSC with an (α,β)-order

A GSC is one of the lattice games, which describes a situation that each player intends to move in the same direction as their opponents. If opponents increase, then the best choice is to increase as well for a given player. SCP and QSMF are used in a GSC to characterize when maximizers are increasing in parameters in an optimization problem, which can be seen as a mathematical foundation for strategic complements [37]. When considering uncertainty in games, interval games arise. Specifically, an interval GSC (denoted by I_{LU}GSC in our paper) is proposed in [5] by using LU-order. In this section, we study a new interval GSC by using an (α,β)-order.

Definition 5.1. *An interval GSC with an* (α,β)*-order (*$I_{(\alpha,\beta)}$*GSC) is a lattice game* $\Gamma = (X_i, u_i)_{i=1}^N$*, where:*

1) Finitely many players are considered and indexed by $i \in \{1, 2, \cdots, N\}$*.*

2) For every player i, X_i*, called the action space, is a nonempty and a sub-complete lattice in* $\Re^{n_i}$ *with the usual order.* $X = X_1 \times X_2 \times \cdots \times X_n$ *is the space of profiles of actions with the product partial order. We denote* $X = X_i \times X_{-i}$ *for convenience, where* $X_{-i} = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_N$*. An element in X is* $x = (x_1, x_2, \cdots, x_n)$ *or* $x = (x_i, x_{-i})$*.*

3) For every player i, the payoff function $u_i(x_i, x_{-i})$ *is interval-valued in* $I(\Re)$*, in which (a) for* $\alpha, \beta \in [0, 1], \alpha \neq \beta$*,* $k_\alpha \circ u_i$*, and* $k_\beta \circ u_i$ *are upper semicontinuous on X and continuous on* X_{-i} *and (b)* u_i *has* $I_{(\alpha,\beta)}$*SCP in* (x_i, x_{-i})*, and for each* x_{-i}*,* $u_i(\cdot, x_{-i})$ *is an* $I_{(\alpha,\beta)}$*QSMF in* x_i*.*

In an $I_{(\alpha,\beta)}$GSC, $B_i(x_{-i}) = \arg\max_{x_i\in X_i} u_i(x_i, x_{-i})$ is the best response set of player i to opponent actions x_{-i} and $B(x) = B_1(x_{-1}) \times B_2(x_{-2}) \times \cdots \times B_N(x_{-N})$ is the joint best response correspondence. If $x* \in B(x*)$, $x*$ is a Nash equilibrium.

The following is an example of $I_{(\alpha,\beta)}$GSC, which is transformed from a classical coordination game.

Example 5.1. *Suppose that two players are considering working together by adopting a technology. The technology to be selected is either technology a or technology b. One player prefers technology a to technology b, the other prefers b to a. It is meaningless for collaboration if they choose different technologies. In this case, each player considers not only his own choice but also choice of another player. The best result is to move in the direction of choosing the same technology as another player. Based on uncertainty, we use an interval to represent the payoff to a player from the collective choices of both players.*

The related data is listed in Table 5.1.

Table 5.1. Interval-valued coordination game.

P_1 \ P_2	a	b
a	[1,3] [0,2]	[0,0] [0,1]
b	[0,1] [0,0]	[1,2] [2,3]

The first interval in each cell is payoff for player 1 and the second interval is the payoff for player 2. For example, when player 1 chooses a and player 2 chooses a, the first interval, [1,3], in the corresponding cell is player 1's payoff from this collective choice (a, b), that is, $u_1(a,a) = [1,3]$. *The second interval, [0,2], is player 2's payoff from this collective choice, denoted* $u_2(a,a) = [0,2]$. *Here,* $X = X_1 \times X_2$, $X_1 = X_2 = \{a,b\}$, *and suppose* $a \preceq b$.

It is clear that for $\alpha, \beta \in [0,1], \alpha \neq \beta$, $k_\alpha \circ u_i$ *and* $k_\beta \circ u_i$ *are upper semicontinuous on X and continuous on* $X_{-i} (i = 1,2)$ *because* $u_i (i = 1,2)$ *is discrete. Furthermore, we can verify that* u_1 *and* u_2 *have* $I_{(\alpha,\beta)}$*SCP in* (x_1, x_2) *and* (x_2, x_1)*, respectively. For* x_2, $u_1(\cdot, x_2)$ *is an* $I_{(\alpha,\beta)}$*QSMF in* x_1 *owing to the linearity of* X_1 *, and a similar result is for* x_1*. According to Definition 5.1, this is an* $I_{(\alpha,\beta)}$*GSC.*

In the $I_{(\alpha,\beta)}$*GSC,* $B_1(a) = \{a\}$, $B_1(b) = \{b\}$, $B_2(a) = \{a\}$, *and* $B_2(b) = \{b\}$. *Furthermore,* $B(a,a) = \{(a,a)\}$, $B(a,b) = \{(b,a)\}$, $B(b,b) = \{(b,b)\}$, *and* $B(b,a) = \{(a,b)\}$. *So the Nash equilibrium set is* $\{(a,a),(b,b)\}$. *Indeed, the best result is to move in the direction of choosing the same technology as the other player, just like in a classical coordination game.*

Specifically, if we use the lexicographical order 1 in Definition 5.1, we get an I_{Lex1}GSC. In the following, we take Bertrand oligopoly as an example to show that it is just an I_{Lex1}GSC and its Nash equilibrium is unique.

Bertrand oligopoly is a model of price competition in economics. It emphasizes that oligopolies make decisions at the same time as other firms without any historical decisions. It is worth it to note that the model is based on the following assumptions. First, oligopoly firms compete through price selection. Second, the products produced by each oligopoly are homogeneous, such as branded handbags, bottles of perfume, running shoes, and so on. Finally, there is no formal or informal collusion between oligopoly firms. It can be realized that demand for the product of one firm is inverse to the price it charges but positive to the price other firms charge for their own product.

Example 5.2. *(Interval-valued Bertrand oligopoly) We can formalize Bertrand oligopoly as a game as follows. N firms are considered and indexed* $i \in \{1, 2, \cdots, N\}$. *The unit price of firm i is* $x_i \in X_i = [0, a] \subset \Re$, *in which a is a possible maximum unit price. Let* q_i *be quantity of demand for the product of firm i and* $q_i = f(x_i) + g(x_{-i})$. f *is an interval-valued function from* X_i *to* $I(\Re)$, *reflecting uncertain demand conditions of firm i. We denote* $f(x_i) = [\underline{f}(x_i), \overline{f}(x_i)]$, *where* $0 \leq \underline{f}(x_i) \leq \overline{f}(x_i)$ *and they are decreasing. Similarly,* $g(x_{-i}) = [\underline{g}(x_{-i}), \overline{g}(x_{-i})]$ *reflects demand conditions of opponents, where* $0 \leq \underline{g}(x_{-i}) \leq \overline{g}(x_{-i})$, *and they are increasing. Let c be marginal cost of production, considered a constant with* $0 < c < a$. *The profit to firm i is total revenue minus total cost for a profile of other firms prices* x_{-i}, *that is,* $u_i(x_i, x_{-i}) = q_i x_i - c q_i = (x_i - c)(f(x_i) + g(x_{-i}))$.

Specifically, we suppose $\underline{f}(x_i) = \underline{r} - \underline{b} x_i$ *and* $\overline{f}(x_i) = \overline{r} - \overline{b} x_i$. *The constants* $\underline{r}, \overline{r}, \underline{b}, \overline{b} > 0$, *and* $2\underline{b} > (N-1)\underline{\gamma}$, *which carry uncertain demand information in the industry. Then,* $\underline{f}$ *and* $\overline{f}$ *are strictly* *decreasing and twice differentiable. Let* $\underline{g}(x_{-i}) = \underline{\gamma}(\sum_{k\neq i} x_k)$ *and* $\overline{g}(x_{-i}) = \overline{\gamma}(\sum_{k\neq i} x_k)$*, in which* $\underline{\gamma}, \overline{\gamma} > 0$ *and carry uncertain information of opponents. Here,* $\underline{g}$ *and* $\overline{g}$ *are strictly increasing in each* $x_k (k \neq i)$ *and continuously differentiable. Therefore,* $\underline{u_i}(x_i, x_{-i}) = (x_i - c)(\underline{r} - \underline{b}x_i + \underline{\gamma}(\sum_{k\neq i} x_k))$, $\overline{u_i}(x_i, x_{-i}) = (x_i - c)(\overline{r} - \overline{b}x_i + \overline{\gamma}(\sum_{k\neq i} x_k))$.

Because $\frac{\partial^2 \underline{u_i}}{\partial x_i \partial x_k} = \underline{\gamma} > 0$ *and* $\frac{\partial^2 \overline{u_i}}{\partial x_i \partial x_k} = \overline{\gamma} > 0$ *for* $i \neq k$, $\underline{u_i}$ *and* $\overline{u_i}$ *have IDP in* (x_i, x_{-i})*, even SCP [26], and they are QSMF in* x_i *since* X_i *is a chain. According to the conclusions in the sections above, we get an* $I_{Lex1}GSC$ *and the best response set of firm i to opponents* x_{-i} *is nondecreasing in theory. Actually,* $\frac{\partial \underline{u_i}}{\partial x_i} = \underline{r} - 2\underline{b}x_i + \underline{\gamma}(\sum_{k\neq i} x_k) + c\underline{b}$*, and* $\frac{\partial^2 \underline{u_i}}{\partial x_i^2} = -2\underline{b} < 0$ *in the game. So the best response set of firm i to opponents* x_{-i} *is* $B_i(x_{-i}) = \{\frac{1}{2\underline{b}}(\underline{r} + c\underline{b} + \underline{\gamma}(\sum_{k\neq i} x_k))\}$*. Clearly, the set is a singleton for each* x_{-i} *owing to the strictly concave property of* $\underline{u_i}$ *in* x_i*.*

We can more intuitively perceive the relationship between the best response set of firm i and opponents x_{-i} *in Figure 5.1. Here, we let* $N = 5$, $c = 1$, $\underline{r} = 2$, $\underline{b} = 3$*, and* $\underline{\gamma} = 1$*, then* $B_i(x_{-i}) = \{\frac{6}{5+\sum_{k\neq i} x_k}\}$*. Note that the graph in Figure 5.1 is not the set graph of* $B_i(x_{-i})$*, which only shows the changes of* $B_i(x_{-i})$ *as* $\sum_{k\neq i} x_k$ *undergoes changes.*

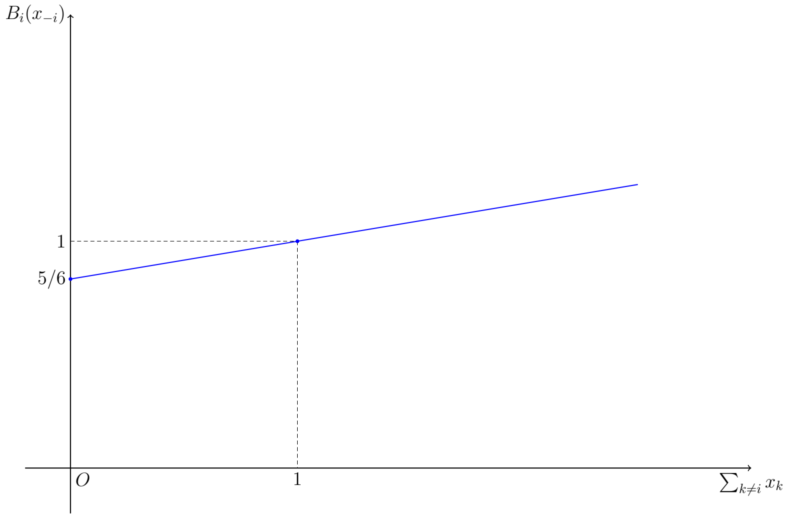

Figure 5.1. The relationship between the best response set of firm i and opponents x_{-i}.

In fact, when opponents increase their prices, the best action of a given player is to increase as well. Note that there is a unique Nash equilibrium $x_i* = \frac{\underline{r}+c\underline{b}}{2\underline{b}-(N-1)\underline{\gamma}}$ $(i = 1, \cdots, N)$ *in the game.*

It is worth it to point out that using the lexicographical order 1 in this game is superior to using LU-order. We need to compare $\underline{u_i}$ only based on the strictly concave $\underline{u_i}$, rather than to compare both $\underline{u_i}$ and $\overline{u_i}$. Another point is that the maximum points in $\underline{u_i}$ and $\overline{u_i}$ are not necessarily the same, that is, finding the maximum interval by LU-order is more difficult.

We may view the cost c as a parameter, which reflects changes in manufacturing technology. Then, we get a parameterized game. Generally speaking, there are strategic complements at a parameter $t \in T$ for each player in such a game, in which each player tends to take a higher action in t [5,35].

Definition 5.2. *A parameterized* $I_{(\alpha,\beta)}GSC$ $\Gamma = (X_i, u_i)_{i=1}^N, T)$ *is an* $I_{(\alpha,\beta)}GSC$ *first for each* $t \in T$*, in* *which T is a poset, called the parameter space. An additional condition is $u_i(x_i, x_{-i}, t)$, the payoff function of player i, has $I_{(\alpha,\beta)}$SCP in (x_i, t) for x_{-i}.*

Similarly, we know what a parameterized I_{Lex1}GSC is. Let the cost be $c \in T = [0, a)$ in the Example 5.2, then we get a parameterized interval-valued Bertrand oligopoly model.

Example 5.3. *(Parameterized interval-valued Bertrand oligopoly) The additional condition for the parameter $c \in T$ holds since $\frac{\partial^2 \underline{u_i}}{\partial x_i \partial c} = -\underline{f'}(x_i) > 0$ and $\frac{\partial^2 \overline{u_i}}{\partial x_i \partial c} = -\overline{f'}(x_i) > 0$. We obtain a parameterized I_{Lex1}GSC. Actually, the best response set of firm i to opponents x_{-i} and the parameter c, $B_i(x_{-i}, c) = \{\frac{1}{2\underline{b}}(\underline{r} + c\underline{b} + \underline{\gamma}(\sum_{j\neq i} x_j))\}$, is nondecreasing. When the costs increase, the best action is to increase its price for the given firm i supposing the prices of opponents were fixed.*

In Examples 5.2 and 5.3, using the lexicographical order 1 to resolve the model is convenient, and the case using the lexicographical order 2 is completely similar, but their results only rely on the endpoints of the related interval-valued functions. Considering the information on uncertainty should be more fully used, we substitute XY-order for the lexicographical order 1 in Example 5.2.

Example 5.4. *Denote $k_{\frac{1}{2}} \circ u_i(x) = u_{ic}(x) = \frac{1}{2}(\underline{u_i}(x) + \overline{u_i}(x))$. We continue to use the specific case in Example 5.1. We can verify u_{ic} has IDP in (x_i, x_{-i}) since $\frac{\partial^2 u_{ic}}{\partial x_i \partial x_j} = \frac{1}{2}(\underline{\gamma} + \overline{\gamma}) > 0$, and $\overline{u_i}$ has IDP in (x_i, x_{-i}), which is shown in Example 5.1. So, u_i has I_{XY}SCP in (x_i, x_{-i}). u_{ic} and $\overline{u_i}$ are QSMF in x_i clearly. In conclusion, we get an I_{XY}GSC, in which the best response set of firm i to opponents x_{-i} is $B_i(x_{-i}) = \{\frac{1}{2(\underline{b}+\overline{b})}[(\underline{r} + \overline{r}) + c(\underline{b} + \overline{b}) + (\underline{\gamma} + \overline{\gamma})(\sum_{j\neq i} x_j)]\}$. The Nash equilibrium is also unique, that is, $x_i* = \frac{(\underline{r}+\overline{r})+c(\underline{b}+\overline{b})}{2(\underline{b}+\overline{b})-(N-1)(\underline{\gamma}+\overline{\gamma}}(i = 1, \cdots, N)$.*

The result is different from that in Example 5.2, which carries more information on uncertainty clearly. The parameterized case using XY-order is similar and omitted.

6. Conclusions

In view of the importance of admissible orders in theory and application, we generalize four classical concepts of a real-valued function into an interval-valued function by using an (α, β)-order, which is an admissible order. They are $I_{(\alpha,\beta)}$SMF, $I_{(\alpha,\beta)}$IDP, $I_{(\alpha,\beta)}$QSMF, and $I_{(\alpha,\beta)}$SCP. The essential properties to characterize them are proposed by using k_α mappings. These properties in interval-valued situations are not perfect as in real-valued situations, so that we give abundant examples to illustrate all kinds of special cases. Fortunately, when we use the specific (α, β)-orders, i.e., the lexicographical orders, we obtain some modified properties.

The important thing is that $I_{(\alpha,\beta)}$QSMF and $I_{(\alpha,\beta)}$SCP are used as the necessary and sufficient condition for MCS, which is superior to the existing results in interval cases. We take one simple example first to show different results by using the lexicographical orders, XY-order, and LU-order. When you choose an order in uncertain situations, it shows your preferences on uncertainty, which may bring different results. Then, we use the lexicographical order 1 to resolve another example in economics to show how to use MCS to adjust economic behavior.

Considering a natural connection between GSC and MCS, we propose the definition of $I_{(\alpha,\beta)}$GSC and parameterized $I_{(\alpha,\beta)}$GSC. In this part, we transform a classical coordination game into an interval-valued game first. Then, we illustrate that an interval-valued Bertrand oligopoly model is an I_{Lex1}GSC by using the lexicographical order 1. In the interval-valued Bertrand oligopoly, the best response correspondence for player i is nondecreasing and the Nash equilibrium is unique. Furthermore, we discuss a parameterized case in the same game. When we use the lexicographical order 2, the results are similar, but information about uncertainty in a game is carried less by the lexicographical orders than by XY-order. Therefore, we use XY-order in the interval-valued Bertrand oligopoly again to obtain a unique Nash equilibrium, which represents more uncertainty.

In Section 3, sufficient theories and examples illustrate the difficulty in generalizing interval-valued IDP. Although interval IDP can be reduced to interval SCP to be used, interval IDP itself is easier to be verified as an important property to characterize MCS. Hence, it is necessary to continue to explore interval IDP. In fact, the main difficulty in generalization is the limitation of interval subtraction. In interval problems, especially those using interval calculus, interval subtraction cannot be avoided. Therefore, we will systematically research interval subtraction in the future to achieve better results.

Recently, there has been a considerable amount of discussion on game theory and its integration with economics, such as the strategic attitudes of participants in games [59], the propagation of monetary shocks in a general equilibrium economy [60], open data in the digital economy [61], the psychology of individuals in the evolutionary dynamics of populations [62], etc. Given that the specific objects can all be considered from the perspective of uncertainty, we can subsequently introduce intervals to these problems and try our best to make contributions to game theory and economic issues from different aspects.

Author contributions

Xiaojue Ma: Conceptualiztion, formal analysis, methodology and writing-original draft; Chang Zhou: Formal analysis and writing-review and editing; Lifeng Li: Funding acquistion and investigation; Jianke Zhang: Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

We declare we have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

We declare that we have no conflict of interest.

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