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Research article

A characterization of the reachable profiles of entropy solutions for the elementary wave interaction problem of convex scalar conservation laws

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Abstract: In this paper, we analyze and characterize the set $\mathcal{A}_T$ which consists of all possible profiles at a fixed time of the entropy solution of the elementary wave interaction problem in a bounded domain for a convex scalar conservation law. The elementary wave interaction problem is the initial and boundary value problem for a scalar conservation law, where the flux is a strictly convex function, and the initial and boundary data are constant functions. In the first main result of the article, we state and prove that $\mathcal{A}_T$ is a subset of the set of piecewise functions that are constant on each subdomain, or there is a subdomain where the function is strictly increasing. We prove the result by applying the method of characteristics in three steps: the Riemann problem solution, the entropy solution of the interaction of two Riemann problems, and restriction of the entropy solution to the spatial bounded domain. Moreover, we characterize the strictly increasing part of the solution's profile regarding the flux function. In the second result, which is stated as an application of the first result, we introduce the conditions for ill-posedness and local flux identification from the knowledge of the entropy solution's profile.

Keywords: elementary wave interaction; Riemann problem; entropy solutions; reachable profiles
Mathematics Subject Classification: 35L02, 35L04, 35L65

1. Introduction

The study of conservation laws was initially oriented toward analyzing problems related to gas dynamics, but afterward, it was considered in the study of phenomena that occurr in several areas, such as electrophoresis, chromatography, sedimentation, combustion, elasticity, secondary oil recovery [1–5], and others [6]. There are new applications, including the study of biology and even social models [7–9]. There has been excellent progress in understanding the functional framework for studying scalar conservation laws. A particular challenge in the context of conservation laws is the solution of inverse problems and optimal control problems. In a broad sense, the functional spaces have a low regularity which does not permit a straightforward application of the standard methods developed to analyze the optimal control theory of elliptic or parabolic equations [10–12]. In particular, an interesting aspect in control theory is the characterization of the attainable profiles of the solution [13–17]. The knowledge of the set of possible profiles is an interesting aspect to develop in the functional framework when studyng the inverse problem of flux identification and the convergence of numerical methods for parameter identification in chromatography [18] and sedimentation [19–23]. Hence, in the present article, we apply the method of characteristics to construct the entropy solution of the elementary wave interaction problem and introduce a characterization of the reachable profiles of the solution.

The elementary wave interaction problem is defined as the following initial boundary value problem:

$$u_t + f(u)_x = 0, \qquad (x,t) \in Q := (0,1) \times \mathbb{R}_+, \tag{1.1}$$
$$u(x,0) = u_L, \qquad x \in (0,1), \tag{1.2}$$
$$u(0,t) = u_0, \qquad t \in \mathbb{R}_+, \tag{1.3}$$
$$u(1,t) = u_R, \qquad t \in \mathbb{R}_+, \tag{1.4}$$

when $(u_L, u_0, u_R) \in \mathbb{R}^3$ and the flux function f is a strictly convex function

$$f \in U_{ad} := \{f \in C^2(\mathbb{R}) \,:\, f''(u) > 0,\ u \in \mathbb{R},\ \text{and}\ f(0) = f'(0) = 0\}. \tag{1.5}$$

We note that the assumption $f(0) = f'(0) = 0$, as is given in (1.5), is not a restrictive assumption, since we can apply a change of variable to rewrite any general convex flux function in the form of a flux function belonging to U_{ad} [24].

We observe that (1.1)–(1.4) is a particular case of the Cauchy–Dirichlet problem for scalar conservation laws on a spatially bounded domain given by

$$u_t + f(u)_x = 0, \qquad (x,t) \in Q_T := (0,1) \times (0,T), \tag{1.6}$$
$$u(x,0) = \phi(x), \qquad x \in (0,1), \tag{1.7}$$
$$u(0,t) = g(t), \qquad t \in (0,T), \tag{1.8}$$
$$u(1,t) = h(t), \qquad t \in (0,T). \tag{1.9}$$

The well-posedness of (1.6)–(1.9) was investigated by Bardos, Le Roux, and Nedelec in [25]. By applying Kružkov's theory [26], the authors obtained results concerning the existence, uniqueness, and stability of the entropy solutions. We recall that an entropy solution of the initial-boundary value problem (1.6)–(1.9) is a function $u \in L^1_{loc}(Q_T)$ that satisfies the inequality

$$\iint_{Q_T} \Big\{|u-k|\varphi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\varphi_x\Big\} dxdt + \int_0^1 |\phi - k|\varphi(x,0)dx$$

$$+\int_0^T \operatorname{sgn}(g(t)-k)(f(u(0,t))-f(k))\varphi_x(0,t)dt \tag{1.10}$$
$$-\int_0^T \operatorname{sgn}(h(t)-k)(f(u(1,t))-f(k))\varphi_x(1,t)dt \geq 0,$$

which holds for all $\varphi \in C_0^\infty(Q_T)$ with $\varphi \geq 0$ and for each $k \in \mathbb{R}$. Here, $\operatorname{sgn}(\cdot)$ denotes the sign function. Additionally, it is well known that the boundary conditions (1.8)–(1.9) require a definition, which is given by introducing a boundary inequality that consistently selects the boundary conditions (see Item (3) of Proposition 4.1 given below).

The main result of the paper is given by the following theorem.

Theorem 1.1. *Let us consider the notations* $\operatorname{Rang}(f)$ *for the range of a function* $f : \mathbb{R} \to \mathbb{R}$, $u_m = \min\{u_L, u_0, u_R\}$, $u_M = \min\{u_L, u_0, u_R\}$, *and* $\mathcal{A}_T$ *the set of reachable profiles of the entropy solutions of the elementary wave interaction problem* (1.1)–(1.4) *at a fixed time* $t = T \in (0, \infty)$ *defined by*

$$\mathcal{A}_T = \Big\{ u(x,T) \ : \ u \text{ is the entropy solution of the problem (1.1)–(1.4)} \Big\}.$$

Given $T > 0$, $u(x,T) \in \mathcal{A}_T$ has the form

$$u(x,T) = \begin{cases} \rho_1, & 0 \leq x < x_1, \\ \rho_2, & x_1 \leq x < x_2, \\ \psi(x), & x_2 \leq x < x_3, \\ \rho_3, & x_3 \leq x < x_4, \\ \rho_4, & x_4 \leq x \leq 1, \end{cases} \tag{1.11}$$

with $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3, \rho_4) \in [u_m, u_M]^4$, $\mathbf{x} = (x_1, x_2, x_3, x_4) \in [0,1]^4$, and ψ is a constant or a strictly increasing function with $\operatorname{Rang}(\psi) \subset [u_m, u_M]$. Moreover, in the strictly increasing case, the correspondence rule for ψ is defined by $\psi(x) = \big(f'\big)^{-1}\big((x-1)/T\big)$ when $u_0 < 0$ or $\psi(x) = \big(f'\big)^{-1}\big(x/T\big)$ when $u_0 > 0$.

The proof of Theorem 1.1 is given by introducing a systematic construction of the entropy solution of (1.1)–(1.4) and characterizing the possible profiles. Let us consider the Riemann problem

$$u_t + f(u)_x = 0, \qquad (x,t) \in \mathbb{R} \times (b, \infty), \tag{1.12}$$

$$u(x,b) = \begin{cases} \alpha, & x \leq a, \\ \beta, & x > a, \end{cases} \qquad x \in \mathbb{R}, \tag{1.13}$$

when $f \in U_{ad}$, $a \in \mathbb{R}$ and $b \in \mathbb{R}_+$ are given. The analysis begins by reformulating the solution of the Riemann problem (1.13) with a new notation. Specifically, we introduce a partition of $\mathbb{R}^2$ (see (2.11)) such that the possible constant solutions and shock or rarefaction waves are characterized by α and β (see Proposition 2.1). We then study all possible shock and rarefaction interactions by considering the construction of the entropy solution of the following initial value problem

$$w_t + f(w)_x = 0, \qquad (x,t) \in \mathbb{R} \times (b, \infty), \tag{1.14}$$

$$w(x,b) = \begin{cases} \alpha, & x \leq a, \\ \beta, & a < x < c, \\ \gamma, & x \geq c, \end{cases} \qquad x \in \mathbb{R}. \tag{1.15}$$

The solution of the initial value problem (1.14)–(1.15) is constructed by analyzing the solution of two Riemann problems, one centered at (a, b) separating the states (α, β) and another centered at (c, b) separating the states (β, γ). We introduce a partition of $\mathbb{R}^3$, characterizing the solution of the Cauchy problem (1.14)–(1.15) in terms of constant states (α, β, γ) and we also give a connection with the partition $\mathbb{R}^2$ of the Riemann problem. Afterward, we restrict w to $x \in [0, 1]$ and, using the characterization of the entropy solution of scalar conservation laws on bounded domains, we deduce all possible types of entropy solutions for the initial boundary value problem (1.1)–(1.4). Hence, by analyzing each type of solution, we construct a characterization of the set $\mathcal{A}_T$ containing the elementary wave interaction problem solution profiles at the fixed time $t = T$.

Related results are given in [27–29]. In those articles, the authors use some results of advanced mathematical analysis, such as the method of generalized characteristics and the semigroup theory, to characterize the attainable sets for scalar conservation laws and balance laws.

A particular application of the main result of this work is the analysis and numerical solution of the inverse problem of flux identification from knowledge of an observed model solution's profile in a fixed time. We note that the identification of a nonlinear flux function in scalar conservation laws is a relevant inverse problem in many interesting applications of conservation laws theory, for instance, in sedimentation and chromatography (see [18–20]). In these applications, the reconstruction of the flux function should be done by observing the end-time profile of the forward problem solution. In [18], the authors formulate the inverse problem as an optimal control problem and, using this formulation, they introduce the convergence framework for a numerical method applied to the solution of the parameter identification problem. Advances and improvements on the numerical solution and the analysis of the flux identification problem are given in [19–23] and [30–32], respectively. Nevertheless, despite these results, the mathematical framework for the well-posedness of the inverse problem still needs to be clarified. The major drawback, which prevents the direct application of optimal control theory on the inverse problem's analysis, is the formation of discontinuities in the direct problem solution, even when the initial–boundary and flux functions are smooth [34, 35]. Concerning the well–posdennes analysis, we note that the existence of solutions for the flux identification problem was obtained in [18]. This result is based on the continuous dependence of the entropy solution with respect to the flux function (see [33]). An alternative proof of existence can be obtained by adaptation of the technique introduced by Ulbrich in [14]. However, to the best of our knowledge, results on the uniqueness of solutions for the flux identification problem have yet to be published. Hence, this paper contributes to answering the uniqueness question of flux identification, since we introduce the conditions for ill–posedness and the local flux identification in a particular and representative case of the direct problem: the elementary wave interaction problem.

The paper is organized as follows. In Section 2, we construct the entropy solution of the Riemann problem (1.12)–(1.13) and give a partition of $\mathbb{R}^2$ characterizing the type of entropy solution. In Section 3, we present the systematic construction of the entropy solution of the Cauchy problem (1.14)–(1.15). In Section 4, we present the entropy solution of the elementary wave interaction problem (1.1)–(1.4). In Section 5, we prove Theorem 1.1. In Section 6, we present the applications of the result to the flux identification problem. Finally, in Section 7, we give the conclusions and some future research perspectives.

2. The entropy solution of the Riemann problems (1.12)–(1.13)

We recall the results related to the entropy solution of the Riemann problem (1.12)–(1.13) and introduce an appropriate notation which will be used below in the solution of the elementary wave interaction problem. In a broad sense, the entropy solution of the problem (1.12)–(1.13) can be characterized in terms of the constant states α and β and is given by two possible types of waves called shock waves and rarefactions.

Definition 2.1. *Let $a, \alpha, \beta \in \mathbb{R}, b \in \mathbb{R}_+, f \in U_{ad}$, and*

$$\sigma(\alpha,\beta) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}, \quad \beta \neq \alpha, \tag{2.1}$$

we define the shock and rarefaction waves as follows:

(i) A shock wave originating at (a, b) and between states α and β is denoted by $S^{(\alpha,\beta)}_{(a,b)}$ and is defined as the function of $\mathbb{R} \times [b, \infty)$ to $\mathbb{R}$ given by

$$S^{(\alpha,\beta)}_{(a,b)}(x,t) = \begin{cases} \alpha, & x - a \leq \sigma(\alpha,\beta)(t-b), \\ \beta, & \text{otherwise.} \end{cases}$$

Furthermore, we will use $\overrightarrow{S}^{(\alpha,\beta)}_{(a,b)}$, $\overleftrightarrow{S}^{(\alpha,\beta)}_{(a,b)}$, and $\overleftarrow{S}^{(\alpha,\beta)}_{(a,b)}$ to denote the shock waves such that the propagation speed is positive, negative, or zero, i.e., $\sigma > 0$, $\sigma = 0$, or $\sigma < 0$, respectively.

(ii) A rarefaction wave centered at (a, b) and separating the steady states α and β is denoted by $R^{(\alpha,\beta)}_{(a,b)}$ and defined as the continuous function of $\mathbb{R} \times [b, \infty)$ to $\mathbb{R}$ given by

$$R^{(\alpha,\beta)}_{(a,b)}(x,t) := \begin{cases} \alpha, & x - a < f'(\alpha)(t-b), \\ (f')^{-1}\Big(\dfrac{x-a}{t-b}\Big), & f'(\alpha)(t-b) \leq x - a \leq f'(\beta)(t-b), \\ \beta, & \text{otherwise.} \end{cases}$$

Furthermore, we use $\overrightarrow{R}^{(\alpha,\beta)}_{(a,b)}$, $\overleftrightarrow{R}^{(\alpha,\beta)}_{(a,b)}$, and $\overleftarrow{R}^{(\alpha,\beta)}_{(a,b)}$ to denote the rarefactions such as $0 < f'(\alpha) < f'(\beta)$, $f'(\alpha) \leq 0 \leq f'(\beta)$, and $f'(\alpha) < f'(\beta) < 0$, respectively.

Proposition 2.1. *Consider the notation of Definition 2.1. Moreover, consider the sets*

$$C = \{(x,y) \in \mathbb{R}^2 \ : \ x = y\}, \ \mathcal{S} = \{(x,y) \in \mathbb{R}^2 \ : \ x > y\}, \ \mathcal{R} = \{(x,y) \in \mathbb{R}^2 \ : \ x < y\}, \tag{2.2}$$

defining a partition of $\mathbb{R}^2$. The entropy solution of (1.12)–(1.13) is defined analytically as

$$u(x,t) = \begin{cases} \alpha, & \text{when } (\alpha,\beta) \in C, \text{ i.e., } \alpha = \beta, \\ S^{(\alpha,\beta)}_{(a,b)}(x,t), & \text{when } (\alpha,\beta) \in \mathcal{S}, \text{ i.e., } \alpha > \beta, \\ R^{(\alpha,\beta)}_{(a,b)}(x,t), & \text{when } (\alpha,\beta) \in \mathcal{R}, \text{ i.e., } \alpha < \beta, \end{cases}$$

for $(x, t) \in \mathbb{R} \times (b, \infty)$.

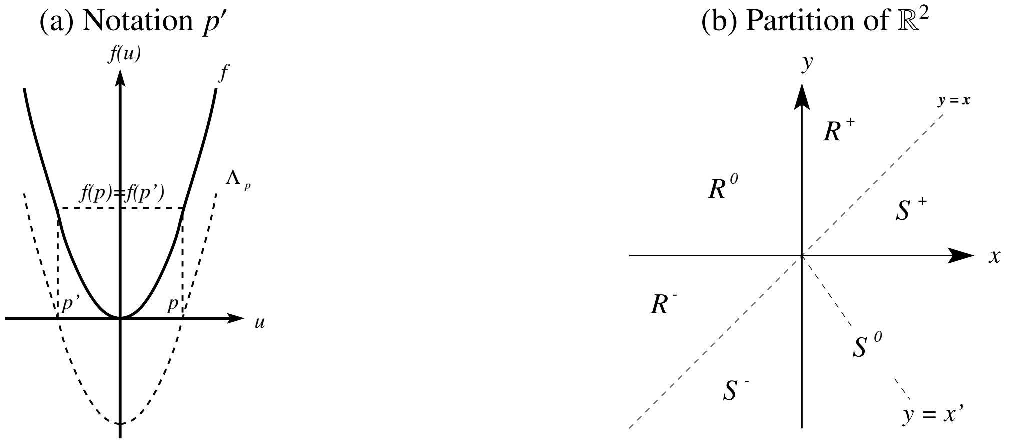

Figure 1. Notation p' to characterize the orientation of a shock curve's orientation and the partition of $\mathbb{R}^2$ to characterize the entropy solution of (1.12)–(1.13) in terms of the initial states α and β, including the orientation of shocks and rarefactions. We note that $y = x'$ is reduced to $y = -x$ when f is an even function.

We introduce a suggested notation to characterize the orientation of different waves in terms of the states defining the initial condition. We begin by considering the partition of the plane $\mathbb{R}^2$ by the sets $\mathcal{C}, \mathcal{S}$, and $\mathcal{R}$ as given in Proposition 2.1. We continue by introducing partitions of $\mathcal{S}$ and $\mathcal{R}$ by considering the orientation of shocks and rarefactions. First, we consider the set $\mathcal{S}$, where there are three possible types of shock waves, depending on the sign of the shock's velocity: $\operatorname{sgn}(\sigma)$. To make this fact more precise, we introduce the function $\Lambda_p(u) = f(u) - f(p)$, where $p \in \mathbb{R}$ is fixed. It is observed that the set of roots of Λ_p consists of two elements, p and another one that will be denoted by p'; that is, p' is defined as $p' \in \mathbb{R} - \{p\}$ and is such that $\Lambda_p(p') = 0$. Furthermore, since $f \in U_{ad}$, it is noted that $pp' < 0$; that is, both p and p' have different signs. Note that in the case where f is an even function, $p' = -p$; see Figure 1(a). Then, when $(\alpha,\beta) \in \mathcal{S}$, the solution of (1.12)–(1.13) is a shock wave with a constant positive speed when $f(\alpha) > f(\beta)$ and $\alpha > 0$; equivalently $\alpha > 0 \ \wedge \ \beta \in [\min\{\alpha', \alpha\}, \max\{\alpha', \alpha\}]$. Similarly, we can characterize the shock waves with a constant negative speed, and zero velocity. More precisely, if we define the notation

$$\mathcal{S}^- = \{(x, y) \in \mathcal{S} \ : \ y < \min\{x', x\} < 0\}, \tag{2.3}$$

$$\mathcal{S}^+ = \{(x, y) \in \mathcal{S} \ : \ x > 0 \ \ \wedge \ \ y \in (x', x)\}, \tag{2.4}$$

$$\mathcal{S}^0 = \{(x, y) \in \mathcal{S} \ : \ x > 0 \ \ \wedge \ \ y = x'\}, \tag{2.5}$$

we have

$$S^{(\alpha,\beta)}_{(a,b)} := \begin{cases} \overrightarrow{S}^{(\alpha,\beta)}_{(a,b)}, & \sigma(\alpha,\beta) > 0 \ \Leftrightarrow \ \alpha > 0 \ \wedge \ \beta \in (\alpha', \alpha) \ \Leftrightarrow \ (\alpha,\beta) \in \mathcal{S}^+, \\ \overleftrightarrow{S}^{(\alpha,\beta)}_{(a,b)}, & \sigma(\alpha,\beta) = 0 \ \Leftrightarrow \ \alpha > 0 \ \wedge \ \beta = \alpha' \ \Leftrightarrow \ (\alpha,\beta) \in \mathcal{S}^0, \\ \overleftarrow{S}^{(\alpha,\beta)}_{(a,b)}, & \sigma(\alpha,\beta) < 0 \ \Leftrightarrow \ \beta < \min\{\alpha, \alpha'\} < 0 \ \Leftrightarrow \ (\alpha,\beta) \in \mathcal{S}^-, \end{cases} \tag{2.6}$$

see Figure 2(a)–(c).

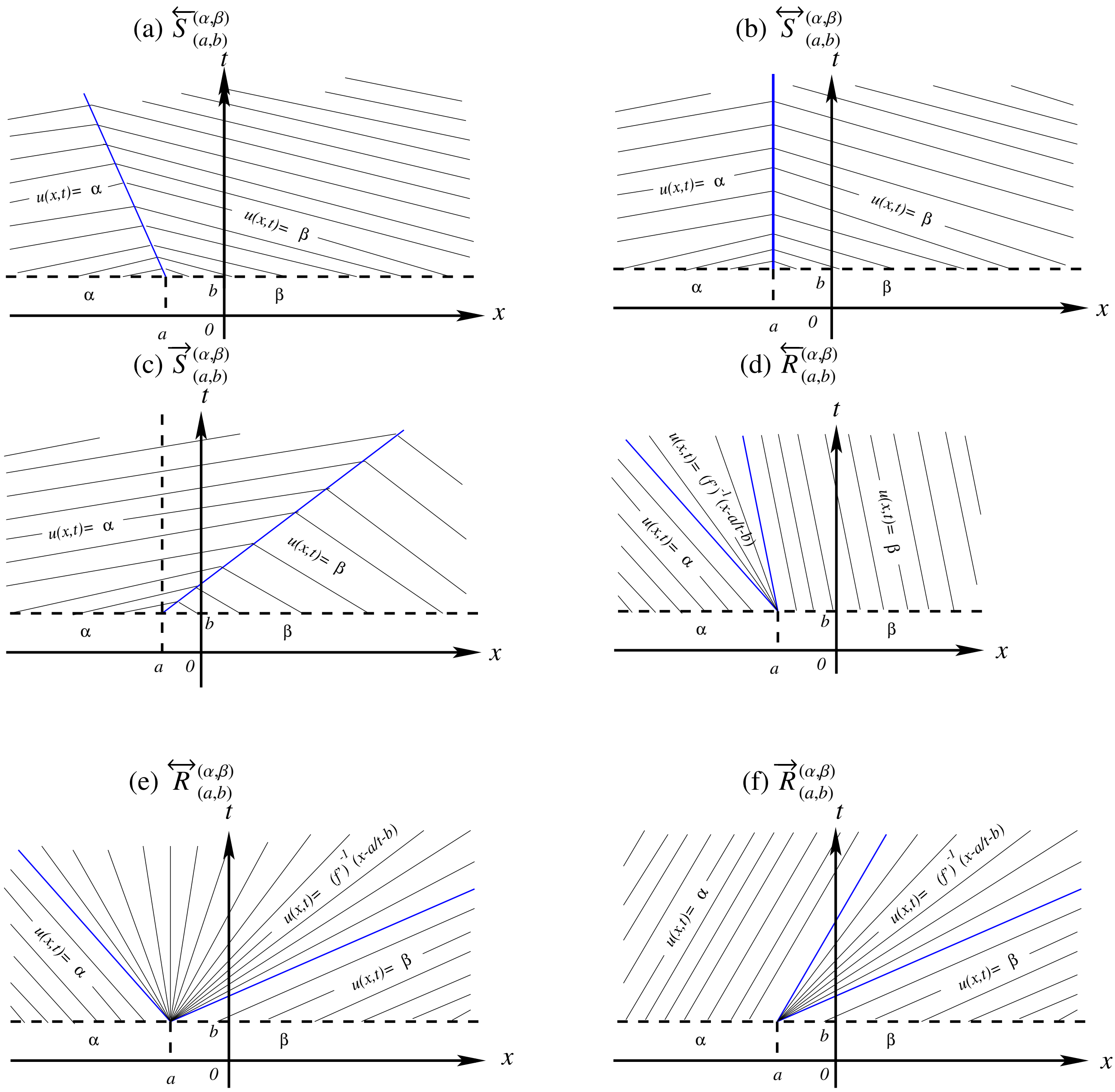

Figure 2. Characteristics for the entropy solution of (1.12)–(1.13) including the orientation of shocks and rarefactions in terms of the partition (2.11). (a)–(c) Shock waves when $(\alpha, \beta) \in \mathcal{S}^-$, $\mathcal{S}^0$, and $\mathcal{S}^+$, respectively. (d)–(f) Rarefaction waves when $(\alpha, \beta) \in \mathcal{R}^-$, $\mathcal{R}^0$, and $\mathcal{R}^+$, respectively.

Second, we consider the set $\mathcal{R}$ and we note that when $(\alpha, \beta) \in \mathcal{R}$, the solution of (1.12)–(1.13) is a rarefaction wave with three possible types of behavior, depending on the sign of the characteristic velocity of the edge of the fans, i.e., $\operatorname{sgn}(f'(\alpha))$ and $\operatorname{sgn}(f'(\beta))$. More precisely, if we introduce the following sets:

$$\mathcal{R}^+ = \{(x, y) \in \mathcal{R} \;:\; y > x > 0\}, \tag{2.7}$$

$$\mathcal{R}^0 = \{(x, y) \in \mathcal{R} \;:\; y \geq 0 \geq x,\; x \neq y\}, \tag{2.8}$$

$$\mathcal{R}^- = \{(x,y) \in \mathcal{R} \ : \ 0 > y > x\}, \tag{2.9}$$

we obtain the following characterization:

$$R^{(\alpha,\beta)}_{(a,b)} := \begin{cases} \overrightarrow{R}^{(\alpha,\beta)}_{(a,b)}, & f'(\beta) > f'(\alpha) > 0 \ \Leftrightarrow \ \beta > \alpha > 0 \Leftrightarrow \ (\alpha,\beta) \in \mathcal{R}^+, \\ \overleftrightarrow{R}^{(\alpha,\beta)}_{(a,b)}, & f'(\beta) \geq 0 \geq f'(\alpha) \ \Leftrightarrow \ \beta \geq 0 \geq \alpha \Leftrightarrow \ (\alpha,\beta) \in \mathcal{R}^0, \\ \overleftarrow{R}^{(\alpha,\beta)}_{(a,b)}, & 0 > f'(\beta) > f'(\alpha) \ \Leftrightarrow \ 0 > \beta > \alpha \Leftrightarrow \ (\alpha,\beta) \in \mathcal{R}^-; \end{cases} \tag{2.10}$$

see Figure 2(d)–(f). Hence, using the sets defined in (2.4)–(2.3) and (2.7)–(2.9), we have

$$\left\{ C, \mathcal{S}^-, \mathcal{S}^0, \mathcal{S}^+, \mathcal{R}^-, \mathcal{R}^0, \mathcal{R}^+ \right\} \quad \text{is a partition of } \mathbb{R}^2, \tag{2.11}$$

which characterizes the solution of the Riemann problem (1.12)–(1.13).

3. Entropy solution of (1.14)–(1.15)

To construct the entropy solution w of (1.14)–(1.15) for $t > b$, we require the solution of two Riemann problems: one centered at (a,b) separating the states (α,β) (see (1.12)–(1.13)) and another at (c,b) separating the states (β,γ) of the form

$$u_t + f(u)_x = 0, \qquad (x,t) \in \mathbb{R} \times (b,\infty) \tag{3.1}$$

$$u(x,b) = \begin{cases} \beta, & x \leq c, \\ \gamma, & x > c, \end{cases} \qquad x \in \mathbb{R}. \tag{3.2}$$

From Proposition 2.1, the solution of (1.12)–(1.13) can be a constant function, a shock $S^{(\alpha,\beta)}_{(a,b)}$, or a rarefaction $R^{(\alpha,\beta)}_{(a,b)}$, depending on the states (α,β). Similarly, the solution of (3.1) and (3.2) can be a constant function when $(\beta,\gamma) \in C$, a shock $S^{(\beta,\gamma)}_{(c,b)}$ when $(\beta,\gamma) \in \mathcal{S}$, or a rarefaction $R^{(\beta,\gamma)}_{(c,b)}$ when $(\beta,\gamma) \in \mathcal{R}$. We then have nine possible combinations of the sets containing the states (α,β) and (β,γ) as summarized in Table 1, where we use the set notation

$$\mathbb{E}_{cc} = \{(x,y,z) \in \mathbb{R}^3 \ : \ x = y = z\}, \quad \mathbb{E}_{cs} = \{(x,y,z) \in \mathbb{R}^3 \ : \ x = y > z\}, \tag{3.3}$$

$$\mathbb{E}_{cr} = \{(x,y,z) \in \mathbb{R}^3 \ : \ x = y < z\}, \quad \mathbb{E}_{sc} = \{(x,y,z) \in \mathbb{R}^3 \ : \ x > y = z\}, \tag{3.4}$$

$$\mathbb{E}_{rc} = \{(x,y,z) \in \mathbb{R}^3 \ : \ x < y = z\}, \quad \mathbb{E}_{ss} = \{(x,y,z) \in \mathbb{R}^3 \ : \ x > y > z\}, \tag{3.5}$$

$$\mathbb{E}_{sr} = \{(x,y,z) \in \mathbb{R}^3 \ : \ x = y = z\}, \quad \mathbb{E}_{rs} = \{(x,y,z) \in \mathbb{R}^3 \ : \ x = y = z\}, \tag{3.6}$$

$$\mathbb{E}_{ss} = \{(x,y,z) \in \mathbb{R}^3 \ : \ x < y < z\}. \tag{3.7}$$

Table 1. Possible interactions of solutions for (1.12)–(1.13) and (3.1)–(3.2).

	$(\beta,\gamma) \in C$	$(\beta,\gamma) \in \mathcal{S}$	$(\beta,\gamma) \in \mathcal{R}$
$(\alpha,\beta) \in C$	$(\alpha,\beta,\gamma) \in \mathbb{E}_{cs}$	$(\alpha,\beta,\gamma) \in \mathbb{E}_{cs}$	$(\alpha,\beta,\gamma) \in \mathbb{E}_{cr}$
$(\alpha,\beta) \in \mathcal{S}$	$(\alpha,\beta,\gamma) \in \mathbb{E}_{sc}$	$(\alpha,\beta,\gamma) \in \mathbb{E}_{ss}$	$(\alpha,\beta,\gamma) \in \mathbb{E}_{sr}$
$\alpha,\beta) \in \mathcal{R}$	$(\alpha,\beta,\gamma) \in \mathbb{E}_{rc}$	$(\alpha,\beta,\gamma) \in \mathbb{E}_{rs}$	$(\alpha,\beta,\gamma) \in \mathbb{E}_{ss}$

We notice that

$$\left\{\mathbb{E}_{cc}, \mathbb{E}_{cs}, \mathbb{E}_{cr}, \mathbb{E}_{sc}, \mathbb{E}_{rc}, \mathbb{E}_{ss}, \mathbb{E}_{sr}, \mathbb{E}_{rs}, \mathbb{E}_{rr}\right\} \quad \text{is a partition of } \mathbb{R}^3. \tag{3.8}$$

Hence, the analysis of all possible wave interactions is equivalent to constructing the solution for the sets of the partition of $\mathbb{R}^3$ given in (3.8). To be more precise, we consider five cases:

Case (i): $(\alpha,\beta,\gamma) \in \mathbb{E}_{cc} \cup \mathbb{E}_{cs} \cup \mathbb{E}_{cr} \cup \mathbb{E}_{sc} \cup \mathbb{E}_{rc}$,
Case (ii): $(\alpha,\beta,\gamma) \in \mathbb{E}_{ss}$,
Case (iii): $(\alpha,\beta,\gamma) \in \mathbb{E}_{sr}$,
Case (iv): $(\alpha,\beta,\gamma) \in \mathbb{E}_{rs}$,
Case (v): $(\alpha,\beta,\gamma) \in \mathbb{E}_{rr}$.

It is clear that we cover all possible interactions by analyzing all five cases.

Case (i). Let $(\alpha,\beta,\gamma) \in \mathbb{E}_{cc} \cup \mathbb{E}_{cs} \cup \mathbb{E}_{cr} \cup \mathbb{E}_{sc} \cup \mathbb{E}_{rc}$. We observe that $(\alpha,\beta,\gamma) \in \mathbb{E}_{cc}$ is equivalent to $\alpha = \beta = \gamma$, w is the constant function, $(\alpha,\beta,\gamma) \in \mathbb{E}_{cs} \cup \mathbb{E}_{cr}$ is reduced to requiring the algebraic restriction of the states $\alpha = \beta$ and $\beta \neq \gamma$, and w is the solution of the Riemann problem (3.1)–(3.2). For $(\alpha,\beta,\gamma) \in \mathbb{E}_{sc} \cup \mathbb{E}_{rc}$, we find that $\alpha \neq \beta$ and $\beta = \gamma$, and w is the solution of the Riemann problem (1.12)–(1.13). Thus, we deduce that the entropy solution of the Cauchy problem (1.14)-(1.15) is given by

$$w(x,t) = \begin{cases} \alpha, & (\alpha,\beta,\gamma) \in \mathbb{E}_{cc}, \\ S^{(\beta,\gamma)}_{(c,b)}(x,t), & (\alpha,\beta,\gamma) \in \mathbb{E}_{cs}, \\ R^{(\beta,\gamma)}_{(c,b)}(x,t), & (\alpha,\beta,\gamma) \in \mathbb{E}_{cr}, \\ S^{(\alpha,\beta)}_{(a,b)}(x,t), & (\alpha,\beta,\gamma) \in \mathbb{E}_{sc}, \\ R^{(\alpha,\beta)}_{(a,b)}(x,t), & (\alpha,\beta,\gamma) \in \mathbb{E}_{rc}, \end{cases} \tag{3.9}$$

for $(x,t) \in \mathbb{R} \times [b,\infty)$. We observe that if we consider the orientation of the characteristics in the case of $(\alpha,\beta,\gamma) \in \mathbb{E}_{cc}$, we have three cases: $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$. Similarly, from (2.6) and (2.10), we have three subclasses in each set: $\mathbb{E}_{cs}$, $\mathbb{E}_{cr}$, $\mathbb{E}_{sc}$, $\mathbb{E}_{rc}$, depending on the orientation of the shocks and rarefactions. For instance, in the case of $\mathbb{E}_{cs}$, we can distinguish the cases $\overrightarrow{S}^{(\beta,\gamma)}_{(c,b)}$ when $(\beta,\gamma) \in \mathcal{S}^-$, $\overleftrightarrow{S}^{(\beta,\gamma)}_{(c,b)}$ when $(\beta,\gamma) \in \mathcal{S}^0$, and $\overleftarrow{S}^{(\beta,\gamma)}_{(c,b)}$ when $(\beta,\gamma) \in \mathcal{S}^+$. Moreover, for notational convenience, we define the set $A \boxtimes B$ as follows:

$$A \boxtimes B = \left\{(x,y,z) \in \mathbb{R}^3 \quad : \quad (x,y) \in A \subset \mathbb{R}^2 \quad \text{and} \quad (y,z) \in B \subset \mathbb{R}^2\right\}. \tag{3.10}$$

Then, using the partition (2.11), we have

$$\begin{aligned} &\mathbb{E}_{cc} = C \boxtimes C, \quad \mathbb{E}_{cs} = (C \boxtimes \mathcal{S}^-) \cup (C \boxtimes \mathcal{S}^0) \cup (C \boxtimes \mathcal{S}^+), \\ &\mathbb{E}_{cr} = (C \boxtimes \mathcal{S}^-) \cup (C \boxtimes \mathcal{S}^0) \cup (C \boxtimes \mathcal{S}^+), \quad \mathbb{E}_{sc} = (\mathcal{S}^- \boxtimes C) \cup (\mathcal{S}^0 \boxtimes C) \cup (\mathcal{S}^+ \boxtimes C), \\ &\mathbb{E}_{rc} = (\mathcal{R}^- \boxtimes C) \cup (\mathcal{R}^0 \boxtimes C) \cup (\mathcal{R}^+ \boxtimes C); \end{aligned}$$

and we can rewrite the function in (3.9) as follows:

$$w = \begin{cases} \alpha, \ (\alpha,\beta,\gamma) \in C \boxtimes C; \\ \overleftarrow{S}^{(\beta,\gamma)}_{(c,b)}, \ (\alpha,\beta,\gamma) \in (C \boxtimes \mathcal{S}^-); \quad \overleftrightarrow{S}^{(\beta,\gamma)}_{(c,b)}, \ (\alpha,\beta,\gamma) \in (C \boxtimes \mathcal{S}^0); \quad \overrightarrow{S}^{(\beta,\gamma)}_{(c,b)}, \ (\alpha,\beta,\gamma) \in (C \boxtimes \mathcal{S}^+); \\ \overleftarrow{R}^{(\beta,\gamma)}_{(c,b)}, \ (\alpha,\beta,\gamma) \in (C \boxtimes \mathcal{S}^-); \quad \overleftrightarrow{R}^{(\beta,\gamma)}_{(c,b)}, \ (\alpha,\beta,\gamma) \in (C \boxtimes \mathcal{S}^0); \quad \overrightarrow{R}^{(\beta,\gamma)}_{(c,b)}, \ (\alpha,\beta,\gamma) \in (C \boxtimes \mathcal{S}^+); \\ \overleftarrow{S}^{(\alpha,\beta)}_{(a,b)}, \ (\alpha,\beta,\gamma) \in (\mathcal{S}^- \boxtimes C); \quad \overleftrightarrow{S}^{(\alpha,\beta)}_{(a,b)}, \ (\alpha,\beta,\gamma) \in (\mathcal{S}^0 \boxtimes C); \quad \overrightarrow{S}^{(\alpha,\beta)}_{(a,b)}, \ (\alpha,\beta,\gamma) \in (\mathcal{S}^+ \boxtimes C); \\ \overleftarrow{R}^{(\alpha,\beta)}_{(a,b)}, \ (\alpha,\beta,\gamma) \in (\mathcal{R}^- \boxtimes C); \quad \overleftrightarrow{R}^{(\alpha,\beta)}_{(a,b)}, \ (\alpha,\beta,\gamma) \in (\mathcal{R}^0 \boxtimes C); \quad \overrightarrow{R}^{(\alpha,\beta)}_{(a,b)}, \ (\alpha,\beta,\gamma) \in (\mathcal{R}^+ \boxtimes C). \end{cases} \tag{3.11}$$

Case (ii). Let $(\alpha,\beta,\gamma) \in \mathbb{E}_{ss}$. In this case, we have the interaction of two shock waves, one originating at (a,b), given by $x = a+\sigma(\alpha,\beta)(t-b)$, and another originating at (c,b), given by $x = c+\sigma(\beta,\gamma)(t-b)$. The intersection of the shock curves is defined by

$$\bar{x}_{ss} = a + \frac{\sigma(\alpha,\beta)(c-a)}{\sigma(\alpha,\beta)-\sigma(\beta,\gamma)} = c + \frac{\sigma(\alpha,\beta)(c-a)}{\sigma(\alpha,\beta)-\sigma(\beta,\gamma)}, \quad \bar{t}_{ss} = b + \frac{c-a}{\sigma(\alpha,\beta)-\sigma(\beta,\gamma)}. \tag{3.12}$$

We notice that for $t > \bar{t}_{ss}$, w is defined by the shock wave $S^{(\alpha,\gamma)}_{(\bar{x}_{ss},\bar{t}_{ss})}$. In this case, the entropy solution of the Cauchy problem (1.14)–(1.15) is given by

$$w(x,t) = \begin{cases} \alpha, & x \le a+\sigma(\alpha,\beta)(t-b), \quad t \le \bar{t}_{ss}, \\ \beta, & a+\sigma(\alpha,\beta)(t-b) < x \le c+\sigma((\beta,\gamma)(t-b), \quad t \le \bar{t}_{ss}, \\ \gamma, & x > c+\sigma((\beta,\gamma)(t-b), \quad t \le \bar{t}_{ss}, \\ S^{(\alpha,\gamma)}_{(\bar{x}_{ss},\bar{t}_{ss})}(x,t), & t \ge \bar{t}_{ss}, \end{cases} \tag{3.13}$$

for $(x,t) \in \mathbb{R}\times[b,\infty)$. We notice that when $(\alpha,\beta) \in \mathcal{S}^-$ and $(\beta,\gamma) \in \mathcal{S}^-$, we have $\gamma < \beta < \min\{\alpha',\alpha\} < 0$, which implies that $\bar{x}_{ss} < a$, and $\bar{t}_{ss} > b$, since $\sigma(\alpha,\beta) < 0$ and $\sigma(\beta,\gamma) < \sigma(\alpha,\beta)$, and also implies $\sigma(\alpha,\gamma) < 0$. We can develop a similar analysis for the other cases, getting the results given in Table 2.

Table 2. Possible solutions for $(\alpha,\beta,\gamma) \in \mathbb{E}_{ss}$ with the algebraic relations for the states (α,β,γ), the intersection of the shocks $(\bar{x}_{ss}\bar{t}_{ss})$ defined in (3.12), and the characterization of $\sigma(\alpha,\gamma)$.

	$(\beta,\gamma) \in \mathcal{S}^-$	$(\beta,\gamma) \in \mathcal{S}^0$	$(\beta,\gamma) \in \mathcal{S}^+$
$(\alpha,\beta) \in \mathcal{S}^-$	$\gamma < \beta < \min\{\alpha',\alpha\} < 0$ $\bar{x}_{ss} < a,\ \bar{t}_{ss} > b$ $\sigma(\alpha,\gamma) < 0$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{S}^- \boxtimes \mathcal{S}^-$		
$(\alpha,\beta) \in \mathcal{S}^0$	$\gamma < \beta = \alpha' < 0 < \alpha$ $\bar{x}_{ss} = a,\ \bar{t}_{ss} > b$ $\sigma(\alpha,\gamma) < 0$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{S}^0 \boxtimes \mathcal{S}^-$		
$(\alpha,\beta) \in \mathcal{S}^+$	$\alpha > 0,\ \beta \in (\alpha',\alpha),\ \gamma \in (\beta',\beta)$ $a < \bar{x}_{ss} < b,\ \bar{t}_{ss} > b$ $\sigma(\alpha,\gamma) \in \mathbb{R}$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{S}^+ \boxtimes \mathcal{S}^-$	$\alpha > 0,\ \alpha > \beta > \alpha',\ \gamma = \beta'$ $\bar{x}_{ss} = c,\ \bar{t}_{ss} > b$ $\sigma(\alpha,\gamma) > 0$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{S}^+ \boxtimes \mathcal{S}^0$	$\beta' < \gamma < 0 < \beta < \alpha$ $\bar{x}_{ss} > c,\ \bar{t}_{ss} > b$ $\sigma(\alpha,\gamma) > 0$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{S}^+ \boxtimes \mathcal{S}^+$

Case (iii). Let $(\alpha,\beta,\gamma) \in \mathbb{E}_{sr}$. In this case, the shock curve originating in (a,b), given by $x = a + \sigma(\alpha,\beta)(t-b)$, intersects the left hand of the rarefaction centered in (c,b) with the equation $x = c + f'(\beta)(t-b)$, which is given by

$$\bar{x}_{sr} = a + \frac{\sigma(\alpha,\beta)(c-a)}{\sigma(\alpha,\beta)-f'(\beta)} = c + \frac{f'(\beta)(c-a)}{\sigma(\alpha,\beta)-f'(\beta)}, \quad \bar{t}_{sr} = b + \frac{c-a}{\sigma(\alpha,\beta)-f'(\beta)}. \tag{3.14}$$

At $(\overline{x}_{sr}, \overline{t}_{sr})$ emanates a shock with a curve $(X^+(t), t)$ that satisfies the following ordinary differential equation:

$$\frac{dX^+}{dt} = \sigma\Big(u(X^+(t), t), \alpha\Big), \quad f'(u) = \frac{X^+ - c}{t - b}, \quad f'(\beta) \leq \frac{X^+ - c}{t - b} \leq f'(\gamma), \tag{3.15}$$

$$X^+(\overline{t}_{sr}) = \overline{x}_{sr}. \tag{3.16}$$

We observe that $X^+ = c + f'(u)(t - b)$ implies that $dX^+/dt = f''(u)(t - b)du/dt + f'(u)$, and

$$\int_\beta^u \frac{f''(u)}{\sigma(u, \alpha) - f'(u)} du = \ln\left(\frac{t}{\overline{t}_{sr}}\right), \quad \beta \leq u \leq \gamma.$$

We distinguish three cases: (a) if $\gamma = \alpha$, the shock curve $(X^+(t), t)$ is able to cross the whole rarefaction completely only when $t \to \infty$; (b) if $\gamma < \alpha$, the shock curve $(X^+(t), t)$ is able to cross the whole rarefaction completely at

$$\hat{\hat{t}}_{sr} = \overline{t}_{sr} \int_\beta^u \frac{f''(u)}{\sigma(u, \alpha) - f'(u)} du; \quad \beta \leq u \leq \gamma;$$

and (c) if $\gamma > \alpha$, it is impossible for the shock curve $(X^+(t), t)$ to cross the whole rarefaction completely. Then the entropy solution of the Cauchy problem (1.14)–(1.15) has two forms: when $(X^+(t), t)$ is able to cross the whole rarefaction completely

$$w(x, t) = \begin{cases} \alpha, & \Big(x \leq a + \sigma(\alpha, \beta)(t - b) \text{ and } t \leq \overline{t}_{sr}\Big) \text{ or } \Big(x \leq X^+(t) \text{ and } \overline{t}_{sr} \leq t \leq \hat{\hat{t}}\Big) \\ \beta, & a + \sigma(\alpha, \beta)(t - b) < x \leq c + f'(\beta)(t - b) \text{ and } t \leq \overline{t}_{sr}, \\ (f')^{-1}\left(\dfrac{x - c}{t - b}\right), & \Big(c + f'(\beta)(t - b) \leq x \leq c + f'(\gamma)(t - b) \text{ and } t \leq \overline{t}_{sr}\Big) \\ & \quad \text{or } \Big(x \leq X^+(t) \leq x \leq c + f'(\gamma)(t - b) \text{ and } \overline{t}_{sr} \leq t \leq \hat{\hat{t}}_{sr}\Big), \\ \gamma, & x > c + \sigma(\beta, \gamma)(t - b) \text{ and } t \leq \hat{\hat{t}}_{sr}, \\ S^{(\alpha,\gamma)}_{(\hat{\hat{x}}_{sr}, \hat{\hat{t}}_{sr})}(x, t), & t \geq \hat{\hat{t}}_{sr}, \end{cases} \tag{3.17}$$

when $\gamma < \alpha$, or

$$w(x, t) = \begin{cases} \alpha, & \Big(x \leq a + \sigma(\alpha, \beta)(t - b) \text{ and } t \leq \overline{t}_{sr}\Big) \text{ or } \Big(x \leq X^+(t) \text{ and } t \geq \overline{t}_{sr}\Big) \\ \beta, & a + \sigma(\alpha, \beta)(t - b) < x \leq c + f'(\beta)(t - b) \text{ and } t \leq \overline{t}_{sr}, \\ (f')^{-1}\left(\dfrac{x - c}{t - b}\right), & \Big(c + f'(\beta)(t - b) \leq x \leq c + f'(\gamma)(t - b) \text{ and } t \leq \overline{t}_{sr}\Big) \\ & \quad \text{or } \Big(x \leq X^+(t) \leq x \leq c + f'(\gamma)(t - b), \end{cases} \tag{3.18}$$

when $\gamma \geq \alpha$, for $(x, t) \in \mathbb{R} \times [b, \infty)$. In Table 3, a classification of several possible cases is presented.

Table 3. Possible solutions for $(\alpha,\beta,\gamma) \in \mathbb{E}_{sr}$ with the algebraic relations for the states (α,β,γ), the intersection of shocks $(\bar{x}_{sr}\bar{t}_{sr})$ defined in (3.14), and the characterization of $\sigma(\alpha,\gamma)$.

	$(\beta,\gamma) \in \mathcal{R}^-$	$(\beta,\gamma) \in \mathcal{R}^0$	$(\beta,\gamma) \in \mathcal{R}^+$
$(\alpha,\beta) \in \mathcal{S}^-$	$\beta < \min\{\alpha',\alpha\} < 0,\ \beta < \gamma < 0$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{S}^- \boxtimes \mathcal{R}^-$; $\bar{x}_{sr} < a,\ \bar{t}_{sr} > b$	$\beta < \min\{\alpha',\alpha\} < 0 \leq \gamma$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{S}^- \boxtimes \mathcal{R}^0$; $\bar{x}_{sr} < a,\ \bar{t}_{sr} > b$	
$(\alpha,\beta) \in \mathcal{S}^0$	$\beta = \alpha' < \gamma < 0 < \alpha$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{S}^0 \boxtimes \mathcal{R}^-$; $\bar{x}_{sr} = a,\ \bar{t}_{sr} > b$	$\beta = \alpha' < 0 < \alpha,\ \gamma \geq 0$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{S}^0 \boxtimes \mathcal{R}^0$; $\bar{x}_{sr} = a,\ \bar{t}_{sr} > b$	
$(\alpha,\beta) \in \mathcal{S}^+$	$\alpha' < \beta < \gamma < 0 < \alpha$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{S}^+ \boxtimes \mathcal{R}^-$; $a < \bar{x}_{sr} < b,\ \bar{t}_{sr} > b$	$\alpha' < \beta \leq 0 < \alpha,\ \gamma \geq 0$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{S}^+ \boxtimes \mathcal{R}^0$; $\bar{x}_{sr} = c,\ \bar{t}_{sr} > b$	$0 < \beta < \alpha,\ \beta < \gamma$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{S}^+ \boxtimes \mathcal{R}^+$; $\bar{x}_{sr} > c,\ \bar{t}_{sr} > b$

Case (iv). Let $(\alpha,\beta,\gamma) \in \mathbb{E}_{rs}$. This case is similar to Case (iii). The shock curve originating in (c,b), given by $x = c + \sigma(\beta,\gamma)(t-b)$, intersecting the right-hand of the rarefaction centered in (a,b) with equation $x = a + f'(\beta)(t-b)$, at $(\bar{x}_{rs}, \bar{x}_{rs})$, defined by

$$\bar{x}_{rs} = a + \frac{f'(\beta)(c-a)}{f'(\beta) - \sigma(\beta,\gamma)} = c + \frac{\sigma(\beta,\gamma)(c-a)}{f'(\beta) - \sigma(\beta,\gamma)}, \quad \bar{t}_{rs} = b + \frac{c-a}{f'(\beta) - \sigma(\beta,\gamma)}. \tag{3.19}$$

At $(\bar{x}_{rs}, \bar{t}_{rs})$, a shock emanates with the curve $(X^-(t), t)$, which is defined by the solution of

$$\frac{dX^-}{dt} = \sigma\Big(u(X^-(t),t),\gamma\Big), \quad f'(u) = \frac{X^- - a}{t-b}, \quad f'(\alpha) \leq \frac{X^- - a}{t-b} \leq f'(\beta), \tag{3.20}$$

$$X^-(\bar{t}_{rs}) = \bar{x}_{rs}. \tag{3.21}$$

From $X^- = a + f'(u)(t-b)$, we get $dX^-/dt = f''(u)(t-b)du/dt + f'(u)$; consequently

$$\int_\alpha^u \frac{f''(u)}{\sigma(u,\gamma) - f'(u)} du = \ln\left(\frac{t}{\bar{t}_{rs}}\right), \quad \alpha \leq u \leq \beta.$$

Moreover, we find that $(X^-(t), t)$ is able to cross the whole rarefaction completely only when $t \to \infty$ and $\gamma = \alpha$, and it can cross the whole rarefaction completely at a finite time

$$\hat{\hat{t}}_{rs} = \bar{t}_{rs} \int_\alpha^u \frac{f''(u)}{\sigma(u,\gamma) - f'(u)} du, \quad \alpha \leq u \leq \beta;$$

when $\gamma < \alpha$; it is impossible to cross the whole rarefaction when $\gamma > \alpha$. Therefore, the entropy solution of the Cauchy problem (1.14)–(1.15) has two forms: when $(X^-(t), t)$ is able to cross the whole rarefaction completely

$$
w(x,t) = \begin{cases} \alpha, & x \le a + f'(\alpha)(t-b) \text{ and } t \le \hat{\hat{t}}_{rs}, \\ (f')^{-1}\left(\dfrac{x-c}{t-b}\right), & \left(c + f'(\beta)(t-b) \le x \le c + f'(\gamma)(t-b) \text{ and } t \le \bar{t}_{sr}\right) \\ & \quad \text{or } \left(a + f'(\alpha)(t-b) \le x \le X^-(t) \text{ and } \bar{t}_{rs} \le t \le \hat{\hat{t}}_{rs}\right), \\ \beta, & a + f'(\beta)(t-b) < x \le c + \sigma(\beta,\gamma)(t-b) \text{ and } t \le \bar{t}_{rs}, \\ \gamma, & \left(x > c + \sigma(\beta,\gamma)(t-b) \text{ and } t \le \bar{t}_{rs}\right) \text{ or } \left(x > X^-(t) \text{ and } \bar{t}_{rs} \le t \le \hat{\hat{t}}_{rs}\right), \\ S^{(\alpha,\gamma)}_{(\hat{\hat{x}}_{rs},\hat{\hat{t}}_{rs})}(x,t), & t \ge \hat{\hat{t}}_{rs}, \end{cases} \tag{3.22}
$$

when $\gamma < \alpha$, or

$$
w(x,t) = \begin{cases} \alpha, & x \le a + f'(\alpha)(t-b), \\ (f')^{-1}\left(\dfrac{x-c}{t-b}\right), & \left(c + f'(\beta)(t-b) \le x \le c + f'(\gamma)(t-b) \text{ and } t \le \bar{t}_{sr}\right) \\ & \quad \text{or } \left(a + f'(\alpha)(t-b) \le x \le X^-(t) \text{ and } t \ge \bar{t}_{rs}\right), \\ \beta, & a + f'(\beta)(t-b) < x \le c + \sigma(\beta,\gamma)(t-b) \text{ and } t \le \bar{t}_{rs}, \\ \gamma, & \left(x > c + \sigma(\beta,\gamma)(t-b) \text{ and } t \le \bar{t}_{rs}\right) \text{ or } \left(x > X^-(t) \text{ and } t \ge \bar{t}_{rs}\right), \end{cases} \tag{3.23}
$$

when $\gamma \ge \alpha$, for $(x,t) \in \mathbb{R} \times [b,\infty)$. A specific notation and details of all possible cases are presented in Table 4.

Table 4. Possible solutions for $(\alpha,\beta,\gamma) \in \mathbb{E}_{rs}$ with the algebraic relations for the states (α,β,γ), the intersection of shocks $(\bar{x}_{sr}\bar{t}_{sr})$ defined in (3.19), and the characterization of $\sigma(\alpha,\gamma)$.

	$(\beta,\gamma) \in \mathcal{S}^-$	$(\beta,\gamma) \in \mathcal{S}^0$	$(\beta,\gamma) \in \mathcal{S}^+$
$(\alpha,\beta) \in \mathcal{R}^-$	$\beta < \min\{\alpha',\alpha\} < 0,\ \beta < \gamma < 0$ $\bar{x}_{sr} < a,\ \bar{t}_{sr} > b$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{R}^- \boxtimes \mathcal{S}^-$;	$\beta < \min\{\alpha',\alpha\} < 0 \le \gamma$ $\bar{x}_{sr} < a,\ \bar{t}_{sr} > b$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{R}^- \boxtimes \mathcal{S}^0$;	
$(\alpha,\beta) \in \mathcal{R}^0$	$\beta = \alpha' < \gamma < 0 < \alpha$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{R}^0 \boxtimes \mathcal{S}^-$; $\bar{x}_{sr} = a,\ \bar{t}_{sr} > b$	$\beta = \alpha' < 0 < \alpha,\ \gamma \ge 0$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{R}^0 \boxtimes \mathcal{S}^0$; $\bar{x}_{sr} = a,\ \bar{t}_{sr} > b$	
$(\alpha,\beta) \in \mathcal{R}^+$	$\alpha' < \beta < \gamma < 0 < \alpha$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{R}^+ \boxtimes \mathcal{S}^-$; $a < \bar{x}_{sr} < b,\ \bar{t}_{sr} > b$	$\alpha' < \beta \le 0 < \alpha,\ \gamma \ge 0$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{R}^+ \boxtimes \mathcal{S}^0$; $\bar{x}_{sr} = c,\ \bar{t}_{sr} > b$	$0 < \beta < \alpha,\ \beta < \gamma$ $\Updownarrow$ $(\alpha,\beta,\gamma) \in \mathcal{R}^+ \boxtimes \mathcal{S}^+$; $\bar{x}_{sr} > c,\ \bar{t}_{sr} > b$

Case (v). Let $(\alpha,\beta,\gamma) \in \mathbb{E}_{rr}$. In this case, we have the interaction of two rarefaction waves, one originating at (a,b) and separating the states (α,β), and another originating at (c,b) and separating the states (β,γ). We observe that $x = a + f'(\beta)(t-b)$ and $x = c + f'(\beta)(t-b)$ are parallel. In this case, w

is defined as follows:

$$w(x,t)=\begin{cases}\alpha, & x<a+f'(\alpha)(t-b),\\ (f')^{-1}\left(\dfrac{x-a}{t-b}\right), & a+f'(\alpha)(t-b)\leq x\leq a+f'(\beta)(t-b),\\ \beta, & a+f'(\beta)(t-b)<x<c+f'(\beta)(t-b),\\ (f')^{-1}\left(\dfrac{x-c}{t-b}\right), & c+f'(\beta)(t-b)\leq x\leq c+f'(\gamma)(t-b),\\ \gamma, & t>c+f'(\gamma)(t-b),\end{cases} \tag{3.24}$$

for $(x,t)\in\mathbb{R}\times[b,\infty)$. We notice that when $(\alpha,\beta)\in\mathcal{R}^-$ and $(\beta,\gamma)\in\mathcal{R}^-$, we have $\alpha<\beta<\gamma<0$. We can develop a similar analysis for the other cases, obtaining the results in Table 5.

Table 5. Summary of possible solutions for $(\alpha,\beta,\gamma)\in\mathbb{E}_{rr}$ in terms of α,β, and γ.

	$(\beta,\gamma)\in\mathcal{R}^-$	$(\beta,\gamma)\in\mathcal{R}^0$	$(\beta,\gamma)\in\mathcal{R}^+$
$(\alpha,\beta)\in\mathcal{R}^-$	$\alpha<\beta<\gamma<0$ $\Updownarrow$ $(\alpha,\beta,\gamma)\in\mathcal{R}^-\boxast\mathcal{R}^-$	$\alpha<\beta<0\leq\gamma$ $\Updownarrow$ $(\alpha,\beta,\gamma)\in\mathcal{R}^-\boxast\mathcal{R}^0$	
$(\alpha,\beta)\in\mathcal{R}^0$		$\alpha<\beta=0<\gamma$ $\Updownarrow$ $(\alpha,\beta,\gamma)\in\mathcal{R}^0\boxast\mathcal{R}^0$	$\alpha\leq 0\leq\beta<\gamma,\ \alpha\neq\beta$ $\Updownarrow$ $(\alpha,\beta,\gamma)\in\mathcal{R}^0\boxast\mathcal{R}^+$
$(\alpha,\beta)\in\mathcal{R}^+$			$0<\alpha<\beta<\gamma$ $\Updownarrow$ $(\alpha,\beta,\gamma)\in\mathcal{R}^+\boxast\mathcal{R}^+$

Summarizing, in Cases (i)–(v), we have proved the following theorem.

Theorem 3.1. *Consider the partition of* $\mathbb{R}^3$ *given in* (3.8). *In this case,*

$$w:\mathbb{R}\times[b,\infty)\to\mathbb{R}\ \textit{given by}\ \begin{cases}(3.9), \textit{for } (\alpha,\beta,\gamma)\in\mathbb{E}_{cc}\cup\mathbb{E}_{cs}\cup\mathbb{E}_{cr}\cup\mathbb{E}_{sc}\cup\mathbb{E}_{rc},\\ (3.13), \textit{for } (\alpha,\beta,\gamma)\in\mathbb{E}_{ss},\\ (3.17)\ \textit{when } \gamma<\alpha \textit{ or } (3.18)\ \textit{when } \gamma\geq\alpha, \textit{for } (\alpha,\beta,\gamma)\in\mathbb{E}_{sr},\\ (3.22)\ \textit{when } \gamma<\alpha \textit{ or } (3.23)\ \textit{when } \gamma\geq\alpha, \textit{for } (\alpha,\beta,\gamma)\in\mathbb{E}_{rs},\\ (3.24), \textit{for } (\alpha,\beta,\gamma)\in\mathbb{E}_{rr},\end{cases} \tag{3.25}$$

defines the entropy solution of the Cauchy problem (1.14)–(1.15).

4. Entropy solution of the initial value problem (1.1)–(1.4)

We note that the initial value problem (1.1)–(1.4) is well-posed in the sense of the entropy solutions (see [25]). The inequality (1.10) provides the natural condition, which implies uniqueness. As pointed out in [25], we note that the boundary condition (1.9) may not be satisfied in an almost everywhere* pointwise sense and becomes a boundary inequality. In general, the definition of this weak

*Let (X,Σ,μ) a measure space, a property P is said to hold almost everywhere in X, if there exists a measurable set $N\in\Sigma$ with $\mu(N)=0$ and all $x\in N-X$ have the property P.

boundary sense requires additional assumptions on the coefficients, initial, and boundary conditions. However, the simplification introduced by the hypothesis (1.5) and the constant values of the initial and boundary value functions (see (1.2)–(1.4)) allows a simpler characterization of the entropy solution. For the completeness of this presentation, we write a result presented by Liu and Pan in the following proposition; see [24] for details.

Proposition 4.1. *Let $u(x,t)$ be a piecewise $C^1(Q_T)$ function. Then, under the assumption (1.5), $u(x,t)$ is an entropy solution of* (1.1)–(1.4) *if and only if the following four conditions are satisfied:*

1) $u(x,t)$ is a classical solution of (1.6) in the domains where it is C^1.
2) If $x = X(t)$ is a discontinuity curve of u, then the Rankine–Hugoniot

$$\frac{dX(t)}{dt} = \sigma\Big(u(X(t)-0,t), u(X(t)+0,t)\Big),$$

and the Lax shock condition $u(X(t)-0,t) > u(X(t)+0,t)$, are satisfied.

3) The entropy boundary inequalities

$$\begin{aligned} u(0,t) = u_L \text{ or } \sigma\Big(u(0,t),k\Big) \le 0,\ \ k \in I(u(0,t),u_L),\ \ k \neq u(0,t), \\ u(1,t) = u_R \text{ or } \sigma\Big(u(1,t),k\Big) \ge 0,\ \ k \in I(u(1,t),u_R),\ \ k \neq u(1,t), \end{aligned}$$

holds, for almost every $t \ge 0$. Here, $I(a,b) := [\min\{a,b\}, \max\{a,b\}]$.

4) $u(x,0) = u_0$ for almost every $x \in (0,1)$.

We obtain the entropy solution of the initial-boundary value problem (1.1)–(1.4) in two steps. First, we construct the entropy solution of the Cauchy problem

$$v_t + f(v)_x = 0, \quad v(x,0) = \begin{cases} u_L, & x \le 0, \\ u_0, & 0 < x < 1, \\ u_R, & x \ge 1. \end{cases} \tag{4.1}$$

Then we apply Proposition 4.1 to restrict the solution to $x \in [0,1]\times\mathbb{R}$. We find that the entropy solution of (4.1) can be obtained by applying the previous analysis of the Cauchy problem (1.14)–(1.15) with $a = 0, b = 0, c = 1, \alpha = u_L, \beta = u_0$, and $\gamma = u_R$. More precisely, by application of Theorem 3.1, we find that the entropy solution of the problem (4.1) is given by (3.25) or, more precisely

$$\begin{aligned} v(x,t) &= w(x,t),\ \text{given by (3.25) with } (a,b,c) = (0,0,1) \text{ and } (\alpha,\beta,\gamma) = (u_L,u_0,u_R) \\ &= w(x,t),\ \text{given by} \\ &\begin{cases} \text{(3.9) with } (a,b,c) = (0,0,1) \text{ and } (\alpha,\beta,\gamma) = (u_L,u_0,u_R) \in \mathbb{E}_{cc} \cup \mathbb{E}_{cs} \cup \mathbb{E}_{cr} \cup \mathbb{E}_{sc} \cup \mathbb{E}_{rc}, \\ \text{(3.13) with } (a,b,c) = (0,0,1) \text{ and } (\alpha,\beta,\gamma) = (u_L,u_0,u_R) \in \mathbb{E}_{ss}, \\ \text{(3.17) with } (a,b,c) = (0,0,1), (\alpha,\beta,\gamma) = (u_L,u_0,u_R) \in \mathbb{E}_{sr} \text{ and } u_L < u_R, \\ \qquad \text{or (3.18) with } (a,b,c) = (0,0,1), (\alpha,\beta,\gamma) = (u_L,u_0,u_R) \in \mathbb{E}_{sr} \text{ and } u_L \ge u_R, \\ \text{(3.22) with } (a,b,c) = (0,0,1), (\alpha,\beta,\gamma) = (u_L,u_0,u_R) \in \mathbb{E}_{rs} \text{ and } u_L < u_R, \\ \qquad \text{or (3.23) with } (a,b,c) = (0,0,1), (\alpha,\beta,\gamma) = (u_L,u_0,u_R) \in \mathbb{E}_{rs} \text{ and } u_L \ge u_R, \\ \text{(3.24) with } (a,b,c) = (0,0,1) \text{ and } (\alpha,\beta,\gamma) = (u_L,u_0,u_R) \in \mathbb{E}_{rr}. \end{cases} \end{aligned} \tag{4.2}$$

Then we apply Proposition 4.1 to analyze the restriction of (4.2) to $(x,t) \in [0,1]\times\mathbb{R}$ by separating the five cases used in the analysis given in Section 3 to construct w.

Case (i). Let $(u_L,u_0,u_R) \in \mathbb{E}_{cc}\cup\mathbb{E}_{cs}\cup\mathbb{E}_{cr}\cup\mathbb{E}_{sc}\cup\mathbb{E}_{rc}$. From (4.2), we observe that v is defined by (3.9). However, to distinguish the different cases, we consider (3.11) instead of (4.2). Let us consider $(u_L,u_0,u_R)\in \mathcal{C}\boxplus\mathcal{C}$; when we restrict v to $x\in[0,1]$, we obtain

$$u_1(x,t) = u_0, \quad (x,t)\in[0,1]\times\mathbb{R}. \tag{4.3}$$

If $(u_L,u_0,u_R)\in\mathbb{E}_{cs}$, we observe that the restriction of $\overleftarrow{S}^{(u_0,u_R)}_{(1,0)}$ to $x\in[0,1]$ is given by

$$u_2(x,t) = \begin{cases} u_0, & 0\le x\le \sigma(u_0,u_R)t+1,\\ u_R, & \text{otherwise,}\end{cases} \tag{4.4}$$

and the restriction of $\overleftrightarrow{S}^{(u_0,u_R)}_{(1,0)}$ and $\overrightarrow{S}^{(u_0,u_R)}_{(1,0)}$ is given by (4.3). Then, for $(u_L,u_0,u_R)\in\mathbb{E}_{cs}$, the entropy solution of the problem (1.1)–(1.4) is defined by (4.3) when $(u_L,u_0,u_R)\in(\mathcal{C}\boxplus\mathcal{S}^0)\cup(\mathcal{C}\boxplus\mathcal{S}^+)$ and by (4.4) when $(u_L,u_0,u_R)\in\mathcal{C}\boxplus\mathcal{S}^-$. Let us remember that for $(u_L,u_0,u_R)\in\mathbb{E}_{cr}$, we see that $\overleftarrow{R}^{(u_0,u_R)}_{(1,0)}$ and $\overleftrightarrow{R}^{(u_0,u_R)}_{(1,0)}$ restricted to $x\in[0,1]$ defines the functions

$$u_3(x,t) = \begin{cases} u_0, & 0\le x\le f'(u_0)t+1,\\ (f')^{-1}\left(\dfrac{x-1}{t}\right), & f'(u_0)t+1\le x\le f'(u_R)t+1,\\ u_R, & \text{otherwise,}\end{cases} \tag{4.5}$$

$$u_4(x,t) = \begin{cases} u_0, & 0\le x\le f'(u_0)t+1,\\ (f')^{-1}\left(\dfrac{x-1}{t}\right), & \text{otherwise,}\end{cases} \tag{4.6}$$

respectively; and $\overrightarrow{R}^{(u_0,u_R)}_{(1,0)}$ restricted to $x\in[0,1]$ is given by (4.3). Then, if $(u_L,u_0,u_R)\in\mathbb{E}_{cs}$, we find that u is defined by (4.5) when $(u_L,u_0,u_R)\in\mathcal{C}\boxplus\mathcal{R}^-$, by (4.6) when $(u_L,u_0,u_R)\in\mathcal{C}\boxplus\mathcal{R}^0$, and by (4.3) when $(u_L,u_0,u_R)\in\mathcal{C}\boxplus\mathcal{R}^+$. Similarly, analyzing the restriction of $S^{(u_L,u_0)}_{(0,0)}$ and $R^{(u_L,u_0)}_{(0,0)}$ to $x\in[0,1]$, we get the following three types of entropy solutions:

$$u_5(x,t) = \begin{cases} u_L, & 0\le x\le \sigma(u_L,u_0)t\\ u_0, & \text{otherwise,}\end{cases} \tag{4.7}$$

$$u_6(x,t) = \begin{cases} (f')^{-1}\left(\dfrac{x}{t}\right), & 0\le x\le f'(u_0)t,\\ u_0, & \text{otherwise,}\end{cases} \tag{4.8}$$

$$u_7(x,t) = \begin{cases} u_L, & 0\le x\le f'(u_L)t,\\ (f')^{-1}\left(\dfrac{x}{t}\right), & f'(u_L)t\le x\le f'(u_R)t,\\ u_0, & \text{otherwise;}\end{cases} \tag{4.9}$$

deducing that u is defined by u_5, u_6, u_7, and u_1 when (u_L,u_0,u_R) belongs to $\mathcal{S}^+\boxplus\mathcal{C}$, $\mathcal{R}^0\boxplus\mathcal{C}$, $\mathcal{R}^+\boxplus\mathcal{C}$, and

$(\mathcal{S}^- \boxtimes C) \cup (\mathcal{S}^0 \boxtimes C) \cup (\mathcal{R}^- \boxtimes C)$, respectively. In this case,

$$u(x,t) = \begin{cases} u_1(x,t), & (u_L,u_0,u_R) \in (C \boxtimes C) \cup (C \boxtimes \mathcal{S}^0) \cup (C \boxtimes \mathcal{S}^+) \cup (C \boxtimes \mathcal{R}^+) \\ & \qquad \cup (\mathcal{S}^0 \boxtimes C) \cup (\mathcal{S}^- \boxtimes C) \cup (\mathcal{R}^- \boxtimes C), \\ u_2(x,t), & (u_L,u_0,u_R) \in C \boxtimes \mathcal{S}^-, \\ u_3(x,t), & (u_L,u_0,u_R) \in C \boxtimes \mathcal{R}^-, \\ u_4(x,t), & (u_L,u_0,u_R) \in C \boxtimes \mathcal{R}^0, \\ u_5(x,t), & (u_L,u_0,u_R) \in \mathcal{S}^+ \boxtimes C, \\ u_6(x,t), & (u_L,u_0,u_R) \in \mathcal{R}^0 \boxtimes C, \\ u_7(x,t), & (u_L,u_0,u_R) \in \mathcal{R}^+ \boxtimes C, \end{cases} \tag{4.10}$$

defines the entropy solution of the initial-boundary value problem (1.1)–(1.4) in Case (i).

Case (ii). Let $(u_L,u_0,u_R) \in \mathbb{E}_{ss}$. From (4.2), (3.13), and the summary of all possible cases given in Table 2, we study three cases separately. First, we consider $(u_L,u_0,u_R) \in (\mathcal{S}^- \boxtimes \mathcal{S}^-) \cup (\mathcal{S}^0 \boxtimes \mathcal{S}^-)$ and observe that u is defined by u_2, as given in (4.4). Second, if $(u_L,u_0,u_R) \in (\mathcal{S}^+ \boxtimes \mathcal{S}^0) \cup (\mathcal{S}^+ \boxtimes \mathcal{S}^+)$, we find that u is given by u_5, as defined in (4.7). Third, for $(u_L,u_0,u_R) \in (\mathcal{S}^+ \boxtimes \mathcal{S}^-)$, we have the following type of solutions

$$u_8(x,t) = \begin{cases} u_L, & 0 \le x \le \sigma(u_L,u_0)t \text{ for } t \le \bar{t}_{ss} \text{ or } 0 \le x \le \sigma(u_L,u_R)(t-\bar{t}_{ss}) + \bar{x}_{ss} \text{ for } t > \bar{t}_{ss}, \\ u_0, & \sigma(u_L,u_0)t \le x \le \sigma(u_0,u_R)t + 1 \text{ and } t \le \bar{t}_{ss}, \\ u_R, & \text{otherwise}, \end{cases} \tag{4.11}$$

$$u_9(x,t) = \begin{cases} u_L, & 0 \le x \le \sigma(u_L,u_0)t \text{ for } t \le \bar{t}_{ss} \text{ or } 0 \le x \le \bar{x}_{ss} \text{ for } t > \bar{t}_{ss}, \\ u_0, & \sigma(u_L,u_0)t \le x \le \sigma(u_0,u_R)t + 1 \text{ for } t \le \bar{t}_{ss}, \\ u_R, & \text{otherwise}, \end{cases} \tag{4.12}$$

$$u_{10}(x,t) = \begin{cases} u_L, & 0 \le x \le \sigma(u_L,u_0)t \text{ for } t \le \bar{t}_{ss} \\ & \qquad \text{or } 0 \le x \le \sigma(u_L,u_R)(t-\bar{t}_{ss}) + \bar{x}_{ss} \le 1 \text{ for } \bar{t}_{ss} \ge \bar{t}_{ss}, \\ u_0, & \sigma(u_L,u_0)t \le x \le \sigma(u_0,u_R)t + 1 \text{ for } t \le \bar{t}_{ss}, \\ u_R, & \text{otherwise}, \end{cases} \tag{4.13}$$

when $\sigma(u_L,u_R) < 0$, $\sigma(u_L,u_R) = 0$, and $\sigma(u_L,u_R) > 0$, respectively. Consequently, we find that

$$u(x,t) = \begin{cases} u_2(x,t), & (u_L,u_0,u_R) \in (\mathcal{S}^- \boxtimes \mathcal{S}^0) \cup (\mathcal{S}^0 \boxtimes \mathcal{S}^-), \\ u_5(x,t), & (u_L,u_0,u_R) \in (\mathcal{S}^+ \boxtimes \mathcal{S}^0) \cup (\mathcal{S}^+ \boxtimes \mathcal{S}^+), \\ u_8(x,t), & (u_L,u_0,u_R) \in (\mathcal{S}^+ \boxtimes \mathcal{S}^-) \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \sigma(\alpha,\gamma) < 0\Big\}, \\ u_9(x,t), & (u_L,u_0,u_R) \in (\mathcal{S}^+ \boxtimes \mathcal{S}^-) \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \sigma(\alpha,\gamma) = 0\Big\}, \\ u_{10}(x,t), & (u_L,u_0,u_R) \in (\mathcal{S}^+ \boxtimes \mathcal{S}^-) \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \sigma(\alpha,\gamma) > 0\Big\}, \end{cases} \tag{4.14}$$

is the entropy solution of the problem (1.1)–(1.4) when $(u_L,u_0,u_R) \in \mathbb{E}_{ss}$.

Case (iii). Let $(u_L,u_0,u_R) \in \mathbb{E}_{sr}$. From (4.2), (3.17), or (3.18) and Table 3, we distinguish seven cases and four new functions labeled as u_{11}, u_{12}, u_{13}, and u_{14}, which are defined by the following correspondence rules:

$$u_{11}(x,t) = \begin{cases} u_0 & 0 \le x \le f'(u_0)t + 1 \\ (f')^{-1}\left(\dfrac{x-1}{t}\right), & f'(u_0)t \le x \le 1 \text{ for } X^+(t) \le t, \\ u_L, & \text{otherwise}, \end{cases} \qquad \text{when } u_L \le 0, \tag{4.15}$$

$$u_{12}(x,t) = \begin{cases} u_0 & 0 \le x \le f'(u_0)t + 1 \\ (f')^{-1}\left(\dfrac{x-1}{t}\right), & f'(u_0)t \le x \le 1 \text{ for } X^{++}(t) \le t, \\ u_L, & \text{otherwise,} \end{cases} \qquad \text{when } u_L > 0, \tag{4.16}$$

$$u_{13}(x,t) = \begin{cases} u_0 & \sigma(u_L,u_0)t \le x \le f'(u_0)t + 1, \\ (f')^{-1}\left(\dfrac{x-1}{t}\right), & f'(u_0)t \le x \le f'(u_R)t + 1 \text{ for } X^{+}(t) \le t, \\ u_R & f'(u_R)t + 1 \le x \le 1 \text{ for } t \le \hat{\hat{t}}_{sr} \\ & \qquad \text{or } \sigma(u_L,u_R)t + 1 \le x \le 1 \text{ for } t > \hat{\hat{t}}_{sr}, \\ u_L, & \text{otherwise,} \end{cases} \tag{4.17}$$

$$u_{14}(x,t) = \begin{cases} u_0 & \sigma(u_L,u_0)t \le x \le f'(u_0)t + 1, \\ (f')^{-1}\left(\dfrac{x-1}{t}\right), & f'(u_0)t \le x \le 1 \text{ for } X^{+}(t) \le t, \\ u_L, & \text{otherwise,} \end{cases} \tag{4.18}$$

where $(X^{++}(t), t)$ is the shock curve defined as the solution of

$$\frac{dX^{++}}{dt} = \sigma\Big(u(X^{++}(t),t), u_L\Big), \quad f'(u) = \frac{X^{++}-1}{t}, \quad f'(u_L') \le \frac{X^{++}-1}{t} \le f'(u_L),$$
$$X^{++}(-1/f'(u_L')) = 0.$$

The seven cases are detailed as follows: In the first case, we consider $(u_L, u_0, u_R) \in \mathcal{S}^- \boxtimes \mathcal{R}^-$ and observe that u is defined by u_3, given in (4.5). In the second case, we consider $(u_L, u_0, u_R) \in \mathcal{S}^- \boxtimes \mathcal{R}^0$ and note that u is given by u_{11} or u_{12}, defined in (4.15) and (4.16). Similarly, we can deduce the other five cases. If we select $(u_L, u_0, u_R) \in \mathcal{S}^0 \boxtimes \mathcal{R}^-$, $(\mathcal{S}^0 \boxtimes \mathcal{R}^0)$, $\mathcal{S}^+ \boxtimes \mathcal{R}^-$, $\mathcal{S}^+ \boxtimes \mathcal{R}^0$, and $\mathcal{S}^+ \boxtimes \mathcal{R}^+$, we see that u is given by u_{13} with $\sigma(u_L, u_0) = 0$, u_{14} with $\sigma(u_L, u_0) = 0$, u_{13} with $\sigma(u_L, u_0) > 0$, u_{14} with $\sigma(u_L, u_0) > 0$, and u_5. We then deduce that

$$u(x,t) = \begin{cases} u_3(x,t), & (u_L,u_0,u_R) \in \mathcal{S}^- \boxtimes \mathcal{R}^-, \\ u_5(x,t), & (u_L,u_0,u_R) \in \mathcal{S}^+ \boxtimes \mathcal{R}^+, \\ u_{11}(x,t), & (u_L,u_0,u_R) \in (\mathcal{S}^- \boxtimes \mathcal{R}^0) \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \alpha \le 0\Big\}, \\ u_{12}(x,t), & (u_L,u_0,u_R) \in (\mathcal{S}^- \boxtimes \mathcal{R}^0) \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \alpha > 0\Big\}, \\ u_{13}(x,t), & (u_L,u_0,u_R) \in \Big[(\mathcal{S}^0 \boxtimes \mathcal{R}^0) \cup (\mathcal{S}^+ \boxtimes \mathcal{R}^0)\Big] \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \sigma(\alpha,\gamma) \ge 0\Big\}, \\ u_{14}(x,t), & (u_L,u_0,u_R) \in \Big[(\mathcal{S}^0 \boxtimes \mathcal{R}^0) \cup (\mathcal{S}^+ \boxtimes \mathcal{R}^0)\Big] \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \sigma(\alpha,\gamma) > 0\Big\}, \end{cases} \tag{4.19}$$

is the entropy solution of the initial-boundary value problem (1.1)–(1.4) when $(u_L, u_0, u_R) \in \mathbb{E}_{sr}$.

Case (iv). Let $(u_L, u_0, u_R) \in \mathbb{E}_{rs}$. From (4.2), (3.22), or (3.23) and Table 4, similarly to Case (iii), we have seven cases and the four functions u_{15}, u_{16}, u_{17}, and u_{18}, defined by

$$u_{15}(x,t) = \begin{cases} (f')^{-1}\left(\dfrac{x}{t}\right), & 0 \le x \le f'(u_0)t \text{ for } X^{-}(t) \le t, \\ u_0 & f'(u_0)t \le x \le \sigma(u_0,u_R)t + 1, \\ u_R, & \text{otherwise,} \end{cases} \tag{4.20}$$

$$u_{16}(x,t) = \begin{cases} (f')^{-1}\left(\dfrac{x}{t}\right), & 0 \le x \le f'(u_0)t \text{ for } X^{-}(t) \le t, \\ u_0, & f'(u_0)t \le x \le 1, \\ u_R, & \text{otherwise,} \end{cases} \qquad \text{when } u_R \le 0, \tag{4.21}$$

$$u_{17}(x,t)=\begin{cases}(f')^{-1}\left(\dfrac{x}{t}\right), & 0\le x\le f'(u_0)t \text{ for } X^{--}(t)\le t,\\ u_0, & f'(u_0)t\le x\le 1,\\ u_R, & \text{otherwise,}\end{cases}\qquad \text{when } u_R>0, \tag{4.22}$$

$$u_{18}(x,t)=\begin{cases}u_L & 0\le x\le f'(u_L)t \text{ for } t\le \hat{\hat{t}}_{rs} \text{ or } 0\le x\le \sigma(u_L,u_R)t \text{ for } t>\hat{\hat{t}}_{rs},\\ (f')^{-1}\left(\dfrac{x}{t}\right), & f'(u_L)t\le x\le f'(u_0)t \text{ for } X^{-}(t)\le t,\\ u_0 & f'(u_0)t\le x\le \sigma(u_L,u_0)t,\\ u_R, & \text{otherwise,}\end{cases} \tag{4.23}$$

where $(X^{--}(t),t)$ is the shock curve defined by the solution of

$$\frac{dX^{--}}{dt}=\sigma\Big(u(X^{--}(t),t),u_R\Big),\quad f'(u)=\frac{X^{--}}{t},\quad f'(u_R')\le\frac{X^{--}}{t}\le f'(u_R),$$
$$X^{--}(1/f'(u_R'))=0.$$

The restrictions in the seven cases are detailed as follows. In the first case, we see that $(u_L,u_0,u_R)\in\mathcal{R}^-\boxtimes\mathcal{S}^-$ and observe that u is defined by u_2, given in (4.4). Second, for $(u_L,u_0,u_R)\in\mathcal{R}^0\boxtimes\mathcal{S}^-$, we see that u is given by u_{15} with $\sigma(u_0,u_R)<0$. Similarly, we can deduce the other five cases. To summarize, we deduce that

$$u(x,t)=\begin{cases}u_2(x,t), & (u_L,u_0,u_R)\in\mathcal{R}^-\boxtimes\mathcal{S}^-,\\ u_7(x,t), & (u_L,u_0,u_R)\in\mathcal{R}^+\boxtimes\mathcal{S}^+,\\ u_{15}(x,t), & (u_L,u_0,u_R)\in\Big[(\mathcal{R}^0\boxtimes\mathcal{S}^-)\cup(\mathcal{R}^0\boxtimes\mathcal{S}^0)\Big]\cap\Big\{(\alpha,\beta,\gamma)\in\mathbb{R}^3\ :\ \sigma(\alpha,\gamma)\ge 0\Big\},\\ u_{16}(x,t), & (u_L,u_0,u_R)\in(\mathcal{R}^0\boxtimes\mathcal{S}^+)\cap\Big\{(u_L,u_0,u_R)\in\mathbb{R}^3\ :\ u_R\le 0\Big\},\\ u_{17}(x,t), & (u_L,u_0,u_R)\in(\mathcal{R}^0\boxtimes\mathcal{S}^+)\cap\Big\{(u_L,u_0,u_R)\in\mathbb{R}^3\ :\ u_R> 0\Big\},\\ u_{18}(x,t), & (u_L,u_0,u_R)\in\Big[(\mathcal{R}^0\boxtimes\mathcal{S}^-)\cup(\mathcal{R}^0\boxtimes\mathcal{S}^0)\Big]\cap\Big\{(\alpha,\beta,\gamma)\in\mathbb{R}^3\ :\ \sigma(\alpha,\gamma)\ge 0\Big\},\end{cases} \tag{4.24}$$

is the entropy solution of the problem (1.1)–(1.4) when $(u_L,u_0,u_R)\in\mathbb{E}_{rs}$.

Case (v). Let $(u_L,u_0,u_R)\in\mathbb{E}_{rr}$. From (4.2), (3.24), Table 5, and a simple restriction to $x\in[0,1]$, we find that

$$u(x,t)=\begin{cases}u_1(x,t), & (u_L,u_0,u_R)\in\mathcal{R}^0\boxtimes\mathcal{R}^0,\\ u_3(x,t), & (u_L,u_0,u_R)\in\mathcal{R}^-\boxtimes\mathcal{R}^-,\\ u_4(x,t), & (u_L,u_0,u_R)\in\mathcal{R}^-\boxtimes\mathcal{R}^0,\\ u_6(x,t), & (u_L,u_0,u_R)\in\mathcal{R}^0\boxtimes\mathcal{R}^+,\\ u_7(x,t), & (u_L,u_0,u_R)\in\mathcal{R}^+\boxtimes\mathcal{R}^+,\end{cases} \tag{4.25}$$

is the entropy solution of the initial-boundary value problem (1.1)–(1.4) when $(u_L,u_0,u_R)\in\mathbb{E}_{rr}$.

From (4.10), (4.14), (4.19), (4.24), and (4.25), we can construct Table 6 to summarize all possible forms of the entropy solution for (1.1)–(1.4) (see also Figures 3–5). Furthermore, we observe that we can define the sets

$$
\left.
\begin{aligned}
&\mathbb{T}_1 = (C \boxtimes C) \cup (C \boxtimes \mathcal{S}^0) \cup (C \boxtimes \mathcal{S}^+) \cup (C \boxtimes \mathcal{R}^+) \cup (\mathcal{S}^+ \boxtimes C) \\
&\qquad \cup (\mathcal{S}^0 \boxtimes C) \cup (\mathcal{R}^- \boxtimes C) \cup (\mathcal{R}^0 \boxtimes \mathcal{R}^0), \\
&\mathbb{T}_2 = (C \boxtimes \mathcal{S}^-) \cup (\mathcal{S}^- \boxtimes \mathcal{S}^0) \cup (\mathcal{S}^0 \boxtimes \mathcal{S}^-) \cup (\mathcal{R}^- \boxtimes \mathcal{S}^-), \\
&\mathbb{T}_3 = (C \boxtimes \mathcal{R}^-) \cup (\mathcal{S}^- \boxtimes \mathcal{R}^-) \cup (\mathcal{R}^- \boxtimes \mathcal{R}^-), \\
&\mathbb{T}_4 = (C \boxtimes \mathcal{R}^0) \cup (\mathcal{R}^- \boxtimes \mathcal{R}^0), \quad \mathbb{T}_5 = \mathcal{S}^+ \boxtimes C \cup (\mathcal{S}^+ \boxtimes \mathcal{S}^0) \cup (\mathcal{S}^+ \boxtimes \mathcal{S}^+) \cup (\mathcal{S}^+ \boxtimes \mathcal{R}^+), \\
&\mathbb{T}_6 = (\mathcal{R}^0 \boxtimes C) \cup (\mathcal{R}^0 \boxtimes \mathcal{R}^+), \quad \mathbb{T}_7 = (\mathcal{R}^+ \boxtimes C) \cup (\mathcal{R}^+ \boxtimes \mathcal{S}^+) \cup (\mathcal{R}^0 \boxtimes \mathcal{R}^+), \\
&\mathbb{T}_8 = (\mathcal{S}^+ \boxtimes \mathcal{S}^-) \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \sigma(\alpha,\gamma) < 0\Big\}, \\
&\mathbb{T}_9 = (\mathcal{S}^+ \boxtimes \mathcal{S}^-) \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \sigma(\alpha,\gamma) = 0\Big\}, \\
&\mathbb{T}_{10} = (\mathcal{S}^+ \boxtimes \mathcal{S}^-) \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \sigma(\alpha,\gamma) > 0\Big\}, \\
&\mathbb{T}_{11} = (\mathcal{S}^- \boxtimes \mathcal{R}^0) \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \alpha \le 0\Big\}, \\
&\mathbb{T}_{12} = (\mathcal{S}^- \boxtimes \mathcal{R}^0) \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \alpha > 0\Big\}, \\
&\mathbb{T}_{13} = \Big[(\mathcal{S}^0 \boxtimes \mathcal{R}^0) \cup (\mathcal{S}^+ \boxtimes \mathcal{R}^0)\Big] \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \sigma(\alpha,\gamma) \ge 0\Big\}, \\
&\mathbb{T}_{14} = \Big[(\mathcal{S}^0 \boxtimes \mathcal{R}^0) \cup (\mathcal{S}^+ \boxtimes \mathcal{R}^0)\Big] \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \sigma(\alpha,\gamma) > 0\Big\}, \\
&\mathbb{T}_{15} = \Big[(\mathcal{R}^0 \boxtimes \mathcal{S}^-) \cup (\mathcal{R}^0 \boxtimes \mathcal{S}^0)\Big] \cap \Big\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \ : \ \sigma(\alpha,\gamma) \ge 0\Big\}, \\
&\mathbb{T}_{16} = (\mathcal{R}^0 \boxtimes \mathcal{S}^+) \cap \Big\{(u_L, u_0, u_R) \in \mathbb{R}^3 \ : \ u_R \le 0\Big\}, \\
&\mathbb{T}_{17} = (\mathcal{R}^0 \boxtimes \mathcal{S}^+) \cap \Big\{(u_L, u_0, u_R) \in \mathbb{R}^3 \ : \ u_R > 0\Big\}, \\
&\mathbb{T}_{18} = \Big[(\mathcal{R}^0 \boxtimes \mathcal{S}^-) \cup (\mathcal{R}^0 \boxtimes \mathcal{S}^0)\Big],
\end{aligned}
\right\}
\tag{4.26}
$$

we obtain the following theorem:

Theorem 4.1. *Consider the partition* $\Big\{\mathbb{T}_1, \ldots, \mathbb{T}_{18}\Big\}$ *of* $\mathbb{R}^3$ *given in* (4.26). *In this case,*

$$u(x,t) = u_j(x,t), \quad (x,t) \in [0,1] \times \mathbb{R}, \qquad (u_L, u_0, u_R) \in \mathbb{T}_j, \quad j = 1, \ldots, 18,$$

defines the entropy solution of the initial-boundary value problem (1.1)–(1.4).

Table 6. Summary of possible solutions to the initial-boundary value problem (1.1)–(1.4) in terms of $(u_L, u_0, u_R) \in \mathbb{R}^3$ using the partition (2.11).

(u_L,u_0) in \ (u_L,u_0) in	C	$\mathcal{S}^-$	$\mathcal{S}^0$	$\mathcal{S}^+$	$\mathcal{R}^-$	$\mathcal{R}^0$	$\mathcal{R}^+$
C	u_1	u_2	u_1	u_1	u_3	u_4	u_1
$\mathcal{S}^-$	u_1	u_2			u_3	u_{11}, u_{12}	
$\mathcal{S}^0$	u_1	u_2			u_{13}	u_{14}	
$\mathcal{S}^+$	u_5	u_8, u_9, u_{10}	u_5	u_5	u_{13}	u_{14}	u_5
$\mathcal{R}^-$	u_1	u_2			u_3	u_4	
$\mathcal{R}^0$	u_6	u_{15}	u_{15}	u_{16}, u_{17}		u_1	u_6
$\mathcal{R}^+$	u_7	u_{18}	u_{18}	u_7			u_7

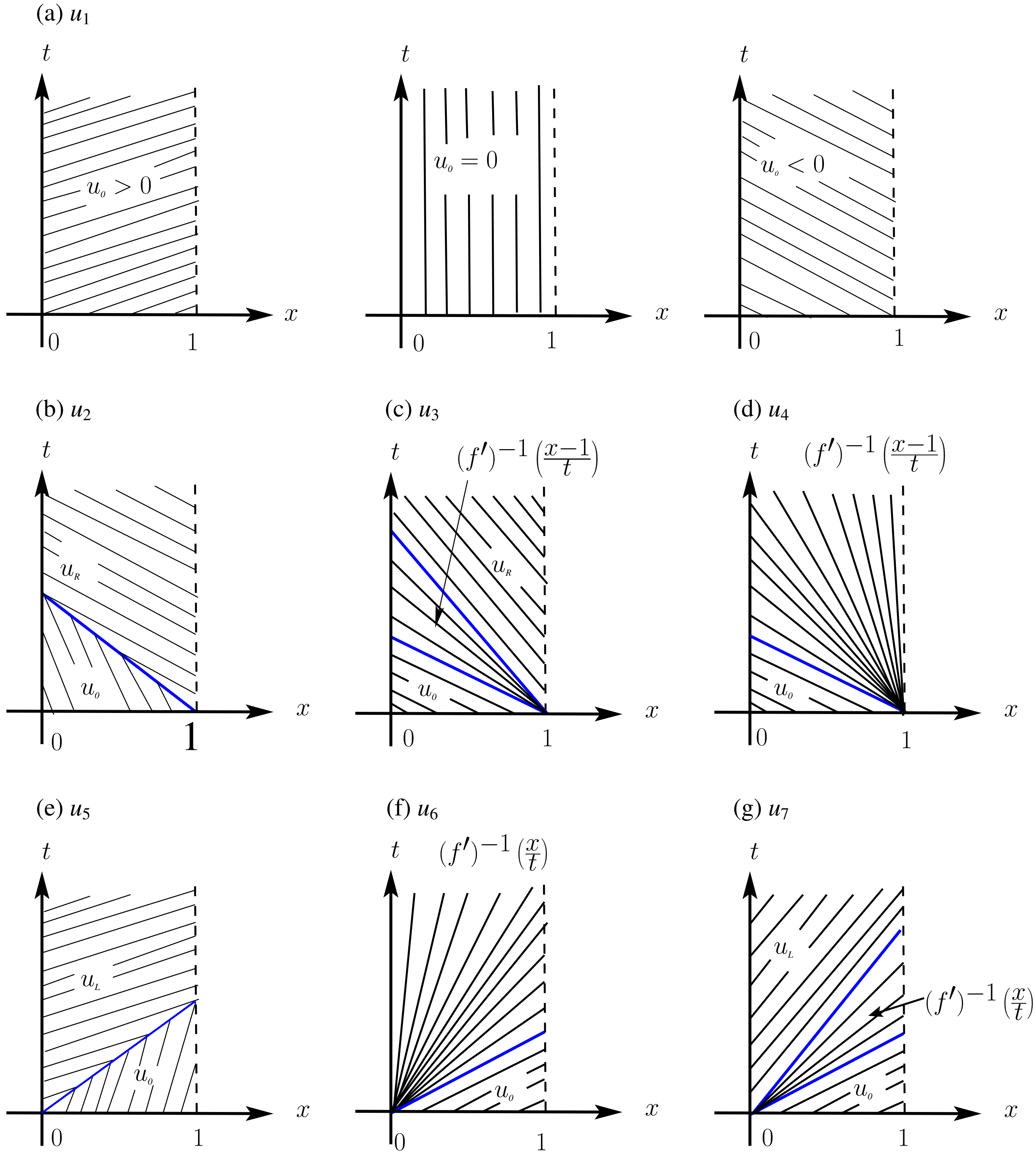

Figure 3. Characteristics for u_1, u_2, u_3, u_4, u_5, u_6, and u_7, the possible entropy solution of the elementary wave interaction problem (1.1)–(1.4) (see Table 6).

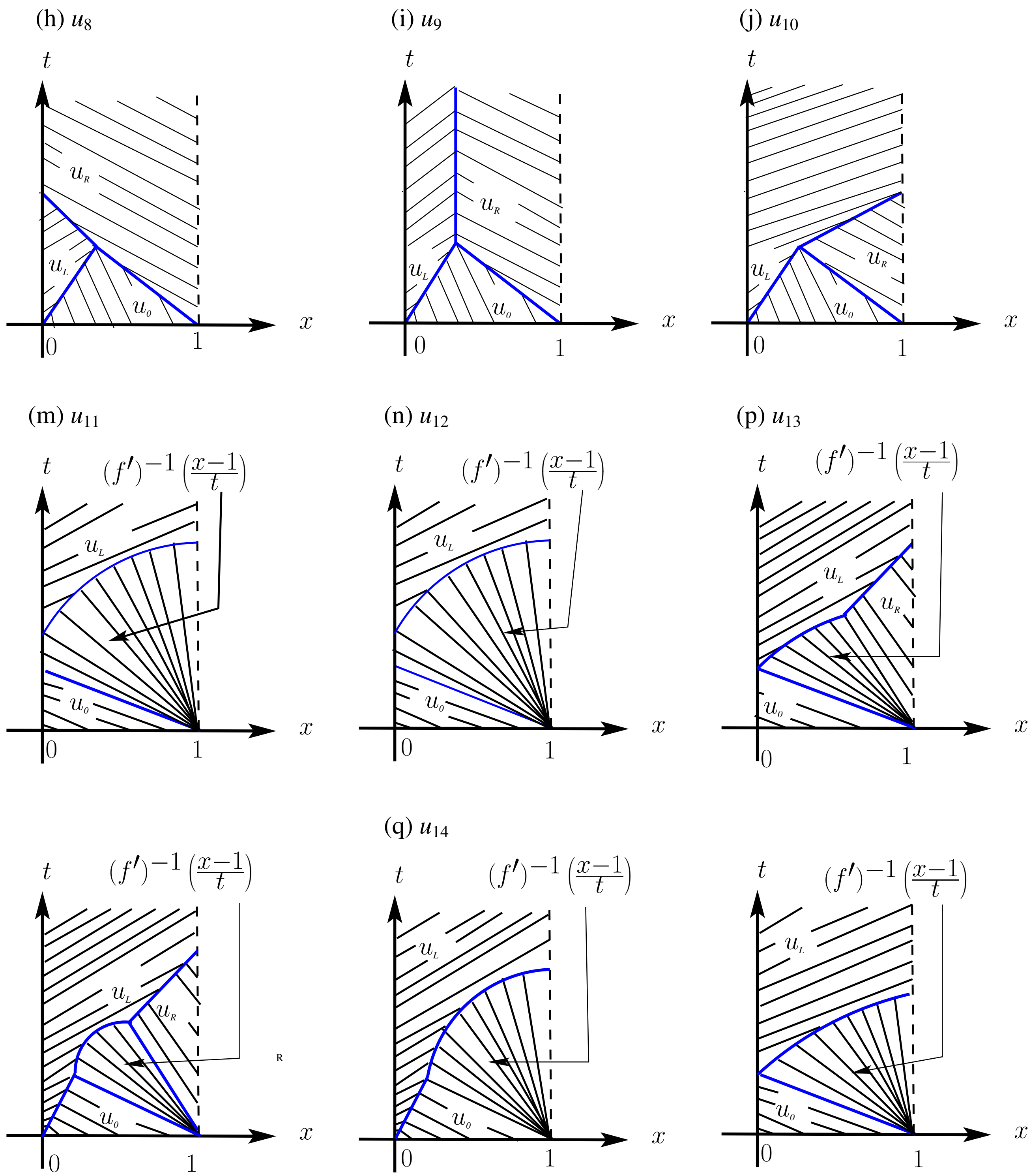

Figure 4. Characteristics for u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, and u_{14}, the possible entropy solution of the elementary wave interaction problem (1.1)–(1.4) (see Table 6).

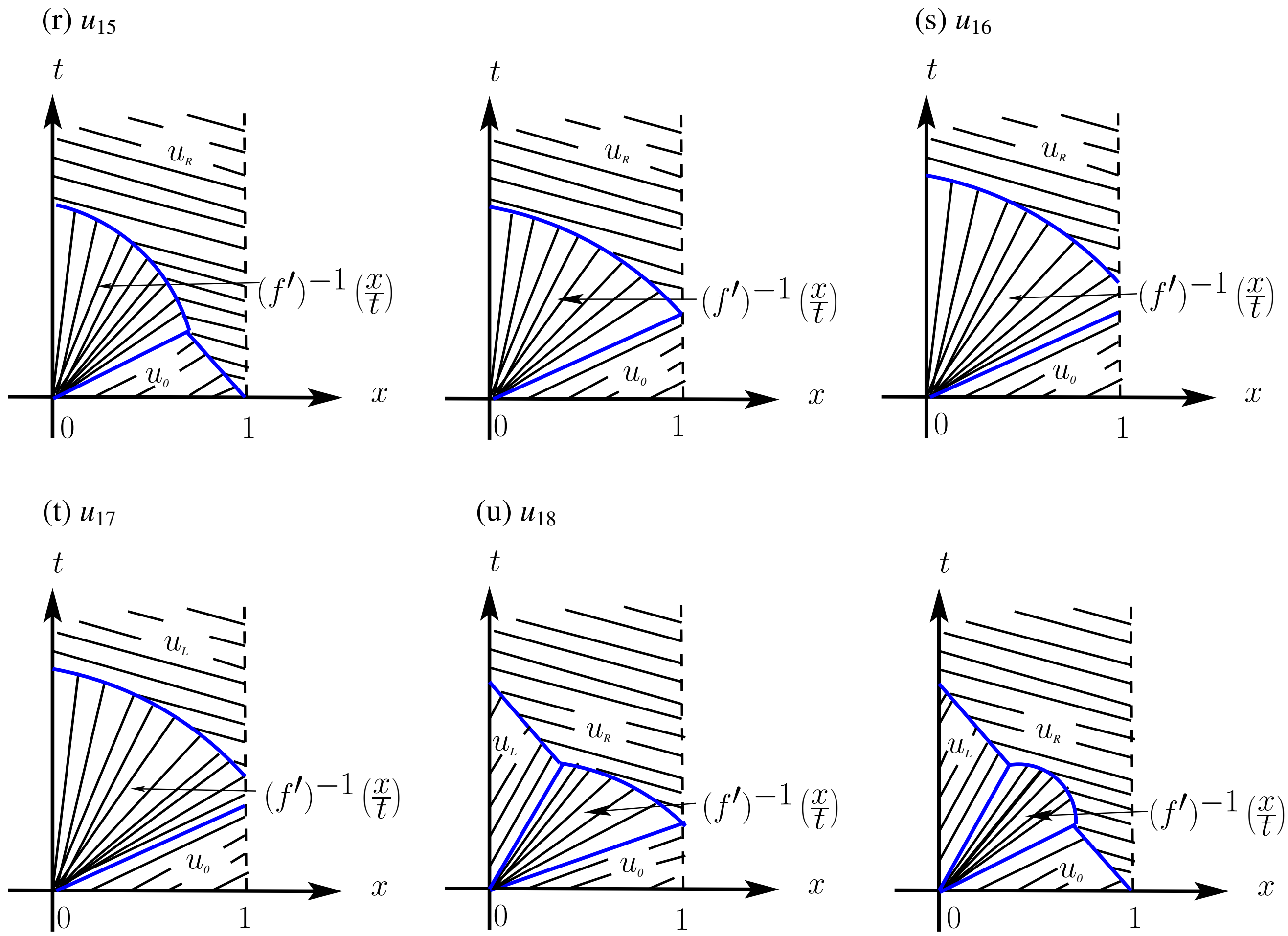

Figure 5. Characteristics for u_{15}, u_{16}, u_{17}, and u_{18}, the possible entropy solution of the elementary wave interaction problem (1.1)–(1.4) (see Table 6).

5. Proof of Theorem 1.1

From Theorem 4.1 (see also Table 6 and Figures 3–5), we see that the entropy solution of (1.1)–(1.4) is given by u_i for $i = 1, \dots, 18$. We can then construct $\mathcal{A}_T$ by analyzing the profiles of each case. We only consider the cases of u_1, u_2, and u_3 in detail, since the other cases can be followed analogously. In the case of u_1, we find that for any $T > 0$ the profiles are given by $u(x, T)$ is of the form given in (1.11) with $\boldsymbol{\rho} = \big(u_0, u_0, u_0, u_0\big)$, $\mathbf{x} = \big(0, 0, 1, 1\big)$, and $\psi(x) = u_0$. If the entropy solution is of the form u_2, we distinguish two types of profiles, both of the form given in (1.11). First, if $T > -1/\sigma(u_0, u_R)$, we observe that $u(x, T)$ is of the form (1.11) with $\boldsymbol{\rho} = \big(u_R, u_R, u_R, u_R\big)$, $\mathbf{x} = \big(0, 0, 1, 1\big)$, and $\psi(x) = u_R$. Second, if $T < -1/\sigma(u_0, u_R)$, we find that $u(x, T)$ is of the form given in (1.11) with $\boldsymbol{\rho} = \big(u_0, u_0, u_R, u_R\big)$, $\mathbf{x} = \big(0, 0, \sigma(u_0, u_R)T + 1, 1\big)$, and $\psi(x) = u_0$. Meanwhile, if the entropy solution of (1.1)–(1.4) is given by u_2, we see that $u(x, T)$ is of the form given in (1.11) with the following information

$$\text{if } T > -\frac{1}{f'(u_R)},\ \boldsymbol{\rho} = \big(u_0, u_0, u_0, u_0\big),\ \mathbf{x} = \big(0, 0, 1, 1\big),\ \text{and } \psi(x) = u_0;$$

$$\text{if } T \in \left[-\frac{1}{f'(u_0)}, -\frac{1}{f'(u_R)}\right],\ \boldsymbol{\rho} = \big(u_0, u_0, u_R, u_R\big),\ \mathbf{x} = \big(0, 0, f'(u_R)T + 1, 1\big),\ \text{and } \psi(x) = \big(f'\big)^{-1}\left(\frac{x-1}{T}\right);$$

$$\text{if } T < -\frac{1}{f'(u_0)}, \boldsymbol{\rho} = \big(u_0, u_0, u_R, u_R\big),\ \mathbf{x} = \big(0, 0, f'(u_R)T + 1, 1\big),\ \text{and } \psi(x) = \big(f'\big)^{-1}\left(\frac{x-1}{T}\right).$$

Analogously, we deduce the profiles for the other entropy solutions.

6. Application of Theorem 1.1 to the inverse problem of flux identification

In this section, we restrict our attention to the inverse problem of flux identification. We begin by presenting the terminology used in the context of inverse problems theory.

1) Direct problem: Given $T > 0$, the initial, boundary, and flux functions satisfying (1.5), find the entropy solution $u(x, T)$ of (1.6)–(1.9).
2) Inverse problem: Given $u^{obs}(x)$ at fixed time $T > 0$ and the initial and boundary functions, find the flux function that belongs to U_{ad} (see (1.5)), such that the entropy solution $u(x, T)$ of (1.6)–(1.9) is "as close as possible" to $u^{obs}(x)$.

If we consider that $u^{obs} \in \mathcal{A}_T$ for almost every $x \in [0, 1]$ or $u^{obs} = u_f(\cdot, T)$ for some $f \in U_{ad}$, from the application of Theorem 1.1, we distinguish the cases of ill-posedness and local uniqueness of the flux identification problem. More precisely, we have the following result:

Theorem 6.1. *Assume that $(u_L, u_0, u_R) \in \mathbb{R}^3$ and $T > 0$ are given and assume that u^{obs} is an attainable profile of a direct problem solution at a fixed time. Then the following two assertions are satisfied:*

(i) If $(u^{obs})'(x) = 0$, almost everywhere in $x \in [0, 1]$, the inverse problem is ill-posed.
(ii) If $(u^{obs})'(x) > 0$, for $x \in]x_2, x_3[\subseteq [0, 1]$ the inverse problem is locally well-posed, in the sense that

the flux function is uniquely defined by

$$f(u) = \begin{cases} \dfrac{1}{T}\left(x_2 + \displaystyle\int_{u^{obs}(x_2^+)}^{u} (u^{obs})^{-1}(s)ds\right), & u_0 > 0, \\ \\ \dfrac{1}{T}\left(x_2 + u^{obs}(x_2^+) - u + \displaystyle\int_{u^{obs}(x_2^+)}^{u} (u^{obs})^{-1}(s)ds\right), & u_0 < 0, \end{cases} \tag{6.1}$$

for $u \in [u^{obs}(x_2^+), u^{obs}(x_3^-)]$.

Proof. The proof is constructive and based on the characterization of $\mathcal{A}_T$ given in Theorem 1.1; equivalently, using the fact that $u^{obs} \in \mathcal{A}_T$, we deduce that u^{obs} is of the form given in (1.11) with ψ being a constant or a strictly increasing function. If ψ is constant, we have Case (i), and when ψ is strictly increasing, we have Case (ii). Equivalently, in a more extensive way, we see that u^{obs} is one of the profiles of entropy solution types $u_i,\ i = 1, \dots, 18$, as detailed in Theorem 4.1. Hence, for the convenience of presentation, we restrict our attention to the assertions in (i) and (ii).

Proof of (i). The assumption that $(u^{obs})'(x) = 0$, almost everywhere in $x \in [0, 1]$ is equivalent to considering that u^{obs} is a constant or a piecewise constant function on $[0, 1]$. By observing the profiles of $u_i,\ i = 1, \dots, 18$, we can subdivide the analysis into two sub-cases:

(a) Let us consider that u^{obs} is constant, i.e., $u^{obs}(x) = C$ for $x \in [0, 1]$ with $C \in \{u_L, u_0, u_R\}$. The profiles of this form are attainable in most situations when T is large enough or, to be precise, in all cases except u_3, u_4, u_6, and u_{11}. Moreover, this profile is attainable for any $f \in U_{ad}$.

(b) Let us consider that u^{obs} is piecewise constant. We distinguish two types of profiles. First, we have the profiles where the constant states are separated by a single discontinuity, as given in u_8, u_9, u_{10}, u_{13}, and u_{18}. Second, two discontinuities are separating the constant states when T is small in u_8, u_9, and u_{10}. We observe that the profiles of this type are attainable for any $f \in U_{ad}$.

Consequently, we note that these types of observation functions are obtained for any $f \in U_{ad}$ or, to be more precise, the solution of the inverse problem can be stated as "either f is arbitrary and belongs to U_{ad}, or f belongs to a subset of U_{ad} defined by a very weak restriction". A weak restriction means an algebraic restriction that does not imply uniqueness. For instance, if $u^{obs}(x) = u_L$ and $u_L = u_0 > 0 > u_R$, is enough to consider that f satisfies $f(u_R) \le f(u_0) = f(u_L)$, where it is clear that $f(u_0) = f(u_L)$ is always true (since $u_L = u_0$), as well as the inequality $f(u_R) \le f(u_0)$ is widely satisfied, we can check that there is more than one solution to the inverse problem. Hence, to conclude Case (i), we see that the inverse problem is ill-posed in uniqueness.

Proof of (ii). Let us assume that $(u^{obs})'(x) > 0$ for $x \in [0, 1]$. To be precise, in the analysis, we assume the more simple case, given by the fact that u^{obs} is a continuous function with a strictly increasing part. For instance, when $u^{obs}(x) = u_7(x, T)$, as given in (4.9), u^{obs} is of the following form:

$$u^{obs}(x) := \begin{cases} u_L, & 0 \le x \le f'(u_L)T, \\ (f')^{-1}(x/T), & f'(u_L)T < x < f'(u_0)T, \\ u_0, & f'(u_0)T \le x \le 1. \end{cases}$$

From the injectivity of u^{obs} for $x \in [f'(u_L)T, f'(u_0)T]$, we deduce that

$$f(u) = f(u_L) + \frac{1}{T}\int_{u_L}^{u} (u^{obs})^{-1}(s)ds, \quad u \in [u_L, u_0]. \tag{6.2}$$

Analogously, for $u^{obs}(x) = u_3(x, T)$ (see (4.5)), we have the following relation

$$u^{obs}(x) := \begin{cases} u_0, & 0 \le x \le f'(u_0)T + 1, \\ (f')^{-1}\Big((x-1)/T\Big), & f'(u_0)T + 1 < x < f'(u_R)T + 1, \\ u_R, & f'(u_R)T + 1 \le x \le 1, \end{cases}$$

which implies that the flux function is given by

$$f(u) = f(u_0) + \frac{u_0 - u}{T} + \frac{1}{T}\int_{u_0}^{u} (u^{obs})^{-1}(s)ds. \quad u \in [u_0, u_R]. \tag{6.3}$$

The other cases of u^{obs} with a locally strictly increasing behavior are analyzed by similar arguments, deducing the formula given in (6.1). Hence, in this case, we deduce the conclusion of local uniqueness of the inverse problem. □

Example 6.1. *Example of the application of Theorem 6.1 (i)*

We begin by assuming that $u_L = 3$, $u_0 = 2$, and $u_R = 1$. Then the entropy solution of the direct problem (1.1)–(1.4) is of the form given by the type u_5; to be precise

$$u(x,t) = \begin{cases} 3, & 0 \le x \le \sigma(3,2)t \\ 2, & \text{otherwise.} \end{cases}$$

In the case of the inverse problem, we consider $T = 1/2$ and analyze two types of profile observations. First, if we assume that $u_1^{obs}(x) = 3$ for $x \in [0, 1]$, we deduce that $u^{obs}(x) = u_f(x, 1/2)$ for any $f \in U_{ad}$ satisfying the restriction $f(3) \ge f(2) + 2$. We deduce the condition for f from the intersection of the shock curve $x = \sigma(3,2)t$, and the right boundary of the domain $x = 1$ is defined by $t^* = 1/\sigma(3,2)$. Then u_1^{obs} is attainable when the observation time T is such that $T > t^*$; equivalently $f(3) > f(2) + 2$, since $T = 1/2$ and $\sigma(3,2) = f(3) - f(2) > 0$. As a second type of observation, we consider

$$u_2^{obs}(x) = \begin{cases} 3, & x \in [0, 1/2], \\ 2, & x \in]1/2, 1]. \end{cases}$$

By similar arguments, we deduce that the inverse problem is ill-posed, since $u_2^{obs}(x) = u_f(x, 1/2)$ for any $f \in U_{ad}$ satisfying the restriction $f(3) = f(2) + 1$. In this case, the restriction for f is deduced from the the shock curve $x = \sigma(3,2)T$, the observation time $T = 1/2$, and the discontinuity location $x = 1/2$.

Example 6.2. *Example of the application of Theorem 6.1 (ii).*

We give two cases with applications of Theorem 6.1 (ii). First, we consider $u_L = 1$, $u_0 = 2$, $u_R = 2$, $T = 1/2$, and the observation given by

$$u_3^{obs}(x) = \begin{cases} 1/4, & x \in [0, 1/2], \\ x^2, & x \in]1/2, 1]. \end{cases}$$

From Theorem 6.1 (ii), the local inverse problem solution is given by

$$f(u) = 2\left(\frac{1}{2} + \int_{1/4}^{u} \sqrt{s}ds\right) = \frac{5 + 8u^{3/2}}{6} \quad \text{for } u \in [1/4, 1].$$

This relation follows by an application of Theorem 6.1 (ii). Second, we assume that $u_L = -1$, $u_0 = 2$, $u_R = 3$, $T = 1/2$, and the observation defined by

$$u_4^{obs}(x) = \begin{cases} -1, & x \in [0, 1/4[, \\ e^x, & x \in [1/4, 1/2[, \\ 2, & x \in [1/2, 3/4[, \\ 3, & x \in [3/4, 1]. \end{cases}$$

Theorem 6.1 (ii) means that

$$f(u) = 2\left(\frac{1}{4} + \int_{e^{1/4}}^{u} \ln s ds\right) = 2u(\ln u - 1) + \frac{1}{2}(3e^{1/4} + 1), \quad u \in [e^{1/4}, e^{1/2}]$$

is the local flux identification of the flux.

7. Conclusions and future work

We have applied the characteristic method to solve the elementary wave interaction for a convex scalar conservation law in a bounded domain. We have proved that the set of attainable profiles of the entropy solution at a fixed time is defined by the union of the sets of piecewise constant functions on the spatial bounded domain and locally increasing functions on the spatial bounded domain. Moreover, we have introduced an application result for the flux identification problems from observations of the direct problem solution profile at a fixed time. We have proved that when the observation function is piecewise constant, the inverse problem is ill-posed, and when the observation function has a strictly increasing part, the inverse problem is locally well-posed in uniqueness. Thus, the main contributions of this paper are the characterization of the set of the attainable profiles of the elementary wave interaction for a convex scalar conservation law and the application to define the appropriate framework where we can analyze the uniqueness of the flux identification problem in scalar conservation laws. Furthermore, we have introduced an analytical formula for the flux function identification problem.

Research into inverse and optimal control problems in partial differential equations is an active area with increasing contributions in the last decade. However, several open questions are still part of the challenges and can be formulated as improvements to the present work. At least we can indicate the following issues that need more exploration:

(i) *Applications to the study of real phenomena.* The present work is reduced to the case of a scalar convex flux function, which limits the industrial or laboratory applications of the results, since, in several applications, the flux function is a non-convex function. For instance, in sedimentation [23], the flux function is a non-convex function with one or three inflection points, and in flow through porous media [5, 36, 38, 46], the flux function has one inflection point and has a S-shape form. Therefore, improving the result to characterize the attainable set profiles in the case of non-convex flux functions is a possible topic to research. Moreover, the analysis of the uniqueness of the inverse problem in the case of non-convex flux functions is an open problem.

(ii) *Generalization of the results to general flux functions.* Another possible topic of research is the characterization of the attainable set profile of an entropy solution of a scalar conservation law with a more general flux function by applying the generalized characteristics method, as developed in [39]. Other possible extensions of the results of this paper are the improvement of the results to more general flux functions as developed recently in [40–44].

(iii) *Applications to optimal control theory.* In identifying flux from experimental observations, we need to reformulate the inverse problem as an optimal control problem, where the cost function compares the observation profile with the entropy solution profile of the direct problem. In this sense, the characterization of the entropy solution profile can be used to define a second term of the cost function, which incorporates regularization. Regularization can help to achieve the convergence of numerical methods in the case of parameter identification problems.

Furthermore, advanced numerical methods for flux identification and control problems, in the case of experimental data are required.

Author contributions

A. Coronel: Supervision, Methodology, Writing-original draft, Visualization, Conceptualization, Formal analysis, Investigation, Writing-review and editing, Funding acquisition; A. Tello: Conceptualization, Formal analysis, Investigation, Writing-review and editing, Funding acquisition; F. Huancas: Supervision, Methodology, Writing-original draft, Visualization, Conceptualization, Formal analysis, Investigation, Writing-review and editing, Funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that they have no conflict of interest.

References

1. D. Serre, Systems of conservation laws: A challenge for the XXIst century, In: *Mathematics unlimited–2001 and beyond*, Berlin, Heidelberg: Springer, 2001, 1061–1080. https://doi.org/10.1007/978-3-642-56478-9_54
2. A. Aw, M. Rascle, Resurrection of "second order" models of traffic flow, *SIAM J. Appl. Math.*, **60** (2000), 916–938. https://doi.org/10.1137/S0036139997332099
3. T. Seo, A. M. Bayen, T. Kusakabe, Y. Asakura, Traffic state estimation on highway: A comprehensive survey, *Annu. Rev. Control*, **43** (2017), 128–151. https://doi.org/10.1016/j.arcontrol.2017.03.005
4. B. L. Keyfitz, H. C. Kranzer, A System of non-strictly hyperbolic conservation laws arising in elasticity theory, *Arch. Rational Mech. Anal.*, **72** (1980), 219–241. https://doi.org/10.1007/BF00281590
5. Y. Wang, C. Y. Kao, Central schemes for the modified Buckley–Leverett equation, *J. Comput. Sci.*, **4** (2013), 12–23. https://doi.org/10.1016/j.jocs.2012.02.001
6. G. B. Whitham, *Linear and nonlinear waves*, New York: Wiley, 1999. http://doi.org/10.1002/9781118032954
7. B. Perthame, *Transport equations in biology*, Basel: Birkhäuser, 2007. https://doi.org/10.1007/978-3-7643-7842-4
8. A. Friedman, Conservation laws in mathematical biology, *Discrete Cont. Dyn.*, **32** (2012), 3081–3097. https://doi.org/10.3934/dcds.2012.32.3081
9. B. Andreianov, C. Donadello, M. D. Rosini, Crowd dynamics and conservation laws with nonlocal constraints and capacity drop, *Math. Mod. Meth. Appl. S.*, **24** (2014), 2685–2722. https://doi.org/10.1142/S0218202514500341
10. A. Bayen, M. L. D. Monache, M. Garavello, P. Goatin, B. Piccoli, *Control problems for conservation laws with traffic applications: modeling, analysis, and numerical methods*, Cham: Birkhäuser, 2002. https://doi.org/10.1007/978-3-030-93015-8
11. C. Prieur, Control of systems of conservation laws with boundary errors, *Netw. Heterog. Media*, **4** (2009), 393–407. https://doi.org/10.3934/nhm.2009.4.393
12. V. Rasvan, Control of conservation laws-an application, *Tatra Mountains Mathematical Publications*, **71** (2018), 155–174. https://doi.org/10.2478/tmmp-2018-0014
13. G. Chavent, Identification of distributed parameters: about the ouput least square method, its implementation, and identiafibility, *IFAC Proceedings Volumes*, **12** (1979), 85–97. https://doi.org/10.1016/S1474-6670(17)65413-2
14. S. Ulbrich, On the existence and approximation of solutions for the optimal control of nonlinear hyperbolic conservation laws, In: *Optimal control of partial differential equations*, Basel: Birkhäuser, 1999, 287–299. https://doi.org/10.1007/978-3-0348-8691-8_25

15. F. Ancona, M. T. Chiri, Attainable profiles for conservation laws with flux function spatially discontinuous at a single point, *ESAIM: Control, Optimisation and Calculus of Variations*, **26** (2020), 124. https://doi.org/10.1051/cocv/2020044

16. N. D. Nitti, E. Zuazua, On the controllability of entropy solutions of scalar conservation laws at a junction via Lyapunov methods, *Vietnam J. Math.*, **51** (2023), 71–88. https://doi.org/10.1007/s10013-022-00598-9

17. N. D. Nitti, Analysis, control, and singular limits for hyperbolic conservation laws, PhD Thesis, Friedrich-Alexander-Universitaet Erlangen-Nuernberg, Erlangen-Nuernberg, Germany, 2023.

18. F. James, M. Sepúlveda, Convergence results for the flux identification in a scalar conservation law, *SIAM J. Control Optim.*, **37** (1999), 869–891. https://doi.org/10.1137/S0363012996272722

19. A. Coronel, F. James, M. Sepúlveda, Numerical identification of parameters for a model of sedimentation processes, *Inverse Probl.*, **19** (2003), 951–972. https://doi.org/10.1088/0266-5611/19/4/311

20. S. Berres, R. Bürger, A. Coronel, M. Sepúlveda, Numerical identification of parameters for a strongly degenerate convection-diffusion problem modelling centrifugation of flocculated suspensions, *Appl. Numer. Math.*, **52** (2005), 311–337. https://doi.org/10.1016/j.apnum.2004.08.002

21. R. Bürger, A. Coronel, M. Sepúlveda, On an upwind difference scheme for strongly degenerate parabolic equations modelling the settling of suspensions in centrifuges and non-cylindrical vessels, *Appl. Numer. Math.*, **56** (2006), 1397–1417. https://doi.org/10.1016/j.apnum.2006.03.021

22. R. Bürger, A. Coronel, M. Sepúlveda, A numerical descent method for an inverse problem of a scalar conservation law modelling sedimentation, In: *Numerical mathematics and advanced applications*, Berlin, Heidelberg: Springer, 2008, 225–232. https://doi.org/10.1007/978-3-540-69777-0_26

23. R. Bürger, A. Coronel, M. Sepúlveda, Numerical solution of an inverse problem for a scalar conservation law modelling sedimentation, In: *Hyperbolic problems: theory, numerics and applications*, Providence: American Mathematical Society, 2009, 445–454. http://doi.org/10.1090/psapm/067.2/2605240

24. H. X. Liu, T. Pan, Interaction of elementary waves for scalar conservation laws on a bounded domain, *Math. Method. Appl. Sci.*, **26** (2003), 619–632. https://doi.org/10.1002/mma.370

25. C. Bardos, A. Y. Leroux, J. C. Nédélec, First order quasilinear equations with boundary conditions, *Commun. Part. Diff. Eq.*, **4** (1979), 1017–1034. https://doi.org/10.1080/03605307908820117

26. S. N. Kružkov, First order quasilinear equations in several independent variables, *Mathematics of the USSR-Sbornik*, **10** (1970), 217–143. https://doi.org/10.1070/SM1970v010n02ABEH002156

27. F. Ancona, A. Marson, On the attainable set for scalar nonlinear conservation laws with boundary control, *SIAM J. Control Optim.*, **36** (1998), 290–312. https://doi.org/10.1137/S0363012996304407

28. M. Corghi, A. Marson, On the attainable set for scalar balance laws with distributed control, *ESAIM: Control, Optimisation and Calculus of Variations*, **22** (2016), 236–266. https://doi.org/10.1051/cocv/2015009

29. B. P. Andreianov, C. Donadello, A. Marson, On the attainable set for a scalar nonconvex conservation law, *SIAM J. Control Optim.*, **55** (2017), 2235–2270. https://doi.org/10.1137/16M1085966
30. M. Herty, A. Kurganov, D. Kurochkin, Numerical method for optimal control problems governed by nonlinear hyperbolic systems of PDEs, *Commun. Math. Sci.*, **13** (2015), 15–48. http://doi.org/10.4310/CMS.2015.v13.n1.a2
31. C. Castro, E. Zuazua, Flux identification for 1-d scalar conservation laws in the presence of shocks, *Math. Comput.*, **80** (2011), 2025–2070. http://doi.org/10.1090/S0025-5718-2011-02465-8
32. Q. Li, J. H. Geng, S. Evje, Identification of the flux function of nonlinear conservation laws with variable parameters, *Physica D*, **451** (2023), 133773. https://doi.org/10.1016/j.physd.2023.133773
33. B. J. Lucier, A moving mesh numerical method for hyperbolic conservation laws, *Math. Comp.*, **46** (1986), 59–69. https://doi.org/10.2307/2008214
34. R. J. LeVeque, *Numerical methods for conservation laws*, 2 Eds., Basel: Birkhäuser, 1992. https://doi.org/10.1007/978-3-0348-8629-1
35. C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, 3 Eds., Berlin: Springer, 2010. https://doi.org/10.1007/3-540-29089-3
36. S. E. Buckley, M. C. Leverett, Mechanism of fluid displacement in sands, *Transactions of the AIME*, **146** (1942), 107–116. http://doi.org/10.2118/942107-G
37. J. Bear, *Dynamics of fluids in porous media*, Amsterdam: American Elsevier, 1972.
38. C. J. van Duijn, L. A. Peletier, I. S. Pop, A new class of entropy solutions of the Buckley–Leverett equation, *SIAM J. Math. Anal.*, **39** (2007), 507–536. https://doi.org/10.1137/05064518X
39. Adimurthi, S. S. Ghoshal, G. D. Veerappa, Finer regularity of an entropy solution for 1-d scalar conservation laws with non uniform convex flux, *Rend. Sem. Mat. Univ. Padova*, **132** (2014), 1–24. https://doi.org/10.4171/RSMUP/132-1
40. S. S. Ghoshal, S. Junca, A. Parmar, Higher regularity for entropy solutions of conservation laws with geometrically constrained discontinuous flux, *SIAM J. Math. Anal.*, **56** (2014), 6121–6136. https://doi.org/10.1137/23M1604199
41. B. Guelmame, S. Junca, D. Clamond, Regularizing effect for conservation laws with a Lipschitz convex flux, *Commun. Math. Sci.*, **17** (2019), 2223–2238 https://doi.org/10.4310/CMS.2019.v17.n8.a6
42. S. S. Ghoshal, S. Junca, A. Parmar, Fractional regularity for conservation laws with discontinuous flux, *Nonlinear Anal.-Real*, **75** (2014), 103960. https://doi.org/10.1016/j.nonrwa.2023.103960
43. C. Caginalp, Minimization solutions to conservation laws with non-smooth and non-strictly convex flux, *AIMS Mathematics*, **3** (2018), 96–130. https://doi.org/10.3934/Math.2018.1.96
44. C. Caginalp, A minimization approach to conservation laws with random initial conditions and non-smooth, non-strictly convex flux, *AIMS Mathematics*, **3** (2018), 148–182. https://doi.org/10.3934/Math.2018.1.148
45. E. Godlewski, P. A. Raviart, *Hyperbolic systems of conservation laws*, Paris: Ellipses, 1991.
46. E. F. Toro, *Riemann solvers and numerical methods for fluid dynamics: a practical introduction*, 3 Eds., Berlin, Heidelberg: Springer Science, 2009. https://doi.org/10.1007/b79761

Appendix

A. Weak solution of partial differential equations: entropy solution in conservation laws

This section discusses the concept of entropy solutions in scalar conservation laws. Our presentation is based mainly on the books [34, 45, 46].

Inherently, having a partial differential equation implies the aim of solving this partial differential equation, but this challenge entails the first obstacle, which is to specify what is to be understood as a solution. Considering the concept of solutions, the modern theory of partial differential equations distinguishes between two large groups known as classical solutions and weak solutions. The concept of the classical solution is defined as follows: A given function is a classical solution of a partial differential equation when this function satisfies the differential equation, including its initial and boundary conditions if this is the case. However, a weak solution cannot be defined as a single concept for any partial differential equation and depends on many aspects, such as the type of differential equation or the type of coefficients involved. Consequently, before beginning to analyze weak solutions, defining the concept of weak solutions is essential. In particular, some types of weak solutions have specific names, such as variational solutions, viscosity solutions, renormalized solutions, entropy solutions, and mild solutions. The most extended concept of weak solutions is the variational solution, which originated in the study of elliptic equations when the coefficients belong to Sobolev spaces. However, the application of that concept cannot be used to analyze all types of partial differential equations, since, in some cases, the uniqueness or the stability of solutions could be lost. This will be specified below for the inviscid Burger equation, where the uniqueness of the variational solutions is lost, motivating the concept of entropy solutions.

Let us consider the case of linear flux in the scalar conservation law (1.2): $f(u) = au$ with a being a constant. We omit the boundary conditions, as we are interested in the following Cauchy problem:

$$u_t + au_x = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{A.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{A.2}$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ is a given function. Equation (A.1) is called the transport equation or the advection equation and is analyzed by the characteristics method. The characteristic curve associated with (A.1) is defined as the function $X : \mathbb{R}_+ \to \mathbb{R}$ satisfying the ordinary differential equation

$$X'(t) = a, \qquad t \in \mathbb{R}_+, \tag{A.3}$$

$$X(0) = x_0, \qquad x_0 \in \mathbb{R}. \tag{A.4}$$

We observe that the solution of (A.3)–(A.4) is given by $X(t) = at + x_0$. If we differentiate u satisfying (A.3)–(A.4) through the characteristic curve $(X(t), t)$, we have

$$\frac{du}{dt}(X(t),t) = u_t(X(t),t) + u_x(X(t),t)X'(t) = u_t(X(t),t) + au_x(X(t),t) = 0.$$

This means that u is constant throughout the characteristic curve $(X(t), t)$ or, more explicitly

$$u(X(t),t) = u(X(0),0) = u(x_0,0) = u_0(x_0) = u_0(X(t) - at).$$

Hence, we see that the function

$$u(x,t) = u_0(x - at), \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$

is the classical solution of (A.1)–(A.2) when u_0 is a C^1 function; otherwise, we need the definition of the "weak solution". However, we avoid that discussion by focusing our presentation on the definition of the entropy solution associated with nonlinear conservation laws.

In the case of non-linear flux, we consider $f(u) = u^2/2 \in U_{ad}$ (see (1.5)) and study the initial value problem

$$u_t + uu_x = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{A.5}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{A.6}$$

with $u_0 : \mathbb{R} \to \mathbb{R}$ being a given function. Equation (A.5) is called the Burger equation or sometimes the inviscid Burger equation. Analogous to the advection equation, the analysis of (A.5) can be developed by the characteristics method. We begin by defining the characteristic curve associated with (A.5) as the solution of the ordinary differential equation

$$X'(t) = u(X(t),t), \qquad t \in \mathbb{R}_+, \tag{A.7}$$

$$X(0) = x_0, \qquad x_0 \in \mathbb{R}. \tag{A.8}$$

A differentiation of the function u, defined by Eqs (A.5) and (A.6), through the characteristic curve $(X(t),t)$, we get

$$\frac{du}{dt}(X(t),t) = u_t(X(t),t) + u_x(X(t),t)X'(t) = u_t(X(t),t) + u(X(t),t)u_x(X(t),t) = 0. \tag{A.9}$$

Similar, to the advection case, we find that (A.9) implies that u, defined by (A.5)–(A.6), is constant through the characteristic curves $(X(t),t)$, i.e. $u(X(t),t) = u(X(0),0) = u(x_0,0) = u_0(x_0)$. The solution of (A.7)–(A.8) is defined by $X(t) = u_0(x_0)t + x_0$. However, unlike the case of the transport equation, if we consider a general initial condition, we cannot find an explicit relation defining x_0 in terms of $X(t)$ and t, since it is not always possible to find and explicit definition of x_0 from the equation $X(t) = u_0(x_0)t + x_0$. A universal exception is the initial conditions, which are piecewise constant.

We consider the following concept of the classical solution.

Definition A.1. *A function u is called a classic solution of the Cauchy problem* (A.5)–(A.6) *if $u \in C^1(\mathbb{R} \times \mathbb{R}_+)$ satisfies* (A.5)–(A.6) *for each* $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

Let us discuss the possibility of the existence of discontinuities in a finite time, independently of the regularity of the initial condition. Let us assume that u_0 is given by

$$u_0(x) = \begin{cases} 1, & x \le 0, \\ 0, & \text{otherwise.} \end{cases} \tag{A.10}$$

We observe that the characteristics curves are defined by

$$X(t) = u_0(x_0)t + x_0 = \begin{cases} t + x_0, & x_0 \le 0, \\ x_0, & \text{otherwise.} \end{cases}$$

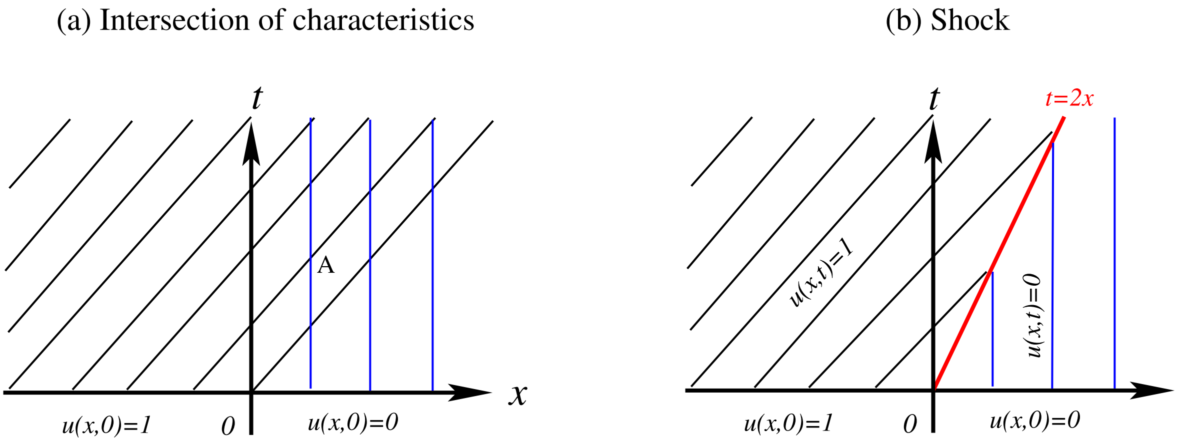

Figure 6. Characteristic curves to construct the weak solution of the Burger equation with the initial condition (A.10).

We observe that in a plane $t - x$, the characteristics $X(t) = t + \xi_1$ for $\xi_1 \leq 0$ and $X(t) = \xi_2$ for $\xi_2 > 0$ have an intersection in $A = (\xi_2, \xi_2 - \xi_1)$; see Figure 6(a). If consider the fact that u is constant through the characteristic curves. We see that in A is a multivalued function, since $u(\xi_2, \xi_2 - \xi_1) = 1$ and $u(\xi_2, \xi_2 - \xi_1) = 0$. It is not consistent with the physical meaning of u, since u models a conserved quantity, like mass density or momentum, which are not multivalued functions. The solution in A must be a discontinuous function and, consequently, a weak solution. In principle, we assume that the weak solution is defined as the variational formulation.

Definition A.2. *A function u is called a weak solution of the Cauchy problem* (A.5)–(A.6) *if*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}}\left(\varphi_t u + \varphi_x \frac{u^2}{2}\right)(x,t)dxdt + \int_{\mathbb{R}} \varphi(x,0)u_0(x)dx = 0$$

holds for all functions $\varphi \in C_0^1(\mathbb{R}\times\mathbb{R})$.

It is well known that the weak solutions are characterized by the following result.

Proposition A.1. *Let u be a piecewise $C^1(\mathbb{R}\times\mathbb{R}_+)$ function. In this case, u is a weak solution of* (A.5)-(A.6) *if and only if the following two conditions are satisfied:*

(i) u is a classical solution of (A.5)–(A.6) in the domains where it is C^1.
(ii) If $x = \Sigma(t)$ is a discontinuity curve of u, then the Rankine–Hugoniot jump condition

$$\frac{d\Sigma(t)}{dt} = \frac{u(\Sigma(t)-0,t) + u(\Sigma(t)+0,t)}{2}$$

is satisfied.

Hence, in the case of u_0, as given in (A.10), we can construct the weak solution as follows. Applying, Proposition (A.1) and using the fact that $\Sigma(0) = 0$, we deduce that the curve's discontinuity is given by $\Sigma(t) = t/2$, since $\sigma(1,0) = 1/2$. In this case, the weak solution of the initial value problem (A.5)–(A.10) is defined by

$$u_0(x) = \begin{cases} 1, & x \leq t/2, \\ 0, & \text{otherwise}, \end{cases}$$

which is known as a shock wave; see Figure 6(b). We remark that the formation of shocks appears even in the case of a smooth initial condition. For instance, let us assume that $X(t) = x_0 + u_0(x_0)t$ and $X(t) = x_1 + u_0(x_1)t$ are the characteristics through $(x_1, 0)$ and $(x_2, 0)$ with $x_1 < x_2$, and also assume that $u \in C^\infty(\mathbb{R})$ satisfies $u_0(x_1) > u_0(x_2)$. It is easy to deduce that

$$t^* = \frac{x_2 - x_1}{u_0(x_1) - u_0(x_2)} \qquad x^* = \frac{x_2 u_0(x_1) - x_1 u_0(x_2)}{u_0(x_1) - u_0(x_2)}$$

is the intersection of both characteristic curves, implying that there are not a classical solution for $t \geq t^*$.

In order to introduce the definition of the entropy solution, we present two situations that make it clear that the Definition (A.2) and the Rankine–Hugoniot condition are not enough to guarantee the uniqueness of the solution. First, it is observed that if, for example, if the initial condition is given by

$$u_0(x) = 0,$$

we find that $u(x, t) = 0$ is a classical solution, and it is observed that an infinite number of weak solutions in the form

$$u(x,t) = \begin{cases} 0, & x < -pt, \\ -2p, & -pt < x < 0, \\ 2p, & 0 < x < pt, \\ 0, & pt < x, \end{cases} \qquad \text{for each } p \in \mathbb{R}, \tag{A.11}$$

can be constructed by the application of Proposition (A.1). A second case is that manipulating weak solutions can change the velocity of the shock curve defined by the Rankine–Hugoniot jump condition. Multiplying Eq (A.5) by $2u$ gives

$$(u^2)_t + (\frac{2}{3}u^3)_x = 0, \tag{A.12}$$

which is a conservation law for u^2 and not for u and its flux function $f(u) = 2u^{3/2}/3$. Equation (A.12) has the same classical solutions as Eq (A.5), but not the same weak solutions. To fix our ideas, if we consider the initial condition (A.10), we find that the shock wave velocities for (A.5) and (A.12) are given by $\sigma_1(1, 0) = 1/2$ and $\sigma_1(1, 0) = 2/3$, respectively. Therefore, the weak solutions of (A.12) and of (A.5) are different. These behaviors for weak solutions make it clear that an additional condition is required to select the unique weak solution that makes relevant physical sense. This condition is known as the entropy condition.

There are different versions of the definition for entropy conditions, among which, the best known are the Lax entropy condition and Oleinik's entropy condition. Let us consider a discontinuity of the weak solution propagating with velocity σ given by the Rankine–Hugoniot jump condition. The Lax entropy condition condition is given by the inequality

$$f'(u(\Sigma(t) - 0, t)) > \sigma > f'(u(\Sigma(t) + 0, t)). \tag{A.13}$$

Meanwhile, the Oleynik entropy condition is given by the inequality

$$\frac{f(u) - f(\Sigma(t) - 0, t)}{u - u(\Sigma(t) - 0, t)} \geq \sigma \geq \frac{f(u) - f(\Sigma(t) + 0, t)}{u - u(\Sigma(t) + 0, t)}, \quad \forall u \in I, \tag{A.14}$$

with $I = [\min(u(\Sigma(t) - 0, t), u(\Sigma(t) + 0, t)), \max(u(\Sigma(t) - 0, t), u(\Sigma(t) + 0, t))]$. Nowadays, there are other entropy conditions for low-regularity flux functions, but the most widely used in the scalar case is the Kružkov's condition, which is formulated as the inequality

$$\iint_{Q_T} [\varphi_t E(u) + \varphi_x F(u)] dxdt + \int_{\mathbb{R}} \varphi(x, 0) u_0(x) dx \geq 0, \quad \forall \varphi \geq 0 \text{ in } C_0^\infty(Q_T), \tag{A.15}$$

for every convex entropy function E and its respective entropy flux F, i.e., $F' = E'f'$. The function E is called the entropy function, and the function F is called the entropy flux. The origin of these names comes from the theory of gas dynamics, where a physical quantity called entropy exists that has the property of being constant along smooth-flowing particle paths and jumps to increasingly higher values at the moment of crossing a shock wave. We note that all entropy conditions (A.13), (A.14), and (A.15) are equivalent in the case that $f \in U_{ad}$. Hence, we present the definition only for the condition (A.15).

Definition A.3. *A function $u(x, t)$ is an entropy solution of he Cauchy problem* (A.5)–(A.6) *if satisfies the entropy condition* (A.15).

In the case of the weak solution (A.11), we observe that the discontinuity curve $(0, t)$ satisfies the entropy condition (A.13) when $-2p > 0 > 2p$, which is impossible for any $p \in \mathbb{R}$. In this case, (A.11) is not an entropy solution. Moreover, in the case of the conservation laws (A.5) and (A.12), we can prove that both conservation laws have the same entropy solution. Let us consider the convex function $E(u) = u^2$. The entropy flux is given by $F'(u) = E'(u)f'(u) = (u^2)'(u^2/2)' = 2u^2$. Thus, the entropy condition (A.15) can be written as follows:

$$\iint_{Q_T} \left(\varphi_t u^2 + \varphi_x \frac{2u^3}{3}\right)(x, t) dxdt + \int_{\mathbb{R}} \varphi(x, 0) u_0(x) dx \geq 0, \quad \forall \varphi \in C_0^\infty(Q_T),$$

which corresponds to the variational formulation of

$$(u^2)_t + \left(\frac{2}{3} u^3\right)_x \leq 0, \quad u(x, 0) = u_0(x).$$

Consequently, the relation (A.12), which is deduced for smooth solutions, holds as an inequality in the variational sense for entropic solutions.

On the other hand, to conclude, we remark that the entropy solution in the sense of Definition A.3 allows us to select the only weak solution with a physical sense in the context of conservation laws.