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Research article

NE transform of pathway fractional integrals involving S-function

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Abstract: This paper aims to derive the NE transform of fractional integrals in the pathway that incorporates the S-function in the kernel, considering various parameters. Furthermore, by applying these mathematical operators, we have explored and clarified several key findings and corollaries. Our current study highlights the results of the NE transform, combined with the fractional integral formula of the pathway that includes the S-function within the kernel.

Keywords: S-function; NE transform; pathway fractional integral operator; fractional calculus
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1. Introduction

Fractional calculus (F.C) has become a vital tool for modeling and analysis, making substantial contributions across various fields such as material science, biology, mechanics, energy, economics, and control theory. Numerous researchers have explored different aspects of F.C operators, thoroughly investigating their properties, implementation techniques, and complex adaptations [1, 2]. These solutions have been fully developed and are practically utilized in multiple areas [3, 4]. A deep understanding of classical functions is grounded in fundamental special functions, including the beta, gamma, and hypergeometric functions [5, 6]. Importantly, many special functions are defined in terms of the Gamma function [7–9]. Over the last few years, the field of classical functions, particularly the generalization of these functions, has experienced rapid growth in mathematical theory [10]. Scientists have recognized the importance of analyzing the properties of these functions as their domains undergo modifications [11, 12]. Consequently, numerous researchers and scientists have been consistently engaged in this evolving field, with the aim of exploring and understanding the implications of these changes [13].

1.1. S-function

In [14], Daiya and Saxena introduced and examined a novel function known as the S-function. The author investigated its relationships with several other special functions, including the generalized K-function (K.F), M-series (M.S), k-Mittag-Leffler function (k-M.L.F), Mittag-Leffler type functions (M.L), and various other special functions. These distinctive functions have proven to be crucial in the address of challenges in applied sciences, biology, physics, and engineering, showcasing their growing importance and widespread applications. It is defined for $\varrho, \sigma, \gamma, \mu \in C, R(\varrho) > 0, l \in R, R(\varrho) > lR(\mu)$,

$$S_{(m.n)}^{\varrho,\sigma,\gamma,\mu,l}[w_1, w_2, ..., w_i; x_1, x_2, ..., x_j; z] = \sum_{d=0}^{\infty} \frac{(w_1)_d.....(w_i)_d(\gamma)_{d\mu,l}}{(x_1)_d.....(x_j)_d\Gamma_k(d\varrho + \sigma)} \frac{z^d}{d!}. \tag{1.1}$$

k-pochhammer (k-P.S) symbol is defined as

$$(\gamma)_{n,k} = \gamma(\gamma + k)......(\gamma + (n-1)k) \quad (n \in N, \gamma \in C). \tag{1.2}$$

Also, k-gamma is [15],

$$\Gamma_k(\gamma) = k^{(\frac{\gamma}{k})-1}\Gamma(\frac{\gamma}{k}). \tag{1.3}$$

Here, γ belongs to the complex number set, k is a real number, and n is a natural number, as introduced by Diaz and Pariguan [16]. Several significant special cases of the S-function are outlined below:

(1) If $m = n = 0$, the generalized k-M.L function from Saxena et al. [17–19] defines S-function as

$$\begin{aligned} \varepsilon_{l,\varrho,\sigma}^{\gamma,\mu} &= S_{(0,0)}^{\varrho,\sigma,\gamma,\mu,l}[w_1, w_2, ..., w_i; x_1, x_2, ..., x_j; z] \\ &= \sum_{d=0}^{\infty} \frac{(\gamma)_{d\mu,l}}{\Gamma_k(d\varrho + \sigma)} \frac{z^d}{d!}, \qquad R\left(\frac{\varrho}{l} - \mu\right) > m - n. \end{aligned} \tag{1.4}$$

(2) If $l = \mu = 1$, the S-function is the generalized K.F, introduced by Sharma et al. [20, 21] is defined as

$$\begin{aligned} &K_{m,n}^{\varrho,\sigma,\gamma}[w_1, w_2, ..., w_i; x_1, x_2, ..., x_j; z] \\ =&S_{m,n}^{\varrho,\sigma,\gamma,1,1}[w_1, w_2, ..., w_i; x_1, x_2, ..., x_j; z] \\ =&\sum_{d=0}^{\infty} \frac{(w_1)_d(w_2)_d.....(w_i)_d(\gamma)_{d\mu,l}}{(x_1)_d(x_2)_d, ...(x_j)_d\Gamma_k(d\varrho + \sigma)} \frac{z^d}{d!}, \qquad R(\alpha) > m - n. \end{aligned} \tag{1.5}$$

(3) If $\varrho = l = \gamma = 1$, then the S-function reduces to the generalized M.S function introduced by Sharma and Jain [22] is defined as

$$\begin{aligned} &M_{m,n}^{\varrho,\sigma}[w_1, w_2, ..., w_i; x_1, x_2, ..., x_j; z] \\ =&S_{m,n}^{\varrho,\sigma,\gamma,1,1}[w_1, w_2, ..., w_i; x_1, x_2, ..., x_j; z] \\ =&\sum_{d=0}^{\infty} \frac{(w_1)_d(w_2)_d.....(w_i)_d z^d}{(x_1)_d(x_2)_d....(x_j)_d\Gamma_k(d\varrho + \sigma)}, \qquad R(\varrho) > m - n - 1. \end{aligned} \tag{1.6}$$

1.2. The pathway fractional integral formula

Nair [23] introduced an expanding pathway fractional integral (PFI) operator, previously defined by Mathai and Haubold [24–26]. This operator is defined as follows:

$$(P_{0+}^{\mu',\xi}g)(x) = x^{\mu'}\int_0^{[x/(b(1-\xi))]}\left(1-\frac{b(1-\xi)\lambda}{x}\right)^{\frac{\mu'}{1-\xi}} f(\lambda)d\lambda. \tag{1.7}$$

Given a Lebesgue measurable function g in the interval (c, d), where c and d are real or complex terms, and μ' is a complex number with a positive real part, and ξ is a pathway parameter with $\xi < 1$. A model is presented for a real scalar ξ and scalar random variables within the pathway, illustrated through the probability density function in the following fashion:

$$g(x) = \frac{w}{|x|^{1-\varsigma}}[1-b(1-\xi)|x|^{\rho}]^{\mu'/(1-\xi)}, \tag{1.8}$$

where $x \in (-\infty, \infty); \mu' > 0; \rho > 0; [1-b(1-\xi)|x|^{\rho}]^{\mu'/(1-\xi)}; \varsigma > 0$ and ξ and w denote the pathway parameter and normalizing constant, respectively. Additionally, for $\xi \in R$ we can express the normalizing constant as

$$w = \begin{cases} \frac{1}{2}[b(1-\xi)]^{\frac{\varsigma}{\rho}}\Gamma\left(\frac{\varsigma}{\rho}+\frac{\mu'}{1-\xi}+1\right), & (\xi < 1). \\ \frac{1}{2}\frac{\rho[b(1-\xi)]^{\frac{\varsigma}{\rho}}\Gamma\left(\frac{\mu'}{1-\xi}\right)}{\Gamma\left(\frac{\varsigma}{\rho}\right)\Gamma\left(\frac{\nu}{1-\alpha}+\frac{\varsigma}{\rho}\right)}, & \left(\frac{1}{\xi-1}-\frac{\varsigma}{\rho}, \delta < 1\right). \\ \frac{1}{2}\frac{[b\nu]^{\varsigma/\rho}}{\Gamma\left(\frac{\varsigma}{\rho}\right)}, & (\xi \longleftrightarrow 1). \end{cases} \tag{1.9}$$

It is defined that if $\xi < 1$, finite range density with $[1-b(1-\xi)|x|^{\rho}]^{\mu'/(1-\xi)}$ and $g(x)$ mentioned above in Eq (1.8) can be considered as the member of the extended generalized type-1 beta family. In addition to the probability density function specified for $g(x)$ for the variable ξ, several specific instances, such as the triangular density, uniform density, extended type-1 beta density, and various other probability density functions, are precise special cases. For example, if $\xi > 1$ and by setting $(1-\xi) = -(\xi-1)$ in Eq (1.7), then we have

$$(P_{0+}^{\mu',\xi}g)(x) = x^{\nu}\int_0^{[x/(-b(1-\alpha))]}\left(1+\frac{b(1-\xi)\lambda}{x}\right)^{\frac{\mu'}{-1+\xi}} f(\lambda)d\lambda, \tag{1.10}$$

$$g(x) = \frac{w}{|x|^{1-\varsigma}}[1+b(1-\xi)|x|^{\rho}]^{\mu'/-(1-\xi)}. \tag{1.11}$$

Provided that $x \in (-\infty, \infty); \rho > 0$; and $\xi > 1$ characterize the extended generalized type-2 beta model for real x. The specific cases of **d.f** in Eq (1.11) include the type-2 beta **d.f.**, the probability density function, and the student's t-density function. For $\delta \longleftrightarrow 1$, Eq (1.7), diminishes to the Laplace integral operator. In a similar way, if $\xi = 0, a = 1$, and μ' takes the place of $\mu' - 1$, then it diminishes to the familiar Riemann Liouvelle fractional operator $(P_{0+}^{\mu'}g)$

$$(P_{0+}^{\mu'-1,0}g)(x) = \Gamma(\nu)(P_{0+}^{\mu'}g)(x) \qquad (R(\mu') > 1). \tag{1.12}$$

The PFI operator, as defined in Eq (1.7), offers access to a range of captivating instances, encompassing F.C. connections to probability density functions and their significance in statistical theory [27–29]. Currently, various researchers are investigating PFI formulas associated with diverse special functions [30–32]. Recently, Amsalu et al. [33] have examined formulas for PFI involving S-function, as follows:

$$P_{0+}^{\mu',\xi}\left[\eta^{\frac{\varepsilon}{l}}S_{m,n}^{\varrho,\sigma,\gamma,\mu,l}[w_1,w_2,....,w_i;x_1,x_2,.....,x_j;r\eta^{\frac{\varrho}{l}}]\right](k)$$
$$=\frac{k^{\mu'+\frac{\varepsilon}{l}}l^{\left(l+\frac{\mu'}{1-\xi}\right)}\Gamma\left(\frac{\mu'}{1-\xi}+1\right)}{(b(1-\xi))^{\frac{\varepsilon}{l}}}\times S_{m,n}^{\varrho,\sigma+\left(1+\frac{\mu'}{1-\xi}\right)l,\gamma,\mu,l}\left[w_1,w_2,....,w_i;x_1,x_2,....,x_j;\frac{rk^{\frac{\varrho}{l}}}{(b(1-\xi))^{\frac{\varrho}{l}}}\right]. \tag{1.13}$$

Here$r,l\in R,\varrho,\sigma,\gamma,\mu\in C,R(\varrho)>0,R(\mu')>0,R(\varrho)>lR(\mu)$, and $m<n+1,R\left(\frac{\mu'}{(1-\xi)}\right)>-1;\xi<1$

$$P_{0+}^{\mu',\xi}\left[\eta^{\frac{\varepsilon}{l}}S_{m,n}^{\varrho,\sigma,\gamma,\mu,l}[w_1,w_2,....,w_i;x_1,x_2,.....,x_j;r\eta^{\frac{\varrho}{l}}]\right](k)$$
$$=\frac{k^{\mu'+\frac{\varepsilon}{l}}l^{\left(l-\frac{\mu'}{\xi-1}\right)}\Gamma\left(1-\frac{\mu'}{\xi-1}\right)}{(-b(\xi-1))^{\frac{\varepsilon}{l}}}\times S_{m,n}^{\varrho,\sigma+\left(1-\frac{\mu'}{\xi-1}\right)l,\gamma,\mu,l}\left[w_1,w_2,....,w_i;x_1,x_2,....,x_j;\frac{rk^{\frac{\varrho}{l}}}{(-b(\xi-1))^{\frac{\varrho}{l}}}\right]. \tag{1.14}$$

Here $r,l\in R,\varrho,\sigma,\gamma,\mu\in C,R(\varrho)>0,R(\mu')>0,R(\varrho)>lR(\mu)$.

1.3. NE transform

For many years, integral transforms (I.T) have been instrumental in solving a variety of ‘differential’ and ‘integral’ equations. The use of appropriate I.T is crucial in converting differential and integral operators in a given domain into multiplication operators in a different domain, simplifying the problem-solving process. One method for returning the manipulated solution to the intended solution of the original problem within its original domain is to solve the transformed problem in the new domain and then apply the inverse transform. The traditional I.T commonly utilized for addressing differential equations, integral equations, and in analysis and function theory encompasses the Laplace transform applied in engineering and real-life problems and the Hankel’s and Weierstrass transforms used in heat and diffusion equations. Furthermore, the natural transform and the Yang transform, both applied across diverse fields in physical science and engineering, are also noteworthy [34]. If we take a function $h(t)$ that is of exponential order $\frac{1}{k}$, if there exists a positive constant P and Q such that $|h(t)|\leq Pe^{\frac{q}{k}}$ for all $q\geq Q$. For any function $h(t)$, we assume that an integral equation exists. NE integral transform is denoted by the operator $E(.)$ and is defined by the integral equation,

$$E(s,u)=Nh(t)=\frac{1}{2}\int_0^{\infty}e^{-st}h(ut)dt. \tag{1.15}$$

NE Transform of a function $h(t)=x^n$ given as

$$N(x^n)=\frac{u^n}{s^{n+2}}\Gamma(n+1). \tag{1.16}$$

1.4. The gamma identities

The mathematical function known as the gamma function, or $\Gamma(z)$, extends the idea of a factorial to non-integer values. It is defined for complex numbers with a positive real portion. The gamma function formula is as follows [35–37]:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \tag{1.17}$$

For complex numbers z with a positive real portion, this integral converges. Numerous uses of the gamma function can be found in physics, engineering, and mathematics. The generalization of the gamma function is the k-gamma function and is defined as:

$$\Gamma_k(z) = \int_0^\infty t^{kz-1} e^{-t^k} dt, \tag{1.18}$$

k is a positive constant in this case. The k-gamma function adds the parameter k to the exponent yet nevertheless extends the idea of a factorial to non-integer values. This function appears in many contexts in mathematics and science, especially those involving power transformations. Applications for the gamma and k-gamma functions can be found in statistical physics, number theory, complex analysis, and probability theory. They are essential for more broadly describing and resolving problems involving products of factorials or powers.

This paper broadly consists of 3 sections. Section 1 gives a brief introduction about S-function, NE transform, and PFI formulas, along with PFI, along with S-function and the gamma identities. In Section 2, we have discussed the main results of the paper in which we have derived the NE transform of the PFI that incorporates the S-function in the kernel with various parameters; additionally; by applying a mathematical operator, we have also explored several corollaries. In Section 3, conclusion and future work of the paper are given.

2. Main results

In this investigation, we have examined the NE integral transform of PFIs that incorporate the S-function in the kernel, as defined in Eqs (1.13) and (1.14).

The NE transform for PFI formula involving S-function given as follows:

Theorem 2.1. *Let $l \in R, \varrho, \sigma, \gamma, \mu \in C, R(\varrho) > 0, R(\mu^{'}) > 0, R(\varrho) > lR(\mu)$ and $m < n+1, R\left(\frac{\mu^{'}}{1-\xi}\right) > -1; \xi < 1, |r| < 1$, then,*

$$N\left[r.\left(P_{0+}^{\mu^{'},\xi}\eta^{\left(\frac{\xi}{l}\right)-1} S_{m,n}^{\varrho,\sigma,\gamma,\mu,l}\left[w_1, w_2, ...w_i; x_1, x_2, ..., x_j; r\eta^{\frac{\varrho}{l}}\right]\right)(k)\right]$$
$$= \frac{uk^{\mu^{'}+\frac{\sigma}{l}} l^{\left(1+\frac{\mu^{'}}{1-\xi}\right)} \Gamma\left(\frac{\mu^{'}}{1-\xi}+1\right)}{s^3(q(1-\xi))^{\frac{\sigma}{l}}} \times S_{(m,n)}^{\varrho,\sigma+\left(1+\frac{\mu^{'}}{1-\xi}\right)l,\gamma,\mu,l}\left[w_1, w_2, ..., w_i; x_1, x_2, ..., w_j; 2; \left(\frac{u}{s}\left(\frac{k}{q(1-\xi)}\right)^{\frac{\varrho}{l}}\right)\right]. \tag{2.1}$$

Proof. Taking into account the left-hand side of Eq (1.13) and representing it as L_1, we can use the result in Eq (2.1) to obtain,

$$L_1 = N\left[\frac{rk^{\mu'+\frac{\sigma}{l}}l^{\left(1+\frac{\mu'}{1-\xi}\right)}\Gamma\left(\frac{\mu'}{1-\xi}+1\right)}{(q(1-\xi))^{\frac{\sigma}{l}}}\right]\times\left[S_{(m,n)}^{\varrho,\sigma+\left(1+\frac{\mu'}{1-\xi}\right)l,\gamma,\mu,l}\left[w_1,w_2,......,w_i;x_1,x_2,.....,w_j;2;\frac{rk^{\left(\frac{\varrho}{l}\right)}}{(q(1-\xi))^{\frac{\varrho}{l}}}\right]\right].$$

By using the definition of S-function defined in Eq (1.1), we obtain

$$L_1 = N\left[\frac{rk^{\mu'+\frac{\sigma}{l}}l^{\left(1+\frac{\mu'}{1-\xi}\right)}\Gamma\left(\frac{\mu'}{1-\xi}+1\right)}{(q(1-\xi))^{\frac{\sigma}{l}}}\right]\times\left[\sum_{d=0}^{\infty}\frac{(w_1)_d....(w_i)_d(\gamma)_{d\mu,l}}{(x_1)_d....(x_j)_d\Gamma_k\left(d\varrho+\sigma+1+\frac{\mu'}{1-\xi}\right)}\frac{\left(\frac{rk^{\frac{\varrho}{l}}}{(q(1-\xi))^{\frac{\varrho}{l}}}\right)^d}{d!}\right].$$

Then by the definition of NE transform defined in Eq (2.1), we obtain

$$L_1 = \frac{k^{\mu'+\left(\frac{\sigma}{l}\right)}l^{\left(1+\frac{\mu'}{1-\xi}\right)}\Gamma\left(\frac{\mu'}{1-\xi}+1\right)}{(q(1-\xi))^{\frac{\sigma}{l}}}\times\sum_{d=0}^{\infty}\frac{(w_1)_d....(w_i)_d(\gamma)_{d\mu,l}}{(x_1)_d....(x_j)_d\Gamma_k\left(d\varrho+\sigma+1+\frac{\mu'}{1-\xi}\right)}\frac{\left(\frac{k^{\frac{\varrho}{l}}}{(q(1-\xi))^{\frac{\varrho}{l}}}\right)^d}{d!}\times N[r^{d+1}].$$

Further solving we obtain

$$L_1 = \frac{k^{\mu'+\left(\frac{\sigma}{l}\right)}l^{\left(1+\frac{\mu'}{1-\xi}\right)}\Gamma\left(\frac{\mu'}{1-\xi}+1\right)}{(q(1-\xi))^{\frac{\sigma}{l}}}\times\sum_{d=0}^{\infty}\frac{(w_1)_d....(w_i)_d(\gamma)_{d\mu,l}}{(x_1)_d....(x_j)_d\Gamma_k\left(d\varrho+\sigma+1+\frac{\mu'}{1-\xi}\right)}\frac{\left(\frac{k^{\frac{\varrho}{l}}}{(q(1-\xi))^{\frac{\varrho}{l}}}\right)^d}{d!}\frac{u^{d+1}\Gamma(d+2)}{s^{d+3}}.$$

We simplify the above equation by using the properties of the Pochhammer symbol and gamma function,

$$L_1 = \frac{uk^{\mu'+\left(\frac{\sigma}{l}\right)}l^{\left(1+\frac{\mu'}{1-\xi}\right)}\Gamma\left(\frac{\mu'}{1-\xi}+1\right)}{s^3(q(1-\xi))^{\frac{\sigma}{l}}}\times\sum_{d=0}^{\infty}\frac{(w_1)_d....(w_i)_d(\gamma)_{d\mu,l}(2)_d}{(x_1)_d....(x_j)_d\Gamma_k\left(d\varrho+\sigma+1+\frac{\mu'}{1-\xi}\right)}\frac{\left(\frac{uk^{\frac{\varrho}{l}}}{(q(1-\xi))^{\frac{\varrho}{l}}}\right)^d}{d!}.$$

Then by using some definition of S-function, we obtain the result. □

Corollary 2.2. *Let* $l \in R, \varrho, \sigma, \gamma, \mu \int C, R(\varrho) > 0, R(\mu') > 0, R(\varrho) > lR(\mu)$ *and* $R\left(\frac{\mu'}{1-\xi}\right) > -1; \xi < 1, |r| < 1,$

$$N\left[r.\left(P_{0+}^{\mu',\xi}\eta^{\left(\frac{\sigma}{l}\right)-1}E_{\mu,l}^{\varrho,\sigma,\gamma}[r\eta^{\frac{\varrho}{l}}]\right)(k)\right]$$
$$=\frac{uk^{\mu'+\left(\frac{\sigma}{l}\right)}l^{\left(1+\frac{\mu'}{1-\xi}\right)}\Gamma\left(\frac{\mu'}{1-\xi}+1\right)}{s^3(q(1-\xi))^{\frac{\sigma}{l}}}\times E_{\mu,l}^{\varrho,\sigma+\left(1+\frac{\mu'}{1-\xi}\right)l,\gamma}\left(2;\frac{u}{s}\left(\frac{k}{q(1-\xi)}\right)^{\frac{\varrho}{l}}\right). \quad (2.2)$$

Proof. If we put $m = n = 0$ in Eq (1.13), we obtain the result for the generalized k-M.L function as given in [33]. The result obtained in Eq (2.2), is NE integral transform of the PFI involving generalized k-M.L function. □

Corollary 2.3. *Let* $l \in R, \varrho, \sigma, \gamma, \mu \int C, R(\varrho) > 0, R(\mu') > 0, R(\varrho) > lR(\mu)$ *and* $R\left(\frac{\mu'}{1-\xi}\right) > -1; \xi < 1, |r| < 1,$

$$N\left[r.\left(P_{0+}^{\mu',\xi}\eta^{\sigma-1}K_{m,n}^{\varrho,\sigma,\gamma}[w_1, w_2, ..., w_i; x_1, x_2, ..., x_j; r\eta^{\varrho}]\right)(k)\right]$$
$$= \frac{uk^{\mu'+\sigma}\Gamma\left(\frac{\mu'}{1-\xi}+1\right)}{s^3(q(1-\xi))^{\sigma}} \times K_{m,n}^{\varrho,\sigma+\left(1+\frac{\mu'}{1-\xi}\right),\gamma}\left[w_1, w_2, ..., w_i; x_1, x_2, ..., x_j; 2; \left(\frac{u}{s}\left(\frac{k}{q(1-\xi)}\right)^{\varrho}\right)\right]. \quad (2.3)$$

Proof. If we put $l = \mu = 1$ in Eq (1.13), we obtain the result for generalized K.F as given in [33]. The result obtained in Eq (2.3), is the NE integral transform of the PFI that is involved in generalized K.F. □

Corollary 2.4. *Let* $l \in R, \varrho, \sigma, \gamma, \mu \int C, R(\varrho) > 0, R(\mu') > 0, R(\varrho) > lR(\mu)$ *and* $R\left(\frac{\mu'}{1-\xi}\right) > -1; \xi < 1, |r| < 1,$

$$N\left[r.\left(P_{0+}^{\mu',\xi}\eta^{\sigma-1}M_{m,n}^{\varrho,\sigma}[w_1, w_2, ..., w_i; x_1, x_2, ..., x_j; r\eta^{\varrho}]\right)(k)\right]$$
$$= \frac{uk^{\mu'+\sigma}\Gamma\left(\frac{\mu'}{1-\xi}+1\right)}{s^3(q(1-\xi))^{\sigma}} \times M_{m,n}^{\varrho,\sigma+\left(1+\frac{\mu'}{1-\xi}\right)}\left[w_1, w_2, ..., w_i; x_1, x_2, ..., x_j; 2; \left(\frac{u}{s}\left(\frac{k}{q(1-\xi)}\right)^{\varrho}\right)\right]. \quad (2.4)$$

Proof. If we put $l = \mu = \gamma = 1$ in Eq (1.13), we obtain the result for generalized M.S as given in [33]. The result obtained in Eq (2.4) is the NE integral transform of the PFI involving generalized M.S. □

Theorem 2.5. *The NE for the PFI formula involving the* S*-function is given as follows: Let* $l \in R, \varrho, \sigma, \gamma, \mu \in C, R(\varrho) > 0, R(\mu') > 0, R(\varrho) > lR(\mu)$ *and* $m < n+1, R\left(\frac{\mu'}{1-\xi}\right) > -1; \xi < 1, |r| < 1$ *then,*

$$N\left[r.\left(P_{0+}^{\mu',\xi}\eta^{\left(\frac{\varepsilon}{l}\right)-1}S_{m,n}^{\varrho,\sigma,\gamma,\mu,l}\left[w_1, w_2, ...w_i; x_1, x_2, ..., x_j; r\eta^{\frac{\varrho}{l}}\right]\right)(k)\right]$$
$$= \frac{uk^{\mu'+\frac{\sigma}{l}}l^{\left(1-\frac{\mu'}{\xi-1}\right)}\Gamma\left(1-\frac{\mu'}{\xi-1}\right)}{s^3(-q(\xi-1))^{\frac{\sigma}{l}}} \times S_{(m,n)}^{\varrho,\sigma+\left(1-\frac{\mu'}{\xi-1}\right)l,\gamma,\mu,l}\left[w_1, w_2, ..., w_i; x_1, x_2, ..., w_j; 2; \left(\frac{u}{s}\left(\frac{k}{-q(\xi-1)}\right)^{\frac{\varrho}{l}}\right)\right]. \quad (2.5)$$

Proof. Taking into account the left-hand side of the Eq (2.5) and representing it as L_2, we can use the result defined in Eq (1.14) to obtain,

$$L_2 = N\left[\frac{rk^{\mu'+\frac{\sigma}{l}}l^{\left(1-\frac{\mu'}{\xi-1}\right)}\Gamma\left(1-\frac{\mu'}{\xi-1}\right)}{(-q(\xi-1))^{\frac{\sigma}{l}}}\right] \times S_{(m,n)}^{\varrho,\sigma+\left(1-\frac{\mu'}{\xi-1}\right)l,\gamma,\mu,l}\left[w_1, w_2, ...w_i; x_1, x_2, ...x_j; \frac{rk^{\frac{\varrho}{l}}}{(-q(\xi-1))^{\frac{\varrho}{l}}}\right].$$

By using the definition of S-function defined in Eq (1.1), we obtain

$$L_2 = N\left[\frac{rk^{\mu'+\frac{\sigma}{l}}l^{\left(1-\frac{\mu'}{\xi-1}\right)}\Gamma\left(1-\frac{\mu'}{\xi-1}\right)}{(-q(\xi-1))^{\frac{\sigma}{l}}}\right] \times \sum_{d=0}^{\infty} \frac{(w_1)_d, ..., (w_i)_d(\gamma)_{d\mu,l}}{(x_1)_d, ..., (x_j)_d\Gamma_k\left(d\varrho+\sigma+1-\frac{\mu'}{\xi-1}\right)} \frac{\left(\frac{rk^{\frac{\varrho}{l}}}{(-q(\xi-1))^{\frac{\varrho}{l}}}\right)^d}{d!}.$$

Then by the definition of NE transform defined in Eq (1.16), we obtain

$$L_2 = \frac{k^{\mu' + \frac{\sigma}{l}} l^{\left(1-\frac{\mu'}{\xi-1}\right)} \Gamma\left(1 - \frac{\mu'}{\xi-1}\right)}{(-q(\xi-1))^{\frac{\sigma}{l}}} \times \sum_{d=0}^{\infty} \frac{(w_1)_d, ..., (w_i)_d (\gamma)_{d\mu,l}}{(x_1)_d, ..., (x_j)_d \Gamma_k \left(d\varrho + \sigma + 1 - \frac{\mu'}{\xi-1}\right)} \frac{\left(\frac{k^{\frac{\varrho}{l}}}{(-q(\xi-1))^{\frac{\varrho}{l}}}\right)^d}{d!} N(r^{d+1}).$$

Further solving we obtain

$$L_2 = \frac{k^{\mu' + \frac{\sigma}{l}} l^{\left(1-\frac{\mu'}{\xi-1}\right)} \Gamma\left(1 - \frac{\mu'}{\xi-1}\right)}{(-q(\xi-1))^{\frac{\sigma}{l}}} \times \sum_{d=0}^{\infty} \frac{(w_1)_d, ..., (w_i)_d (\gamma)_{d\mu,l}}{(x_1)_d, ..., (x_j)_d \Gamma_k \left(d\varrho + \sigma + 1 - \frac{\mu'}{\xi-1}\right)} \frac{\left(\frac{k^{\frac{\varrho}{l}}}{(-q(\xi-1))^{\frac{\varrho}{l}}}\right)^d}{d!} \frac{u^{d+1}\Gamma(d+2)}{s^{d+3}}.$$

From the properties of the Pochhamer symbol and gamma function and simplifying the above equation, we obtain

$$L_2 = \frac{uk^{\mu' + \frac{\sigma}{l}} l^{\left(1-\frac{\mu'}{\xi-1}\right)} \Gamma\left(1 - \frac{\mu'}{\xi-1}\right)}{s^3(-q(\xi-1))^{\frac{\sigma}{l}}} \times \sum_{d=0}^{\infty} \frac{(w_1)_d, ..., (w_i)_d (\gamma)_{d\mu,l} (2)_d}{(x_1)_d, ..., (x_j)_d \Gamma_k \left(d\varrho + \sigma + 1 - \frac{\mu'}{\xi-1}\right)} \frac{\left(\frac{uk^{\frac{\varrho}{l}}}{(-q(\xi-1))^{\frac{\varrho}{l}}}\right)^d}{d!}.$$

Then by using some results of the PFI operator on the S-function as mentioned in Eqs (1.1) and (1.14), we get the desired result. □

Corollary 2.6. *Let* $l \in R, \varrho, \sigma, \gamma, \mu \in C, R(\varrho) > 0, R(\mu') > 0, R(\varrho) > lR(\mu)$ *and* $m < n + 1, R\left(\frac{\mu'}{1-\xi}\right) > -1; \xi < 1, |r| < 1$ *then,*

$$N\left[r.\left(P_{0+}^{\mu',\xi} \eta^{\left(\frac{\sigma}{l}\right)-1} E_{\mu,l}^{\varrho,\sigma,\gamma}\left[r\eta^{\left(\frac{\varrho}{l}\right)}\right]\right)(k)\right]$$
$$= \frac{uk^{\mu+\frac{\sigma}{l}} l^{\left(1-\frac{\mu'}{\xi-1}\right)} \Gamma\left(1-\frac{\mu'}{\xi-1}\right)}{s^3(-q(\xi-1))^{\left(\frac{\sigma}{l}\right)}} \times E_{\mu,l}^{\varrho,\sigma+\left(1-\frac{\mu'}{\xi-1}\right)l,\gamma}\left[2;\left(\frac{u}{s}\left(\frac{k}{-q(\xi-1)}\right)^{\frac{\varrho}{l}}\right)\right]. \tag{2.6}$$

Proof. If we put $m = n = 0$ in Eq (1.14), we obtain the result for the generalized k-M.L function as given in [33]. The result obtained in Eq (2.6) is the NE integral transform of the PFI involving the generalized k-M.L function. □

Corollary 2.7. *Let* $l \in R, \varrho, \sigma, \gamma, \mu \in C, R(\varrho) > 0, R(\mu') > 0, R(\varrho) > lR(\mu)$ *and* $m < n + 1, R\left(\frac{\mu'}{1-\xi}\right) > -1; \xi < 1, |r| < 1$ *then,*

$$N\left[r.\left(P_{0+}^{\mu',\xi} \eta^{\sigma-1} K_{m,n}^{\varrho,\sigma,\gamma}\left[w_1, w_2, ..., w_i; x_1, x_2, ..., x_j; r\eta^{\sigma}\right]\right)(k)\right]$$
$$= \frac{uk^{\mu+\sigma}\Gamma\left(1-\frac{\mu'}{\xi-1}\right)}{s^3(-q(\xi-1))^{\sigma}} \times K_{m,n}^{\varrho,\sigma+\left(1-\frac{\mu'}{\xi-1}\right),\gamma}\left[w_1, w_2, ..., w_i; x_1, x_2, ..., x_j; 2; \left(\frac{u}{s}\left(\frac{k}{-q(\xi-1)}\right)^{\sigma}\right)\right]. \tag{2.7}$$

Proof. If we put $l = \mu = 1$ in Eq (1.14), we obtain the result for generalized K.F as given in [33]. The result obtained in Eq (2.7) is the NE integral transform of the PFI involving generalized K.F. □

Corollary 2.8. *Let $l \in R, \varrho, \sigma, \gamma, \mu \in C, R(\varrho) > 0, R(\mu^{'}) > 0, R(\varrho) > lR(\mu)$ and $m < n+1, R\left(\frac{\mu^{'}}{1-\xi}\right) > -1; \xi < 1, |r| < 1$ then,*

$$N\left[r.\left(P_{0+}^{\mu^{'},\xi}\eta^{\sigma-1}M_{m,n}^{\varrho,\sigma}\left[w_1, w_2, ..., w_i; x_1, x_2, ..., x_j; r\eta^{\sigma}\right]\right)(k)\right]$$
$$=\frac{uk^{\mu+\sigma}\Gamma\left(1-\frac{\mu^{'}}{\xi-1}\right)}{s^3(-q(\xi-1))^{\sigma}} \times M_{m,n}^{\varrho,\sigma+\left(1-\frac{\mu^{'}}{\xi-1}\right)}\left[w_1, w_2, ..., w_i; x_1, x_2, ..., x_j; 2; \left(\frac{u}{s}\left(\frac{k}{-q(\xi-1)}\right)^{\sigma}\right)\right]. \quad (2.8)$$

Proof. If we put $l = \mu = \gamma = 1$ in Eq (1.14), we obtain the result for generalized M.S as given in [33]. The result obtained in Eq (2.8) is the NE integral transform of the PFI involving generalized M.S. □

3. Conclusions

Our main research objective was to apply the NE transform to a PFI formula containing an S-function in the kernel. By utilizing these mathematical operators, we have investigated and elucidated various important discoveries and corollaries. Our current analysis emphasizes that the outcomes obtained through the NE transform, in conjunction with the PFI formula incorporating the S-function in the kernel, are highly significant, broadly applicable, and well-suited for integral transform techniques. Consequently, the results from our study are expected to have potential applications in diverse fields such as geophysics, earth sciences, engineering, and chemical physics.

Author contributions

S. Mishra: Conceptualization, formal analysis, investigation, methodology, validation, original draft writing; H. Nagar: data curation, formal analysis, methodology, visualization; N. Mani: review and editing, data curation, formal analysis, methodology, visualization; R. Shukla: formal analysis, investigation, project administration, resources, supervision, validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they did not use Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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