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Research article

On the relationship between dominance order and θ-dominance order on multipartitions

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Abstract: Many cellular bases have been constructed for the cyclotomic Hecke algebras of $G(\ell, 1, n)$. For example, with dominance order on multipartitions, Dipper, James, and Mathas constructed a cellular basis $\{m_{\mathfrak{st}}\}$ and Hu, Mathas constructed a graded cellular basis $\{\psi_{\mathfrak{st}}\}$. With θ-dominance order on multipartitions, Bowman constructed integral cellular basis $\{c^{\theta}_{\mathfrak{st}}\}$. Following Graham and Lehrer's cellular theory, different constructions of cellular basis may determine different parameterizations of simple modules of the cyclotomic Hecke algebras of $G(\ell, 1, n)$. To study the relationship between these parameterizations, it is necessary to understand the relationship between dominance order and θ-dominance order on multipartitions. In this paper, we define the weak θ-dominance order and give a combinatorial description of the neighbors with weak θ-dominance order. Then we prove weak θ-dominance order is equivalent to dominance order whenever the loading θ is strongly separated. As a corollary, we give the relationship between weak θ-dominance order, θ-dominance order, and dominance order on multipartitions.

Keywords: cyclotomic Hecke algebras; multipartitions; dominance order; θ-dominance order; weak θ-dominance order
Mathematics Subject Classification: 05E10, 06A07, 20C08

1. Introduction

The theory of cellular algebra was first introduced by Graham and Lehrer [1]. König and Xi [2] later gave a more structural equivalent definition of cellular theory. Suppose K is a field and A is an associative unital free K-algebra. In the sense of Graham and Lehrer, A is cellular if it has a K-basis $\{c_{\mathfrak{s},\mathfrak{t}} | \lambda \in \Lambda,\ \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$, where $(\Lambda, \geq)$ is a poset (partially ordered set) and $T(\lambda)$ are finite index sets, such that

(*i*) The K-linear map $* : A \longrightarrow A$ defined by $c_{\mathfrak{st}} \mapsto c_{\mathfrak{ts}}$ for all $\lambda \in \Lambda$, $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ is an anti-isomorphism of A.

(ii) For any $\lambda \in \Lambda$, $\mathfrak{t} \in T(\lambda)$, and $a \in A$, there exists $r_{\mathfrak{t}\mathfrak{v}}^{a} \in K$ such that for all $\mathfrak{s} \in T(\lambda)$,

$$c_{\mathfrak{s}\mathfrak{t}}a \equiv \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{t}\mathfrak{v}}^{a} c_{\mathfrak{s}\mathfrak{v}} \qquad \mod A^{>\lambda}.$$

The basis $\{c_{\mathfrak{s}\mathfrak{t}} | \lambda \in \Lambda,\ \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$ is the so called cellular basis. The existence of a cellular basis implies rich information on representations of A. One of the main uses of a cellular basis is to give the complete set of simple modules of A. According to Graham and Lehrer's theory, the cellular basis determines a cell filtration (a two-sided ideal filtration) $A(\lambda_1) \subset A(\lambda_2) \subset \cdots \subset A(\lambda_k)$ of A with respect to a total ordering $\lambda_1, \lambda_2, \ldots, \lambda_k$ of the poset Λ. As an A-module, each quotient $A(\lambda_i)/A(\lambda_{i-1})$ of the filtration is a direct sum of $|T(\lambda_i)|$ copies of cell module $C(\lambda_i)$. Moreover, for each $\lambda \in \Lambda$, the cellular basis attaches $C(\lambda)$ a bilinear form $\langle , \rangle_\lambda$ such that $C(\lambda)/\operatorname{rad}\langle , \rangle_\lambda$ is either 0 or an irreducible module. Denote by $D(\lambda)$ the quotient $C(\lambda)/\operatorname{rad}\langle , \rangle_\lambda$; all the nonzero $D(\lambda)$s consist of a complete set of non-isomorphic simple A-modules. For a cellular algebra, it may possess different constructions of cellular bases. By Graham and Lehrer's theory, different cellular bases may determine different parameterizations of simple modules. So the study of the relationship between different parameterizations of simple modules becomes an interesting topic.

In this paper, we fix n as a natural number and ℓ a positive integer. The cyclotomic Hecke algebras of $G(\ell, 1, n)$ was introduced by Ariki, Koike [3] and Broué, Malle [4] independently. Many authors have constructed different cellular bases of cyclotomic Hecke algebras of $G(\ell, 1, n)$. For example, Dipper, James, and Mathas [5] constructed the cellular basis $\{m_{\mathfrak{s}\mathfrak{t}} | \boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ and $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})\}$ with respect to the poset $(\mathcal{P}_n^\ell, \trianglerighteq)$, where $\mathcal{P}_n^\ell$ is the set of ℓ-partitions of n and $\trianglerighteq$ is the dominance order on $\mathcal{P}_n^\ell$. Through the cellular basis $m_{\mathfrak{s}\mathfrak{t}}$, Ariki [6] proved that the simple modules of cyclotomic Hecke algebras of $G(\ell, 1, n)$ are paramaterized by Kleshchev multipartitons. By Brundan–Kleshchev's isomorphism [7], Hu and Mathas [8] constructed the graded cellular basis $\{\psi_{\mathfrak{s}\mathfrak{t}} | \boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ and $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})\}$ of cyclotomic Hecke algebras of $G(\ell, 1, n)$ with respect to the poset $(\mathcal{P}_n^\ell, \trianglerighteq)$. Different from $m_{\mathfrak{s}\mathfrak{t}}$ and $\psi_{\mathfrak{s}\mathfrak{t}}$, Bowman [9] constructed an integral graded cellular basis $\{c_{\mathfrak{s}\mathfrak{t}}^\theta | \boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ and $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}_\theta(\boldsymbol{\lambda})\}$ of cyclotomic Hecke algebras of $G(\ell, 1, n)$ with respect to the poset $(\mathcal{P}_n^\ell, \trianglerighteq_\theta)$, where $\trianglerighteq_\theta$ is the θ-dominance order on $\mathcal{P}_n^\ell$. Corresponding to Bowman's basis, the simple modules of cyclotomic Hecke algebra of $G(\ell, 1, n)$ are labeled by Uglov multipartitions. We want to study the relationship between these different paramaterizations of simple modules of cyclotomic Hecke algebra of $G(\ell, 1, n)$. To this aim, it's necessary for us to understand the relationship between dominance order and θ-dominance order on $\mathcal{P}_n^\ell$.

The content of this paper is organized as follows; In Section 2, we introduce some notations and definitions. In Section 3, we give a combinatorial description of the neighbors with weak θ-dominance order whenever the loading θ is strongly separated. In Section 4, we give the main results of this paper: The relationship between weak θ-dominance order, θ-dominance order, and dominance order. Throughout this paper, we denote by $\mathbb{N}$ the set of natural numbers and $\mathbb{Z}$ the set of integers.

2. Notations and definitions

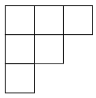
A partition of n is a finite non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers with $|\lambda| = \sum_i \lambda_i = n$. If λ is a partition of n, we write $\lambda \vdash n$. Let $\mathcal{P}_n$ be the set of partitions of n. The Young diagram of λ is a set

$$[\lambda] = \{(i, j) | 1 \leq j \leq \lambda_i, \forall i \geq 1\}.$$

The elements of $[\lambda]$ are called the nodes of λ. The Young diagram can be identified with a tableau. For example, $\lambda = (3, 2, 1)$ is a partition of 6; its Young diagram

$$[\lambda] = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\},$$

it can be identified with the tableau

$[\lambda] =$  ,

where (i, j) corresponds to the box in the i-th row and j-th column. For a partition λ, define its height $h(\lambda) = \max\{k \in \mathbb{N} | \lambda_k \neq 0\}$.

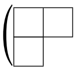
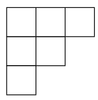
A multipartition of n with ℓ components is an ordered sequence $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ of partitions such that $|\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| = n$. We denote by $\mathcal{P}_n^\ell$ the set of multipartitions of n with ℓ components. For $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$, we write $\boldsymbol{\lambda} \vdash_\ell n$ and call $\boldsymbol{\lambda}$ an ℓ-partition of n. When $\ell = 1$, it is clear $\mathcal{P}_n^1 = \mathcal{P}_n$. The Young diagram of $\boldsymbol{\lambda}$ is a set

$$[\boldsymbol{\lambda}] = \{(i, j, s) | 1 \leq s \leq \ell, 1 \leq j \leq \lambda_i^{(s)}, \forall i \geq 1\}.$$

The elements of $[\boldsymbol{\lambda}]$ are called the nodes of $\boldsymbol{\lambda}$. The Young diagram $[\boldsymbol{\lambda}]$ can be identified with an ordered sequence of tableaux. For example, $\boldsymbol{\lambda} = ((2, 1), \emptyset, (3, 2, 1))$ is a 3-partition of 9, the Young diagram

$$[\boldsymbol{\lambda}] = \{(1, 1, 1), (1, 2, 1), (2, 1, 1), (1, 1, 3), (1, 2, 3), (1, 3, 3), (2, 1, 3), (2, 2, 3), (3, 1, 3)\},$$

it can be identified with the following ordered sequence of tableaux;

$[\boldsymbol{\lambda}] = \Big($  , $\emptyset$,  $\Big)$,

where (i, j, s) corresponds to the box in the i-th row and j-th column of the s-th tableau. For simplicity, we identify $\boldsymbol{\lambda}$ with its Young diagram $[\boldsymbol{\lambda}]$.

Suppose $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. If $\alpha \in [\boldsymbol{\lambda}]$ and $[\boldsymbol{\lambda}] \setminus \{\alpha\}$ is a Young diagram of ℓ-partition of $n - 1$, then we call α a removable node of $\boldsymbol{\lambda}$. If $\beta \notin [\boldsymbol{\lambda}]$ and $[\boldsymbol{\lambda}] \cup \{\beta\}$ is a Young diagram of ℓ-partition of $n + 1$, then we call β an addable node of $\boldsymbol{\lambda}$.

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}), \boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(\ell)})$ be ℓ-partitions of n; write $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$ if

$$\sum_{a=1}^{s-1} |\mu^{(a)}| + \sum_{i=1}^{t} \mu_j^{(s)} \geq \sum_{a=1}^{s-1} |\lambda^{(a)}| + \sum_{i=1}^{t} \lambda_j^{(s)} \quad \forall\, 1 \leq s \leq \ell \;\; \forall\, t \geq 1.$$

If $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$ and $\boldsymbol{\mu} \neq \boldsymbol{\lambda}$, we write $\boldsymbol{\mu} \triangleright \boldsymbol{\lambda}$. In particular, for $\lambda, \mu \in \mathcal{P}_n$, write $\mu \trianglerighteq \lambda$ if

$$\sum_{i=1}^{t} \mu_i \geq \sum_{i=1}^{t} \lambda_i \quad \forall\, t \geq 1.$$

We call $\trianglerighteq$ the **dominance order**.

Let $\mathcal{N}_n^\ell = \{(r, c, l) | r, c, l \in \mathbb{N}_{\geq 1}, r + c \leq 2(n + 1), 1 \leq l \leq \ell\}$. The elements of $\mathcal{N}_n^\ell$ are also called nodes and the subsets of $\mathcal{N}_n^\ell$ are called configurations of nodes. By definition, the Young diagrams of ℓ-partitions of n are configurations of nodes.

We fix e an element in $\mathbb{N}_{\geq 2} \cup \{\infty\}$ and $I = \mathbb{Z}/e\mathbb{Z}$, where $I = \mathbb{Z}$ whenever $e = \infty$. An e-multicharge is a sequence $(\kappa_1, \kappa_2, \ldots, \kappa_\ell) \in I^\ell$. For $\alpha = (r, c, l) \in \mathcal{N}_n^\ell$, we define its residue to be $\mathrm{res}(\alpha) = c - r + \kappa_l \in I$. A loading is a sequence of integers $\theta = (\theta_1, \ldots, \theta_\ell)$ such that $\theta_i - \theta_j \notin \ell\mathbb{Z}$ for $i < j$.

Definition 2.1. *[9, Definition 1.2] Let $\alpha = (r, c, l), \alpha' = (r', c', l') \in \mathcal{N}_n^\ell$. We write $\alpha' <_\theta \alpha$ if either*

(*i*) $\theta_l + \ell(r - c) < \theta_{l'} + \ell(r' - c')$ *or*
(*ii*) $\theta_l + \ell(r - c) = \theta_{l'} + \ell(r' - c')$ *and* $r + c < r' + c'$.

Moreover, if $\mathrm{res}(\alpha) = \mathrm{res}(\alpha')$*, then we write* $\alpha' \lhd_\theta \alpha$.

Definition 2.2. *[9, Definition 1.2] Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$, we write $\boldsymbol{\mu} \unlhd_\theta \boldsymbol{\lambda}$ if*

$$|\{\beta \in \boldsymbol{\mu} | \gamma \lhd_\theta \beta\}| \leq |\{\beta \in \boldsymbol{\lambda} | \gamma \lhd_\theta \beta\}| \quad \forall \gamma \in \mathcal{N}_n^\ell.$$

We call $\unlhd_\theta$ the θ**-dominance order**.

Deleting the residue condition in the definition of θ-dominance order, we can get a weak version of it.

Definition 2.3. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$, we write $\boldsymbol{\mu} \leq_\theta \boldsymbol{\lambda}$ if*

$$|\{\beta \in \boldsymbol{\mu} | \gamma <_\theta \beta\}| \leq |\{\beta \in \boldsymbol{\lambda} | \gamma <_\theta \beta\}| \quad \forall \gamma \in \mathcal{N}_n^\ell.$$

We call $\leq_\theta$ the **weak** θ**-dominance order**.

3. A combinatorial description of weak θ-dominance order on multipartitions

Fix a loading $\theta = (\theta_1, \ldots, \theta_\ell)$, if $\theta_{i+1} - \theta_i > \ell n$ for each $i = 1, 2, \ldots, \ell - 1$, then we call θ a strongly separated loading.

Let $\boldsymbol{\lambda}$ be a configuration of nodes. For $i \in \mathbb{Z}$, we call $\{(r, c, l) \in \boldsymbol{\lambda} | \theta_l + \ell(r - c) = i\}$ the i-diagonal of $\boldsymbol{\lambda}$ and $d_i^{\boldsymbol{\lambda}} = |\{(r, c, l) \in \boldsymbol{\lambda} | \theta_l + \ell(r - c) = i\}|$ the length of the i-diagonal. Let (r, c, l) be a node in the i-diagonal of $\boldsymbol{\lambda}$. We call (r, c, l) the **terminal node** (respectively, **head node**) in the i-diagonal of $\boldsymbol{\lambda}$ if $r' + c' \leq r + c$ (respectively, $r' + c' \geq r + c$) for each (r', c', l) in the i-diagonal of $\boldsymbol{\lambda}$.

We give a rough description of the weak θ-dominance order by the length of diagonals.

Lemma 3.1. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$, then $\boldsymbol{\mu} \leq_\theta \boldsymbol{\lambda}$ if and only if*

$$\sum_{i=-\infty}^{t} d_i^{\boldsymbol{\lambda}} \geq \sum_{i=-\infty}^{t} d_i^{\boldsymbol{\mu}} \quad \forall t \in \mathbb{Z}.$$

Proof. Firstly, let us prove the necessity. Assume t to be an integer such that

$$\sum_{i=-\infty}^{t} d_i^{\boldsymbol{\lambda}} < \sum_{i=-\infty}^{t} d_i^{\boldsymbol{\mu}}.$$

Since $|\boldsymbol{\lambda}| = |\boldsymbol{\mu}| = n$, hence there exists an integer $t' > t$ such that

- the t'-diagonal of $\boldsymbol{\lambda}$ is non-empty, and
- $\forall t < t'' < t'$, the t''-diagonal of $\boldsymbol{\lambda}$ is empty.

Let α be the head node in the t'-diagonal of $\boldsymbol{\lambda}$; then we have

$$|\{\beta \in \boldsymbol{\lambda} | \alpha <_\theta \beta\}| = \sum_{i=-\infty}^{t} d_i^{\boldsymbol{\lambda}} < \sum_{i=-\infty}^{t} d_i^{\boldsymbol{\mu}} \leq |\{\beta \in \boldsymbol{\mu} | \alpha <_\theta \beta\}|.$$

This contradicts to $\boldsymbol{\lambda} \geq_\theta \boldsymbol{\mu}$.

Next, let us prove the sufficiency. Suppose $\sum_{i=-\infty}^{t} d_i^{\boldsymbol{\lambda}} \geq \sum_{i=-\infty}^{t} d_i^{\boldsymbol{\mu}}$ for all $t \in \mathbb{Z}$. Let $\gamma = (r, c, l)$ be a node in the t-diagonal of $\mathcal{N}_n^\ell$ and (r', c', l') be the head node in the t-diagonal of $\mathcal{N}_n^\ell$. If $\gamma \notin \boldsymbol{\lambda} \cup \boldsymbol{\mu}$, then

$$|\{\alpha \in \boldsymbol{\lambda} | \gamma <_\theta \alpha\}| = \sum_{i=-\infty}^{t} d_i^{\boldsymbol{\lambda}} \geq \sum_{i=-\infty}^{t} d_i^{\boldsymbol{\mu}} = |\{\beta \in \boldsymbol{\mu} | \gamma <_\theta \beta\}|.$$

If $\gamma \in \boldsymbol{\lambda} \setminus \boldsymbol{\mu}$, then

$$|\{\alpha \in \boldsymbol{\lambda} | \gamma <_\theta \alpha\}| = r - r' + \sum_{i=-\infty}^{t-1} d_i^{\boldsymbol{\lambda}} \geq r - r' + \sum_{i=-\infty}^{t-1} d_i^{\boldsymbol{\mu}} \geq |\{\beta \in \boldsymbol{\mu} | \gamma <_\theta \beta\}|.$$

If $\gamma \in \boldsymbol{\mu} \setminus \boldsymbol{\lambda}$, then

$$|\{\alpha \in \boldsymbol{\lambda} | \gamma <_\theta \alpha\}| = \sum_{i=-\infty}^{t} d_i^{\boldsymbol{\lambda}} \geq \sum_{i=-\infty}^{t} d_i^{\boldsymbol{\mu}} > r - r' + \sum_{i=-\infty}^{t-1} d_i^{\boldsymbol{\mu}} = |\{\beta \in \boldsymbol{\mu} | \gamma <_\theta \beta\}|.$$

If $\gamma \in \boldsymbol{\lambda} \cap \boldsymbol{\mu}$, then

$$|\{\alpha \in \boldsymbol{\lambda} | \gamma <_\theta \alpha\}| = r - r' + \sum_{i=-\infty}^{t-1} d_i^{\boldsymbol{\lambda}} \geq r - r' + \sum_{i=-\infty}^{t-1} d_i^{\boldsymbol{\mu}} = |\{\beta \in \boldsymbol{\mu} | \gamma <_\theta \beta\}|.$$

Therefore, $\boldsymbol{\lambda} \geq_\theta \boldsymbol{\mu}$. □

Remark 3.2. *Suppose $\theta = (\theta_1, \ldots, \theta_\ell)$ are strongly separated and $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \mathcal{P}_n^\ell$. Let $\alpha = (r, c, s)$ be a node in the i-diagonal of $\boldsymbol{\lambda}$ and $\alpha' = (r', c', s+1)$ be a node in the i'-diagonal of $\boldsymbol{\lambda}$. Then $i' > i$. In fact, $\boldsymbol{\lambda}$ is a multipartition of n, hence*

$$i' - i = \theta_{s+1} + \ell(r' - c') - (\theta_s + \ell(r - c)) = (\theta_{s+1} - \theta_s) + \ell(r' + c - c' - r) > \ell n + \ell(2 - n) = 2\ell > 0.$$

That is, the s-component $\lambda^{(s)}$ is completely separated from the $(s+1)$-component $\lambda^{(s+1)}$.

For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$, we say that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are neighbors with the weak θ-dominance order if $\boldsymbol{\mu} >_\theta \boldsymbol{\lambda}$ and there is no $\boldsymbol{\gamma} \in \mathcal{P}_n^\ell$ such that $\boldsymbol{\mu} >_\theta \boldsymbol{\gamma} >_\theta \boldsymbol{\lambda}$.

In [[10], Theorem 1.4.10], there is a characterization of partitions that are neighbors with the usual dominance order. In the following lemma, let us prove a similar combinatorial description of neighbors with weak θ-dominance order on partitions. Consequently, it will be clear that the weak θ-dominance order coincides with the usual dominance order on partitions.

Lemma 3.3. *Suppose $\lambda, \mu \in \mathcal{P}_n$ and $\mu >_\theta \lambda$, then λ, μ are neighbors with the weak θ-dominance order if and only if there exist positive integers $r < r'$ such that one of the following (a) and (b) occurs, where*

(a) $r' = r + 1, \quad \mu_r = \lambda_r + 1, \quad \mu_{r+1} = \lambda_{r+1} - 1 \quad$ *and* $\quad \mu_t = \lambda_t \quad \forall t \neq r, r+1,$
(b) $\lambda_r = \lambda_{r'}, \quad \mu_r = \lambda_r + 1, \quad \mu_{r'} = \lambda_{r'} - 1 \quad$ *and* $\quad \mu_t = \lambda_t \quad \forall t \neq r, r'.$

Proof. Let $\ell = 1$ and $\theta, \theta' \in \mathbb{Z}$ be different integers. For $\alpha, \alpha' \in \mathcal{N}_n^{\ell}$, we have $\alpha >_\theta \alpha'$ if and only if $\alpha >_{\theta'} \alpha'$. Therefore, for each $\lambda, \mu \in \mathcal{P}_n$, we have $\mu >_\theta \lambda$ if and only if $\mu >_{\theta'} \lambda$. Hence, for simplicity, we assume $\theta = 0$. By this assumption, node (a, b) lies in the $(a-b)$-diagonal.

First, let us prove the necessity. Assume $\mu >_\theta \lambda$ to be partitions of n and there exist no $\gamma \in \mathcal{P}_n$ such that $\mu >_\theta \gamma >_\theta \lambda$. Define $i := \min\{k \in \mathbb{Z} | d_k^\mu \neq d_k^\lambda\}$. Then i is well defined since $\mu \neq \lambda$. Define $i' := \min\Big\{k \in \mathbb{Z} \Big| \sum_{t=-\infty}^{k} d_t^\mu = \sum_{t=-\infty}^{k} d_t^\lambda, i < k\Big\}$. Then i' is well defined since $|\lambda| = |\mu| = n$. By definition, $-n < i < i' < n$. Combining with Lemma 3.1, we derive

$$0 \leq d_i^\lambda < d_i^\mu, \quad d_{i-1}^\mu = d_{i-1}^\lambda \tag{3.4}$$

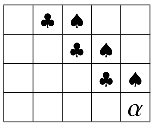
and

$$d_{i'}^\lambda > d_{i'}^\mu \geq 0, \quad d_{i'+1}^\mu \geq d_{i'+1}^\lambda. \tag{3.5}$$

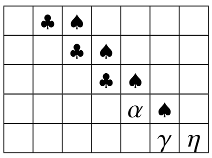
We will give the proof of necessity in 3 steps:

Step 1. Let $\alpha = (r, c)$ be the terminal node in the i-diagonal of μ. Let us prove $\lambda_{r-1} \geq \lambda_r + 1$ and (r, c) is the last node in the r-th row of μ.

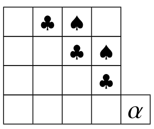
If $i \leq 0$, let us prove $d_{i-1}^\mu = d_i^\mu - 1$. We should prove the i-diagonal and $(i-1)$-diagonal of μ is like

$\mu \supseteq$  ,

where ♣ and $\alpha = (r, c)$ are the i-diagonal of μ and ♠ are the $(i-1)$-diagonal of μ. If $d_{i-1}^\mu > d_i^\mu$, then we derive $\eta = (r+1, c+2) \in \mu$, hence $\gamma = (r+1, c+1) \in \mu$. Then the i-diagonal and $(i-1)$-diagonal of μ is like

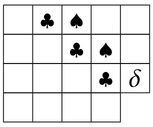
$\mu \supseteq$  ,

where ♣, $\alpha = (r, c)$ and $\gamma = (r+1, c+1)$ are the i-diagonal of μ, while ♠ and $\eta = (r+1, c+2)$ are the $(i-1)$-diagonal of μ. This contradicts that $\alpha = (r, c)$ is the terminal node in the i-diagonal of μ; therefore, $d_{i-1}^\mu \leq d_i^\mu$. Similarly, one can prove $d_{i-1}^\lambda \leq d_i^\lambda$. If $d_{i-1}^\mu = d_i^\mu$, by (3.4), we have $d_{i-1}^\lambda = d_{i-1}^\mu = d_i^\mu > d_i^\lambda$; this contradicts $d_{i-1}^\lambda \leq d_i^\lambda$, hence $d_{i-1}^\mu < d_i^\mu$. If $d_{i-1}^\mu < d_i^\mu - 1$, then $(r-1, c) \notin \mu$. Then the i-diagonal and $(i-1)$-diagonal of μ are like

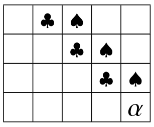
$\mu \supseteq$  ,

where ♣ and $\alpha = (r, c)$ are the i-diagonal of μ and ♠ are the $(i-1)$-diagonal of μ. This contradicts to $\alpha = (r, c) \in \mu$ and $\mu_{r-1} \geq \mu_r$. Therefore, $d_{i-1}^\mu = d_i^\mu - 1$. By (3.4), we have $d_{i-1}^\lambda = d_{i-1}^\mu = d_i^\mu - 1 > d_i^\lambda - 1$, hence $d_i^\lambda = d_{i-1}^\lambda = d_{i-1}^\mu = d_i^\mu - 1$. So we derive $\alpha = (r, c) \notin \lambda$ and $\delta = (r-1, c) \in \lambda$. Hence, the

i-diagonal and $(i-1)$-diagonal of λ are like

$\lambda \supseteq$  ,

where ♣ are the i-diagonal of λ and ♠, $\delta = (r-1, c)$ are the $(i-1)$-diagonal of λ. The i-diagonal and $(i-1)$-diagonal of μ are like

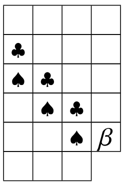
$\mu \supseteq$  ,

where ♣, $\alpha = (r, c)$ are the i-diagonal of μ and ♠ are the $(i-1)$-diagonal of μ. Therefore $\lambda_{r-1} \geq \lambda_r + 1$. Moreover, (r, c) is the last node in the r-th row of μ.

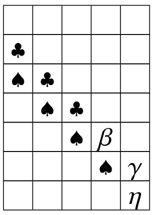
For the case when $i > 0$, the discussion is tedious and similar to that of $i \leq 0$, so we don't show it here again.

Step 2. Let $\beta = (r', c')$ be the terminal node in the i'-diagonal of λ. Let us prove $\lambda_{r'} - 1 \geq \lambda_{r'+1}$ and $r' > r$.

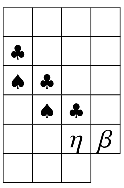
If $i' \geq 0$, let us prove $d^{\lambda}_{i'+1} = d^{\lambda}_{i'} - 1$. We should prove the i'-diagonal and $(i'+1)$-diagonal of λ are like

$\lambda \supseteq$  ,

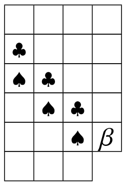
where ♣ and $\beta = (r', c')$ are the i'-diagonal of λ and ♠ are the $(i'+1)$-diagonal of λ. If $d^{\lambda}_{i'+1} > d^{\lambda}_{i'}$, then $\eta = (r'+2, c'+1) \in \lambda$, hence $\gamma = (r'+1, c'+1) \in \lambda$. The i'-diagonal and $(i'+1)$-diagonal of λ are like

$\lambda \supseteq$  ,

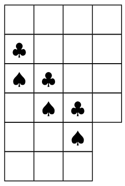
where ♣, $\beta = (r', c')$ and $\gamma = (r'+1, c'+1)$ are the i'-diagonal of λ and ♠, $\eta = (r'+2, c'+1)$ are the $(i'+1)$-diagonal of λ. This contradicts that $\beta = (r', c')$ is the terminal node in the i'-diagonal of λ. Therefore, $d^{\lambda}_{i'+1} \leq d^{\lambda}_{i'}$. Similarly, we can prove $d^{\mu}_{i'+1} \leq d^{\mu}_{i'}$. If $d^{\lambda}_{i'+1} = d^{\lambda}_{i'}$, by (3.5), we have $d^{\mu}_{i'+1} \geq d^{\lambda}_{i'+1} = d^{\lambda}_{i'} > d^{\mu}_{i'}$. This contradicts to $d^{\mu}_{i'+1} \leq d^{\mu}_{i'}$. If $d^{\lambda}_{i'+1} < d^{\lambda}_{i'} - 1$, then $\eta = (r', c'-1) \notin \lambda$. Then the i'-diagonal and $(i'+1)$-diagonal of λ are like

$\lambda \supseteq$  $\setminus \{\boxed{\eta}\}$

where ♣ and $\beta = (r', c')$ are the i'-diagonal of λ and ♠ are the $(i'+1)$-diagonal of λ. This contradicts to $\beta = (r', c') \in \lambda$. Therefore, $d^{\lambda}_{i'+1} = d^{\lambda}_{i'} - 1$, by (3.5), we have $d^{\mu}_{i'+1} \geq d^{\lambda}_{i'+1} = d^{\lambda}_{i'} - 1 > d^{\mu}_{i'} - 1$. Therefore, we derive $d^{\mu}_{i'} = d^{\mu}_{i'+1} = d^{\lambda}_{i'+1} = d^{\lambda}_{i'} - 1$. Hence, $(r'+1, c') \notin \lambda$, $\quad \beta = (r', c') \notin \mu$. Then the i'-diagonal and $(i'+1)$-diagonal of λ are like

$\lambda \supseteq$ 

where ♣ and $\beta = (r', c')$ are the i'-diagonal of λ and ♠ are the $(i'+1)$-diagonal of λ. The i'-diagonal and $(i'+1)$-diagonal of μ are like

$\mu \supseteq$  ,

where ♣ are the i'-diagonal of μ and ♠ are the $(i'+1)$-diagonal of μ. Therefore, we have $\lambda_{r'} - 1 \geq \lambda_{r'+1}$. Next, let us prove $r' > r$. If $r' = r$, since $(r', c') \notin \mu$ and (r, c) is the last node in the r-th row of μ, so we have $c' > c$, hence $i' = r - c' < r - c = i$, this contradicts to $i' > i$. If $r' < r$, since $r' - c' = i' > i = r - c$, then $c' + s < c$, where $s = r - r' > 0$. Since (r, c) is the last node in the r-th row of μ, so $(r, c' + s) \in \mu$. Moreover, since $r - (c' + s) = r' - c' = i'$, so $(r, c' + s)$ lies in the i'-diagonal of μ and $(r, c' + s) <_{\theta} (r', c')$, this contradicts to $(r', c') \notin \mu$. Therefore, we derive $r' > r$.

For the case when $i < 0$, the discussion is also tedious and similar to that of $i \geq 0$, so we do not show it here again.

Step 3. Now we have proved $r' > r$, $\lambda_{r-1} \geq \lambda_r + 1$ and $\lambda_{r'} - 1 \geq \lambda_{r'+1}$. Hence

$$\gamma = (\lambda_1, \ldots, \lambda_{r-1}, \lambda_r + 1, \lambda_{r+1}, \ldots, \lambda_{r'-1}, \lambda_{r'} - 1, \lambda_{r'+1}, \ldots)$$

is a partition of n. Let $\alpha' = (r, \lambda_r + 1)$, $\beta' = (r', \lambda_{r'})$ and $j = r - (\lambda_r + 1)$, $j' = r' - \lambda_{r'}$. Then α' is the terminal node in the j-diagonal of γ, and β' is the terminal node in the j'-diagonal of λ. We can obtain γ from λ by removing β' to α'. Since $r' > r$, $\lambda_r + 1 > \lambda_{r'}$, hence $j' = r' - \lambda_{r'} > r - (\lambda_r + 1) = j$ and $\beta' = (r', \lambda_{r'}) <_{\theta} (r, \lambda_r + 1) = \alpha'$. So we have $\gamma >_{\theta} \lambda$. Next, let us prove $\mu \geq_{\theta} \gamma$. Since $\alpha = (r, c) \notin \lambda$, so $\lambda_r + 1 \leq c$, hence $j = r - (\lambda_r + 1) \geq r - c = i$ and $\alpha' = (r, \lambda_r + 1) \leq_{\theta} (r, c) = \alpha$. Since $\beta = (r', c') \in \lambda$, so $\lambda_{r'} \geq c'$ and $j' = r' - \lambda_{r'} \leq r' - c' = i'$. Hence $\beta = (r', c') \leq_{\theta} (r', \lambda_{r'}) = \beta'$. Therefore,

$$\alpha = (r, c) \geq_{\theta} \alpha' = (r, \lambda_r + 1) >_{\theta} \beta' = (r', \lambda_{r'}) \geq_{\theta} \beta = (r', c') \quad \text{and} \quad i \leq j < j' \leq i'.$$

Combining with the choice of i, i' and Lemma 3.1, we derive $\mu \geq_{\theta} \gamma$. Hence $\mu \geq_{\theta} \gamma >_{\theta} \lambda$. Since λ and μ are neighbors with $\geq_{\theta}$, so we have $\mu = \gamma$.

Finally, let us prove $r = r' - 1$ or $\lambda_r = \lambda_{r'}$. Otherwise, suppose $r \neq r' - 1$ and $\lambda_r \neq \lambda_{r'}$, then $r < r' - 1$ and $\lambda_r > \lambda_{r'}$. Let $t = 1 + \min\{k | \lambda_k > \lambda_{k+1}, r \leq k < r'\}$. If $t = r'$, then $\lambda_r = \cdots = \lambda_{r'-1} > \lambda_{r'} > 0$, let $\nu = (\lambda_1, \ldots, \lambda_{r-1}, \lambda_r + 1, \lambda_{r+1}, \ldots, \lambda_{r'-2}, \lambda_{r'-1} - 1, \lambda_{r'}, \ldots)$. Since $(r', \lambda_{r'}) <_{\theta} (r' - 1, \lambda_{r'-1}) <_{\theta} (r, \lambda_r + 1)$, hence$\mu >_{\theta} \nu >_{\theta} \lambda$, this contradicts that μ and λ are neighbors with $\geq_{\theta}$. If $t < r'$, then $\lambda_r = \cdots = \lambda_{t-1} > \lambda_t \geq \cdots \geq \lambda_{r'} > 0$, let

$$\eta := (\lambda_1, \ldots, \lambda_{t-1}, \lambda_t + 1, \ldots, \lambda_{r'-1}, \lambda_{r'} - 1, \lambda_{r'+1}, \ldots).$$

Since $(r', \lambda_{r'}) <_{\theta} (t, \lambda_t + 1) <_{\theta} (r, \lambda_r + 1)$, then $\mu >_{\theta} \eta >_{\theta} \lambda$, this contradicts that μ and λ are neighbors with $\geq_{\theta}$. Therefore, $r = r' - 1$ or $\lambda_r = \lambda_{r'}$. Now we complete the proof of necessity.

Next, let us prove the sufficiency. Suppose $\lambda, \mu \in \mathcal{P}_n$ and there exist positive integers $r < r'$ satisfying

(a) $r' = r + 1, \quad \mu_r = \lambda_r + 1, \quad \mu_{r+1} = \lambda_{r+1} - 1 \quad$ and $\quad \mu_t = \lambda_t \quad \forall t \neq r, r+1$, or
(b) $\lambda_r = \lambda_{r'}, \quad \mu_r = \lambda_r + 1, \quad \mu_{r'} = \lambda_{r'} - 1 \quad$ and $\quad \mu_t = \lambda_t \quad \forall t \neq r, r'$.

Let $\nu \in \mathcal{P}_n^\ell$ such that $\mu \geq_\theta \nu >_\theta \lambda$ and λ, ν are neighbors with $\geq_\theta$. Let us prove $\mu = \nu$. Let $i = r - (\lambda_r + 1)$, $i' = r' - \lambda_{r'}$, it is clear $i < i'$. By assumption, μ can be obtained from λ by removing $(r', \lambda_{r'})$ to $(r, \lambda_r + 1)$. By Lemma 3.1 and the the choice of ν, we have

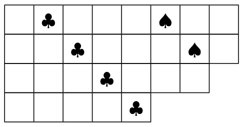
$$d_t^\mu = d_t^\nu = d_t^\lambda, \qquad \text{for all } t < i \text{ or } t > i'. \tag{3.6}$$

By the necessity of this lemma, there exist integers $s < s'$ such that

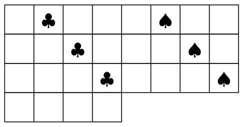
$$\nu_s = \lambda_s + 1, \quad \nu_{s'} = \lambda_{s'} - 1, \quad \nu_t = \lambda_t \quad \forall t \neq s, s'.$$

Let $j = s - (\lambda_s + 1)$, $j' = s' - \lambda_{s'}$, it is clear $j < j'$. In other words, ν can be obtained from λ by removing $(s', \lambda_{s'})$ to $(s, \lambda_s + 1)$. Combining with (3.6), we know $i \leq j < j' \leq i'$.

If (a) occurs, $r = r' - 1$, the i-diagonal and i'-diagonal of λ are like

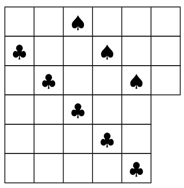
$\lambda \supseteq$ 

where ♠ are the i-diagonal and ♣ are the i'-diagonal. The i-diagonal and i'-diagonal of μ are like

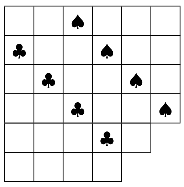
$\mu \supseteq$ 

where ♠ are the i-diagonal and ♣ are the i'-diagonal. By the above arguments, we have $s' - \lambda_{s'} = j' = i' = r' - \lambda_{r'}$, $s - \lambda_s - 1 = j = i = r - \lambda_r - 1$. Since the addable node and removable node are unique for the i-diagonal and i'-diagonal of λ, respectively. Hence, $s = r$, $s' = r'$ and $\mu = \nu$.

If (b) occurs, $r < r' - 1$ and $\lambda_r = \lambda_{r+1} = \cdots = \lambda_{r'}$, the i-diagonal and i'-diagonal of λ are like

$\lambda \supseteq$ 

where ♠ are the i-diagonal and nodes ♣ are the i'-diagonal. The i-diagonal and i'-diagonal of λ are like

$\mu \supseteq$ 

where ♠ are the i-diagonal and ♣ are the i'-diagonal. By the above arguments, we have $s' - \lambda_{s'} = j' = i' = r' - \lambda_{r'}$, $\quad s - \lambda_s - 1 = j = i = r - \lambda_r - 1$. Since the addable node and removable node are unique for the i-diagonal and i'-diagonal of λ respectively. Hence, $s = r$, $s' = r'$, and $\mu = \nu$. □

Now we can give a combinatorial description of neighbors with weak θ-dominance order on multipartitions.

Proposition 3.7. *Suppose $\theta = (\theta_1, \ldots, \theta_\ell)$ be strongly separated, $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(\ell)})$ be ℓ-partitions of n with $\boldsymbol{\mu} >_\theta \boldsymbol{\lambda}$. Then $\boldsymbol{\lambda}, \boldsymbol{\mu}$ are neighbors with the weak θ-dominance order if and only if one of (a), (b), and (c) occurs, where*

(a) *there exists $s < \ell$ such that*

$$\begin{aligned} &\mu^{(r)} = \lambda^{(r)} \quad \forall r \neq s, s+1 \\ &\mu^{(s)} = (\lambda^{(s)}_1, \ldots, \lambda^{(s)}_{k_s}, 1) \quad \textit{where} \quad k_s = h(\lambda^{(s)}) \\ &\mu^{(s+1)}_1 = \lambda^{(s+1)}_1 - 1 \quad \textit{and} \quad \mu^{(s+1)}_j = \lambda^{(s+1)}_j \quad \forall j > 1. \end{aligned}$$

(b) *there exist s and i such that*

$$\begin{aligned} &\mu^{(r)} = \lambda^{(r)} \quad \forall r \neq s \\ &\mu^{(s)}_j = \lambda^{(s)}_j \quad \forall j \neq i, i+1 \\ &\mu^{(s)}_i = \lambda^{(s)}_i + 1 \quad \textit{and} \quad \mu^{(s)}_{i+1} = \lambda^{(s)}_{i+1} - 1. \end{aligned}$$

(c) *there exist s and $i < i'$ such that*

$$\begin{aligned} &\mu^{(r)} = \lambda^{(r)} \quad \forall r \neq s \\ &\mu^{(s)}_j = \lambda^{(s)}_j \quad \forall j \neq i, i' \\ &\mu^{(s)}_i - 1 = \mu^{(s)}_{i'} + 1 = \lambda^{(s)}_i = \lambda^{(s)}_{i'}. \end{aligned}$$

Proof. Let us prove the necessity. We assume $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(\ell)})$ are ℓ-partitions of n with $\boldsymbol{\mu} >_\theta \boldsymbol{\lambda}$ and there exists no $\boldsymbol{\gamma} = (\gamma^{(1)}, \ldots, \gamma^{(\ell)}) \in \mathcal{P}^\ell_n$ such that $\boldsymbol{\mu} >_\theta \boldsymbol{\gamma} >_\theta \boldsymbol{\lambda}$. Define

$$s := \min\{k | \mu^{(k)} \neq \lambda^{(k)}\},$$

then $\lambda^{(s)} \neq \mu^{(s)}$ and $\lambda^{(r)} = \mu^{(r)}$ for all $r < s$. Since $\boldsymbol{\mu} >_\theta \boldsymbol{\lambda}$ and θ is strongly separated, combining with Lemma 3.1 and Remark 3.2, we have

$$\sum_{k=1}^{t} |\mu^{(k)}| \geq \sum_{k=1}^{t} |\lambda^{(k)}| \quad \forall 1 \leq t \leq \ell. \tag{3.8}$$

Hence $|\mu^{(s)}| \geq |\lambda^{(s)}|$.

Suppose $|\mu^{(s)}| > |\lambda^{(s)}|$, then $s < \ell$. Set $m_s = h(\mu^{(s)})$, let us prove $\mu^{(s)}_{m_s} = 1$. Otherwise, assume $\mu^{(s)}_{m_s} > 1$. Let $\boldsymbol{\gamma} = (\gamma^{(1)}, \ldots, \gamma^{(s)}, \ldots, \gamma^{(\ell)})$, where

$$\begin{aligned} &\gamma^{(r)} = \mu^{(r)} \quad \forall r \neq s, \\ &\gamma^{(s)} = (\mu^{(s)}_1, \ldots, \mu^{(s)}_{m_s - 1}, \mu^{(s)}_{m_s} - 1, 1). \end{aligned}$$

Since $(m_s, \mu^{(s)}_{m_s}, s) >_\theta (m_s + 1, 1, s)$, so $\boldsymbol{\mu} >_\theta \boldsymbol{\gamma}$. Let us prove $\boldsymbol{\gamma} >_\theta \boldsymbol{\lambda}$. Let

$$u = \theta_s + \ell(1 - \mu^{(s)}_1), \ p = \theta_s + \ell(m_s - \mu^{(s)}_{m_s}), \ q = \theta_s + \ell(m_s + 1 - 1) = \theta_s + \ell m_s.$$

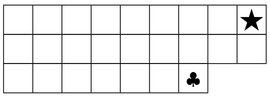
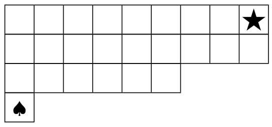
Where $u \le p < q$. Define

$$\bigstar := (1, \mu_1^{(s)}, s) \text{ is the unique node in the } u\text{-diagonal of } \boldsymbol{\mu}$$

$$\clubsuit := (m_s, \mu_{m_s}^{(s)}, s) \text{ is the terminal node in the } p\text{-diagonal of } \boldsymbol{\mu}$$

$$\spadesuit := (m_s + 1, 1, s) \text{ is the unique node in the } q\text{-diagonal of } \boldsymbol{\gamma}.$$

$\boldsymbol{\gamma}$ can be obtained from $\boldsymbol{\mu}$ by removing $\clubsuit$ to $\spadesuit$. The Young diagrams of $\mu^{(s)}$ and $\gamma^{(s)}$ are like

$\mu^{(s)} =$  $\gamma^{(s)} =$ 

By the construction of $\boldsymbol{\gamma}$, we have $d_t^{\boldsymbol{\mu}} = d_t^{\boldsymbol{\gamma}}$ for all $t \neq p, q$. By Remark 3.2 and the choice of s, we have

$$\sum_{t=-\infty}^{u-1} d_t^{\boldsymbol{\gamma}} = \sum_{t=-\infty}^{u-1} d_t^{\boldsymbol{\mu}} = \sum_{t=1}^{s-1} |\mu^{(t)}| = \sum_{t=1}^{s-1} |\lambda^{(t)}| = \sum_{t=-\infty}^{u-1} d_t^{\boldsymbol{\lambda}}.$$

By Lemma 3.1, we derive

$$\sum_{t=u}^{k} d_t^{\boldsymbol{\gamma}} = \sum_{t=u}^{k} d_t^{\boldsymbol{\mu}} \ge \sum_{t=u}^{k} d_t^{\boldsymbol{\lambda}} \quad \forall u \le k < p \tag{3.9}$$

$$1 + \sum_{t=u}^{k} d_t^{\boldsymbol{\gamma}} = \sum_{t=u}^{k} d_t^{\boldsymbol{\mu}} \ge \sum_{t=u}^{k} d_t^{\boldsymbol{\lambda}} \quad \forall p \le k < q \tag{3.10}$$

and

$$\sum_{t=-\infty}^{k} d_t^{\boldsymbol{\gamma}} = \sum_{t=-\infty}^{k} d_t^{\boldsymbol{\mu}} \ge \sum_{t=-\infty}^{k} d_t^{\boldsymbol{\lambda}} \quad \forall k \ge q.$$

If $\boldsymbol{\gamma} \not\trianglerighteq_\theta \boldsymbol{\lambda}$, then there exists $p \le \epsilon < q$ such that

$$\sum_{t=u}^{\epsilon-1} d_t^{\boldsymbol{\gamma}} \ge \sum_{t=u}^{\epsilon-1} d_t^{\boldsymbol{\lambda}}, \qquad \sum_{t=u}^{\epsilon} d_t^{\boldsymbol{\gamma}} < \sum_{t=u}^{\epsilon} d_t^{\boldsymbol{\lambda}}. \tag{3.11}$$

If $\epsilon = p$, by (3.9)–(3.11), we have

$$\sum_{t=u}^{p-1} d_t^{\boldsymbol{\gamma}} = \sum_{t=u}^{p-1} d_t^{\boldsymbol{\mu}} \ge \sum_{t=u}^{p-1} d_t^{\boldsymbol{\lambda}} \quad \text{and} \quad 1 + \sum_{t=u}^{p} d_t^{\boldsymbol{\gamma}} = \sum_{t=u}^{p} d_t^{\boldsymbol{\mu}} = \sum_{t=u}^{p} d_t^{\boldsymbol{\lambda}},$$

hence $d_p^{\boldsymbol{\lambda}} \ge d_p^{\boldsymbol{\mu}}$. So $\clubsuit = (m_s, \mu_{m_s}^{(s)}, s) \in \lambda^{(s)}$ and hence $|\lambda^{(s)}| \ge |\mu^{(s)}|$, this contradicts to $|\mu^{(s)}| > |\lambda^{(s)}|$. If $p < \epsilon < q$, by (3.9)–(3.11), we have

$$\sum_{t=u}^{\epsilon-1} d_t^{\boldsymbol{\mu}} > \sum_{t=u}^{\epsilon-1} d_t^{\boldsymbol{\gamma}} \ge \sum_{t=u}^{\epsilon-1} d_t^{\boldsymbol{\lambda}}, \qquad \sum_{t=u}^{\epsilon} d_t^{\boldsymbol{\mu}} = 1 + \sum_{t=u}^{\epsilon} d_t^{\boldsymbol{\gamma}} = \sum_{t=u}^{\epsilon} d_t^{\boldsymbol{\lambda}}$$

then $d_\epsilon^{\boldsymbol{\lambda}} > d_\epsilon^{\boldsymbol{\mu}}$ and $|\lambda^{(s)}| > |\mu^{(s)}|$; this contradicts to $|\mu^{(s)}| > |\lambda^{(s)}|$. Therefore $\boldsymbol{\gamma} \geq_\theta \boldsymbol{\lambda}$. Since $|\gamma^{(s)}| = |\mu^{(s)}| > |\lambda^{(s)}|$, hence $\boldsymbol{\gamma} \neq \boldsymbol{\lambda}$. So we derive

$$\boldsymbol{\mu} >_\theta \boldsymbol{\gamma} >_\theta \boldsymbol{\lambda}.$$

This contradicts that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are neighbors with $\geq_\theta$. So we have $\mu_{m_s}^{(s)} = 1$.

Let $\boldsymbol{\nu} := (\nu^{(1)}, \ldots, \nu^{(s)}, \nu^{(s+1)}, \ldots, \nu^{(\ell)})$, where

$$\nu^{(r)} = \mu^{(r)} \quad \forall r \neq s, s+1,$$
$$\nu^{(s)} = (\mu_1^{(s)}, \ldots, \mu_{m_s-1}^{(s)})$$

and $\nu^{(s+1)} = (\nu_1^{(s+1)}, \nu_2^{(s+1)} \ldots)$, where

$$\nu_1^{(s+1)} = \mu_1^{(s+1)} + 1,\ \nu_t^{(s+1)} = \mu_t^{(s+1)} \quad t > 1.$$

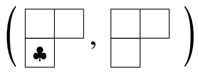
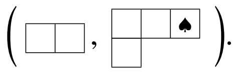
Let

$$\clubsuit := (m_s, 1, s), \quad \spadesuit := (1, \nu_1^{(s+1)}, s+1)$$

and

$$u' := \theta_s + \ell(m_s - 1),$$
$$q' := \theta_{s+1} + \ell(1 - \nu_1^{(s+1)}),$$
$$p' := \theta_{s+1} + \ell(1 - \lambda_1^{(s+1)}).$$

Let $r' := \min\{q', p'\}$, by Remark 3.2, we have $u' < r'$. Since $\mu_{m_s}^{(s)} = 1$, so $\clubsuit$ is the unique node in the u'-diagonal of $\boldsymbol{\mu}$ and $\spadesuit$ is the unique node in the q'-diagonal of $\boldsymbol{\nu}$. That is, $\boldsymbol{\nu}$ can be obtained from $\boldsymbol{\mu}$ by removing the node $\clubsuit$ to $\spadesuit$. From the point of Young's diagram

$(\mu^{(s)}, \mu^{(s+1)}) =$  $\quad (\nu^{(s)}, \nu^{(s+1)}) =$ .

By Lemma 3.1, we have $\boldsymbol{\mu} >_\theta \boldsymbol{\nu}$. Next, let us prove $\boldsymbol{\nu} \geq_\theta \boldsymbol{\lambda}$. By Lemma 3.1, we have

$$\sum_{t=-\infty}^{k} d_t^{\boldsymbol{\nu}} = \sum_{t=-\infty}^{k} d_t^{\boldsymbol{\mu}} \geq \sum_{t=-\infty}^{k} d_t^{\boldsymbol{\lambda}} \quad \text{where } k < u' \text{ or } k \geq q'.$$

Moreover, by (3.8) and the assumption $|\mu^{(s)}| > |\lambda^{(s)}|$, we have

$$\sum_{t=1}^{s} |\mu^{(t)}| > \sum_{t=1}^{s} |\lambda^{(t)}|.$$

Combining with Remark 3.2 and the definition of u', r', we derive

$$\sum_{t=-\infty}^{k} d_t^{\boldsymbol{\nu}} = \sum_{t=-\infty}^{k} d_t^{\boldsymbol{\mu}} - 1 = \sum_{t=1}^{s} |\mu^{(t)}| - 1 \geq \sum_{t=1}^{s} |\lambda^{(t)}| \geq \sum_{t=-\infty}^{k} d_t^{\boldsymbol{\lambda}}$$

where $u' \leq k < r'$. If $r' = q'$, then we have proved

$$\sum_{t=-\infty}^{k} d_t^{\boldsymbol{\nu}} \geq \sum_{t=-\infty}^{k} d_t^{\boldsymbol{\lambda}} \quad \forall k \in \mathbb{Z}.$$

Therefore, $\boldsymbol{\nu} \geq_\theta \boldsymbol{\lambda}$. If $r' = p' < q'$, suppose $\boldsymbol{\nu} \ngeq_\theta \boldsymbol{\lambda}$, there exists some $r' \leq \epsilon' < q'$ such that

$$\sum_{t=-\infty}^{\epsilon'} d_t^{\boldsymbol{\nu}} < \sum_{t=-\infty}^{\epsilon'} d_t^{\boldsymbol{\lambda}}, \quad \sum_{t=-\infty}^{\epsilon'+1} d_t^{\boldsymbol{\nu}} \geq \sum_{t=-\infty}^{\epsilon'+1} d_t^{\boldsymbol{\lambda}}.$$

On the other hand, by the definition of p' and q', we have

$$\sum_{t=-\infty}^{\epsilon'+1} d_t^{\boldsymbol{\nu}} \leq 1 + \sum_{t=-\infty}^{\epsilon'} d_t^{\boldsymbol{\nu}} < 1 + \sum_{t=-\infty}^{\epsilon'} d_t^{\boldsymbol{\lambda}} \leq \sum_{t=-\infty}^{\epsilon'+1} d_t^{\boldsymbol{\lambda}},$$

this contradicts to the choice of ϵ'.

So we derive $\boldsymbol{\nu} \geq_\theta \boldsymbol{\lambda}$, then $\boldsymbol{\mu} >_\theta \boldsymbol{\nu} \geq_\theta \boldsymbol{\lambda}$. Since $\boldsymbol{\lambda}, \boldsymbol{\mu}$ are neighbors with $\geq_\theta$, we derive $\boldsymbol{\nu} = \boldsymbol{\lambda}$. That is, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ satisfy (a).

Suppose $|\lambda^{(s)}| = |\mu^{(s)}|$. Let $\boldsymbol{\nu} = (\mu^{(1)}, \dots, \mu^{(s)}, \lambda^{(s+1)}, \dots, \lambda^{(\ell)})$, then $\boldsymbol{\mu} \geq_\theta \boldsymbol{\nu} >_\theta \boldsymbol{\lambda}$, so $\boldsymbol{\nu} = \boldsymbol{\mu}$ since $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are neighbors with $\geq_\theta$. Hence $\mu^{(r)} = \lambda^{(r)}, \forall r \neq s$. Let $m := |\lambda^{(s)}| = |\mu^{(s)}|$, then $\mu^{(s)}$ and $\lambda^{(s)}$ are partitions of m with $\mu^{(s)} >_\theta \lambda^{(s)}$. If there exist partition η of m with $\mu^{(s)} >_\theta \eta >_\theta \lambda^{(s)}$, then $\boldsymbol{\eta} = (\lambda^{(1)}, \dots, \lambda^{(s-1)}, \eta, \lambda^{(s+1)}, \dots, \lambda^{(\ell)})$, satisfy $\boldsymbol{\mu} >_\theta \boldsymbol{\eta} >_\theta \boldsymbol{\lambda}$, this contradicts that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are neighbors with $\geq_\theta$. So $\mu^{(s)}$ and $\lambda^{(s)}$ are neighbors with $\geq_\theta$. Applying the necessity of Lemma 3.3 to $\lambda^{(s)}$ and $\mu^{(s)}$, we derive that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ satisfy either (b) or (c).

Next, let us prove the sufficiency. Suppose $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are ℓ-partitions of n with $\boldsymbol{\mu} >_\theta \boldsymbol{\lambda}$ and one of $(a), (b), (c)$ holds. Suppose $\boldsymbol{\nu}$ be a ℓ-partition of n with $\boldsymbol{\mu} \geq_\theta \boldsymbol{\nu} >_\theta \boldsymbol{\lambda}$ and $\boldsymbol{\lambda}, \boldsymbol{\nu}$ are neighbors with $\geq_\theta$. Now let us prove $\boldsymbol{\mu} = \boldsymbol{\nu}$.

If (a) holds, let $p = \theta_s + \ell(k_s + 1 - 1) = \theta_s + \ell k_s$ and $q = \theta_{s+1} + \ell(1 - \lambda_1^{(s+1)})$. We have $d_p^{\boldsymbol{\mu}} = d_q^{\boldsymbol{\lambda}} = 1$. Moreover, by Remark 3.2, Lemma 3.1, and the choice of $\boldsymbol{\nu}$, we derive

$$\begin{aligned} & d_t^{\boldsymbol{\mu}} = d_t^{\boldsymbol{\lambda}} = d_t^{\boldsymbol{\nu}} \quad t < p \text{ or } t > q \\ & d_{t'}^{\boldsymbol{\mu}} = d_{t''}^{\boldsymbol{\lambda}} = 0 \quad p < t' \leq q, \ p \leq t'' < q. \end{aligned}$$

Therefore,

$$\sum_{p \leq t \leq q} d_t^{\boldsymbol{\nu}} = \sum_{p \leq t \leq q} d_t^{\boldsymbol{\mu}} = \sum_{p \leq t \leq q} d_t^{\boldsymbol{\lambda}} = 1. \tag{3.12}$$

We claim $d_t^{\boldsymbol{\nu}} = 0$ for all $q < t < p$; otherwise, there must be $d_p^{\nu^{(s)}} \neq 0$ or $d_q^{\nu^{(s+1)}} \neq 0$, this contradicts (3.12). If $d_q^{\boldsymbol{\nu}} = 1$, then $d_q^{\nu^{(s+1)}} = 1$ and $\boldsymbol{\nu} = \lambda$; this contradicts $\boldsymbol{\lambda} \neq \boldsymbol{\nu}$. Therefore $d_p^{\boldsymbol{\nu}} = 1$, hence $d_p^{\nu^{(s)}} = 1$, and $\boldsymbol{\mu} = \boldsymbol{\nu}$.

If (b) or (c) holds. Combining with the choice of $\boldsymbol{\nu}$, we have

$$\begin{aligned} & \mu^{(r)} = \nu^{(r)} = \lambda^{(r)} \quad \text{where } r \neq s, \\ & \mu^{(s)} \geq_\theta \nu^{(s)} >_\theta \lambda^{(s)}. \end{aligned}$$

Apply the sufficiency of Lemma 3.3 to $\mu^{(s)}$ and $\lambda^{(s)}$; we derive $\mu^{(s)}$ and $\lambda^{(s)}$ are neighbors with $\geq_\theta$; hence, $\mu^{(s)} = \nu^{(s)}$ and $\boldsymbol{\mu} = \boldsymbol{\nu}$. □

4. Main results

Now we can give the relationship between dominance order and weak θ-dominance order on multipartitions.

Theorem 4.1. *Suppose $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$ and $\theta = (\theta_1, \ldots, \theta_\ell)$ are strongly separated. Then $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$ if and only if $\boldsymbol{\mu} \geq_\theta \boldsymbol{\lambda}$.*

Proof. The conclusion is clear by Proposition 3.7 and [11, Lemma 6.3]. □

For $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$, we define $\operatorname{res}(\boldsymbol{\lambda}) = \{\operatorname{res}(\alpha) | \alpha \in \boldsymbol{\lambda}\}$ to be a multi-set. According to Definitions 2.2 and 2.3, by a trivial discussion, one can prove $\boldsymbol{\mu} \geq_\theta \boldsymbol{\lambda}$ and $\operatorname{res}(\boldsymbol{\mu}) = \operatorname{res}(\boldsymbol{\lambda})$ whenever $\boldsymbol{\mu} \trianglerighteq_\theta \boldsymbol{\lambda}$. Finally, as a corollary of Theorem 4.1, we obtain the relationship between dominance order and θ-dominance order.

Theorem 4.2. *Suppose $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$ and $\theta = (\theta_1, \ldots, \theta_\ell)$ be strongly separated. If $\boldsymbol{\lambda} \trianglelefteq_\theta \boldsymbol{\mu}$, then $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$ and* $\operatorname{res}(\boldsymbol{\lambda}) = \operatorname{res}(\boldsymbol{\mu})$.

We point out that the inverse of Theorem 4.2 is not true. We can give a counterexample as follows:

Example 4.3. *Let $\ell = 2$, $n = 6$, $(\sigma_1, \sigma_2) = (0, 1)$, $\theta = (0, 25)$, θ is strongly separated. Let $\boldsymbol{\lambda} = ((2, 1), (2, 1))$, $\boldsymbol{\mu} = ((3), (3))$, the Young diagrams with residue are as follows:*

$$\boldsymbol{\mu} = \left(\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline \end{array}\,, \begin{array}{|c|c|c|}\hline 1 & 0 & 1 \\ \hline \end{array} \right) \qquad \boldsymbol{\lambda} = \left(\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 \\ \cline{1-1} \end{array}\,, \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 \\ \cline{1-1} \end{array} \right).$$

On one hand, $\boldsymbol{\mu} \triangleright \boldsymbol{\lambda}$ and $\operatorname{res}(\boldsymbol{\lambda}) = \operatorname{res}(\boldsymbol{\mu})$. *On another hand, let $\gamma = (3, 2, 1)$; we have* $\operatorname{res}(\gamma) = 1$ *and*

$$|\{\alpha \in \boldsymbol{\lambda} | \gamma \triangleleft_\theta \alpha\}| = |\{(1, 2, 1), (2, 1, 1)\}| > |\{\beta \in \boldsymbol{\mu} | \gamma \triangleleft_\theta \beta\}| = |\{(1, 2, 1)\}|$$

hence $\boldsymbol{\mu} \not\trianglerighteq_\theta \boldsymbol{\lambda}$.

5. Conclusions

In this paper, we prove that the weak θ-dominance order coincides with the dominance order on multipartitions, whenever the loading θ is strongly separated. As a corollary, we prove that the θ-dominance order is stronger than the usual dominance order on multipartitions, whenever the loading θ is strongly separated.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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