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Research article

Modeling default risk charge (DRC) with intensity probability theory

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Abstract: The latest regulation [1] of the fundamental review of the trading book (FRTB) proposes replacing incremental risk charge (IRC) with default risk charge (DRC). Accordingly, many studies were implemented to analyze this change and its impact. Current modeling practices test several assumptions during conception and implementation. However, these assumptions impact model output and sometimes do not reflect market behavior. Two common assumptions used in DRC modeling in the literature are: (i) the default is implemented in a structural model (e.g., the Merton model) and (ii) correlations between issuers follow the Gaussian copula. Notably, the Merton model does not pick up defaults for positions with a very small probability of default or instant default. Therefore, the structural approach results in a model risk that is not conservative enough to cover the DRC risk. In this paper, we compared an intensity model (CreditRisk+) to a structural model (Merton) to assess their impact on DRC and quantify the risk generated by the first assumption.

Keywords: credit risk; market risk; fundamental review of the trading book (FRTB); default risk charge; model risk; mathematical finance modeling; statistics; probability
Mathematics Subject Classification: 37M05, 46N30, 47N30, 62D05, 62E10

1. Introduction

The fundamental review of the trading book (FRTB) defines default risk charge (DRC) as a new measure of default to replace incremental risk charge (IRC) that considers only the default state and equity scope. DRC measures risk as a one-year 99.9% value at risk computed weekly. Its capital requirement is represented as follows:

$$KDRC = \max\left(\frac{1}{N}\sum_{i=1}^{N} DRC_i\,, DRC_{N+1}\right); N = 12.$$

The FRTB prescribes the standardized approach (SA) and internal model approach (IMA) for DRC computation. The SA calculates DRC by applying risk weights to jump to default (JTD) for each obligor by rating. However, the literature defines two approaches to implement the default model under IMA:

- **Structural approach**: The Merton model [2] is the most used model in the banking industry.
- **Intensity approach**: Financial institutions also use the CreditRisk+ model [3].

Four components need to be calibrated and modeled to evaluate DRC under IMA:

- Obligor correlation: Initially, the FRTB allows the use of historical data related to credit spread or listed equity price. This historical data must span at least 10 years and a stressed period, as defined in the ES model, with a liquidity horizon of one year. However, equities have a minimum liquidity horizon of 60 days. Portfolios should have a high correlation that includes short and long positions. On the other hand, a low correlation is assigned to portfolios that contain only long exposures. Next, obligor default must be modeled using two types of systematic factors to deduce model correlation. Finally, correlation measurement must be done on a one-year liquidity horizon.
- Probability of default (PD): The FRTB defines some conditions and priorities for PD estimation. The first two conditions are: (1) market PDs are not allowed, and (2) all default probabilities are floored to 0.03%. When the model is validated, internal ratings-based (IRB) PDs are the obvious choice. Otherwise, a conformity method in line with the IRB approach has to be developed. Market PD data should not be included in the calibration process; instead, a historical default from a 5-year observation must be used as a minimum for the calibration period. Banks could also use external ratings provided by rating agencies (e.g., S&P, Fitch, or Moody's) to estimate PDs. In this case, banks must define a priority ranking for the choice of PDs used for modeling.
- Loss given default (LGD) model: The LGD model must capture the correlation between recovery and systematic factors. If the institution already has a homologated model, the model has to be calibrated with IRB data, and historical data should be relevant for accurate estimates. All LGDs must be floored to zero, and external LGDs can also be used, respective to some defined ranking choice.
- Jump to default (JTD) model: The JTD model must catch each obligor's long and short positions. Set assets must contain credit (i.e., sovereign and corporate credit) and equity exposures. This measure can be defined as a function of LGD and exposure at the default (EAD) for credit assets. However, it must measure equity P&L when the default occurs since the LGD equals 100% for equity assets. The model must also include the valuation of equity derivatives with zero value of the spot price. The JTD of non-linear products must integrate the multi-default obligors in the case of the derivative products with multiple underlyings. A linear approach, like the sensitivities approach, could be used for these products based solely on obligor default subject to supervisor approval.

Various papers in the literature suggest DRC modeling. For instance, Laurent, Sestier, and

Thomas (2016) [4] use the Hoeffding decomposition to explain the loss function. Other examples include Wilkens and Predescu (2017) [5,6], who proposed a complete framework to build the DRC model, and Angelo (2023) [7], who proposed DRC modeling on emerging markets. All of these works use the Merton model in a multi-factor setting with a structural approach. This model assumption could induce some obligor default on the loss distribution tail, especially for those with a good rating. In fact, obligors having a very good rating, with PDs floored to 0.03% per the regulation, have a higher chance of appearing in the extreme end of the Merton model's loss distribution tail under the Monte Carlo simulation. The DRC may not catch this since the quantile fixed for the VaR is 99.9%. Furthermore, the literature could be expended to other models using machine and deep learning for default modeling that was used in the banking book context. Indeed, Mestiri (2024) [8] suggested many approaches for default modeling like support vector machines (SVM) and deep neural networks (DNN). On the other hand, Pourkermani (2024) [9] proposed to use BRV (binary response VaR method) to estimate the VaR. Hence, this approach could also be used to compute DRC as an alternative approach. However, we will focus only on the intensity models in this paper and others could be subject to upcoming research works.

Some issuers could default instantly, even with a good rating. This is especially possible in emerging markets like the Gulf Cooperation Council (GCC), which includes the UAE. They will not appear in the Merton model's loss distribution tail. On the other hand, intensity models can resolve this issue and catch these extreme events. In this paper, we will use CreditRisk+ as an intensity model and compare it to the Merton model. The following section describes the two models and their differences.

2. DRC framework modeling

2.1. Merton and CreditRisk+

The literature defines two approaches for modeling obligor default: the structural approach and the intensity approach. The most widely used structural model is the Merton model, which defines default at maturity when the asset value is less than the value of the liability. This condition allows writing the default variable for an obligor as $D = \mathbb{1}_{\{X<\Phi^{-1}(PD)\}}$, where PD represents the obligor's non-conditional probability of default. The obligor's PD depends on its rating and reflects its financial situation. X defines the asset return value following a Gaussian distribution. Alternatively, we have the CreditRisk+ intensity model, which defines the default variable as $D = \mathbb{1}_{\{N\geq 1\}}$, with N being the default frequency following a Poisson distribution. Based on the definition of default, the first difference between the two models is the type of variable used for default simulation. The Merton model does not catch defaults for obligors with a high rating because the PD is very close to zero and defaults occur rarely when the Monte Carlo simulation is used. However, defaults appear frequently in the CreditRisk+ model because the rating is not used as a variable to generate them.

We also have the same behavior for low-maturity obligors. Specifically, we can use the distance to the default formula for this conclusion, which results in the following by (2.1):

$$\begin{cases} \lim_{T\to 0} \frac{P(\tau\le T)}{T} = 0 \; Merton\ model \\ \lim_{T\to 0} \frac{P(\tau\le T)}{T} = \lambda > 0 \; Credit\ Risk + model \end{cases}, \tag{2.1}$$

where τ represents the default time, and λ is the default intensity.

Proof for Merton model. The Merton model defines the obligor asset value V_t using geometric brownian motion (GBM):

$$\frac{dV_t}{V_t} = (r-k)dt + \sigma dW_t,$$

where r is the risk-free rate, k is the payout ratio, σ is the volatility of the asset, and W_t defines the Brownian motion.

The solution of this stochastic differential equation (SDE) at maturity is:

$$\ln\left(\frac{V(T)}{V(0)}\right) = \left(r-k-\frac{\sigma^2}{2}\right)\times T + \sigma W(T).$$

The strong assumption of a default event in the Merton model could happen only at maturity under the following condition:

$$\{V(T) \le L\} = \{\ln(V(T)) \le \ln(L)\}. \tag{2.2}$$

As the logarithm is an increasing function and L defines the obligor's liability. Hence, we obtain the following default condition by replacing the value of $\ln(V(T))$ in (2.1):

$$\{\ln(V(T)) \le \ln(L)\} = \left\{\ln(V(0)) + \left(r-k-\frac{\sigma^2}{2}\right)\times T + \sigma W(T) \le \ln(L)\right\}$$

$$= \left\{W(T) \le \frac{\ln(L) - \ln(V(0)) - \left(r-k-\frac{\sigma^2}{2}\right)\times T}{\sigma}\right\}.$$

Given that $W(T)\sim N(0,T)$, we have the following result:

$$P(\{\ln(V(T)) \le \ln(L)\}) = P(\tau \le T) = \Phi\left(\frac{\ln\left(\frac{L}{V(0)}\right) - \left(r-k-\frac{\sigma^2}{2}\right)\times T}{\sigma\times\sqrt{T}}\right).$$

The distance to default in the Merton model is given by:

$$\lim_{T\to 0}\frac{P(\tau\le T)}{T} = \lim_{T\to 0}\frac{\Phi\left(\frac{\ln\left(\frac{L}{V(0)}\right) - \left(r-k-\frac{\sigma^2}{2}\right)\times T}{\sigma\times\sqrt{T}}\right)}{T}.$$

The limit theorem of De L'Hopital is used to compute this limit:

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}.$$

Let us define $f(x) = \Phi\left(\frac{\ln\left(\frac{L}{V(0)}\right)-\left(r-k-\frac{\sigma^2}{2}\right)\times x}{\sigma\times\sqrt{x}}\right)$ and $g(x) = x$. The derivatives of these two functions are:

$$\begin{cases} f'(x) = \varphi\left(\frac{A}{\sqrt{x}} - B \times \sqrt{x}\right) \times \left(-\frac{A}{2x\sqrt{x}} + \frac{B}{\sqrt{x}}\right) \\ g'(x) = 1 \end{cases},$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $A = \frac{1}{\sigma}\ln\left(\frac{L}{V(0)}\right)$, $B = \frac{\left(r-k-\frac{\sigma^2}{2}\right)}{\sigma}$.

If we set $y = \frac{1}{\sqrt{x}}$, the derivative in (1.2) becomes:

$$f'(x) = f'\left(\frac{1}{y^2}\right) = \varphi\left(A \times y - B \times \frac{1}{y}\right) \times \left(-\frac{A}{2} \times y^3 + B \times y\right) = \frac{\left(-\frac{A}{2}\times y^3 + B\times y\right)}{\sqrt{2\pi}\times e^{\frac{\left(A\times y - B\times\frac{1}{y}\right)^2}{2}}}.$$

Then we reach the following result:

$$f'(x) = f'\left(\frac{1}{y^2}\right) = \frac{e^{\left(A\times B - \frac{B^2}{2y^2}\right)}}{\sqrt{2\pi}} \times \frac{\left(-\frac{A}{2}\times y^3 + B\times y\right)}{e^{\frac{A^2\times y^2}{2}}},$$

when $x \to 0$ then $y \to +\infty$ and $\lim_{y\to+\infty} \frac{e^{\left(A\times B - \frac{B^2}{2y^2}\right)}}{\sqrt{2\pi}} = \frac{e^{(A\times B)}}{\sqrt{2\pi}}$.

Consequently, the limit becomes:

$$\lim_{x\to 0} f'(x) = \frac{e^{(A\times B)}}{\sqrt{2\pi}} \times \left(\lim_{y\to+\infty} \frac{-\frac{A}{2}\times y^3}{e^{\frac{A^2\times y^2}{2}}} + \lim_{y\to+\infty} \frac{B\times y}{e^{\frac{A^2\times y^2}{2}}}\right) = 0.$$

The first limit is equal to zero by applying the De L'Hopital theorem two times. The second one will also be zero by applying the same theorem once. Thus, we can conclude that the result of the distance to default of this model is $\lim_{T\to 0} \frac{P(\tau\leq T)}{T} = 0$.

Proof for CreditRisk+ model. The intensity model's distance to default is defined as:

$$\frac{P(\tau\leq T)}{T} = \frac{1-e^{-\lambda\times T}}{T}.$$

The result $\lim_{T\to 0} \frac{P(\tau\leq T)}{T} = \lim_{T\to 0} \frac{1-e^{-\lambda\times T}}{T} = \lambda,$ is straightforward since the exponential first derivative is $f(T) = e^{-\lambda\times T}, f'(T) = -\lambda \times e^{-\lambda\times T}$. Indeed, $\lim_{T\to 0} \frac{1-e^{-\lambda\times T}}{T} = -f'(0) = \lambda$.

Hence, the Merton model may not catch instant defaults, potentially leading to wrong conclusions for some obligors with good ratings that cannot survive extreme market conditions.

These cases are more observable in emerging markets like the UAE.

The second difference comes from how the systematic factors are defined. The Merton model supposes these factors are independent and follow the Gaussian distribution. The asset return is given by (2.2):

$$X = \beta Z' + \sigma \times \varepsilon, \tag{2.3}$$

where β represents the implied correlation vector between the obligor and the systematic factors $Z \sim N(0,1)$, $\sigma = \sqrt{1-\beta\beta'}$, β' is the transposed vector, and $\varepsilon \sim N(0,1)$ is the specific risk.

However, the CreditRisk+ model considers that these factors are independent but follow the Gamma distribution. Default intensity is written as in (2.3):

$$\lambda^Y = \lambda \times (\omega^0 + \omega Y'), \tag{2.4}$$

where $\lambda > 0$ gives the non-conditional obligor intensity, $\sum_k \omega^k = 1, \omega^0 = 1 - \sum_{k \neq 0} \omega^k$, $Y \sim Gamma\left(\alpha, \theta, \alpha = {}^{1}/_{\theta}\right)$, and Y' is the transposed vector.

Default variables for both models are defined as follows by (2.4):

$$\begin{cases} D = \mathbb{1}_{\{X < \Phi^{-1}(PD)\}} \; Merton\ model \\ D = \mathbb{1}_{\{N \geq 1\}} \; Credit\ Risk + model \end{cases}, \tag{2.5}$$

where N is the number of defaults and follows the Poisson distribution.

Therefore, the conditional default probability of the systematic factors has the following formula for each model by (2.5):

$$\begin{cases} PD(Z) = \Phi\left(\frac{\Phi^{-1}(PD) - \beta Z'}{\sigma}\right) Merton\ model \\ PD(Y) = 1 - \exp(-\lambda^Y) \; Credit\ Risk + model \end{cases}, \tag{2.6}$$

where PD defines the non-conditional default probability linked to the obligor rating and Z' is the transposed vector of Z.

Proof for Merton Model. The conditional default probability of the systematic factors using the default variable is equal to:

$$PD(z) = \mathbb{P}(D = 1 | Z = z) = \mathbb{P}(\beta Z' + \sigma \times \varepsilon \leq \Phi^{-1}(PD) | Z = z)$$

$$= \mathbb{P}\left(\beta z + \sigma \times \varepsilon \leq \Phi^{-1}(PD)\right)$$

$$= \mathbb{P}\left(\varepsilon \leq \frac{\Phi^{-1}(PD) - \beta z}{\sigma}\right)$$

$$= \Phi\left(\frac{\Phi^{-1}(PD) - \beta z}{\sigma}\right),$$

since $\varepsilon \sim N(0,1)$.

Proof for CreditRisk+ model. The conditional default probability of the systematic factors using the default variable is equal to:

$$PD(y) = \mathbb{P}(D = 1|Y = y) = \mathbb{P}(N \geq 1|Y = y).$$

N follows the Poisson distribution with λ^Y as intensity:

$$\mathbb{P}(N = k|Y = y) = \frac{(\lambda^y)^k}{k!}\exp(-\lambda^y), k = 0,1,2 \ldots$$

Then, we obtain the following result:

$$PD(y) = \mathbb{P}(N \geq 1|Y = y)$$
$$= 1 - \mathbb{P}(N = 0|Y = y)$$
$$= 1 - \exp(-\lambda^y).$$

Consequently, the non-conditional default probability $PD \to 0$ for obligors with high rating quality. However, the non-conditional intensity of default is still small but not zero. Indeed, $\exists \epsilon > 0$, when $PD \to 0$ $then\ \lambda \to \epsilon\ and\ \lambda^Y \to \lambda_\epsilon^Y \neq 0$. Hence, we have the following result in (2.6):

$$\lim_{PD\to 0} PD(z) = 0; \lim_{PD\to 0} PD(y) = \lambda_\epsilon^Y. \quad (2.7)$$

Proof.

$$\lim_{PD\to 0} PD(z) = \lim_{PD\to 0} \mathbb{P}(D = 1|Z = z) = \lim_{PD\to 0} \Phi\left(\frac{\Phi^{-1}(PD)-\beta z}{\sigma}\right).$$

We know that $\lim_{PD\to 0} \Phi^{-1}(PD) = -\infty \Rightarrow \lim_{PD\to 0} \frac{\Phi^{-1}(PD)-\beta z}{\sigma} = -\infty$ and Φ is an increasing function. Thus, we deduce the result of $\lim_{PD\to 0} \Phi\left(\frac{\Phi^{-1}(PD)-\beta z}{\sigma}\right) = 0$,

$$\lim_{PD\to 0} PD(y) = \lim_{\lambda\to\epsilon} PD(y)$$
$$= \lim_{\lambda^Y \to \lambda_\epsilon^Y} (1 - \exp(-\lambda^Y))$$
$$= 1 \approx 1 - (1 - \lambda_\epsilon^Y)$$
$$= \lambda_\epsilon^Y.$$

Indeed, for a small value of x, we have the following approximation: $e^x \approx 1 + x$.

This result proves that obligors with a high rating quality would not default in the Merton model. However, they could default under the CreditRisk+ model.

The following section will focus more on the model definition of DRC in both the Merton and CreditRisk+ approaches. We will also try to prove that defaults for obligors with high rating quality could happen more frequently in the CreditRisk+ model than in the Merton model.

2.2. Model definition

The FRTB requires two types of systematic factors to simulate obligor default. We suggest using the same configuration used in [10]. Therefore, we deem two types of factors: (1) global factors and (2) sectorial factors. The first set of factors is built by one global factor and two global asset types: (1) sovereign and (2) corporate. The second asset type contains regional and industrial factors. We denote these sets, respectively, by $\mathrm{GA} = \{\mathrm{GS}, \mathrm{GC}\}$, $\mathrm{R} = \{1 \ldots \mathrm{r}\}$, and $\mathrm{I} = \{1 \ldots \mathrm{s}\}$. In [10], we have proposed a multi-factor Merton (1974) model as a framework, and we will keep the same for our study. Additionally, we build an intensity model based on CreditRisk+ to establish a comparison with the structural Merton model.

The Merton model defines the return variable for an obligor i as in (2.8):

$$X_i = \beta_i \times Z_G + \beta_i^g \times Z_g + \beta_i^j \times Z_j^R + \beta_i^l \times Z_l^I + \sigma_i \varepsilon_i, \tag{2.8}$$

where Z_G, Z_g, Z_j^R, Z_l^I are independent by set and follow $N(0,1)$, with $g \in GA, j \in R, l \in I$. β gives the correlation between obligors and systematic factors, whereas $\varepsilon_i \sim N(0,1)$ represents the specific risk, and they are independent and identically distributed for $i \in \{1 \ldots N\}$, $X_i \sim N(0,1)$, and are independent from all systematic factors. Moreover, the following formula is used to keep $X_i \sim N(0,1)$:

$$\sigma_i = \sqrt{1 - \left({\beta_i}^2 + {\beta_i^g}^2 + {\beta_i^j}^2 + {\beta_i^l}^2\right)}.$$

Therefore, the implied correlation between obligors can be deduced by:

$$\rho^I = \beta \times \rho^F \times \beta' + \sigma^2 \times I,$$

where ρ^I represents the obligor implied correlation matrix, $N \times N$; ρ^F is the systematic factor intra-correlation matrix, $K \times K$ and $K = (3 + r + s)$; β represents the correlation factors between the obligor matrix, $N \times (3 + r + s)$, and the systematic factors; β' represents the transposed matrix; σ^2 is the vector of σ_i^2; and I is the identity matrix.

The CreditRisk+ model uses the intensity default for default simulation as the default frequency follows the Poisson distribution. Therefore, we denote N_i as the number of defaults for an obligor i. We keep the same structure of systematic factors and denote them Y. We consider that the variables of the default numbers between obligors are idiosyncratically independent. The systematic part of these variables conditional on Y, follows a Poisson distribution with the following intensity in (2.9):

$$\lambda_i^Y = \lambda_i \times \left(\omega_i^0 + \omega_i \times Y_G + \omega_i^g \times Y_g + \omega_i^j \times Y_j^R + \omega_i^l \times Y_l^I\right), \tag{2.9}$$

where Y_G, Y_g, Y_j^R, Y_l^I are independent by set and are Gamma distributed given parameters $\left(\alpha^Y, \theta^Y, \alpha^Y = 1/_{\theta^Y}\right)$, with $Y \in \{Y_G, Y_g, Y_j^R, Y_l^I\}$ and $g \in GA, j \in R, l \in I$. ω_i verifies the following condition: $\omega_i^0 + \omega_i + \omega_i^g + \omega_i^j + \omega_i^l = 1$. Finally, λ_i represents the non-conditional intensity for the obligor i.

The first result comes from normalization, and we have $\mathbb{E}[\lambda_i^Y] = \lambda_i$. The second one allows computing the expectation of $N_i, i = 1 \ldots N$, conditionally to Y, $\mathbb{E}[N_i \,|\mathrm{Y}] = \lambda_i^Y$. The covariance between two obligors is given as:

$$COV(N_i, N_j) = \mathbb{E}[N_i \times N_j] - \mathbb{E}[N_i] \times \mathbb{E}[N_j]$$

$$= \mathbb{E}\left[\mathbb{E}[N_i \times N_j|Y]\right] - \mathbb{E}[\mathbb{E}[N_i|Y]] \times \mathbb{E}\left[\mathbb{E}[N_j|Y]\right]$$

$$= \mathbb{E}\left[COV(N_i, N_j|Y) + \mathbb{E}[N_i|Y] \times \mathbb{E}[N_j|Y]\right] - \mathbb{E}[\mathbb{E}[N_i|Y]] \times \mathbb{E}\left[\mathbb{E}[N_j|Y]\right]$$

$$= \mathbb{E}[COV(N_i, N_j|Y)] + \left(\mathbb{E}\left[\mathbb{E}[N_i|Y] \times \mathbb{E}[N_j|Y]\right] - \mathbb{E}[\mathbb{E}[N_i|Y]] \times \mathbb{E}\left[\mathbb{E}[N_j|Y]\right]\right)$$

$$= \mathbb{E}[COV(N_i, N_j|Y)] + COV(\mathbb{E}[N_i\ |\mathrm{Y}], \mathbb{E}[N_j\ |\mathrm{Y}])$$

$$= \mathbb{E}[\mathbb{1}_{\{i=j\}} \times \lambda_i^Y] + COV(\lambda_i^Y, \lambda_j^Y)$$

$$= \mathbb{1}_{\{i=j\}} \times \lambda_i + COV(\lambda_i^Y, \lambda_j^Y).$$

We can then write the implied covariance matrix of obligors as:

$$C^I = \omega \times C^F \times \omega' + \lambda \times I,$$

where C^I represents the implied covariance matrix of obligors; C^F is the systematic factor intra-covariance matrix; ω represents the matrix of ω_i; ω' represents the transposed matrix; λ is the vector of $\lambda_i, i = 1 \dots N$; and I is the identity matrix, $N \times N$.

Using these results, we can deduce the implied correlation between two obligors:

$$\rho_{i,j}^I = \begin{cases} \frac{C_{i,j}^I}{\sqrt{\lambda_i \times \lambda_j}} & i \neq j \\ 1 & i = j \end{cases}.$$

Therefore, the conditional default probability of the systematic factors in both models can be written as follows as defined in the previous section by (2.10):

$$\begin{cases} PD_i(Z) = \Phi\left(\frac{\Phi^{-1}(PD_i) - \beta_i Z'}{\sigma_i}\right) \\ PD_i(Y) = 1 - \exp(-\lambda_i \times (\omega_i^0 + \omega_i Y')) \end{cases}. \quad (2.10)$$

Given that β_i and ω_i are, respectively, the obligor lines of the β and ω matrices, Z' and Y' are the transposed systematic vectors defined as follows: $Z = (Z_G, Z_{GS}, Z_{GC}, Z_1^R, \dots, Z_r^R, Z_1^I, \dots, Z_s^I)$; $Y = (Y_G, Y_{GS}, Y_{GC}, Y_1^R, \dots, Y_r^R, Y_1^I, \dots, Y_s^I)$.

We keep the same model as in our first study [6] for the LGD and the JTD, and consider the following relationship between the LGD and the probability of default conditional on systematic factors in (2.11):

$$\begin{cases} LGD(Z) = 1 - b \times e^{-a \times PD(Z)} \\ LGD(Y) = 1 - b \times e^{-a \times PD(Y)} \end{cases}; a, b \geq 0, \tag{2.11}$$

where

$$\begin{cases} a = -l\,n\left(\frac{1-LGD_{max}}{1-LGD_{min}}\right) \\ b = 1 - LGD_{min} \end{cases}.$$

We use IRB data to calibrate LGD_{min} and LGD_{max} as outlined by the FRTB regulation. The calibration is done for both sovereign and corporate obligors, so we have to define a_{sov}, b_{sov} for sovereign and a_{corp}, b_{corp} for corporate obligors. Calibration can also be done by seniority. However, we keep the sovereign and corporate subdivisions in our case, taking the following values for calibration:

$$\begin{cases} LGD_{min}(SOV) = 0.0, LGD_{max}(SOV) = 0.8 \\ LGD_{min}(CORP) = 0.6, LGD_{max}(CORP) = 0.99 \end{cases}.$$

Given these values, we find our parameters below:

$$\begin{cases} (a_{sov}, b_{sov}) = (1.61, 1.0) \\ (a_{corp}, b_{corp}) = (19.8, 0.4) \end{cases}.$$

The JTD for the obligor $i = 1 \ldots N$ is given by (2.12):

$$\begin{cases} JTD_i(Z) = LGD_i(Z) \times EAD_i^{Credit} + EAD_i^{Equity} \\ JTD_i(Y) = LGD_i(Y) \times EAD_i^{Credit} + EAD_i^{Equity} \end{cases}, \tag{2.12}$$

with $\mathrm{EAD}_\mathrm{i}^{\mathrm{Credit}}$ and $\mathrm{EAD}_\mathrm{i}^{\mathrm{Equity}}$ representing the credit and the equity exposure at default for the given obligor i, respectively.

The following equations give the loss function for each model in (2.13):

$$\begin{cases} L = \sum_{i=1}^{N} JTD_i(Z) \times \mathbb{1}_{\{X_i < \Phi^{-1}(PD_i)\}} & Merton\ model \\ L = \sum_{i=1}^{N} JTD_i(Y) \times \mathbb{1}_{\{N_i \geq 1\}} & CreditRisk+model \end{cases}. \tag{2.13}$$

The loss induced by systematic factors for each model is defined as follows in (2.14):

$$\begin{cases} L_Z = \mathbb{E}[L|Z] = \sum_{i=1}^{N} JTD_i(Z) \times PD_i(Z) \\ L_Y = \mathbb{E}[L|Y] = \sum_{i=1}^{N} JTD_i(Y) \times PD_i(Y) \end{cases}. \tag{2.14}$$

We obtain the following result in case of constant LGD in (2.15) when all obligors have high rating quality $\forall i, \exists \epsilon_i, PD_i \to 0\ then\ \lambda_i \to \epsilon_i\ and\ \lambda_i^y \to \lambda_{\epsilon_i}^Y \neq 0$:

$$\begin{cases} \lim_{PD \to 0} L_Z = 0 \\ \lim_{PD \to 0} L_Y = \sum_{i=1}^{N} JTD_i \times \lambda_{\epsilon_i}^Y \neq 0 \end{cases}. \tag{2.15}$$

Proof. We have under the assumption of constant LGD, $\forall i, JTD_i(Z) = JTD_i(Y) = JTD_i$. We also have proof that $\lim_{PD_i \to 0} PD_i(Z) = 0$ and $\lim_{PD_i \to 0} PD_i(Y) = JTD_i \times \lambda_{\epsilon_i}^Y$ in the previous section's Eq (2.6). Based on this, we obtain:

$$\lim_{PD \to 0} L_Z = \sum_{i=1}^{N} JTD_i \times \lim_{PD_i \to 0} PD_i(Z) = 0$$

and

$$\lim_{PD \to 0} L_Y = \sum_{i=1}^{N} JTD_i \times \lim_{PD_i \to 0} PD_i(Y) = \sum_{i=1}^{N} JTD_i \times \lambda_{\epsilon_i}^Y.$$

Consequently, we proved that losses would occur more in the CreditRisk+ model than the Merton model under the assumption of constant LGD and a high-quality rating. However, proving this in the general case where LGD is stochastic and obligor ratings are not all high quality is not straightforward. The next section presents the results of model calibration. We use a numerical approach based on the Monte Carlo simulation to compare the DRC values for the two models in the general case. The results of this approach will allow us to draw conclusions regarding model choice.

3. Numerical approach

We deem a set of 1,342 issuers within a 10-year historical spread with monthly observation for the Merton model and intensity for CreditRisk+ linked to 6 regions and 11 industries (input data linked to the paper gives names for these regions and industries). Our population includes 115 obligors with very small PDs that equal 0.03%, which also means that their defaults appear in the Merton model rarely. However, these obligors have a default intensity that could bring them to default frequently under the Poisson distribution. Additionally, the total exposure summing the long and short positions for these obligors is equal to half a million euros so we can assess the difference in magnitude between the two models. Figure 1 gives the exposure density of the portfolio used in this paper:

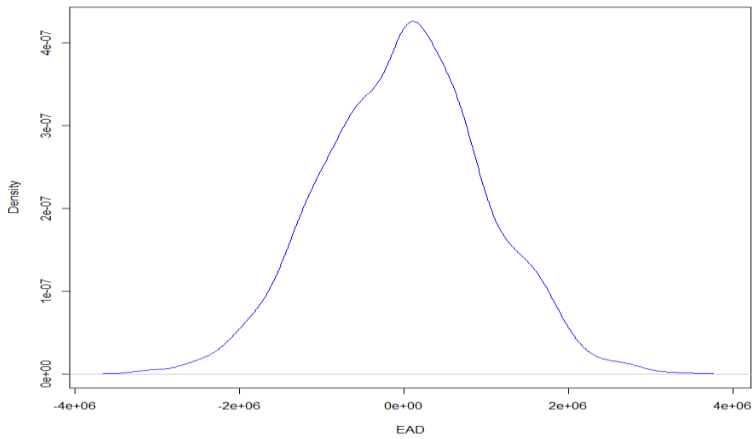

Figure 1. EAD density.

Thereafter, we compute the implied correlation for the two models, comparing it with the historical correlation using the following plots:

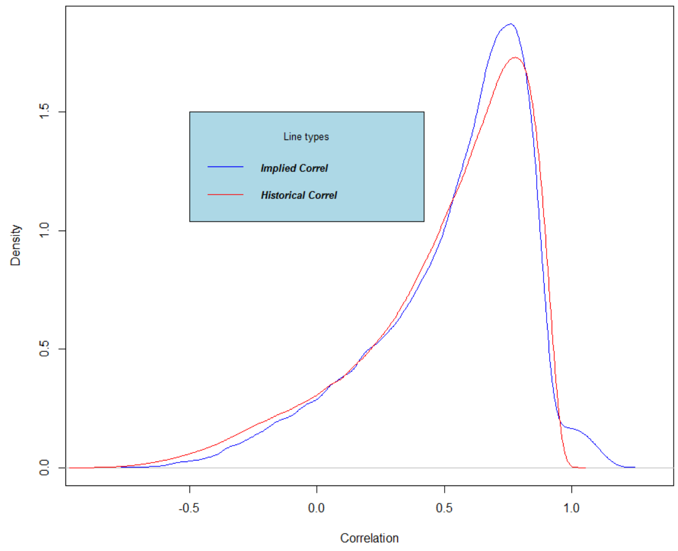

Figure 2. Correlation densities for the Merton model.

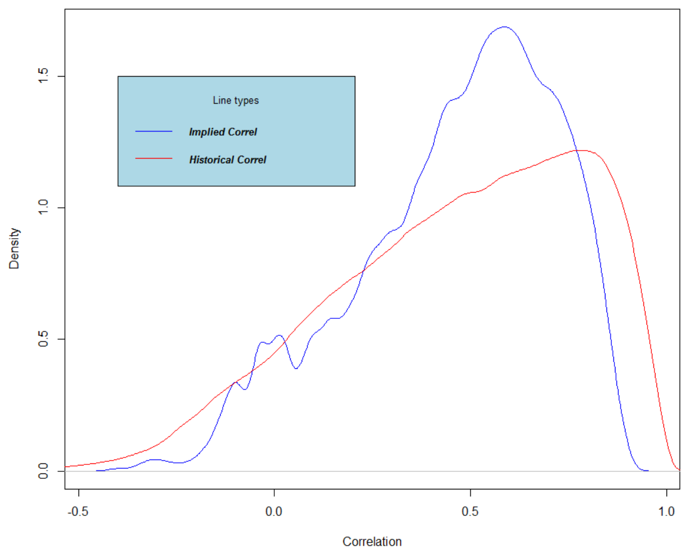

Figure 3. Correlation densities for the Credit Rsik+ model.

Figure 2 represents the Merton model's correlation density, and Figure 3 gives the CreditRisk+ model's correlation density. We observe that the Merton model's correlation fits better than the CreditRisk+ model's correlation, resulting in another modeling risk at this stage. However, we will not explore this risk in this paper.

As we see, the systematic factors Y follow the Gamma distribution with parameters $\left(\alpha, \frac{1}{\alpha}\right)$.

Hence, we should calibrate the factor α for each one using maximum likelihood estimation (MLE). We consider n observations of $Y = (y_1 \dots y_n)$, and the likelihood function is defined by (3.1):

$$l(x,\alpha) = \prod_{i=1}^{n} \frac{\alpha^{\alpha}}{\Gamma(\alpha)} e^{-\alpha y_i} y_i^{\alpha-1} = \left(\frac{\alpha^{\alpha}}{\Gamma(\alpha)}\right)^{n} \times e^{-\alpha \sum_{i=1}^{n} y_i} \times \prod_{i=1}^{n} y_i^{\alpha-1}. \quad (3.1)$$

Let's calculate the first derivative of the logarithmic function (3.1) with respect to alpha in (3.2):

$$L(x,\alpha) = n \times \left(\alpha \times ln(\alpha) - ln\big(\Gamma(\alpha)\big)\right) + \alpha \times \left(\sum_{i=1}^{n}(ln(y_i) - y_i)\right) - \sum_{i=1}^{n} ln(y_i). \quad (3.2)$$

What remains at this point is to develop the first-order derivative on (3.2) to find the maximum. The calculation yields the following results in (3.3):

$$\frac{\partial L(x,\alpha)}{\partial \alpha} = 0 \Rightarrow ln(\hat{\alpha}) + 1 - \frac{\Gamma'(\hat{\alpha})}{\Gamma(\hat{\alpha})} = \frac{1}{n} \times \left(\sum_{i=1}^{n}(y_i - ln(y_i))\right). \quad (3.3)$$

To solve this equation, we use the Stirling approximation:

$$ln\big(\Gamma(\alpha)\big) \approx \left(\alpha - \frac{1}{2}\right) \times ln(\alpha) - \alpha - l\,n\big(\sqrt{2\pi}\big) \Rightarrow \frac{\partial\, ln\big(\Gamma(\alpha)\big)}{\partial \alpha} = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} \approx ln(\alpha) - \frac{1}{2\times\alpha}.$$

We obtain the estimated parameter by replacing in Eq (3.3):

$$\hat{\alpha} = \frac{n}{2} \times \left(\sum_{i=1}^{n}\big((y_i - ln(y_i)) - 1\big)\right)^{-1}.$$

Once the calibration is completed, we launch the computations using the Monte Carlo approach with one million simulations to draw the loss densities for both models. The results are plotted in the following figures:

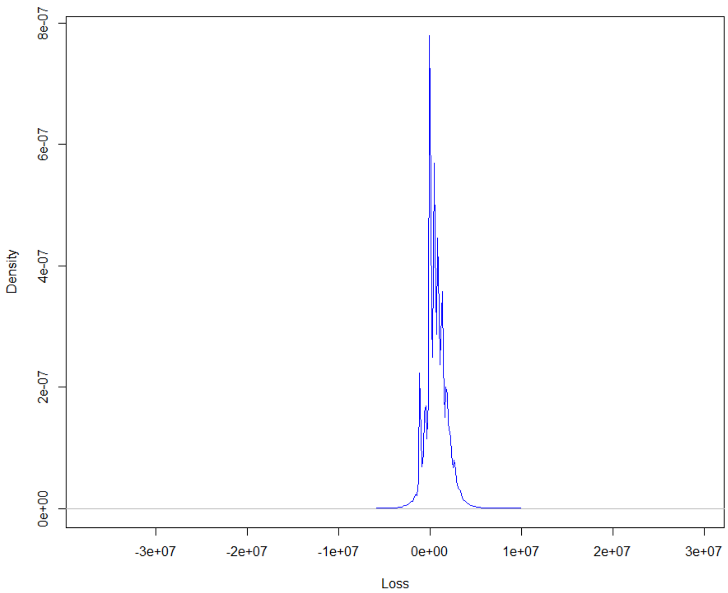

Figure 4. Loss densities for the Merton model.

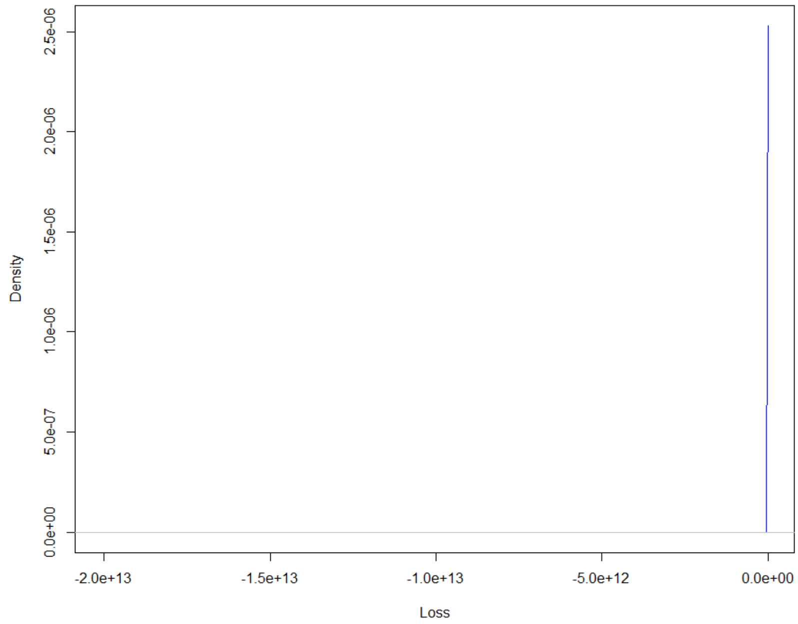

Figure 5. Loss densities for the CreditRisk+ model.

Figures 4 and 5 represent the loss density of the Merton and CreditRisk+ models, respectively. The DRC values equal 7,396,194 for the Merton model and 7,527,208 for CreditRisk+. The value of the relative difference is 1.77% which seems small because of the small number of obligors with small PDs. However, in practice, we could obtain more significant differences between the two models when we can yield important model risk consequences that arise from the model choice assumption. Specially, if we add the impact of environmental, social, and governance (ESG) as a q variable for default modeling as suggested by Mengze and Dejun (2024) [11], and Orlando, Bufalo, Penikas and Zurlo (2022) [12], then it will reduce the default in the Merton model. The difference of 1.77% could serve as a provision to cover model risk raised from the model choice assumption.

4. Conclusions

The literature shows that DRC modeling uses a structural approach via the Merton model. However, this approach is based on two assumptions that may carry risks. The assumptions are that (1) we are in a Merton environment, and (2) the Gaussian copula is used.

This study showed the model risk that could arise from the first assumption. The first section introduced the DRC under IMA, FRTB guidelines. The second section compared the Merton and CreditRisk+ models. We then defined our framework model, followed by an implementation to explain the results. The Merton model is considered a structural approach and, theoretically, does not capture defaults when default probabilities are very small. However, the CreditRisk+ model is an intensity model that captures obligor defaults even if they have very small PDs. Moreover, this model can capture instant defaults for obligors in emerging markets like the UAE. The results of the implementation led to the same conclusion as the theoretical results since we found that the DRC of the CreditRisk+ model was more punitive than the Merton model, which could lead to model risk. Model risk always remains an issue for all internal model approaches, and we have to challenge these models since there are always assumptions that could lead to non-conservative risk measurement as was used in Bhattacharya, Biswas and Mandal (2023) [13]. Therefore, we suggest using other copulas, like the student or Gumbel copula, to study the impact of the second assumption on the obtained results. This could be an open area for future research on DRC modeling.

Author contributions

Badreddine Slime and Jaspreet Singh Sahni: Design and implementation of the research, to the analysis of the results, and to the writing of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

The views expressed in this paper are those of the authors and do not necessarily reflect the views and policies of Emirates NBD.

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