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Research article

Extrapolation methods for solving the hypersingular integral equation of the first kind

Qian Ge and Jin Li*

School of Science, Shandong Jianzhu University, Jinan, 250101, China

* **Correspondence:** Email: lijin@lsec.cc.ac.cn; Tel: +18615186091.

Abstract: Hypersingular integral equations have garnered extensive attention in the context of boundary element methods, particularly within natural boundary element methods. The asymptotic expansion of the composite rectangular rule's error function in Hadamard finite-part integrals yields a hypersingular kernel of $1/\sin^2(x-s)$. An extrapolation algorithm was developed to address this issue. To solve the hypersingular integral equation, we employed superconvergence points as collocation points, thereby constructing an extrapolation algorithm for hypersingular integral equations and establishing its convergence rate. A numerical example was provided to validate the efficacy of the method, corroborated by theoretical results that demonstrate the algorithm's effectiveness.

Keywords: extrapolation method; composite rectangle formula; superconvergence; hypersingular integrals equation
Mathematics Subject Classification: 33F05, 65D05

1. Introduction

We consider the following hypersingular integral equation on a circle:

$$\int_c^{c+2\pi} \frac{f(t)}{\sin^2 \frac{t-s}{2}}\, dt = g(s),\ s \in (c, c+2\pi). \tag{1.1}$$

Equation (1.1), considered the natural integral equation for harmonic problems, aligns with the form derived by Yu [28, 29]. While natural integral equations themselves hold limited direct significance for solving boundary value problems (BVPs), they become instrumental in unbounded domain BVPs [8]. By introducing artificial boundaries such as circles or ellipses and applying corresponding natural integral equations on these boundaries, domain decomposition methods and coupled algorithms can be effectively constructed [29].

Integrals of this kind possess several definitions and we present our definition as follows:

$$\oint_{c}^{c+2\pi} \frac{f(t)}{\sin^2 \frac{t-s}{2}}\, dt = \lim_{\epsilon\to 0}\left\{\left(\int_{c}^{s-\epsilon} + \int_{s+\epsilon}^{c+2\pi}\right)\frac{f(t)}{\sin^2 \frac{t-s}{2}}\, dt - \frac{8f(s)}{\epsilon}\right\} \tag{1.2}$$

and

$$\oint_{c}^{c+2\pi} \frac{f(t)}{\sin^2 \frac{t-s}{2}}\, dt = \lim_{\epsilon\to 0}\left\{\left(\int_{c}^{s-\epsilon} + \int_{s+\epsilon}^{c+2\pi}\right)\frac{f(t)}{\sin^2 \frac{t-s}{2}}\, dt - 4f(s)\cot\frac{\epsilon}{2}\right\}. \tag{1.3}$$

The equivalent of (1.2) and (1.3) can be similarly obtained in [32].

The composite middle rectangle rule remains applicable when the singular point coincides with the midpoint of each subinterval, leading to a superconvergence phenomenon. This allows us to derive the error expansion of the error functional. We leverage these asymptotic expansion results to develop an algorithmic scheme for solving hypersingular integral equations.

The hypersigular integral defined on the integral

$$\oint_{a}^{b} \frac{f(t)}{(t-s)^2} dt = \lim_{\varepsilon\to 0}\left\{\int_{a}^{s-\varepsilon} \frac{f(t)}{(t-s)^2} dt + \int_{s+\varepsilon}^{b} \frac{f(t)}{(t-s)^2} dt - \frac{2f(s)}{\varepsilon}\right\}, s \in (a, b), \tag{1.4}$$

has been paid much attention recently.

Hypersingular integrals frequently arise in boundary element methods and various physical problems, including fracture mechanics, elasticity, acoustics, and electromagnetic scattering [31]. Numerous methods have been devised to handle these integrals, such as Gaussian quadrature [9, 10], Newton-Cotes rules [3, 12, 16, 17, 19, 25–28], transformation methods [5, 7, 11], and others [2, 4, 11, 24, 30]. These studies rigorously analyze trapezoidal-type quadrature formulas for weakly singular, singular (Cauchy principal value), and hypersingular integrals, providing full asymptotic expansions for error analysis [33–37]. Reference [38] explored product integration using a variant of the generalized Euler-Maclaurin summation formula, while [39] applied this formula to study the convergence of weakly singular Fredholm and Volterra integral equations. Hypersingular integral equations in potential theory was studied in [40], and modal analysis of a submerged elastic disk was studied by [43]. Superconvergence results for the hypersingular integral equation was presented in [41]. Singular integral operators [45], finite-part integrals of highly oscillatory functions [42], and hypersingular integrals [44] have been paid much attention in recent years.

The extrapolation method [13, 21, 22] is based on the error function as

$$T(h) - a_0 = a_1 h^2 + a_2 h^4 + a_3 h^6 + \cdots,$$

where $T(0) = a_0$ and a_j are constants independent of h. In [14], a generalized trapezoidal rule for numerical computation of hypersingular integrals on intervals was introduced, with asymptotic error expansion proven. In [15], the composite trapezoidal rule for Hadamard finite-part integrals with the hypersingular kernel $1/\sin^2(x - s)$ was discussed, obtaining the main part of the asymptotic error expansion. In [18], two extrapolation algorithms were presented for hypersingular integrals on intervals, proving their convergence rates, which match those of classical middle rectangle rule approximations. In [20], an extrapolation algorithm for supersingular integrals was introduced.

Before presenting our main results, we first let

$$\phi_{ik}(t) = \begin{cases} \dfrac{1}{(k-i)!} \displaystyle\fint_{-1}^{1} \dfrac{\mathcal{M}_{ik}(\tau,t)}{(\tau-t)^2} d\tau, & |t| < 1, \\ \dfrac{1}{(k-i)!} \displaystyle\int_{-1}^{1} \dfrac{\mathcal{M}_{ik}(\tau,t)}{(\tau-t)^2} d\tau, & |t| > 1, \end{cases} \tag{1.5}$$

where $\tau \in [-1, 1]$ and

$$\mathcal{M}_{ik}(\tau,t) = \tau^i (\tau-t)^{k-i} = F_i(\tau)(\tau-t)^{k-i}. \tag{1.6}$$

Set $J := (-\infty) \bigcup (-1, 1) \bigcup (1, \infty)$, $W : C(J) \to C(-1, 1)$ and let

$$T_{ik}(\tau) := \phi_{ik}(\tau) + \sum_{j=1}^{\infty} \phi_{ik}(2j+\tau) + \sum_{j=1}^{\infty} \phi_{ik}(-2j+\tau), \tag{1.7}$$

be the linear operator.

Compared with Gaussian quadrature and Newton-Cotes rules, the extrapolation method is more effective. In the following, the asymptotic error expansion of the middle rectangle rule for Hadamard finite-part integrals on a circle is given, based on the asymptotic error expansion, and an algorithm to solve the hypersingular integrals equation is presented. The asymptotic error expansion is

$$E_n(f, s) = \sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}} f^{(i+1)}(s) a_i(\tau), \tag{1.8}$$

where $a_i(\tau)$ are functions independent of h defined as

$$a_i(\tau) = \begin{cases} \displaystyle\sum_{k=i-1}^{i} \frac{(-1)^k}{(k+1)!} T_{ki}(\tau), & i \geq 1, \\ -\dfrac{1}{2} T_{ki}(\tau), & i = 0. \end{cases} \tag{1.9}$$

This paper provides the asymptotic error expansion of the middle rectangle rule for Hadamard finite-part integrals on circles. Based on this expansion, an algorithm for solving hypersingular integral equations is proposed. To avoid computing $a_i(\tau)$, an extrapolation algorithm is suggested for a given τ. A series of s_j values approximate the singular point s with mesh refinement. Using extrapolation techniques, we achieve higher-order accuracy and obtain a posteriori error estimates. Additionally, a collocation scheme is constructed to solve hypersingular integral equations via extrapolation methods, with proven convergence rates.

2. Main results

Let $c = t_0 < t_1 < \cdots < t_{n-1} < t_n = c + 2\pi$ be a uniform partition of the interval $[c, c + 2\pi]$ with mesh size $h = 2\pi/n$ and set

$$t_i = t_0 + (i-1)h, i = 1, 2, \cdots, n, \tag{2.1}$$

$$\hat{t}_i = t_i + \frac{h}{2}, i = 1, 2, \cdots, n. \tag{2.2}$$

We define $f_C(t)$, the constant interpolation for $f(t)$, as

$$f_C(t) = f(\hat{t}_i), \quad i = 1, \cdots, n, \tag{2.3}$$

and a linear transformation

$$t = \hat{t}_i(\tau) := (\tau + 1)(t_{i+1} - t_i)/2 + t_i, i = 1, \cdots, n-1, \quad \tau \in [-1, 1], \tag{2.4}$$

from the reference element $[-1, 1]$ to the subinterval $[t_i, t_{i+1}]$.

Replacing $f(t)$ in (1.1) with $f_C(t)$ produces the composite rectangle rule:

$$I_n(f, s) := \mathop{=\!\!\!\!\!\!\int}_c^{c+2\pi} \frac{f_C(t)}{\sin^2 \frac{t-s}{2}} dx = \sum_{i=1}^{n} \omega_i(s) f(t_i) = I(f; s) - E_n(f, s), \tag{2.5}$$

where $\omega_i(s)$ denotes the Cotes coefficients given by

$$\omega_i(s) = 2 \cot\left(\frac{t_{i-1} - s}{2}\right) - 2 \cot\left(\frac{t_i - s}{2}\right) \tag{2.6}$$

and $E_n(f, s)$ is the error functional.

Let

$$\gamma(\tau) = \gamma(h, s) = \min_{1 \le i \le n} \frac{|s - t_i|}{h} = \frac{1 - |\tau|}{2} \tag{2.7}$$

and

$$\mathcal{I}_{n,i}(s) = \begin{cases} \displaystyle\int_{t_{i-1}}^{t_i} \frac{t - \hat{t}_i}{\sin^2 \frac{t-s}{2}} dt, & i \neq m, \\ \displaystyle\mathop{=\!\!\!\!\!\!\int}_{t_{m-1}}^{t_m} \frac{t - \hat{t}_m}{\sin^2 \frac{t-s}{2}} dt, & i = m. \end{cases} \tag{2.8}$$

We also define

$$F_i(\tau) = \tau^i \tag{2.9}$$

and

$$\phi_{i,i+1}(t) = \begin{cases} \displaystyle -\frac{1}{2} \fint_{-1}^{1} \frac{F_i(\tau)}{\tau - t} d\tau, & |t| < 1, \\ \displaystyle -\frac{1}{2} \int_{-1}^{1} \frac{F_i(\tau)}{\tau - t} d\tau, & |t| > 1. \end{cases} \tag{2.10}$$

If $F_i(\tau)$ is replaced by the Legendre polynomial, it is known that $\phi_{i,i+1}$ defines the Legendre function of the second kind [1]. Let

$$\phi_{ii}(t) = \begin{cases} \displaystyle -\frac{1}{2} \mathop{=\!\!\!\!\!\!\int}_{-1}^{1} \frac{F_i(\tau)}{(\tau - t)^2} d\tau, & |t| < 1, \\ \displaystyle -\frac{1}{2} \int_{-1}^{1} \frac{F_i(\tau)}{(\tau - t)^2} d\tau, & |t| > 1 \end{cases} \tag{2.11}$$

and

$$\phi_{ik}(t) = -\frac{1}{2}\int_{-1}^{1} \frac{F_i(\tau)(\tau - t)^{k-i-2}}{(k-i)!} d\tau, \quad k > i + 1. \tag{2.12}$$

Define $K_s(t)$ as

$$K_s(t) = \begin{cases} \dfrac{(t-s)^2}{\sin^2 \frac{t-s}{2}} & t \neq s, \\ 4, & t = s. \end{cases} \tag{2.13}$$

Theorem 2.1. *Assume $f(t) \in C^\infty[c, c + 2\pi]$. For the middle rectangle rule $I_n(f; s)$ defined in (2.5) and $a_i(\tau)$ defined in (1.9), we have*

$$E_n(f, s) = \sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}} f^{(i+1)}(s) a_i(\tau) \tag{2.14}$$

where $s = t_{m-1} + (1 + \tau)h/2, \ \ m = 1, 2, \cdots, n.$

Then the error function can be written as

$$E_n(f, s) = \frac{f'(s)}{2} a_0(\tau) + \frac{hf''(s)}{2^2} a_1(\tau) + \cdots + \frac{h^i}{2^{i+1}} f^{(i+1)}(s) a_i(\tau) + \cdots \tag{2.15}$$

From the error functional, we know that it is not convergence when the first term is $a_0(\tau)$. By the first part of $a_0(\tau) = 0$, we have the convergence rate $O(h)$ which is called the superconvergence phenomenon.

3. Proof of Theorem 2.1

Define

$$\mathcal{M}_{ik}^{j}(t, s) = (t - t_{j-1/2})^i (t - s)^{k-i} = F_i^j(t)(t - s)^{k-i}, \quad k \geq i, \tag{3.1}$$

where

$$F_i^j(t) = (t - t_{j-1/2})^i. \tag{3.2}$$

By (2.4), we have

$$\begin{aligned} \mathcal{M}_{ik}^{j}(t, s) &= \frac{h^{k+2}}{2^{k+2}} \tau^i (\tau - c_j)^{k-i} \\ &= \frac{h^{k+2}}{2^{k+2}} \mathcal{M}_{ik}(\tau, c_j) \\ &= \frac{h^{k+2}}{2^{k+2}} F_i(\tau)(\tau - c_j)^{k-i}, \end{aligned} \tag{3.3}$$

where

$$\mathcal{M}_{ik}(\tau, c_j) = \tau^i (\tau - c_j)^{k-i} = F_i(\tau)(\tau - c_j)^{k-i}, \tag{3.4}$$

$$c_j = 2(s - t_{j-1})/h - 1 \tag{3.5}$$

and $F_i(\tau)$ is defined as in(2.9).

Lemma 3.1. *Let $K_s(t)$ be defined as in (2.13). For $t \in (t_{j-1}, t_j)$, by linear transformation (2.6), we have*

$$K_s(t) = K_{c_j}(\tau), \quad \tau \in (-1, 1) \tag{3.6}$$

where

$$K_{c_j}(\tau) = 4 + 4\sum_{l=1}^{\infty} \frac{(\tau - c_j)^2}{(\tau - c_j - 2ln)^2} + 4\sum_{l=1}^{\infty} \frac{(\tau - c_j)^2}{(\tau - c_j + 2ln)^2} \tag{3.7}$$

and c_j is defined as in (3.5).

Proof. By the identity in [1],

$$\frac{\pi^2}{\sin^2 \pi t} = \sum_{l=-\infty}^{l=\infty} \frac{1}{(t+l)^2}, \tag{3.8}$$

and then we get

$$\frac{1}{\sin^2 \frac{t-s}{2}} = \frac{4}{(t-s)^2} + 4\sum_{l=1}^{\infty} \frac{1}{(t-s-2l\pi)^2} + 4\sum_{l=1}^{\infty} \frac{1}{(t-s+2l\pi)^2} \tag{3.9}$$

and

$$\frac{(t-s)^2}{\sin^2 \frac{t-s}{2}} = 4 + 4\sum_{l=1}^{\infty} \frac{(t-s)^2}{(t-s-2l\pi)^2} + 4\sum_{l=1}^{\infty} \frac{(t-s)^2}{(t-s+2l\pi)^2}. \tag{3.10}$$

Then

$$\begin{aligned} K_s(t) &= \frac{(t-s)^2}{\sin^2 \frac{t-s}{2}} = 4 + 4\sum_{l=1}^{\infty} \frac{(\tau - c_j)^2}{(\tau - c_j - 4l\pi/h)^2} + 4\sum_{l=1}^{\infty} \frac{(\tau - c_j)^2}{(\tau - c_j + 4l\pi/h)^2} \\ &= 4 + 4\sum_{l=1}^{\infty} \frac{(\tau - c_j)^2}{(\tau - c_j - 2ln)^2} + 4\sum_{l=1}^{\infty} \frac{(\tau - c_j)^2}{(\tau - c_j + 2ln)^2} \\ &= K_{c_j}(\tau), \end{aligned}$$

which completes the proof.

Lemma 3.2. *(See [18, Lemma 3.2]) Let $P_n(t), n = 0, 1, \cdots$, be the Legendre function [1] defined as*

$$P_n(t) = \frac{1}{2^n} \sum_{r=0}^{[\frac{n}{2}]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} t^{n-2r}. \tag{3.11}$$

For the polynomial $t^n, n = 0, 1, \cdots$, it can be expanded by the Legendre function as

$$t^n = \frac{n!}{2^n} \sum_{k=0}^{[\frac{n}{2}]} \frac{(2n-4k+1)P_{n-2k}(t)}{k!(\frac{3}{2})_{n-k}}, \tag{3.12}$$

where $(a)_n = (a)(a+1)\cdots(a+n-1)$.

Lemma 3.3. *Let $\phi_{i,i+1}(t)$ and $\phi_{ii}(t)$ be defined as in (2.10) and (2.11), respectively. Then*

$$\phi_{i,i+1}(t) = \begin{cases} \sum_{j=1}^{i_1+1} \omega_{2j-1} Q_{2j-1}(t), & i = 2i_1, \\ \sum_{j=0}^{i_1} \omega_{2j} Q_{2j}(t), & i = 2i_1 - 1 \end{cases} \tag{3.13}$$

and

$$\phi_{ii}(t) = \begin{cases} Q_0'(t) + \sum_{j=1}^{i_1} a_j Q_{2j}(t), & i = 2i_1, \\ Q_1'(t) + \sum_{j=1}^{i_1} b_j Q_{2j-1}(t), & i = 2i_1 - 1, \end{cases} \tag{3.14}$$

where

$$\omega_j = \frac{2j+1}{2} \int_{-1}^{1} F_i(\tau) P_j(\tau) d\tau \tag{3.15}$$

and

$$a_j = -(4j+1) \sum_{k=1}^{j} \omega_{2k-1},$$

$$b_j = -(4j-1) \sum_{k=1}^{j} \omega_{2k-2}.$$

Proof. For $i = 2i_1$,

$$F_i(\tau) = \tau^i$$

leads to

$$F_i(\tau) = \sum_{j=1}^{i_1+1} \omega_{2j-1} P_{2j-1}(\tau), \tag{3.16}$$

where ω_{2j-1} is defined as in (3.15) and $P_j(\tau)$ are Legendre polynomials. The first part of (3.13) follows immediately from the definition of $\phi_{i,i+1}(\tau)$. Since

$$\sum_{j=1}^{i_1+1} \omega_{2j-1} = \sum_{j=1}^{i_1+1} \omega_{2j-1} P_{2j-1}(1) = F_i(1) = 1,$$

then we have

$$\phi_{i,i+1}(t) = \sum_{j=1}^{i_1} \frac{a_j}{4j+1} [Q_{2j+1}(t) - Q_{2j-1}(t)]$$

with $a_j = -(4j+1) \sum_{k=1}^{j} \omega_{2k-1}$, which leads to the first part of (3.14) by using the recurrence relation [1].

$$P_{l+1}'(t) - P_{l-1}'(t) = (2l+1) P_l(t), \quad l = 1, 2, \cdots, \tag{3.17}$$

which completed the proof.

Lemma 3.4. *Assume $s \in (t_{m-1}, t_m)$, for some m, c_j defined as in (3.5). Then, we have*

$$\psi_{ik}(c_j) = \begin{cases} -\dfrac{2^k}{h^{k+1}} \displaystyle\fint_{t_{m-1}}^{t_m} \dfrac{\mathcal{M}_{ik}^j(t,s)}{\sin^2\frac{t-s}{2}} dt, & j = m, \\ -\dfrac{2^k}{h^{k+1}} \displaystyle\int_{t_{j-1}}^{t_j} \dfrac{\mathcal{M}_{ik}^j(t,s)}{\sin^2\frac{t-s}{2}} dt, & j \neq m. \end{cases} \tag{3.18}$$

Proof. By the equation of (1.2),

$$\begin{aligned} & \fint_{t_{m-1}}^{t_m} \frac{\mathcal{M}_{ik}^j(t,s)}{\sin^2\frac{t-s}{2}} dt \qquad (3.19) \\ = & \fint_{t_{m-1}}^{t_m} \frac{F_i^m(t)K_s(t)}{(t-s)^2} dt \\ = & \lim_{\varepsilon\to 0}\left\{ \int_{t_{m-1}}^{s-\varepsilon} \frac{F_i^m(t)K_s(t)}{(t-s)^2} dt + \int_{s+\varepsilon}^{t_m} \frac{F_i^m(t)K_s(t)}{(t-s)^2} dt - \frac{2F_i^m(s)K_s(s)}{\varepsilon} \right\} \\ = & \frac{h^{k+1}}{2^{k+1}} \lim_{\varepsilon\to 0}\left\{ \left(\int_{-1}^{c_m - \frac{2\varepsilon}{h}} + \int_{c_m + \frac{2\varepsilon}{h}}^{1} \right) \frac{F_i(\tau)K_s(\tau)}{(\tau - c_m)^2} d\tau - \frac{hF_i(c_m)K_s(c_m)}{\varepsilon} \right\} \\ = & \frac{h^{k+1}}{2^{k+1}} \fint_{-1}^{1} \frac{F_i(\tau)K_s(\tau)}{(\tau-c_m)^2} d\tau \\ = & -\frac{h^{k+1}}{2^k} \psi_{ii}(c_m). \qquad (3.20) \end{aligned}$$

The second identity can be obtained similarly.

Lemma 3.5. *Under the assumption of Lemma 3.4, then we have*

$$\psi_{ik}(c_j) = \begin{cases} -\dfrac{2^k}{h^{k+1}} \displaystyle\fint_{t_{m-1}}^{t_m} \dfrac{\mathcal{M}_{ik}^m(t,s)}{\sin^2\frac{t-s}{2}} dt, & j = m, \\ -\dfrac{2^k}{h^{k+1}} \displaystyle\int_{t_{j-1}}^{t_j} \dfrac{\mathcal{M}_{ik}^j(t,s)}{\sin^2\frac{t-s}{2}} dt, & j \neq m. \end{cases} \tag{3.21}$$

Proof. We have:

$$\begin{aligned} \fint_{t_{m-1}}^{t_m} \frac{\mathcal{M}_{ik}^m(t,s)}{\sin^2\frac{t-s}{2}} dt &= -\!\!\!\!\!\!\int_{t_{m-1}}^{t_m} \frac{F_i^m(t)K_s(t)}{t-s} dt \\ &= \lim_{\varepsilon\to 0}\left\{ \int_{t_{m-1}}^{s-\varepsilon} \frac{F_i^m(t)K_s(t)}{t-s} dt + \int_{s+\varepsilon}^{t_m} \frac{F_i^m(t)K_s(t)}{t-s} dt \right\} \\ &= \frac{h^{k+1}}{2^{k+1}} \lim_{\varepsilon\to 0}\left\{ \left(\int_{-1}^{c_m-\varepsilon} + \int_{c_m+\varepsilon}^{1} \right) \frac{F_i(\tau)K_s(\tau)}{\tau - c_m} d\tau \right\} \\ &= \frac{h^{k+1}}{2^{k+1}} \fint_{-1}^{1} \frac{F_i(\tau)K_s(\tau)}{\tau - c_m} d\tau \\ &= -\frac{h^{k+1}}{2^k} \psi_{i,i+1}(c_m). \qquad (3.22) \end{aligned}$$

The second part of this Lemma 3.4 can be obtained similarly.

Lemma 3.6. *For $k > i+1$, it holds that*

$$\psi_{ik}(c_j) = -\frac{2^k}{h^{k+1}} ⨎_{t_{j-1}}^{t_j} \frac{\mathcal{M}_{ik}^j(t,s)}{(k-i)!\sin^2\frac{t-s}{2}}dt. \tag{3.23}$$

Proof. The proof of (3.23) can be obtained similarly to Lemma 3.4 or Lemma 3.5.

Lemma 3.7. *Suppose $f(t) \in C^\infty[c, c+2\pi]$. If $s \neq t_j$, for any $j = 1, 2, \cdots, n$, then it holds that*

$$f(t) - f_C(t) = \sum_{i=0}^{\infty}\sum_{k=i}^{\infty} \frac{(-1)^{i+1} f^{(k)}(s)}{i!}\frac{M_{ik}^j(t,s)}{(k-i)!}. \tag{3.24}$$

Proof. By taking the Taylor expansion for $f(t_{j-1/2})$, we have

$$f(t_{j-1/2}) = f(t) + \sum_{i=0}^{\infty}\frac{f^{(i)}(t)}{i!}(t_{j-1/2}-t)^i$$

and thus,

$$f(t) - f_C(t) = \sum_{i=1}^{\infty}\frac{(-1)^{i+1}f^{(i)}(t)}{i!}(t-t_{j-1/2})^i \tag{3.25}$$

and

$$f^{(i+1)}(t) = \sum_{k=i}^{\infty}\frac{f^{(k)}(s)(t-s)^{k-i}}{(k-i)!}. \tag{3.26}$$

Combining (3.25) and (3.26) leads to (3.24).

Define

$$\mathcal{H}_j(t) = f(t) - f_C(t) - \sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\frac{(-1)^{i+1}f^{(k+1)}(s)}{h(i+1)!}\frac{M_{ik}^j(t,s)}{(k-i)!}, \quad t \in (t_{j-1}, t_j). \tag{3.27}$$

3.1. Proof of Theorem 2.1

Proof.

$$\left(\int_c^{t_{m-1}} + \int_{t_m}^{c+2\pi}\right)\frac{f(t)-f_C(t)}{\sin^2\frac{t-s}{2}}dt$$

$$= \sum_{j=1, j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{f(t)-f_C(t)}{\sin^2\frac{t-s}{2}}dt$$

$$= \sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\frac{(-1)^{i+1}f^{(k+1)}(s)}{h(i+1)!(k-i)!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{\mathcal{M}_{ik}^j(t,s)}{\sin^2\frac{t-s}{2}}dt. \tag{3.28}$$

By (3.27) of $\mathcal{H}_j(t)$, we have

$$⨎_{t_{m-1}}^{t_m}\frac{f(t)-f_C(t)}{\sin^2\frac{t-s}{2}}dt = ⨎_{t_{m-1}}^{t_m}\frac{\mathcal{H}_m(t)}{\sin^2\frac{t-s}{2}}dt$$

$$+\sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\frac{(-1)^{i+1}f^{(k+1)}(s)}{h(i+1)!(k-i)!}\fint_{t_{m-1}}^{t_m}\frac{\mathcal{M}_{ik}^{m}(t,s)}{\sin^2\frac{t-s}{2}}\,dt. \tag{3.29}$$

Putting (3.28) and (3.29) together yields

$$\begin{aligned}\fint_{c}^{c+2\pi}\frac{f(t)-f_C(t)}{\sin^2\frac{t-s}{2}}dt &= \sum_{i=1}^{\infty}\sum_{k=i}^{\infty}\frac{(-1)^{i+1}h^k f^{(k+1)}(s)}{(i+1)!(k-i)!}\sum_{j=1}^{n}\psi_{ik}(\tau)\\ &= \sum_{i=0}^{\infty}\frac{h^i}{2^{i+1}}f^{(i+1)}(s)a_i(\tau)\end{aligned} \tag{3.30}$$

where

$$a_i(\tau)=\begin{cases}\displaystyle\sum_{k=i-1}^{i}\frac{(-1)^{k+1}}{(k+1)!}T_{ki}(\tau), & i>0,\\[2ex] \dfrac{1}{2}T_{ki}(\tau), & i=0.\end{cases} \tag{3.31}$$

The proof is complete.

For $i=1$,

$$\begin{aligned}a_0(\tau) &= \sum_{i=1}^{n}\fint_{t_{i-1}}^{t_i}\frac{t-\hat{t}_i}{\sin^2\frac{t-s}{2}}\,dt\\ &= -2h\sum_{k=1}^{\infty}\sum_{i=1}^{n}\{\sin[k(t_i-s)]+\sin[k(t_{i-1}-s)]\}\\ &\quad -4\sum_{k=1}^{\infty}\frac{1}{k}\sum_{i=1}^{n}\{\cos[k(t_i-s)]-\cos[k(t_{i-1}-s)]\}\\ &= -4h\sum_{k=1}^{\infty}\sum_{i=1}^{n}\sin[k(t_i-s)]\\ &= -4h\sum_{j=1}^{\infty}n\sin[nj(t_1-s)]\\ &= 8\pi\sum_{j=1}^{\infty}\sin[j(1+\tau)\pi]\\ &= 4\pi\cot\tfrac{(1+\tau)\pi}{2} = -4\pi\tan\tfrac{\tau\pi}{2},\end{aligned} \tag{3.32}$$

where

$$\sum_{i=1}^{n}\sin[k(t_i-s)]=\begin{cases}n\sin[k(t_1-s)], & k=nj,\\ 0, & \textit{otherwise}\end{cases}$$

has been used. When $\tau=0$, we get $a_0(\tau)=0$ [6]. This is not the same as the case with a singular point located on the interval. The reason is that on the circle each subinterval is equal so we just consider the case of s located at the middle of the interval.

At last, we suggest the modified composite middle rectangle rule, denoting by $\tilde{I}_n(f,s)$, defined by

$$\tilde{I}_n(f,s)=I_n(f,s)-4\pi f'(s)\tan\frac{\tau\pi}{2}. \tag{3.33}$$

4. Extrapolation algorithm

Following asymptotic expansion,

$$E_n(f,s)=\sum_{i=0}^{\infty}\frac{h^i}{2^{i+1}}f^{(i+1)}(s)a_i(\tau). \tag{4.1}$$

For the given s and positive integer n_0, we have

$$m_0:=\frac{n_0(s-c)}{2\pi}.$$

First, we partition $[c,c+2\pi]$ into n_0 equal subintervals to get a mesh denoted by Π_1 with mesh size $h_1=2\pi/n_0$. Then Π_1 is refined to get mesh Π_2 with mesh size $h_2=h_1/2$. In this way, a series of meshes $\{\Pi_j\}(j=1,2,\cdots)$ is obtained in which Π_j is refined from Π_{j-1} with mesh size denoted by h_j. The extrapolation scheme is presented in Table 1.

Table 1. Extrapolation scheme of $T_i^{(j)}$.

$T(h_1)=T_1^{(1)}$				
$T(h_2)=T_1^{(2)}$	$T_2^{(1)}$			
$T(h_3)=T_1^{(3)}$	$T_2^{(2)}$	$T_3^{(1)}$		
$T(h_4)=T_1^{(4)}$	$T_2^{(3)}$	$T_3^{(2)}$	$T_4^{(1)}$	
$T(h_5)=T_1^{(5)}$	$T_2^{(4)}$	$T_3^{(3)}$	$T_4^{(2)}$	$T_5^{(1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Define

$$s_j=s+\frac{\tau+1}{2}h_j,\quad j=1,2,\cdots, \tag{4.2}$$

and

$$T(h_j)=I_{2^{j-1}n_0}(f,s_j). \tag{4.3}$$

The extrapolation algorithm takes the form of:

Step one:

$$Compute\ \ T_1^{(j)}=T(h_j),\quad j=1,\cdots,m.$$

Step two:

$$Compute\ \ T_i^{(j)}=T_{i-1}^{(j+1)}+\tfrac{T_{i-1}^{(j+1)}-T_{i-1}^{(j)}}{2^{i-1}-1},\quad i=2,\cdots,m,\quad j=1,\cdots,m-i.$$

Theorem 4.1. *Under the same condition of Theorem 2.1, for a given τ and (4.2), we have*

$$|I(f,s)-T_i^{(j)}|\le Ch^i \tag{4.4}$$

and

$$\left|\frac{T_{i-1}^{(j+1)}-T_{i-1}^{(j)}}{2^{i-1}-1}\right|\le Ch^{i-1}.$$

Proof. By Eq (4.1), we get

$$\begin{aligned} I(f,s)-T(h_j) &= I(f,s)-I(f,s_j)+I(f,s_j)-T(h_j) \\ &= I(f,s)-I(f,s_j)+\sum_{i=0}^{\infty}\frac{h_j^i}{2^i}a_i(\tau)f^{(i+1)}(s_j). \end{aligned} \tag{4.5}$$

$$\begin{aligned} I(f;s_j) &= I(f;s)+I'(f;s)\frac{\tau+1}{2}h_j+I''(f;s)(\frac{\tau+1}{2}h_j)^2 \\ &+ \cdots+I^{(l-2)}(f;s)(\frac{\tau+1}{2}h_j)^{l-2}+\cdots, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} f^{(i+1)}(s_j) &= f^{(i+1)}(s)+f^{(i+2)}(s)\frac{\tau+1}{2}h_j+\frac{f^{(i+3)}(s)}{2!}(\frac{\tau+1}{2}h_j)^2 \\ &+ \cdots+\frac{f^{(l)}(s)}{(l-i-1)!}(\frac{\tau+1}{2}h_j)^{l-i-1}+\cdots. \end{aligned} \tag{4.7}$$

Putting (4.5)–(4.7) together, we have

$$I(f,s)-T(h_j)=\sum_{i=0}^{\infty}b_i(s,\tau)h_j^i, \tag{4.8}$$

where

$$b_i(s,\tau)=f^{(i+1)}(s)\sum_{k=1}^{i}\frac{a_k(\tau)}{2^k}\Big(\frac{\tau+1}{2}\Big)^{i-k}\frac{1}{(i-k)!}-\frac{(\tau+1)^i}{2^i i!}I^{(i)}(f,s), \tag{4.9}$$

where $b_i(s,\tau)$ is a constant for a given τ . By (4.8), we also have

$$I(f,s)-T(h_{j+1})=\sum_{i=0}^{\infty}b_i(s,\tau)h_{j+1}^i. \tag{4.10}$$

By (4.8) and (4.10), with $h_j=2h_{j+1}$, we have

$$\begin{aligned} I(f,s) &= 2T(h_{j+1})-T(h_j)+\sum_{i=2}^{\infty}b_i(s,\tau)\left(\frac{1}{2^{i-1}}-1\right)h_j^i+O(h_j^{l-1}) \\ &= T_2^{(j)}+\sum_{i=2}^{\infty}b_i(s,\tau)\left(\frac{1}{2^{i-1}}-1\right)h_j^i, \end{aligned} \tag{4.11}$$

which implies

$$I(f,s)-T_2^{(j)}=\sum_{i=2}^{\infty}b_i(s,\tau)\left(\frac{1}{2^{i-1}}-1\right)h_j^i \tag{4.12}$$

and

$$T_2^{(j)}=2T(h_{j+1})-T(h_j). \tag{4.13}$$

The accuracy $O(h^3)$ can be obtained after performing the extrapolation process.

In order to directly use the error expansion of (4.1), we presented the following the new partition as $c = t_{00} < t_{01} < \cdots < t_{0,n-1} < t_{0n} = c + 2\pi$ with $t_{0i} = t_i + \frac{h}{2}, i = 0, 1, \cdots, n-1$ and

$$f_{0C}(t) = f(t_{0i}). \tag{4.14}$$

Then we have the approximate formula:

$$\tilde{I}_n(f; s) := \fint_c^{c+2\pi} \frac{f_{0C}(t)}{\sin^2 \frac{t-s}{2}} dt \tag{4.15}$$

$$= \sum_{j=0}^{n} \tilde{\omega}_j(s) f(t_j) = \fint_c^{c+2\pi} \frac{f(t)}{\sin^2 \frac{t-s}{2}} dt - \tilde{E}_n(f, s), \tag{4.16}$$

and get

$$\tilde{\omega}_i(s) = \begin{cases} 2\cot\left(\frac{t_0 - s}{2}\right) - 2\cot\left(\frac{t_{01} - s}{2}\right), & i = 0, \\ 2\cot\left(\frac{t_{0i} - s}{2}\right) - 2\cot\left(\frac{t_{0,i+1} - s}{2}\right), & 0 < i < n, \\ 2\cot\left(\frac{t_{0n} - s}{2}\right) - 2\cot\left(\frac{t_{n+1} - s}{2}\right), & i = n, \end{cases} \tag{4.17}$$

and then we find the following theorem similarly as with Theorem 2.1.

Theorem 4.2. *Assume $f(t) \in C^\infty[c, c+2\pi]$. For the rectangle rule $\tilde{I}_n(f; s)$ defined in (4.15) and $a_{2i+1}(\tau)$ defined in (1.9), independent of h and s, such that*

$$\tilde{E}_n(f, s) = \fint_c^{c+2\pi} \frac{f(t) - f_{0C}(t)}{\sin^2 \frac{t-s}{2}} dt = \sum_{i=0}^{\infty} \frac{h^{2i}}{2^{2i+1}} f^{(i+1)}(s) a_{2i+1}(\tau) \tag{4.18}$$

where $s = t_{m-1} + (1 + \tau)h/2, \ m = 1, 2, \cdots, n$.

Similarly as extrapolation scheme $T(h_j)$, new extrapolation scheme is presented in Table 2.

$$\tilde{T}(h_j) = \tilde{I}_{2^{j-1}n_0}(f, s_j). \tag{4.19}$$

Table 2. Extrapolation scheme of $\tilde{T}_i^{(j)}$.

$\tilde{T}(h_1) =$	$\tilde{T}_1^{(1)}$				
$\tilde{T}(h_2) =$	$\tilde{T}_1^{(2)}$	$\tilde{T}_2^{(1)}$			
$\tilde{T}(h_3) =$	$\tilde{T}_1^{(3)}$	$\tilde{T}_2^{(2)}$	$\tilde{T}_3^{(1)}$		
$\tilde{T}(h_4) =$	$\tilde{T}_1^{(4)}$	$\tilde{T}_2^{(3)}$	$\tilde{T}_3^{(2)}$	$\tilde{T}_4^{(1)}$	
$\tilde{T}(h_5) =$	$\tilde{T}_1^{(5)}$	$\tilde{T}_2^{(4)}$	$\tilde{T}_3^{(3)}$	$\tilde{T}_4^{(2)}$	$\tilde{T}_5^{(1)}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Theorem 4.3. *Under the asymptotic expansion of Theorem 4.2, we have*

$$|I(f,s)-\tilde{T}_i^{(j)}|\leq Ch^{2i} \tag{4.20}$$

and the posteriori asymptotic error estimate is given by

$$\left|\frac{\tilde{T}_{i-1}^{(j+1)}-\tilde{T}_{i-1}^{(j)}}{4^{i-1}-1}\right|\leq Ch^{2i-2}.$$

The proof of Theorem 4.3 can be similarly obtained as with Theorem 4.1, so we omitted it here.

5. Extrapolation scheme for the hypersingular integral equation

Consider

$$\frac{1}{4\pi}\,=\!\!\!\!\!\!\int_0^{2\pi}\frac{f(t)}{\sin^2\frac{t-s}{2}}\,dt=g(s),\quad s\in(0,2\pi), \tag{5.1}$$

with

$$\int_0^{2\pi}g(t)\,dt=0. \tag{5.2}$$

As in [29], under the compatibility condition (5.2), there exists a unique solution up to an additive constant for (5.1). With periodical condition

$$\int_0^{2\pi}f(t)\,dt=0, \tag{5.3}$$

there is a unique solution of (5.1).

Applying $\tilde{I}_n(f,s)$ to approximate the hypersingular integral in (5.1) and using $\hat{t}_k=t_{k-1}+h/2(k=1,2,\cdots,n)$ of each subinterval to be a collocation point, we get

$$\frac{1}{2\pi}\sum_{m=1}^{n}(\cot\frac{\hat{t}_k-t_m}{2}-\cot\frac{\hat{t}_k-t_{m-1}}{2})f_m=g(\hat{t}_k),\ \ k=1,2,\cdots,n, \tag{5.4}$$

denoted by

$$\mathcal{A}_n\mathbf{F}_n^a=\mathbf{G}_n^e, \tag{5.5}$$

where

$$\begin{aligned}&\mathcal{A}_n=(a_{km})_{n\times n},\\ &a_{km}=\frac{1}{2\pi}(\cot\frac{\hat{t}_k-x_m}{2}-\cot\frac{\hat{t}_k-x_{m-1}}{2}),k,m=1,2,\cdots,n,\\ &\mathbf{F}_n^a=(f_1,f_2,\cdots,f_n)^T,\mathbf{G}_n^e=(g(\hat{t}_1),g(\hat{t}_2),\cdots,g(\hat{t}_n))^T,\end{aligned} \tag{5.6}$$

and $f_k=f(\hat{t}_k)(k=1,2,\cdots,n)$. Obviously, $\mathcal{A}_n$ is a circulant matrix and also a symmetric Toeplitz matrix. Since,

$$\sum_{m=1}^{n}a_{km}=\frac{1}{2\pi}\sum_{m=1}^{n}(\cot\frac{\hat{t}_k-t_m}{2}-\cot\frac{\hat{t}_k-t_{m-1}}{2})=0, \tag{5.7}$$

we see that $\mathcal{A}_n$ is singular.

The regularizing factor γ_{0n} (see Reference [23]) in (5.4) is introduced, which leads to

$$\begin{cases} \gamma_{0n} + \dfrac{1}{2\pi}\displaystyle\sum_{m=1}^{n}(\cot\dfrac{\hat{t}_k - t_m}{2} - \cot\dfrac{\hat{t}_k - t_{m-1}}{2})f_m = g(\hat{t}_k), \quad k = 1, 2, \cdots, n, \\ \displaystyle\sum_{m=1}^{n} f_m = 0, \end{cases} \tag{5.8}$$

where γ_{0n} has the form

$$\gamma_{0n} = \frac{1}{2\pi}\sum_{k=1}^{n} g(\hat{t}_k)h. \tag{5.9}$$

System (5.8) is denoted as

$$\mathcal{A}_{n+1}\mathbf{F}^a_{n+1} = \mathbf{G}^e_{n+1}, \tag{5.10}$$

where

$$\begin{aligned} \mathcal{A}_{n+1} &= \begin{pmatrix} 0 & e_n^T \\ e_n & \mathcal{A}_n \end{pmatrix}, \\ \mathbf{F}^a_{n+1} &= \begin{pmatrix} \gamma_{0n} \\ \mathbf{F}^a_n \end{pmatrix}, \mathbf{G}^e_{n+1} = \begin{pmatrix} 0 \\ \mathbf{G}^e_n \end{pmatrix}, \end{aligned} \tag{5.11}$$

and $e_n = (\underbrace{1, 1, \cdots, 1}_{n})$.

Linear system (5.8) can be rewritten as

$$\begin{cases} \gamma_{0n} + \dfrac{1}{2\pi}\displaystyle\sum_{m=1}^{n} -\dfrac{f_{m+1} - f_m}{h}\cot\dfrac{\hat{t}_k - t_m}{2}h = g(\hat{t}_k), \quad k = 1, 2, \cdots, n, \\ -\dfrac{1}{2\pi}\displaystyle\sum_{m=1}^{n}\dfrac{f_{m+1} - f_m}{h}h = 0, \end{cases} \tag{5.12}$$

where $f_1 = f_{n+1}$ has been used. Let $v_m = -(f_{m+1} - f_m)/h$, and we get

$$\begin{cases} \gamma_{0n} + \dfrac{1}{2\pi}\displaystyle\sum_{m=1}^{n}\cot\dfrac{\hat{t}_k - t_m}{2}v_m h = g(\hat{t}_k), \quad k = 1, 2, \cdots, n, \\ \dfrac{1}{2\pi}\displaystyle\sum_{m=1}^{n} v_m h = 0. \end{cases} \tag{5.13}$$

The following lemma has been proved in [23].

Lemma 5.1. *([23, Theorem 6.2.1, §6.2, Chapter 6]) For Eq (5.13), its solution is*

$$v_m = -\frac{h}{2\pi}\sum_{k=1}^{n}\cot\frac{t_m - \hat{t}_k}{2}f(\hat{t}_k). \tag{5.14}$$

Lemma 5.2. *([6, Lemma 4.2]) Let $\mathcal{B}_{n+1} = (b_{ik})_{(n+1)\times(n+1)}$ be the inverse matrix of $\mathcal{A}_{n+1}$, defined in (5.10). Then,*

(1) $\mathcal{B}_{n+1}$ is expressed as

$$\mathcal{B}_{n+1} = \begin{pmatrix} b_{00} & \boldsymbol{B}_1 \\ \boldsymbol{B}_2 & \mathcal{B}_n \end{pmatrix}, \tag{5.15}$$

where

$$\begin{aligned} \boldsymbol{B}_1 &= (b_{01}, b_{02}, \cdots, b_{0n}), \boldsymbol{B}_2 = (b_{10}, b_{20}, \cdots, b_{n0})^T, & (5.16) \\ b_{i0} &= b_{0i} = \frac{1}{n}, 1 \le i \le n, & (5.17) \\ b_{ik} &= \frac{h^2}{2\pi}\left[\sum_{m=i}^{n-1} \cot\frac{\hat{t}_k - t_m}{2} - \frac{1}{n}\sum_{m=1}^{n-1} m\cot\frac{\hat{t}_k - t_m}{2}\right], & (5.18) \\ &\quad 1 \le i \le n-1, 1 \le k \le n, & \\ b_{nk} &= -\frac{h^2}{2n\pi}\sum_{m=1}^{n-1} m\cot\frac{\hat{t}_k - t_m}{2}, 1 \le k \le n. & (5.19) \end{aligned}$$

(2) $\mathcal{B}_n$ is a Toeplitz and also circulant matrix.

(3) for $i = 1, 2, \cdots, n$, we have

$$\sum_{k=1}^{n} |b_{ik}| \le C. \tag{5.20}$$

5.1. Advantage of circulant matrices

Circulant matrices have unique advantages in matrix operations, such as matrix multiplication. It can greatly save the calculation time and improve the calculation efficiency. The circular matrix only needs to store the elements of the first row (or the first column) to completely determine the entire matrix, because the remaining row (or column) elements are obtained from the cyclic shift of the first row (or column) elements. Circulant matrices have many good mathematical properties, such as the fact that their eigenvalues and eigenvectors can be calculated analytically, which makes it easier to analyze and deal with circulant matrices than general matrices in solving linear equations and eigenvalue problems.

We present the main theorem of this section.

Theorem 5.1. *The solution of linear system (5.8) or (5.10)is*

$$f(\hat{t}_i) - f_i = \sum_{i=0}^{\infty} \frac{h^{2i}}{2^{2i+1}} f^{(i+1)}(s) a_{2i+1}(\tau). \tag{5.21}$$

Proof. Let $\mathbf{F}^e_{n+1} = (0, f(\hat{t}_1), f(\hat{t}_2), \cdots, f(\hat{t}_n))^T$ be the exact vector. Then, from (5.10), we get

$$\mathbf{F}^e_{n+1} - \mathbf{F}^a_{n+1} = \mathcal{B}_{n+1}(\mathcal{A}_{n+1}\mathbf{G}^e_{n+1} - \mathbf{G}^e_{n+1}), \tag{5.22}$$

which implies

$$\begin{aligned} f(\hat{t}_i) - f_i &= b_{i0} \sum_{m=1}^{n} g(\hat{t}_m) + \sum_{k=1}^{n} b_{ik} \tilde{E}_n(f, \hat{t}_k) \\ &= \frac{1}{2\pi} \sum_{m=1}^{n} g(\hat{t}_m) h + \sum_{k=1}^{n} b_{ik} \tilde{E}_n(f, \hat{t}_k), \end{aligned} \tag{5.23}$$

where $\{b_{ik}\}$ are the entries of $\mathcal{B}_{n+1}$ and $\tilde{E}_n(f, s)$ is defined in (4.15).

For the first part of $\frac{1}{2\pi} \sum_{m=1}^{n} g(\hat{t}_m) h$ of the rectangle rule to compute the definite integral $\frac{1}{2\pi} \int_0^{2\pi} g(t) dt$. Then we have

$$\begin{aligned} f(\hat{t}_i) - f_i &= \sum_{k=1}^{n} b_{ik} \tilde{E}_n(f, \hat{t}_k) \\ &= \sum_{k=1}^{n} b_{ik} \sum_{i=0}^{\infty} \frac{h^{2i}}{2^{2i+1}} f^{(i+1)}(\hat{t}_k) a_{2i+1}(\tau) \\ &= \sum_{i=0}^{\infty} \frac{h^{2i}}{2^{2i+1}} a_{2i+1}(\tau) \sum_{k=1}^{n} b_{ik} f^{(i+1)}(\hat{t}_k). \end{aligned} \tag{5.24}$$

By (5.24) and (5.20), we obtain

$$b_{ik} = \frac{c}{n} + O(h^3)$$

and

$$\sum_{k=1}^{n} b_{ik} f^{(i+1)}(\hat{t}_k) = C + O(h^2).$$

Then we have

$$f(\hat{t}_i) - f_i = \sum_{i=0}^{\infty} \frac{h^{2i}}{2^{2i+1}} a_{2i+1}(\tau),$$

where (5.19) and (5.20) has been used.

6. Numerical example

Example 6.1. *We consider (1.1) with density function* $f(t) = 1 + 2\cos(t) + 2\cos(2t)$, $c = -\pi$, $c + 2\pi = \pi$.

Numerical results are presented in Table 3 for $s = t_{[n/4]} + (1 + \tau)h/2$. The right half of Table 3 shows that the accuracy of $\tilde{I}_n(s, f)$ is always $O(h^2)$. However, from the left half, the accuracy is $O(h^2)$ if s is located at the superconvergence point ($\tau = 0$), and if $\tau \neq 0$, the middle rectangle rule is divergent.

Table 3. Errors of $I_n(s, f)$ and $\tilde{I}_n(s, f)$ with $s = t_{[n/4]} + (1 + \tau)h/2$.

	$I_n(f, s)$			$\hat{I}_n(f, s)$	
n	$\tau = 0$	$\tau = 2/3$	$\tau = -2/3$	$\tau = -2/3$	$\tau = 2/3$
32	5.3695e-004	-1.2446e+000	1.0651e+000	1.8350e-003	-7.6074e-004
64	1.2851e-004	-1.1295e+000	1.0421e+000	4.5684e-004	-1.9980e-004
128	3.1416e-005	-1.0741e+000	1.0310e+000	1.1397e-004	-5.1140e-005
512	7.7655e-006	-1.0470e+000	1.0256e+000	2.8464e-005	-1.2933e-005
1024	1.9303e-006	-1.0336e+000	1.0229e+000	7.1123e-006	-3.2516e-006

Example 6.2. *We still consider* $f(t) = 1 + 2\cos(t) + 2\cos(2t), c = -\pi, c + 2\pi = \pi$. *For* $s = -\pi/2$, *and* $g(s) = 5.026548245744136e + 001$, *we use* $s = t_{[n/4]} + (\tau + 1)h/2$ *with* $\tau = 0$ *to compute* $s = -\pi/2$.

For $s = 0$ *and* $g(s) = -7.539822368615504e + 01$, *we also use* $s = t_{[n/2]} + (\tau + 1)h/2$ *with* $\tau = 0$ *to compute* $s = 0$.

In Tables 4 and 6, we present the error of the middle rectangle rule as h^2, h^4, h^6, and h^8, respectively. In Tables 5 and 7, the convergence rates h^2, h^4, and h^6 also agree with the theoretic analysis of Theorem 4.2.

Table 4. Error estimate of the mid-rectangle rule $s = -\pi/2$.

	0		h^4−extra		h^6−extra		h^8−extra	
8	5.0106e+000							
16	1.2820e+000	1.9666	3.9121e-002					
32	3.2236e-001	1.9917	2.4790e-003	3.9801	3.6167e-005			
64	8.0707e-002	1.9979	1.5547e-004	3.9950	5.6990e-007	5.9878	4.8600e-009	
128	2.0184e-002	1.9995	9.7253e-006	3.9988	8.9188e-009	5.9977	1.4339e-011	8.4156
256	5.0464e-003	1.9999	6.0796e-007	3.9997	1.3873e-010	6.0088	-6.3238e-013	4.0491

Table 5. A posteriori estimate of the mid-rectangle rule $s = -\pi/2$.

	0		h^4−extra		h^6−extra		h^8−extra	
8								
16	1.2429e+000							
32	3.1988e-001	1.9581	2.4428e-003					
64	8.0551e-002	1.9896	1.5490e-004	3.9791	5.6504e-007			
128	2.0174e-002	1.9974	9.7164e-006	3.9948	8.9044e-009	5.9877	1.9003e-011	
256	5.0458e-003	1.9993	6.0782e-007	3.9987	1.3937e-010	5.9976	5.8710e-014	8.3268

Table 6. Error estimate estimate of the mid-rectangle rule $s = 0$.

	0		h^4−extra		h^6−extra		h^8−extra	
8	-5.6517e+00							
16	-1.4432e+00	1.9694	-4.0361e-02					
32	-3.6271e-01	1.9924	-2.5567e-03	3.9806	-3.6452e-05			
64	-9.0799e-02	1.9981	-1.6033e-04	3.9951	-5.7436e-07	5.9879	-4.8673e-09	
128	-2.2707e-02	1.9995	-1.0029e-05	3.9988	-9.0075e-09	5.9947	-3.3666e-11	7.1899
256	-5.6773e-03	1.9999	-6.2699e-07	3.9996	-1.6669e-10	5.7493	-2.6361e-11	-

Table 7. A posteriori estimate of the mid-rectangle rule $s_j = s + (\tau + 1)h_j/2$.

	0		h^4−extra		h^6−extra		h^8−extra	
8								
16	-1.4028e+00							
32	-3.6016e-01	1.9616	-2.5203e-03					
64	-9.0638e-02	1.9904	-1.5976e-04	3.9796	-5.6949e-07			
128	-2.2697e-02	1.9976	-1.0020e-05	3.9949	-8.9738e-09	5.9878	-1.8955e-11	
256	-5.6766e-03	1.9994	-6.2682e-07	3.9987	-1.4033e-10	5.9990	-2.8645e-14	9.6078

In Tables 8 and 10, we present the error of the middle rectangle rule as h, h^2, h^3, and h^4, respectively. In Tables 9 and 11, the convergence rates h, h^2, and h^3 also agree with the theoretic analysis of Theorem 2.1.

Table 8. Error estimate of the mid-rectangle rule $s = -\pi/2$.

	0		h^2−extra		h^3−extra		h^4−extra	
32	8.1836e+00							
64	3.7710e+00	1.1178	-6.4164e-01					
128	1.7978e+00	1.0687	-1.7536e-01	1.8714	-1.9940e-02			
256	8.7613e-01	1.0370	-4.5544e-02	1.9450	-2.2699e-03	3.1350	2.5439e-04	
512	4.3227e-01	1.0192	-1.1587e-02	1.9747	-2.6881e-04	3.0779	1.7053e-05	3.8989
1024	2.1467e-01	1.0098	-2.9214e-03	1.9879	-3.2661e-05	3.0410	1.0752e-06	3.9874

Table 9. A posteriori estimate of the mid-rectangle rule $s_j = s + (\tau + 1)h_j/2$.

	0		h^2−extra		h^3−extra		h^4−extra	
32								
64	4.4126e+00							
128	1.9732e+00	1.1611	-1.5542e-01					
256	9.2167e-01	1.0982	-4.3274e-02	1.8447	-2.5243e-03			
512	4.4386e-01	1.0542	-1.1319e-02	1.9348	-2.8587e-04	3.1425	1.5822e-05	
1024	2.1760e-01	1.0284	-2.8887e-03	1.9702	-3.3736e-05	3.0830	1.0652e-06	3.8927

Table 10. Error estimate of the mid-rectangle rule $s = -\pi$.

	0		h^2−extra		h^3−extra		h^4−extra	
32	-4.5518e+00							
64	-1.9715e+00	1.2071	6.0874e-01					
128	-9.0311e-01	1.1263	1.6531e-01	1.8807	1.7498e-02			
256	-4.3016e-01	1.0700	4.2802e-02	1.9494	1.9665e-03	3.1535	-2.5230e-04	
512	-2.0964e-01	1.0369	1.0874e-02	1.9768	2.3104e-04	3.0894	-1.6885e-05	3.9013
1024	-1.0345e-01	1.0190	2.7394e-03	1.9889	2.7950e-05	3.0472	-1.0634e-06	3.9890

Table 11. A posteriori estimate of the mid-rectangle rule $s_j = s + (\tau + 1)h_j/2$.

	0		h^2−extra		h^3−extra		h^4−extra	
32								
64	-2.5803e+00							
128	-1.0684e+00	1.2720	1.4781e-01					
256	-4.7296e-01	1.1757	4.0835e-02	1.8559	2.2188e-03			
512	-2.2051e-01	1.1008	1.0643e-02	1.9400	2.4793e-04	3.1618	-1.5694e-05	
1024	-1.0619e-01	1.0542	2.7115e-03	1.9727	2.9013e-05	3.0951	-1.0548e-06	3.8952

Example 6.3. *In this example, we will not only test our method on a hypersingular integral, but also solve a hypersingular integral equation* $g(s) = -2\cos(2s) - 2\sin(2s), c = -\pi, c + 2\pi = \pi$*, while the analysis solution is* $f(t) = \cos(2t) + \sin(2t)$*. The good numerical results indicate that our quadrature is efficient and accurate, which matches the theoretical analysis.*

In Table 12, we list the numerical results at $s = 1.45122657606971$. Numerical results show that the convergence rate of the hypersingular integral equation has the order $O(h^2)$. With extrapolation methods, the order can reach $O(h^4)$, $O(h^6)$, and $O(h^8)$ in Table12.

Table 12. Error estimate of a hypersingular integral equation.

	h^2−	h^4−extra	h^6−extra	h^8−extra
16	3.70E-02			
32	9.13E-03	1.67E-04		
64	2.27E-03	1.03E-05	-1.70E-07	
128	5.68E-04	6.39E-07	-2.61E-09	4.19E-11
256	1.42E-04	3.99E-08	-4.06E-11	1.72E-13

7. Conclusions

The article mainly focuses on a class of hypersingular integral equations in the boundary element method. A solution centered on the extrapolation algorithm is proposed, which is based on the asymptotic expansion of the error of the composite rectangle rule. By leveraging the asymptotic characteristics of the error of the composite rectangle rule, an extrapolation algorithm is constructed, providing a new approach to solving hypersingular integral equations. The algorithm is verified through in-depth theoretical and numerical examples. It is confirmed that the main part of the error function of the composite rectangle rule has an asymptotic expansion. Numerical experiments show that special functions significantly affect the convergence rate of the algorithm. This discovery has important guiding significance for subsequent algorithm optimization and improvement of computational efficiency. The adoption of this algorithm has two prominent advantages. First, it can obtain high-precision calculation results, meeting the precision requirements of practical applications; second, it is convenient for deriving a posteriori error estimates.

Symbol list

$\oint_a^b$— —- —- hypersingular integrals
$⨍_a^b$— —- —- Cauchy principal integral
$I_n(f;s)$ — —- —- quadrature for hypersingular integrals
$E_n(f;s)$ — —- —- error functional of the quadrature for $I_n(f;s)$
$\gamma(\tau)$ — —- —- distance of a singular point to the mesh point
$\eta(y)$ — —- —- distance of singular point s to the boundary point
$\omega_i(s)$ — —- —- Cote coefficients of $I_n(f;s)$
$\phi_{ik}(t)$ — —- —- special function of $E_n(f;s)$
$a_i(\tau)$ — —- —- special function in $E_n(f;s)$
γ_{0n} — —- —- the regularization factor
$T(h_j)$ — —- —- extrapolation value
$\mathcal{H}_j(t)$ — —- —- errors of

$$\mathcal{H}_j(t) = f(t) - f_C(t) - \sum_{i=0}^{\infty}\sum_{k=i}^{\infty} \frac{(-1)^{i+1} f^{(k+1)}(s)}{h(i+1)!} \frac{M_{ik}^j(t,s)}{(k-i)!}, \quad t \in (t_{j-1}, t_j),$$

subinterval $[t_m, t_{m+1}]$.

Author contributions

Qian Ge performed the data analysis; Jin Li performed the formal analysis and wrote the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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