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Research article

Kink soliton phenomena of fractional conformable Kairat equations

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Abstract: This paper presents a new scientific method to obtain the precise soliton solutions of the fractional conformable Kairat-II (K-IIE) and Kairat-X (K-XE) equations using the Riccati-Bernoulli sub-ODE technique and the Bäcklund transformation. These methods seem most effective compared to the previously used methods as they demonstrate novelty in intensity and potential application. The solutions attained in form of trigonometric, hyperbolic, and rational forms can be used in areas of nonlinear optics, ferromagnetic dynamics, photonic crystals, and optical fibers theory. The most important findings are displayed in simple $2D$ graphs in order to illustrate the nature of these solutions. The use of the flexible fractional derivatives shows that the models are integrated and provides opportunities for studying differential geometry and curve equivalence. Similarly, the ease of the methods underscores applications in other nonlinear partial differential equations, affirming their versatility and importance for the upper level courses.

Keywords: Kairat-II equation (K-IIE); Kairat-X equation (K-XE); Bäcklund transformation; nonlinear differential equations; exact solutions
Mathematics Subject Classification: 34G20, 35A20, 35A22, 35R11

1. Introduction

Fractional partial differential equations (PDEs), which include both spatial and temporal fractional derivatives, play a significant role in many scientific disciplines and are now a key research area [1, 2]. The search for exact solutions is currently a common focus of the scientific research dedicated to PDEs. Several analytical approaches have been developed in order to identify the exact solutions of fractional PDEs, namely the exp-function method [3], (G'/G)-expansion method [4], Kudryashov method [5], first integral method [6], and many others [7, 8].

Applications of solitary theory can be found in the fields of plasma physics, engineering, fluid mechanics, nonlinear optics, biology, economics, chemistry, and beyond [9, 10]. The amazing features of solitons helped develop soliton theory as one of the core theories, particularly their ability to remain intact and undisturbed after interacting with other solitons [11–13]. Solitons assume different forms, each defined by specific individual features like lump, dark, bright, periodic, dark-bright, hump, loop, grey, black, kink, rogue, cuspon, compacton, dromion, anti-kink, and peregrine solitons, justifying the diversity within this phenomena [14, 15]. The simplicity of single partial derivatives is what makes linear evolution equations the preferred mathematical tools for modeling physical phenomena. In mathematical physics, finding exact solutions for the solitary wave problem in nonlinear evolution equations has been a research focus [16, 17]. Modern age computer algebra systems have boosted our capability to handle these complicated equations, thus researchers can now solve physical puzzles with more certainty [18–20]. Recent decades have been marked by new methods of deriving multiple exact solutions to non-linear differential equations. The homogeneous balance method [21], the tanh function method [22], the hyperbolic function method [23], the rational expansion method [24], the sine-cosine method [25] the Jacobi elliptic function method [26], and the exponential function method [27] are among the main approaches. Synonymous for their powerfulness and non-specificity, these techniques undoubtedly constitute a powerful arsenal that tackles various nonlinear issues with great accuracy [28–30].

The K-IIE and K-XE are used for solving the dynamics of optical solitons in fibers and predicting the role of dispersion and nonlinearity as well as perturbations to describe the behavior of optical pulses. These equations provide a theoretical foundation for studying soliton beams, while solutions for trigonometric, hyperbolic, and rational functions indicate periodic oscillations and localized energy packets. Some of their uses are to establish pulse steadiness for appropriate signal transmission via long distance, controlling nonlinear waves in photonic systems, and studying multi-dimensional soliton interconnection. Recent studies have employed different techniques for obtaining soliton solution of Kairat equations, playing a significant role in the analysis of the dynamics of optical pulses in fiber optics. For instance, in [31], authors presented the Riccati modified extended simple equation Method for deriving soliton solutions for the fractional conformable (K-IIE and K-XE) equations. These solutions of trigonometric, hyperbolic, and rational functions showed feasibility for usage in nonlinear optics and photonic systems. [32] applied the improved F-expansion technique to construct various solitary wave solutions of the K-XE, including kink, dark-bright, periodic, and peakon solitons. These works show that analytic tools can provide insights into soliton behavior and its significance in optical networks.

The elliptic, rational, and solitary wave solutions obtained using Hirota's bilinear method have also contributed to crucial area of the study of nonlinear wave phenomena [33]. On this basis, Tipu et al. extended the direct algebraic method and named it extended direct algebraic method (EDAM) to derive specific solutions to the K-XE, for illustrating the use of EDAM for more complicated optical systems [34]. However, versatile approaches with greater than single-wave pulse characteristics are still required to unravel pulse dynamics. The Riccati-Bernoulli method with the Bäcklund transformation technique can be applied in physics, fluid dynamics, biological systems, chemistry, and optical communication [35–37]. This method not only reduces the solution domain of complex nonlinear equations but also has robustness over simple multi-wave solutions. Due to the potential of capturing and analyzing the behavior of multi-soliton encounters, it is a functional asset in enhancing knowledge of optical fiber networks. Moreover, this integration is timely to respond to emerging issues in nonlinear wave theory and provide a starting point for subsequent investigations in this field.

The goal of this work is to introduce of a novel approach, the Riccati-Bernoulli differential sub-ODE method, to solve NPDEs of optical solitons with Eqs (1.1) and (1.2):

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right) - 2\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial^2 f}{\partial x^2}\right) - 4\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial t}\left(\frac{\partial^3 f}{\partial x^3}\right) = 0, \tag{1.1}$$

$$\left(\frac{\partial^2 f}{\partial t^2}\right) - 3\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial t}\left(\frac{\partial^3 f}{\partial x^3}\right) = 0, \tag{1.2}$$

where, $f = f(x, t)$. Solitary solutions to nonlinear equations like the K-XE have been solved several times using techniques such as the inverse scattering transform for lump-kink, kink-breather, and multi-shock forms [38].

Combining this method with the Bäcklund transformation, we showcase the simplification strategy of transforming PDEs into ODEs to comprehensively analyze intricate optical pulse shapes. Our approach provides new perspectives on the relationship between soliton solutions and the transmission characteristics of optical fibers.

Extending these works, this paper derives soliton solutions for the K-XE and K-IIE models using a more general and effective method. This method expands the class of soliton solutions to kink optical solitons, and offers a better understanding of these models for a range of applications to nonlinear and optical sciences. This study validates the theoretical and practical applications of these equations to enhance nonlinear optics as well as optical communication fields. The proposed methodology utilizes Bäcklund transformation for transforming fractional differential equations into ordinary differential equations. As a next step, we will use Riccati-Bernoulli sub-ODEs to solve the NODEs in a series form and have them transformed to an algebraic system of equations. By solving this set of equations, we derive soliton solutions organized into three distinct families: Rational, hyperbolic, and trigonometric. We illustrate the proof of concept by evaluating the capabilities of our proposed approach via the emergence of newly optimized optical soliton arrays for the K-IIE and K-XE models as validated in [31, 32].

2. Riccati-Bernoulli sub-ODE technique

The Riccati-Bernoulli sub-ODE is a rather robust technique to analyze complicated nonlinear PDEs. By converting these PDEs into less complex ODEs, the method proposed here actually increases the problem dimension in terms of the traveling wave transformations. To elucidate the core principles underlying the Riccati-Bernoulli sub-ODE method, we can analyze a specific PDE involving both time and spatial variables, represented mathematically as follows:

$$G_1\left(g, D_t^\alpha(g), D_{y_1}^\alpha(g), D_{y_2}^\alpha(g), gD_{y_1}^\alpha(g), \ldots\right) = 0, \qquad 0 < \alpha \leq 1, \tag{2.1}$$

where $g = g(t, y_1, y_2, y_3, \ldots, y_r)$ is an unknown function with $(r + 1)$ variables. A variable form wave transformation denoted as $f(\xi) = g(t, y_1, y_2, y_3, \ldots, y_r)$ is conducted, where ξ can be expressed in various forms. This transformation allows us to convert the given Eq (2.1) into an ordinary differential equation represented as

$$G_2\left(f(\xi), f'(\xi), f''(\xi), f(\xi)f'(\xi), \ldots\right) = 0. \tag{2.2}$$

Following that, we utilize the solution of the Riccati-Bernaulli approach to propose the subsequent series-based solution to the nonlinear ordinary differential equation (2.2).

$$f(\xi) = \sum_{i=-m}^{m} c_i \phi_i(\xi). \tag{2.3}$$

The constant c_i is to be determined such that it satisfies the conditions $c_m \neq 0$ or $c_{-m} \neq 0$. Meanwhile, the function $\phi(\xi)$ is derived from the subsequent Bäcklund transformation. The positive integer (m) can easily be balanced by equating the coefficient of the highest order derivative term with the coefficient of the nonlinear terms in Eq (2.2) [39].

$$\phi(\xi) = \frac{-ZB + A\Theta(\xi)}{A + B\Theta(\xi)}, \tag{2.4}$$

where constants Z, A, and B are fixed parameters with $B \neq 0$. And the function $\Theta(\xi)$ fullfils the Riccati equation:

$$\frac{d\Theta}{d\xi} = Z + \Theta(\xi)^2. \tag{2.5}$$

According to established literature [40], Eq (2.5) exhibits the following solutions:

$$(i) \quad \text{If} \quad Z < 0, \quad \text{then} \quad \Theta(\xi) = -\sqrt{-Z}\tanh(\sqrt{-Z}\xi). \tag{2.6}$$

$$(ii) \quad \text{If} \quad Z > 0, \quad \text{then} \quad \Theta(\xi) = \sqrt{Z}\tan(\sqrt{Z}\xi). \tag{2.7}$$

$$(iii) \quad \text{If} \quad Z = 0, \quad \text{then} \quad \Theta(\xi) = \frac{-1}{\xi}. \tag{2.8}$$

Substituting Eq (2.3) with Eq (2.5) into Eq (2.2), we combine all terms with the same power of $\Theta(\xi)$ and set them to zero. This approach ends up with a set of algebraic equations, resolvable through Maple, revealing the values of (c_i) and (Z). Consequently, we unveil the precise solutions of the solitary wave for Eq (2.1).

Further, the operator integrating α-derivatives of powers agrees exactly to the idea of conformable fractional derivatives. The main advantage of conformable fractional derivatives is, in its more compact and simplified way, to differentiate the fractional order differentiation, which gives way to the theoretical development. This refinement makes them not only easily adaptable towards the theoretical backgrounds but also structurally applicable to data and algorithm processing. With the fractional differentiation operator behaving in accordance with the fundamental mathematical rules, such as chain rule and Taylors series expansion, its use in mathematical modeling frameworks becomes quite straightforward. Consequently, concordant fractional derivatives have been widely utilized in recent years due to their effectiveness in obtaining exact solutions for conformable fractional nonlinear partial differential equations using multiple Riccati-Bernoulli sub-ODE methods and other analytical techniques. Such an application proves the effectiveness of fractional derivatives in solving different nonlinear problems in various fields of science [41]. The definition of the fractional derivative (2.9) emphasizing its dependence on the parameter α and its significance in describing the behavior of the fractional operator within the framework of conformable fractional nonlinear partial differential equations:

$$D_{\Theta}^{\alpha}Z(\Theta) = \lim_{l\to 0}\frac{Z(l(\Theta)^{1-\alpha} - Z(\theta))}{l}, \quad 0 < \alpha \leq 1, \tag{2.9}$$

with the following properties

$$\begin{cases} D_{\Theta}^{\alpha}\Theta^{m} = m\Theta^{m-\alpha}, \\ D_{\Theta}^{\alpha}\left(m_1\eta(\Theta) \pm m_2 t(\Theta)\right) = m_1 D_{\Theta}^{\alpha}(\eta(\Theta)) \pm m_2 D_{\Theta}^{\alpha}(t(\Theta)), \\ D_{\Theta}^{\alpha}\left[f \circ g\right] = \Theta^{1-\alpha} g\left(\Theta\right) D_{\Theta}^{\alpha} f\left(g(\Theta)\right). \end{cases} \tag{2.10}$$

3. Fractional K-IIE and KE equations

3.1. Fractional Kairat-II equation (K-IIE)

Let us contemplate the fractional K-IIE

$$D_x^{\alpha}D_t^{\alpha}f - 2D_t^{\alpha}f \cdot D_x^{2\alpha}f - 4D_x^{\alpha}f \cdot D_x^{\alpha}D_t^{\alpha}f + D_x^{3\alpha}D_t^{\alpha}f = 0, \qquad 0 < \alpha \leq 1. \tag{3.1}$$

The function $f = f(x,t)$, representing a real wave profile, is governed by Eq (3.1), which belongs to the class of integrable equations. This equation serves as a tool to elucidate the differential geometry of curves and explore equivalence aspects. Consider the following complex wave transformation:

$$f(x,t) = F(\xi), \quad where \quad \xi = \frac{ax^{\alpha}}{\alpha} + \frac{bt^{\alpha}}{\alpha} + \vartheta, \tag{3.2}$$

where a, b, and ϑ are constants. By employing the aforementioned complex wave transformation, Eq (3.1) undergoes a transformation, resulting in the following nonlinear ordinary differential equation (ODE):

$$abF''(\xi) - 2ba^2F'(\xi)F''(\xi) - 4ba^2F'(\xi)F''(\xi) + a^3bF^{iv}(\xi) = 0. \tag{3.3}$$

Re-arranging Eq (3.3) yields

$$ab\left(F''(\xi) - 2aF'(\xi)F''(\xi) - 4aF'(\xi)F''(\xi) + a^2F^{iv}(\xi)\right) = 0. \tag{3.4}$$

Thus, we have

$$F''(\xi) - 6aF'(\xi)F''(\xi) + a^2F^{iv}(\xi) = 0. \tag{3.5}$$

Integration of Eq (3.5), with the constant of integration held at zero,

$$\int F''(\xi)\,d\xi - 6a\int F'(\xi)F''(\xi)\,d\xi + a^2\int F^{(iv)}(\xi)\,d\xi = 0, \tag{3.6}$$

we obtain

$$F'(\xi) - 6a\left(\frac{F'^2(\xi)}{2}\right) + a^2F'''(\xi) = 0. \tag{3.7}$$

That is

$$F'(\xi) - 3aF'^2(\xi) + a^2F'''(\xi) = 0. \tag{3.8}$$

Substituting Eq (2.5), along with Eq (2.9) and Eq (2.3) into Eq (3.8) and collecting the coefficients of $\phi(\xi)$. By solving this system of algebraic equations using Maple, the following results are obtained. For example,

Case 1.

$$c_0 = c_0, c_1 = 2\,a, c_{-1} = -1/8\,a^{-1}, A = A, B = B, a = a, Z = 1/16\,a^{-2}. \tag{3.9}$$

Case 2.

$$c_0 = c_0, c_1 = 2\,a, c_{-1} = 0, A = 0, B = B, a = a, Z = 1/4\,a^{-2}. \tag{3.10}$$

Case 3.

$$c_0 = c_0, c_1 = 0, c_{-1} = -1/2\,a^{-1}, A = 0, B = B, a = a, Z = 1/4\,a^{-2}. \tag{3.11}$$

Solution Set 1. Under the assumption of Case 1, we obtain a diverse array of optical soliton solutions when $Z < 0$ for the fractional K-IIE:

$$\begin{aligned} F_1(x,t) = &-1/8\,\frac{\left(A - 1/4\,B\sqrt{-a^{-2}}\tanh\left(1/4\,\sqrt{-a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)}{a\left(-1/16\,\frac{B}{a^2} - 1/4\,A\sqrt{-a^{-2}}\tanh\left(1/4\,\sqrt{-a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)} + c_0 \\ &+ \frac{2\,a\left(-1/16\,\frac{B}{a^2} - 1/4\,A\sqrt{-a^{-2}}\tanh\left(1/4\,\sqrt{-a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)}{\left(A - 1/4\,B\sqrt{-a^{-2}}\tanh\left(1/4\,\sqrt{-a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)}, \end{aligned} \tag{3.12}$$

or

$$\begin{aligned} F_2(x,t) = &-1/8\,\frac{\left(A - 1/4\,B\sqrt{-a^{-2}}\coth\left(1/4\,\sqrt{-a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)}{a\left(-1/16\,\frac{B}{a^2} - 1/4\,A\sqrt{-a^{-2}}\coth\left(1/4\,\sqrt{-a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)} + c_0 \\ &+ \frac{2\,a\left(-1/16\,\frac{B}{a^2} - 1/4\,A\sqrt{-a^{-2}}\coth\left(1/4\,\sqrt{-a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)}{\left(A - 1/4\,B\sqrt{-a^{-2}}\coth\left(1/4\,\sqrt{-a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)}. \end{aligned} \tag{3.13}$$

Solution Set 2. Under the assumption of Case 1, we obtain a diverse array of optical soliton solutions when $Z > 0$ for the fractional K-IIE:

$$\begin{aligned} F_3(x,t) = &-1/8\,\frac{\left(A + 1/4\,B\,\sqrt{a^{-2}}\tan\left(1/4\,\sqrt{a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)}{a\left(-1/16\,\frac{B}{a^2}+1/4\,A\,\sqrt{a^{-2}}\tan\left(1/4\,\sqrt{a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)} + c_0 \\ &+ \frac{2\,a\left(-1/16\,\frac{B}{a^2}+1/4\,A\,\sqrt{a^{-2}}\tan\left(1/4\,\sqrt{a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)}{\left(A + 1/4\,B\,\sqrt{a^{-2}}\tan\left(1/4\,\sqrt{a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)}, \end{aligned} \tag{3.14}$$

or

$$\begin{aligned} F_4(x,t) = &-1/8\,\frac{\left(A - 1/4\,B\,\sqrt{a^{-2}}\cot\left(1/4\,\sqrt{a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)}{a\left(-1/16\,\frac{B}{a^2}-1/4\,A\,\sqrt{a^{-2}}\cot\left(1/4\,\sqrt{a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)} + c_0 \\ &+ \frac{2\,a\left(-1/16\,\frac{B}{a^2}-1/4\,A\,\sqrt{a^{-2}}\cot\left(1/4\,\sqrt{a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)}{\left(A - 1/4\,B\,\sqrt{a^{-2}}\cot\left(1/4\,\sqrt{a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)}. \end{aligned} \tag{3.15}$$

Solution Set 3. Under the assumption of Case 1, we obtain a diverse array of optical soliton solutions when $Z = 0$ for the fractional K-IIE:

$$F_5(x,t) = -1/8\,\frac{\left(A - B\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)^{-1}\right)}{a\left(-1/16\,\frac{B}{a^2} - A\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)^{-1}\right)} + c_0 + \frac{2\,a\left(-1/16\,\frac{B}{a^2} - A\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)^{-1}\right)}{\left(A - B\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)^{-1}\right)}. \tag{3.16}$$

Solution Set 4. Under the assumption of Case 2, we obtain a diverse array of optical soliton solutions when $Z < 0$ for the fractional K-IIE:

$$F_6(x,t) = c_0 + a^{-1}\frac{1}{\sqrt{-a^{-2}}}\left(\tanh\left(1/2\,\sqrt{-a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)^{-1}, \tag{3.17}$$

or

$$F_7(x,t) = c_0 + a^{-1}\frac{1}{\sqrt{-a^{-2}}}\left(\coth\left(1/2\,\sqrt{-a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)^{-1}. \tag{3.18}$$

Solution Set 5. Under the assumption of Case 2, we obtain a diverse array of optical soliton solutions when $Z > 0$ for the fractional K-IIE:

$$F_8(x,t) = c_0 - a^{-1}\frac{1}{\sqrt{a^{-2}}}\left(\tan\left(1/2\,\sqrt{a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)^{-1}, \tag{3.19}$$

or

$$F_9(x,t) = c_0 + a^{-1}\frac{1}{\sqrt{a^{-2}}}\left(\cot\left(1/2\,\sqrt{a^{-2}}\left(\frac{ax^\alpha}{\alpha}+\frac{bt^\alpha}{\alpha}+\vartheta\right)\right)\right)^{-1}. \tag{3.20}$$

Solution Set 6. Under the assumption of Case 2, we obtain a diverse array of optical soliton solutions when $Z = 0$ for the fractional K-IIE:

$$F_{10}(x,t) = c_0 + 1/2\left(\frac{ax^{\alpha}}{\alpha} + \frac{bt^{\alpha}}{\alpha} + \vartheta\right)a^{-1}. \tag{3.21}$$

Solution Set 7. Under the assumption of Case 3, we obtain a diverse array of optical soliton solutions when $Z < 0$ for the fractional K-IIE:

$$F_{11}(x,t) = -a\sqrt{-a^{-2}}\tanh\left(1/2\sqrt{-a^{-2}}\left(\frac{ax^{\alpha}}{\alpha} + \frac{bt^{\alpha}}{\alpha} + \vartheta\right)\right) + c_0, \tag{3.22}$$

or

$$F_{12}(x,t) = -a\sqrt{-a^{-2}}\coth\left(1/2\sqrt{-a^{-2}}\left(\frac{ax^{\alpha}}{\alpha} + \frac{bt^{\alpha}}{\alpha} + \vartheta\right)\right) + c_0. \tag{3.23}$$

Solution Set 8. Under the assumption of Case 3, we obtain a diverse array of optical soliton solutions when $Z > 0$ for the fractional K-IIE:

$$F_{13}(x,t) = a\sqrt{a^{-2}}\tan\left(1/2\sqrt{a^{-2}}\left(\frac{ax^{\alpha}}{\alpha} + \frac{bt^{\alpha}}{\alpha} + \vartheta\right)\right) + c_0, \tag{3.24}$$

or

$$F_{14}(x,t) = -a\sqrt{a^{-2}}\cot\left(1/2\sqrt{a^{-2}}\left(\frac{ax^{\alpha}}{\alpha} + \frac{bt^{\alpha}}{\alpha} + \vartheta\right)\right) + c_0. \tag{3.25}$$

Solution Set 9. Under the assumption of Case 3, we obtain a diverse array of optical soliton solutions when $Z = 0$ for the fractional K-IIE:

$$F_{15}(x,t) = -2\,a\left(\frac{ax^{\alpha}}{\alpha} + \frac{bt^{\alpha}}{\alpha} + \vartheta\right)^{-1} + c_0. \tag{3.26}$$

3.2. Fractional Kairat-X equation (K-XE)

Let us contemplate the fractional K-XE

$$D_t^{2\alpha} f - 3D_x^{\alpha}\left(D_t^{\alpha} f \cdot D_x^{\alpha} f\right) + D_x^{3\alpha} D_t^{\alpha} f = 0, \qquad 0 < \alpha \leq 1. \tag{3.27}$$

Utilizing the complex wave transformation described in Eqs (3.2) and (3.27) results in a change, yielding the subsequent nonlinear ordinary differential equation (ODE)

$$b^2F''(\xi) - 3a^2bF'(\xi)F''(\xi) - 3a^2bF'(\xi)F''(\xi) + a^3bF^{iv}(\xi) = 0. \tag{3.28}$$

Re-arranging Eq (3.28) gives

$$b\left(bF''(\xi) - 6a^2F'(\xi)F''(\xi) + a^3F^{iv}(\xi)\right) = 0. \tag{3.29}$$

Thus,

$$bF''(\xi) - 6a^2F'(\xi)F''(\xi) + a^3F^{iv}(\xi) = 0. \tag{3.30}$$

Integration of Eq (3.30) with the constant of integration held at zero yields,

$$bF'(\xi) - 3a^2F'^2(\xi) + a^3F'''(\xi) = 0. \tag{3.31}$$

Substituting Eq (2.5), along with Eqs (2.9) and (2.3) into Eq (3.31) and collecting the coefficients of $\phi(\xi)$. By solving this system of algebraic equations using Maple, the following results are obtained for different sets of model parameters. For example,

Case 4.

$$c_0 = c_0, c_1 = 2\,a, c_{-1} = -1/8\,\frac{b}{a^2}, B = B, Z = 1/16\,\frac{b}{a^3}. \tag{3.32}$$

Case 5.

$$c_0 = c_0, c_1 = 2\,a, c_{-1} = 0, B = B, Z = 1/4\,\frac{b}{a^3}. \tag{3.33}$$

Case 6.

$$c_0 = c_0, c_1 = 0, c_{-1} = -1/2\,\frac{b}{a^2}, B = B, Z = 1/4\,\frac{b}{a^3}. \tag{3.34}$$

Solution Set 10. Under the assumption of Case 4, we obtain a diverse array of optical soliton solutions when $Z < 0$ for the fractional K-XE:

$$\begin{aligned} F_{16}(x,t) = &-1/8\,\frac{b\left(A - 1/4\,B\sqrt{-\frac{b}{a^3}}\tanh\left(1/4\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{a\left(-1/16\,\frac{bB}{a^3} - 1/4\,A\sqrt{-\frac{b}{a^3}}\tanh\left(1/4\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)} + c_0 \\ &+ \frac{2\,a\left(-1/16\,\frac{bB}{a^3} - 1/4\,A\sqrt{-\frac{b}{a^3}}\tanh\left(1/4\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{\left(A - 1/4\,B\sqrt{-\frac{b}{a^3}}\tanh\left(1/4\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}, \end{aligned} \tag{3.35}$$

or

$$\begin{aligned} F_{17}(x,t) = &-1/8\,\frac{b\left(A - 1/4\,B\sqrt{-\frac{b}{a^3}}\coth\left(1/4\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{a\left(-1/16\,\frac{bB}{a^3} - 1/4\,A\sqrt{-\frac{b}{a^3}}\coth\left(1/4\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)} + c_0 \\ &+ \frac{2\,a\left(-1/16\,\frac{bB}{a^3} - 1/4\,A\sqrt{-\frac{b}{a^3}}\coth\left(1/4\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{\left(A - 1/4\,B\sqrt{-\frac{b}{a^3}}\coth\left(1/4\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}. \end{aligned} \tag{3.36}$$

Solution Set 11. Under the assumption of Case 4, we obtain a diverse array of optical soliton solutions when $Z > 0$ for the fractional K-XE:

$$F_{18}(x,t) = -1/8\,\frac{b\left(A + 1/4\,B\sqrt{\frac{b}{a^3}}\tan\left(1/4\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{a\left(-1/16\,\frac{bB}{a^3} + 1/4\,A\sqrt{\frac{b}{a^3}}\tan\left(1/4\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)} + c_0 + \frac{2\,a\left(-1/16\,\frac{bB}{a^3} + 1/4\,A\sqrt{\frac{b}{a^3}}\tan\left(1/4\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{\left(A + 1/4\,B\sqrt{\frac{b}{a^3}}\tan\left(1/4\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}, \tag{3.37}$$

or

$$F_{19}(x,t) = -1/8\,\frac{b\left(A - 1/4\,B\sqrt{\frac{b}{a^3}}\cot\left(1/4\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{a\left(-1/16\,\frac{bB}{a^3} - 1/4\,A\sqrt{\frac{b}{a^3}}\cot\left(1/4\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)} + c_0 + \frac{2\,a\left(-1/16\,\frac{bB}{a^3} - 1/4\,A\sqrt{\frac{b}{a^3}}\cot\left(1/4\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{\left(A - 1/4\,B\sqrt{\frac{b}{a^3}}\cot\left(1/4\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}. \tag{3.38}$$

Solution Set 12. Under the assumption of Case 4, we obtain a diverse array of optical soliton solutions when $Z = 0$ for the fractional K-XE:

$$F_{20}(x,t) = -1/8\,\frac{b\left(A - B\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)^{-1}\right)}{a\left(-1/16\,\frac{bB}{a^3} - A\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)^{-1}\right)} + c_0 + \frac{2\,a\left(-1/16\,\frac{bB}{a^3} - A\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)^{-1}\right)}{\left(A - B\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)^{-1}\right)}. \tag{3.39}$$

Solution Set 13. Under the assumption of Case 5, we obtain a diverse array of optical soliton solutions when $Z < 0$ for the fractional K-XE:

$$F_{21}(x,t) = c_0 + \frac{2\,a\left(-1/4\,\frac{bB}{a^3} - 1/2\,A\sqrt{-\frac{b}{a^3}}\tanh\left(1/2\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{\left(A - 1/2\,B\sqrt{-\frac{b}{a^3}}\tanh\left(1/2\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}, \tag{3.40}$$

or

$$F_{22}(x,t) = c_0 + \frac{2\,a\left(-1/4\,\frac{bB}{a^3} - 1/2\,A\sqrt{-\frac{b}{a^3}}\coth\left(1/2\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{\left(A - 1/2\,B\sqrt{-\frac{b}{a^3}}\coth\left(1/2\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}. \tag{3.41}$$

Solution Set 14. Under the assumption of Case 5, we obtain a diverse array of optical soliton solutions when $Z > 0$ for the fractional K-XE:

$$F_{23}(x,t) = c_0 + \frac{2\,a\left(-1/4\,\frac{bB}{a^3} + 1/2\,A\sqrt{\frac{b}{a^3}}\tan\left(1/2\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{\left(A + 1/2\,B\sqrt{\frac{b}{a^3}}\tan\left(1/2\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}, \tag{3.42}$$

or

$$F_{24}(x,t) = c_0 + \frac{2\,a\left(-1/4\,\frac{bB}{a^3} - 1/2\,A\,\sqrt{\frac{b}{a^3}}\cot\left(1/2\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{\left(A - 1/2\,B\,\sqrt{\frac{b}{a^3}}\cot\left(1/2\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}. \tag{3.43}$$

Solution Set 15. Under the assumption of Case 5, we obtain a diverse array of optical soliton solutions when $Z = 0$ for the fractional K-XE:

$$F_{25}(x,t) = c_0 + 2\,a\left(-1/4\,\frac{bB}{a^3} - A\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)^{-1}\right)\left(A - B\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)^{-1}\right)^{-1}. \tag{3.44}$$

Solution Set 16. Under the assumption of Case 6, we obtain a diverse array of optical soliton solutions when $Z < 0$ for the fractional K-XE:

$$F_{26}(x,t) = -1/2\,\frac{b\left(A - 1/2\,B\,\sqrt{-\frac{b}{a^3}}\tanh\left(1/2\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{a\left(-1/4\,\frac{bB}{a^3} - 1/2\,A\,\sqrt{-\frac{b}{a^3}}\tanh\left(1/2\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)} + c_0, \tag{3.45}$$

or

$$F_{27}(x,t) = -1/2\,\frac{b\left(A - 1/2\,B\,\sqrt{-\frac{b}{a^3}}\coth\left(1/2\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{a\left(-1/4\,\frac{bB}{a^3} - 1/2\,A\,\sqrt{-\frac{b}{a^3}}\coth\left(1/2\,\sqrt{-\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)} + c_0. \tag{3.46}$$

Solution Set 17. Under the assumption of Case 6, we obtain a diverse array of optical soliton solutions when $Z > 0$ for the fractional K-XE:

$$F_{28}(x,t) = -1/2\,\frac{b\left(A + 1/2\,B\,\sqrt{\frac{b}{a^3}}\tan\left(1/2\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{a\left(-1/4\,\frac{bB}{a^3} + 1/2\,A\,\sqrt{\frac{b}{a^3}}\tan\left(1/2\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)} + c_0, \tag{3.47}$$

or

$$F_{29}(x,t) = -1/2\,\frac{b\left(A - 1/2\,B\,\sqrt{\frac{b}{a^3}}\cot\left(1/2\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)}{a\left(-1/4\,\frac{bB}{a^3} - 1/2\,A\,\sqrt{\frac{b}{a^3}}\cot\left(1/2\,\sqrt{\frac{b}{a^3}}\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)\right)\right)} + c_0. \tag{3.48}$$

Solution Set 18. Under the assumption of Case 6, we obtain a diverse array of optical soliton solutions when $Z = 0$ for the fractional K-XE:

$$F_{30}(x,t) = -1/2\,b\left(A - B\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)^{-1}\right)a^{-2}\left(-1/4\,\frac{bB}{a^3} - A\left(\frac{ax^\alpha}{\alpha} + \frac{bt^\alpha}{\alpha} + \vartheta\right)^{-1}\right)^{-1} + c_0. \tag{3.49}$$

4. Results and discussion

4.1. Graphical analysis of kink solitons

The Riccati-Bernoulli sub-ODE method successfully visualizes the kink wave structures in the fractional K-IIE and K-XE systems. The 2D and 3D plots show how fractional order (α) influences and how the time (t) plays out the systems. Integer value of α provide details of wave forms in the 3D plots, whereas the 2D plot gives localized information. The derived hyperbolic, trigonometric, and rational soliton solutions help in explaining nonlinear optical systems. These solutions show in optical fibers, photonic crystals, and wave propagation and clearly explain that the above approach can used to study fractional models, nonlinear optics, and other relevant fields.

4.2. Implications for fiber communication

The solutions proposed in this work show the specific possibility of using kink solitons in performing functions and controlling waves, especially in fibers. From the analysis of this work, it is found that these solitons have a stable structure in both shape, temporal as well as in velocity fields over a long range and hence are different from an ordinary soliton. This stability makes them as some of the most suitable in optical communication. Moreover, the dispersion of these waves is considerably lower than that of high-frequency waves in conventional systems, which makes them ideal for the next-generation optical signal processing.

4.3. Practical application and future potential

The kink solitons studied in this paper have enhanced the understanding of wave transmission phenomena. The different mechanical characteristics of these solitons lead to increased stability and structural support of optical communication networks. This work provides robust groundwork for enhancing optical technologies that contribute to enhanced speed communication and accurate signal management. Therefore, these solutions hold the potential of changing the face of nonlinear optics by overcoming dispersion-related hurdles and exploring the characteristics of kink solitons.

The anti-kink soliton solutions in 2D and 3D presented in Figure 1 concern the impact of the fractional-order parameter (α). From the 2D plots, it can be found that the overall profile of the anti-kink soliton is still preserved for all cases, but there is a growing sensitivity to the horizontal upper position as (α) increases, which corresponds to the fractional dispersion effects or higher order nonlinearity. On the other hand, integer-order systems depicted in the 3D plots are classical or stable anti-kink solutions with insignificant memory or fractional characteristics. They propose potential uses in nonlinear optics, for instance in determining refractive index profiles, pulse evolution in fractional nonlocal medium, domain wall dynamics in ferromagnetic systems, and active localized modes in photonic crystals. Hence, we have found that (α) is a crucial factor for designing advanced soliton-based systems, being a controlling parameter.

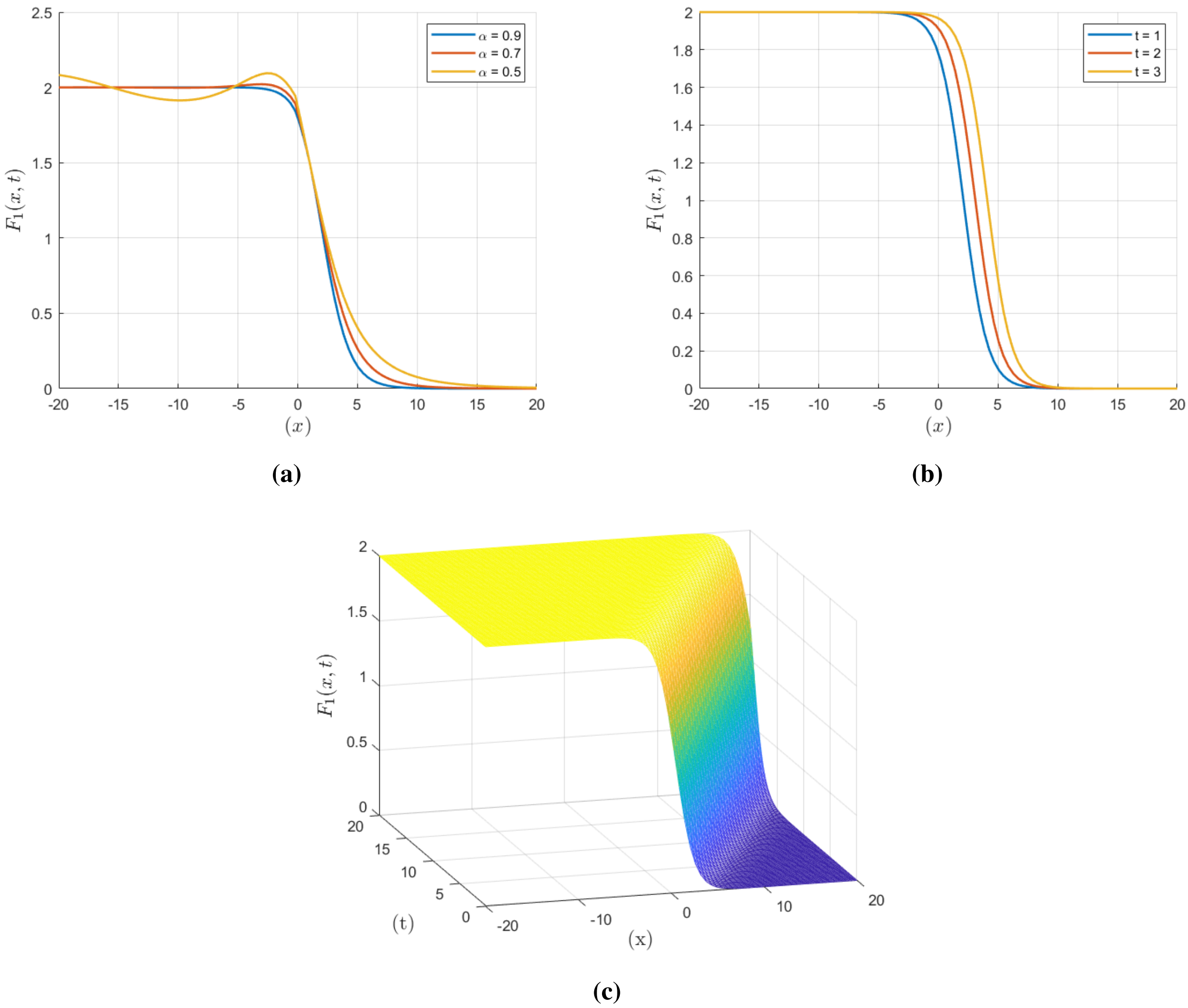


Figure 1. Impact of fractional and temporal parameters, along with $3D$ variation for $\alpha = 1$, $A = 1.2$, $B = 1.1$, $a = 1$, and $b = -1$ on the dynamics of the solution $F_1(x, t)$, highlighting key variations.

As seen in the profile of Figure 2, variations are smoother, thus pointing to a higher ability to discern fractional effects. However, with respect to these perturbations, the temporal evolution continually plots the anti-kink in a smooth and stable manner, further asserting the temporal stability of the soliton structure. These observations show that although the fractional dynamics alter the spatial profile by small fractions, the temporal profile is not significantly affected. This makes these solutions particularly useful in applications where temporal solitons are important, such as in stable signal transmission through optical fibers in nonlinear optics, ferromagnetic systems, and photonic crystals. In these areas, the soliton dynamics have to remain manageable over long times scales; therefore, Figure 2 provides a convenient visualization of the stability and possible applications of fractional soliton solutions.

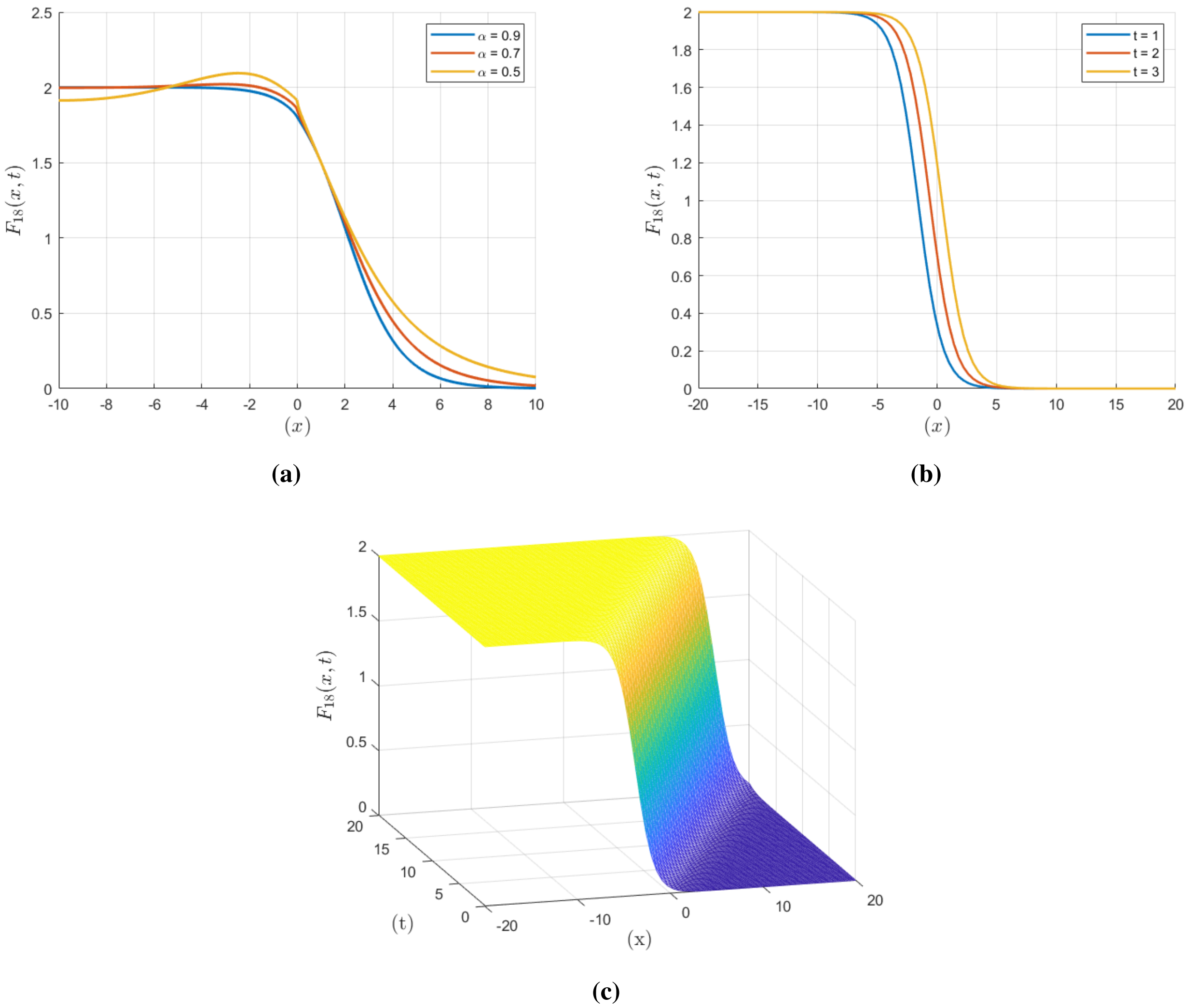
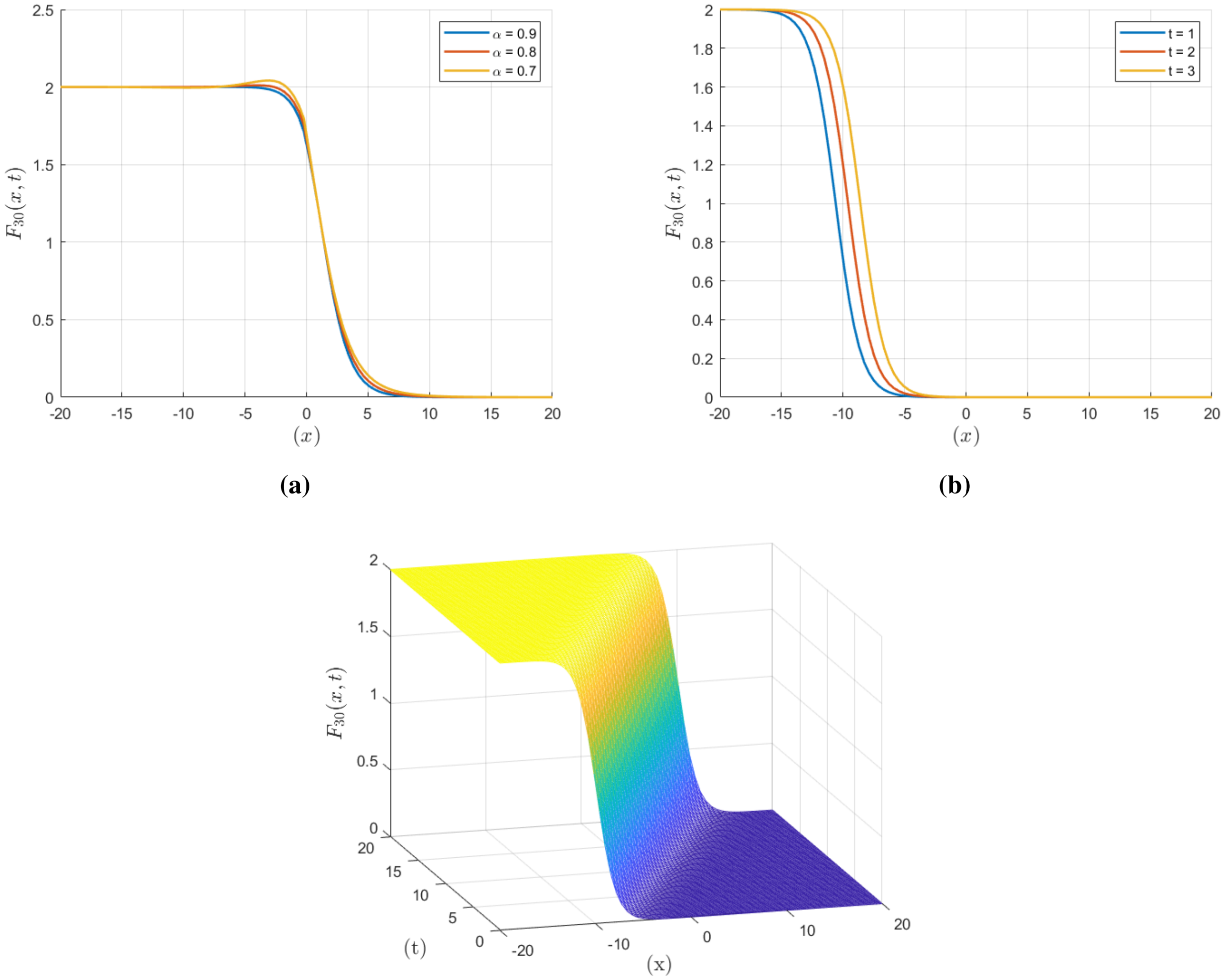


Figure 2. Impact of fractional and temporal parameters, along with $3D$ variation for $\alpha = 1$, $A = 1.1$, $B = 1.2$, $a = 1$, and $b = -1$ on the dynamics of the solution $F_{18}(x, t)$, highlighting key variations.

Figure 3 displays the improved oscillations in the spatial anti-kink profiles, which are created because of either more powerful fractional domination or more nonlinearity. However, superposition of these oscillations with the 3D representation and temporal variations will show that there exist stable anti-kink structures and solutions of the soliton form during the dynamic evolution. Sustaining clear soliton profiles across spatial and temporal scales is fundamental to stable nonlinear pulse transmission, ferromagnetic systems, and photonic crystals applications. In such domains, soliton formation is critical for energy storage and wave propagation along complex processing pathways, as shown by the identical representation in Figure 3 for soliton behavior in nonlinear systems.



(a) **(b)**

(c)

Figure 3. Impact of fractional and temporal parameters, along with $3D$ variation for $\alpha = 1$, $A = 1$, $B = 0.1$, $a = 1$, and $b = -1$ on the dynamics of the solution $F_{30}(x,t)$, highlighting key variations.

From Table 1, it is clear that the present Riccati-Bernoulli sub-ODE method with Bäcklund transformation presents more versatility and solution diversity than the RMESEM method for the K-IIE. While the present method clearly provides solutions in all the cases, presenting hyperbolic and rational forms, the alternative method is limited in this sense and does not give the solutions for $K = 0$. Furthermore, the present method of analysis retains the more detailed features of the solution structure, which is advantageous for addressing nonlinear wave characteristics and for gaining further insights into the K-IIE behavior.

Table 1. Comparison of the fractional K-XE with the F-expansion method [32].

Case	Present Method	F-expansion method
Case I. $Z < 0$ **Case I.** $\Lambda < 0$	$f_{22}(x,t) = \frac{-1/2\, b\left(A - 1/2\, B\sqrt{-\frac{b}{a^3}}\coth\left(1/2\sqrt{-\frac{b}{a^3}}\xi\right)\right)}{a\left(-1/4\,\frac{bB}{a^3} - 1/2\, A\sqrt{-\frac{b}{a^3}}\coth\left(1/2\sqrt{-\frac{b}{a^3}}\xi\right)\right)} + c_0.$	$g_{22}(x,t) = \sqrt[3]{-2}\Lambda^{2/3}\sqrt[3]{\mu}(\sqrt{-\Lambda}\coth(M) - 1).$ $M = \left(\sqrt{-\Lambda}\left(\Theta + \frac{\Gamma(\sigma+1)(\mu t^{\beta} + \kappa x^{\beta})}{\beta}\right)\right)$
Case II. $Z > 0$ **Case II.** $\Lambda > 0$	$f_{24}(x,t) = \frac{-1/2\, b\left(A - 1/2\, B\sqrt{\frac{b}{a^3}}\cot\left(1/2\sqrt{\frac{b}{a^3}}\xi\right)\right)}{a\left(-1/4\,\frac{bB}{a^3} - 1/2\, A\sqrt{\frac{b}{a^3}}\cot\left(1/2\sqrt{\frac{b}{a^3}}\xi\right)\right)} + c_0.$	$g_{24}(x,t) = \sqrt[3]{-2}\Lambda^{2/3}\sqrt[3]{\mu}(\sqrt{-\Lambda}\cot(M) - 1).$ $M = \left(\sqrt{-\Lambda}\left(\Theta + \frac{\Gamma(\sigma+1)(\mu t^{\beta} + \kappa x^{\beta})}{\beta}\right)\right)$
Case III. $Z = 0$ **Case III.** $\Lambda = 0$	$f_{25}(x,t) = \frac{-1/2\, b\left(A - \frac{B}{\xi}\right)}{a\left(-1/4\,\frac{bB}{a^3} - \frac{A}{\xi}\right)} + c_0.$	$g_{25}(x,t) = -\frac{\beta}{\beta\Theta + \Gamma(\sigma+1)(\mu t^{\beta} + \kappa x^{\beta})}.$

With reference to Table 2, it can be concluded that the Riccati-Bernoulli sub-ODE method with the Bäcklund transformation is a more general and broad approach to derive soliton solutions for fractional (K-XE) than the F-expansion method. The present method produces more diversified solutions, trigonometric, hyperbolic, and rational solutions, and is responsive to complex wave features such as kink solitons. On the other hand, the F-expansion method mostly targets single waves such as periodic and peakon solitons. The Riccati-Bernoulli sub-ODE method offers broader opportunities in the investigation of nonlinear wave phenomena in optical and other nonlinear systems due to its increased flexibility compared to other similar methods.

Table 2. Comparison of the fractional K-IIE with the alternative approach, specifically the Riccati modified extended simple equation method [31].

Case	Present Method	Riccati Modified Extended Simple Equation Method
Case I. $Z < 0$ **Case I.** $K < 0$	$f_1(x,t) = -1/8\,\frac{\left(A - 1/4\, B\sqrt{-a^{-2}}\tanh\left(1/4\sqrt{-a^{-2}}\xi\right)\right)a^{-1}}{\left(-1/16\,\frac{B}{a^2} - 1/4\, A\sqrt{-a^{-2}}\tanh\left(1/4\sqrt{-a^{-2}}\xi\right)\right)} +$ $c_0 + \frac{2\, a\left(-1/16\,\frac{B}{a^2} - 1/4\, A\sqrt{-a^{-2}}\tanh\left(1/4\sqrt{-a^{-2}}\xi\right)\right)}{\left(A - 1/4\, B\sqrt{-a^{-2}}\tanh\left(1/4\sqrt{-a^{-2}}\xi\right)\right)}$	$f_{1,1,3}(x,t) = \frac{C_0 - \frac{2\lambda}{\sqrt{-K}}}{\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-K}(\tan(\sqrt{-K}\Lambda) + \sec(\sqrt{-K}\Lambda))}{\nu}\right)}.$
Case II. $Z > 0$ **Case II.** $K > 0$	$f_3(x,t) = -1/8\,\frac{\left(A + 1/4\, B\sqrt{a^{-2}}\tan\left(1/4\sqrt{a^{-2}}\xi\right)\right)}{a^{-1}\left(-1/16\,\frac{B}{a^2} + 1/4\, A\sqrt{a^{-2}}\tan\left(1/4\sqrt{a^{-2}}\xi\right)\right)} +$ $c_0 + \frac{2\, a\left(-1/16\,\frac{B}{a^2} + 1/4\, A\sqrt{a^{-2}}\tan\left(1/4\sqrt{a^{-2}}\xi\right)\right)}{\left(A + 1/4\, B\sqrt{a^{-2}}\tan\left(1/4\sqrt{a^{-2}}\xi\right)\right)}.$	$f_{1,1,6}(x,t) = \frac{C_0 - \frac{2\lambda}{\sqrt{-K}}}{\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{K}(\tanh(\sqrt{K}\Lambda) + i\,\mathrm{sech}(\sqrt{K}\Lambda))}{\nu}\right)}.$
Case III. $Z = 0$ **Case III.** $K = 0$	$f_5(x,t) = -1/8\,\frac{\left(A - \frac{B}{\xi}\right)}{a^{-1}\left(-1/16\,\frac{B}{a^2} - \frac{A}{\xi}\right)} + c_0 + \frac{2\, a\left(-1/16\,\frac{B}{a^2} - \frac{A}{\xi}\right)}{\left(A - \frac{B}{\xi}\right)}.$	No solution exists in this technique corresponding to K=0.

5. Conclusions

The present work utilized the Riccati-Bernoulli sub-ODE method within the context of the Bäcklund transformation to explore the features of optical pulse systems. This approach was beneficial in getting explicit series solutions for nonlinear differential equations, especially for the fractional Kairat equations that describe the propagation of optical solitons. Key findings and implications of the study are summarized as follows:

- **Methodological advancement:**
  - The use of the Riccati-Bernoulli sub-ODE technique provided a stable approach to derive soliton solutions in nonlinear systems. It was a unique revelation of how complexity in physical systems could easily be transformed to analytically soluble forms of equations.
- **Soliton dynamics and characteristics:**
  - The solutions derived explained several types of soliton behaviors such as the hyperbolic, trigonometric, and rational solitons, explained through 2D and 3D plots. These graphs focused on impacts of important parameters that are fractional order (α) and time (t) in relation to the temporal stability as well as spatial flexibility of solitons.
- **Practical relevance:**
  - The conclusions highlighted the importance of these solutions of solitons in the field of nonlinear optics, photonic crystals, and optical fiber communication systems. The fact that solitons are composed by stable and highly resistant structures is of high importance in situations requiring effective signal transmission, energy confinement, and accurate wavefront manipulation.

6. Future directions

Future investigations could further this research by generalizing the obtained fractional Kairat equations either for other physical phenomena or by adding variable coefficients and external fields to capture more dynamic behaviors. Numerical simulations would also assist in confirming the analytical solutions and examine how these solutions behave in configurations of higher dimensions. Also, extending the analysis to other application areas such as metamaterials, biomedical engineering, and quantum information systems may extend the application of the soliton solutions. Exploration of several soliton configurations and their stability or interactions with other solitons will help towards a better understanding of nonlinear waves. Finally, experimental studies that would tend to compare the theoretical findings with the real situation may add functionality to these solitons in optical fiber telecommunications, photonic gadgets, and other inconsonant optics solitons.

Author contributions

M. Mossa Al-Sawalha: Conceptualization, Writing-review & editing; Safyan Mukhtar: Visualization, Writing-review & editing, Data curation; Azzh Saad Alshehry: Data curation, Project administration; Mohammad Alqudah: Resources, Validation, Software; Musaad S. Aldhabani: Formal analysis, Investigation, Resources. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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