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Research article

Constructing boundary layer approximations in rotating magnetohydrodynamic fluids within cylindrical domains

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Abstract: This paper aims to investigate the effects of the Ekman-Hartmann boundary layer on rotating magnetohydrodynamics (MHD) within cylindrical domains, focusing on constructing approximate solutions within the boundary layer. We employed the multiscale analysis method to derive the approximate solutions, emphasizing the solutions at the cylinder's corners and lateral boundaries. Furthermore, we rigorously examined the asymptotic behavior of the rotating MHD flow in the limit case, proving its convergence to a two-dimensional damped and rotating dynamical system. These findings revealed the significant impact of high-speed rotation and strong magnetic fields on the structure and flow characteristics of the boundary layer, providing new insights into the dynamics of rotating MHD flows.

Keywords: magnetohydrodynamics; rotating dynamical system; Ekman-Hartmann boundary layer; cylinder domain; asymptotic behavior
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1. Introduction

Magnetohydrodynamics (MHD) investigates the interplay between conducting fluids and electromagnetic fields. It has extensive potential applications across diverse domains, encompassing energy, materials science, astrophysics, and engineering technologies [19]. In particular, the impacts of rotational and boundary layer effects on MHD are of significant research importance, as detailed in prior studies [1, 2, 4, 9–11, 15, 20].

The classical incompressible MHD equations constitute a set of coupled partial differential equations. Grounded in the fundamental principles of physics, such as the conservation of mass, momentum, and energy, as well as Maxwell's electromagnetic equations, they are employed to depict the behavior of conducting fluids under the influence of electromagnetic fields. The incompressible

MHD equations can be summarized as follows:

$$\begin{cases} \frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p - \boldsymbol{B} \times \boldsymbol{j} + \nu \Delta \boldsymbol{u}, \\ \frac{\partial}{\partial t}\boldsymbol{B} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \Delta \boldsymbol{B}, \\ \boldsymbol{j} = \sigma(\boldsymbol{E} - \boldsymbol{B} \times \boldsymbol{u}), \\ \boldsymbol{E} = \nabla \varphi, \\ \nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{B} = 0, \end{cases}$$

where $\boldsymbol{u}, p, \boldsymbol{j}, \boldsymbol{B}, \boldsymbol{E}, \varphi$ correspond to the fluid velocity, the pressure, the current density, the magnetic field, the electric field, and the electric potential, respectively. The coefficients ν, η, σ are the kinematic viscosity, the magnetic diffusivity, and the electrical conductivity, respectively. $\boldsymbol{B} \times \boldsymbol{j}$ represents the Lorentz force.

In this paper, we consider the MHD equations under the influence of the Coriolis force, with the magnetic field $\boldsymbol{B} = \frac{\beta}{\varepsilon}(0, 0, 1)^T$.

In this scenario, the MHD equations are simplified to include only the momentum equation with the Coriolis force and the current density equation. Specifically,

$$\begin{cases} \partial_t \boldsymbol{u}^\varepsilon + (\boldsymbol{u}^\varepsilon \cdot \nabla)\boldsymbol{u}^\varepsilon - \varepsilon \Delta \boldsymbol{u}^\varepsilon + \frac{\alpha}{\varepsilon} e^3 \wedge \boldsymbol{u}^\varepsilon + \frac{\beta}{\varepsilon} e^3 \wedge \boldsymbol{j}^\varepsilon + \frac{1}{\varepsilon}\nabla p^\varepsilon = 0, \\ \boldsymbol{j}^\varepsilon - \nabla \varphi^\varepsilon + e^3 \wedge \boldsymbol{u}^\varepsilon = 0, \\ \nabla \cdot \boldsymbol{u}^\varepsilon = \nabla \cdot \boldsymbol{j}^\varepsilon = 0, \end{cases} \tag{1.1}$$

where $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \Omega$, $\Omega = S \times [0, 1]$, S is smooth bounded domain of $\mathbb{R}^2$, $\frac{\alpha}{\varepsilon} e^3 \wedge \boldsymbol{u}^\varepsilon$ is the Coriolis force term, and the charge conservation principle requires $\nabla \cdot \boldsymbol{j}^\varepsilon = 0$. We also consider Equation (1.1) under the following initial and boundary conditions:

$$\boldsymbol{u}^\varepsilon(t, \boldsymbol{x})|_{t=0} = \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}), \tag{1.2}$$

$$\boldsymbol{u}^\varepsilon(t, \boldsymbol{x})|_{\partial\Omega} = 0, \quad \boldsymbol{j}^\varepsilon \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \tag{1.3}$$

where $\boldsymbol{n}$ is the normal vector of $\partial\Omega$. Since we are considering a system (1.1) in the region $\Omega = S \times [0, 1]$, the boundary condition of $\boldsymbol{j}^\varepsilon$ is equivalent to

$$j_3^\varepsilon(t, \boldsymbol{x})|_{z=0,1} = 0, \quad \boldsymbol{j}_h^\varepsilon \cdot \boldsymbol{n}_s|_{\partial S} = 0, \tag{1.4}$$

where $\boldsymbol{n}_s$ is the normal vector of ∂S.

It is crucial to acknowledge that boundary layer effects must be considered when examining rotating fluids within bounded regions. The boundary layer concept, originally introduced by the German physicist Ludwig Prandtl, is of paramount importance in fluid dynamics. It delineates the transitional zone wherein the fluid velocity shifts from zero near the solid surface to free-flow velocity due to viscous influences. Extensive experimental and theoretical investigations have established that the flow region adjacent to the solid wall can be bifurcated into two distinct zones: one is a skinny layer near the object, called the boundary layer, where the coefficient of viscosity plays a significant role. The other is the region outside the boundary layer, which has a negligible viscosity coefficient.

In Model (1.1), the parameter $\varepsilon > 0$ is very small ($\sim 10^{-7}$), with $1/\varepsilon$ used to describe the strength of the magnetic field and the rotation rate of the fluid. Therefore, the system in (1.1) describes the dynamic behavior of incompressible fluids with low viscosity and large force terms. Furthermore, the ratio $\beta/\alpha > 0$ represents the Elsasser number utilized to describe the relative strength between the magnetic field and fluid flow in MHD. As the Elsasser number increases, the boundary layer transitions from the Ekman type to the Hartmann type. For example, when the external force term is of the Coriolis type ($\beta/\alpha = 0$), it can simulate rotating fluids in oceans, atmospheres, or containers (see [14, 15]). When magnetic effects are considered ($\beta/\alpha \gg 1$), $e^3 \wedge \boldsymbol{j}^\varepsilon$ represents the Lorentz force. It is linked to $\boldsymbol{u}^\varepsilon$ through Ohm's law, as shown in $(1.1)_2$.

This paper considers a three-dimensional model subject to high-speed rotation and the effect of a high-intensity magnetic field ($\beta/\alpha = O(1)$) within a bounded domain Ω. It is assumed that the direction of the rotation axis aligns with that of the mean magnetic field, both being $e^3 = (0, 0, 1)^T$. The hydrodynamic behavior within this region is profoundly influenced by the magnetic field and rotational effects, displaying characteristics that markedly deviate from those of the interior region. Furthermore, the structure of the boundary layer exerts a substantial impact on the stability and performance of the overall flow system. For further details on MHD layers, please refer to [6–8, 13, 16, 17].

The Ekman-Hartmann layer is crucial in MHD systems with strong magnetic fields and rapid rotation. It impacts ocean currents and wind patterns in geophysics, heat management in engineering, and plasma confinement in fusion reactors. Extensive research has been devoted to the mathematical analysis of the Ekman-Hartmann boundary layer. For instance, in [6], the authors employed a matched asymptotic expansion technique to investigate the boundary layer for the half-space domain and the region between two parallel plates. Their findings revealed that the boundary layer displays nonlinear stability when the characteristic Reynolds number, defined within the boundary layer, falls below a critical threshold. This conclusion was corroborated in [16] under more generalized spectral assumptions. It is noteworthy that the models discussed in [6] and [16] represent generalizations of the system in (1.1), wherein Eq $(1.1)_2$ is replaced by an equation governing the evolution of the magnetic field. For the simplified Model (1.1), [13] introduced a unified approach for boundary layer analysis, with special attention given to the derivation of approximate solutions in scenarios involving rotation (the Ekman layer) or magnetic fields (the Hartmann layer). Furthermore, in the intricate setting characterized by concurrent high-speed rotation and intense magnetic fields, [17] undertook a comprehensive investigation of Model (1.1) under Dirichlet boundary conditions applied to the region bounded by two parallel planes. This study effectively extended the nonlinear stability conclusion established for the Ekman-Hartmann boundary layer in [16] to encompass a broader range of initial value conditions. Subsequently, Rousset [18] proved the nonlinear stability of Ekman-Hartmann boundary layers in a spherical geometry for well-prepared initial data.

Furthermore, investigating rotating fluids within cylindrical domains presents numerous challenges, primarily arising from the intricate interplay among hydrodynamics, rotation, and the container's geometry, particularly in the vicinity of corners and edges. Bresch, Desjardins, and Gérard-Varet [3] addressed these challenges by developing correction terms near the lateral edges while preserving the integrity of the upper and lower boundary terms as well as the interior terms.

1.1. Notation

Before presenting the results, we provide the following definitions for convenience.

Let $\nabla = (\partial_x, \partial_y, \partial_z)^T$, $\nabla_h = (\partial_x, \partial_y)^T$, and $\nabla_h^\perp = (-\partial_y, \partial_x)^T$. We also write $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ and

$\Delta_h = \partial_x^2 + \partial_y^2$, $\boldsymbol{f} = (\boldsymbol{f}_h, f_3)^T$, $\boldsymbol{f}_h = (f_1, f_2)^T$, $\boldsymbol{f}_{h,\perp} = (-f_2, f_1)^T$, and

$$\mathcal{A} = \begin{pmatrix} -\beta & \alpha \\ -\alpha & -\beta \end{pmatrix}.$$

In this paper, we occasionally employ the notation $A \lesssim B$ to denote the equivalence $A \leq CB$, where C is a uniform constant.

1.2. The main result

For a fixed $\varepsilon > 0$, the mathematical behavior of Systems (1.1)–(1.4) closely resembles that of the incompressible Navier–Stokes equations. By the theory of global weak solutions, which is analogous to the Leray solutions of the incompressible Navier–Stokes equations (for further details, see [19]), this paper aims to investigate the asymptotic behavior of the weak solutions as ε approaches zero. The details are as follows.

Theorem 1.1. *Let* $(\boldsymbol{u}^\varepsilon, \boldsymbol{j}^\varepsilon) \in L^\infty(\mathbb{R}_+; L^2(\Omega))$ *be a family of weak solutions of Systems* (1.1)–(1.4) *associated with the initial data* $\boldsymbol{u}_0^\varepsilon(x) \in L^2(\Omega)$. *Under the following well-prepared initial data conditions:* $\boldsymbol{u}_0^\varepsilon = (\boldsymbol{u}_{0,h}^\varepsilon, u_{0,3}^\varepsilon)$ *and* $\bar{\boldsymbol{u}}_{0,h} = \int_0^1 \boldsymbol{u}_{0,h}^\varepsilon \, dz$*, satisfy*

$$\lim_{\varepsilon \to 0^+} \boldsymbol{u}_0^\varepsilon = (\bar{\boldsymbol{u}}_{0,h}, 0) =: \bar{\boldsymbol{u}}_0, \quad in\ L^2(S). \tag{1.5}$$

For an universal constant C_0*,*

$$\|\bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)} < C_0, \tag{1.6}$$

then $\bar{\boldsymbol{u}}(t, x, y) = (\bar{\boldsymbol{u}}_h, 0)$ *satisfies the following two-dimensional (2D) primitive type equations with the initial data* $\bar{\boldsymbol{u}}_0$*:*

$$\begin{cases} \partial_t \bar{\boldsymbol{u}}_h + (\bar{\boldsymbol{u}}_h \cdot \nabla_h)\bar{\boldsymbol{u}}_h + \gamma \bar{\boldsymbol{u}}_h + \eta \bar{\boldsymbol{u}}_{h,\perp} + \nabla_h \bar{p} + \beta \nabla_h^\perp \bar{\varphi} = 0, \\ \nabla_h \cdot \bar{\boldsymbol{u}}_h = 0, \\ \Delta_h \bar{\varphi} = -\frac{2\cos\frac{\tau}{2}}{(\alpha^2+\beta^2)^{\frac14}} \nabla_h^\perp \cdot \bar{\boldsymbol{u}}_h, \\ \bar{\boldsymbol{u}}_h \mid_{\partial S} = 0, \quad \nabla_h \bar{\varphi} \cdot \boldsymbol{n}_s \mid_{\partial S} = 0, \end{cases} \tag{1.7}$$

where γ *and* η *are defined by* $\gamma = \frac{2}{(\alpha^2+\beta^2)^{\frac14}}(\alpha \sin(\frac{\tau}{2}) + \beta \cos(\frac{\tau}{2}))$, $\eta = \frac{2}{(\alpha^2+\beta^2)^{\frac14}}(\alpha \cos(\frac{\tau}{2}) - \beta \sin(\frac{\tau}{2}))$, $\sin(\tau) = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$*, and* $\cos(\tau) = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$*, such that*

$$\lim_{\varepsilon \to 0^+} \|\boldsymbol{u}^\varepsilon - \bar{\boldsymbol{u}}\|_{L^\infty(\mathbb{R}_+; L^2(\Omega))} = 0.$$

Remark 1.1. *We employ strict asymptotic analysis to demonstrate that System* (1.1) *converges to the limiting system* (1.7) *under high rotational conditions, which is a 2D system incorporating both damping and rotational effects, where the term* $\gamma \bar{\boldsymbol{u}}_h$ *represents the damping, and the term* $\eta \bar{\boldsymbol{u}}_{h,\perp}$ *signifies the rotation. The structure of the damping term* $\gamma \bar{\boldsymbol{u}}_h$ *specifically recovers the results obtained by [13], confirming the accuracy and consistency of our analysis. Meanwhile, the term* $\eta \bar{\boldsymbol{u}}_{h,\perp}$ *indicates that the derived limiting state still exhibits rotational effects, aligning with physical expectations and highlighting the enduring impact of rotation on magnetohydrodynamic fluids.*

Remark 1.2. *Addressing the challenges posed by the corners and lateral boundaries of cylindrical domains, we adopt an approach inspired by the work of [3]. We refine the boundary conditions by constructing correction terms in a thin layer near the lateral boundaries, ensuring a more accurate representation of the physical system.*

Remark 1.3. *In Section 2, we derived the structure of the approximate solution for the MHD fluid within a cylindrical domain. This structure provides a more accurate representation of the fluid characteristics in such geometries. By capturing the essential features of the fluid's behavior under the influence of magnetic fields and rotation, our solution offers a robust framework for constructing numerical models in geophysics and related fields. This approach facilitates more precise simulations and predictions.*

This paper is organized as follows: Section 2 constructs approximate solutions order by order through asymptotic expansion and introduces correction terms to satisfy incompressibility and boundary conditions. Section 3 investigates the properties of the 2D limiting equation. Section 4 proves the convergence results for rotating magnetohydrodynamics in the limiting state.

2. Construction of approximate solutions

This section constructs a linear approximate solution $(\boldsymbol{u}_L^{app}, p_L^{app}, \boldsymbol{j}_L^{app}, \varphi_L^{app})$ of the following form:

$$\begin{cases} \boldsymbol{u}_L^{app} = \Sigma_{i=0}^{\infty}\varepsilon^i[\boldsymbol{u}^{i,int}(t,x,y,z) + \boldsymbol{u}^{i,T}(t,x,y,\lambda) + \boldsymbol{u}^{i,B}(t,x,y,\theta)], \\ p_L^{app} = \Sigma_{i=0}^{\infty}\varepsilon^i[p^{i,int}(t,x,y,z) + p^{i,T}(t,x,y,\lambda) + p^{i,B}(t,x,y,\theta)], \\ \boldsymbol{j}_L^{app} = \Sigma_{i=0}^{\infty}\varepsilon^i[\boldsymbol{j}^{i,int}(t,x,y,z) + \boldsymbol{j}^{i,T}(t,x,y,\lambda) + \boldsymbol{j}^{i,B}(t,x,y,\theta)], \\ \varphi_L^{app} = \Sigma_{i=0}^{\infty}\varepsilon^i[\varphi^{i,int}(t,x,y,z) + \varphi^{i,T}(t,x,y,\lambda) + \varphi^{i,B}(t,x,y,\theta)], \end{cases} \tag{2.1}$$

where $\theta = \frac{z}{\varepsilon}$, $\lambda = \frac{1-z}{\varepsilon}$, and the superscripts *int*, T, B, and c represent the interior, top boundary, bottom boundary terms, and correction terms, respectively. We also put a nature boundary condition as follows:

$$\lim_{\lambda\to\infty} \boldsymbol{u}^{i,T}(t,x,y,\lambda) = \lim_{\theta\to\infty} \boldsymbol{u}^{i,B}(t,x,y,\theta) = 0. \tag{2.2}$$

Furthermore, the approximate solution satisfies the following linear approximate equations:

$$\begin{cases} \partial_t \boldsymbol{u}_L^{app} - \varepsilon\Delta \boldsymbol{u}_L^{app} + \frac{\alpha}{\varepsilon}\mathrm{e}^3 \wedge \boldsymbol{u}_L^{app} + \frac{\beta}{\varepsilon}\mathrm{e}^3 \wedge \boldsymbol{j}_L^{app} + \frac{1}{\varepsilon}\nabla p_L^{app} = \boldsymbol{R}_L^{app}, \\ \boldsymbol{j}_L^{app} - \nabla\varphi_L^{app} + \mathrm{e}^3 \wedge \boldsymbol{u}_L^{app} = 0, \\ \nabla\cdot \boldsymbol{u}_L^{app} = \nabla\cdot \boldsymbol{j}_L^{app} = 0, \end{cases} \tag{2.3}$$

where $\boldsymbol{R}_L^{app}$ represents the residual term obtained by substituting the linear approximate solution $(\boldsymbol{u}_L^{app}, p_L^{app}, \boldsymbol{j}_L^{app}, \varphi_L^{app})$ into the original linear system, with the boundary conditions

$$\boldsymbol{u}_L^{app}|_{\partial\Omega} = 0, \quad j_{L,3}^{app}|_{z=0,1} = 0, \quad \boldsymbol{j}_{L,h}^{app}\cdot \boldsymbol{n}_s|_{\partial S} = 0. \tag{2.4}$$

Next, we decide the precise forms of (2.1) by analyzing the order of ε. Moreover, we substitute the approximate forms of (2.1) for the top and bottom boundaries into System (2.3) and analyze its ε^i-order parts ($i = -2, -1, 0, \cdots$).

2.1. Analyzing the internal terms and top and bottom boundary terms

In this subsection, we analyze the part of the linear approximation system of order ε^i and determine the specific form of the linear approximated solution by combining the top and bottom boundary conditions and the incompressibility conditions. We mainly construct the bottom boundary and interior terms in the following section. The construction process of the top boundary layer is similar to that of the bottom boundary.

2.1.1. The $O(\varepsilon^{-2})$-order part in the system.

Through simple computation

$$\partial_\theta p^{0,B} = \partial_\theta^2 \varphi^{0,B} = 0,$$

we obtain the highest order term as ε^{-2}. Clearly, getting $p^{0,B}$ and $\partial_\theta \varphi^{0,B}$ is independent of θ. It is natural to take $p^{0,B} = 0$, implying that the boundary layer's highest order pressure term is vanishing.

Similarly, it can be obtained that $p^{0,T} = 0$, and that $\partial_\lambda \varphi^{0,T}$ is independent of λ.

2.1.2. The $O(\varepsilon^{-1})$-order part in the system.

From the ε^{-1}-order bottom boundary term, we get

$$\begin{cases} \partial_\theta^2 \boldsymbol{u}_h^{0,B} + \mathcal{A}\boldsymbol{u}_h^{0,B} - \beta \nabla_h^\perp \varphi^{0,B} = 0, \\ \partial_\theta^2 u_3^{0,B} - \partial_\theta p^{1,B} = 0, \\ \partial_\theta \varphi^{0,B} = \partial_\theta u_3^{0,B} = \partial_\theta^2 \varphi^{1,B} = 0. \end{cases} \tag{2.5}$$

First, from $(2.5)_3$, we know that $u_3^{0,B}$ is independent of θ. Combining $u_3^{0,B}$ then satisfies the Dirichlet boundary condition, and the Taylor–Proudman theorem leads to the conclusion that $u_3^{0,B} = 0$. Next, due to $\boldsymbol{u}_h^{0,B}$ satisfying the boundary condition, take the limit $\varepsilon \to 0$ for $(2.5)_1$, which gives $\nabla_h^\perp \varphi^{0,B} = 0$. Combined with $\partial_\theta \varphi^{0,B} = 0$ from $(2.5)_3$, this gives $\nabla \varphi^{0,B} = 0$. Moreover, $j_3^{0,B} = 0$ can be obtained from $(2.3)_2$. On this basis, in combination with $(2.5)_2$, we can see that $p^{1,B}$ is also independent of θ.

Similarly, we take the ε^{-1}-order internal terms from the equations as

$$\begin{cases} \partial_x p^{0,int} = \alpha u_2^{0,int} + \beta j_2^{0,int}, \\ \partial_y p^{0,int} = -\alpha u_1^{0,int} - \beta j_1^{0,int}, \\ \partial_z p^{0,int} = 0. \end{cases} \tag{2.6}$$

It is natural to show that $p^{0,int}(t,x,y)$ is independent of z. Combining the incompressible conditions of $\boldsymbol{u}^{0,int}$ and $\boldsymbol{j}^{0,int}$ and $(2.6)_{1,2}$, we get

$$\begin{aligned} &\partial_y \partial_x p^{0,int} - \partial_x \partial_y p^{0,int} \\ =&\alpha(\nabla_h \cdot \boldsymbol{u}_h^{0,int}) + \beta(\nabla_h \cdot \boldsymbol{j}_h^{0,int}) \\ =&-\partial_z(\alpha u_3^{0,int} + \beta j_3^{0,int}) = 0. \end{aligned}$$

Due to $u_3^{0,B} = j_3^{0,B} = 0$ and their boundary conditions in (2.4), we obtain

$$u_3^{0,int}|_{z=0,1} = j_3^{0,int}|_{z=0,1} = (\alpha u_3^{0,int} + \beta j_3^{0,int})|_{z=0,1} = 0. \tag{2.7}$$

According to the Taylor–Proudman theorem, $\partial_z(\alpha u_3^{0,int} + \beta j_3^{0,int}) = 0$ and (2.7); it follows that $u_3^{0,int} = j_3^{0,int} = 0$. Hence, $\partial_z\varphi^{0,int} = 0$ and $\varphi^{0,int}(t,x,y)$ is independent of z. Since $\boldsymbol{j}^{0,int}$ satisfies

$$\boldsymbol{j}^{0,int} = \nabla\varphi^{0,int} - e^3 \wedge \boldsymbol{u}^{0,int},$$

Eq (2.6) can be changed to

$$\begin{cases} \partial_x p^{0,int} = \alpha u_2^{0,int} + \beta\partial_y\varphi^{0,int} - \beta u_1^{0,int}, \\ \partial_y p^{0,int} = -\alpha u_1^{0,int} - \beta\partial_x\varphi^{0,int} - \beta u_2^{0,int}, \\ \partial_z p^{0,int} = 0. \end{cases}$$

Since $p^{0,int}$ and $\varphi^{0,int}$ are independent of z, it follows from expression above that $\boldsymbol{u}_h^{0,int}(t,x,y)$ is also independent of z.

The following inner product of the system in Eq $(2.6)_{1,2}$ and $\boldsymbol{u}_h^{0,int}$, combined with the incompressibility condition for $\boldsymbol{u}_h^{0,int}$, gives

$$\int_S -j_2^{0,int} \cdot u_1^{0,int} + j_1^{0,int} \cdot u_2^{0,int} = 0. \tag{2.8}$$

Note that $\boldsymbol{u}_h^{0,int}$ and $\boldsymbol{j}_h^{0,int}$ satisfy the equations and the boundary condition

$$\begin{cases} \boldsymbol{j}^{0,int} = \nabla\varphi^{0,int} - e^3 \wedge \boldsymbol{u}^{0,int}, \\ \boldsymbol{j}_h^{0,int} \cdot \boldsymbol{n}_s|_{\partial S} = 0. \end{cases} \tag{2.9}$$

Then, by combining (2.8) and (2.9), it can be deduced that

$$\begin{aligned} &\int_S -|\boldsymbol{j}_h^{0,int}|^2 + \boldsymbol{j}_h^{0,int} \cdot \nabla_h\varphi^{0,int} \\ = &-\int_S |\boldsymbol{j}_h^{0,int}|^2 + \int_{\partial S} \boldsymbol{j}_h^{0,int} \cdot \boldsymbol{n}_s \cdot \varphi^{0,int} = 0. \end{aligned}$$

Thus we obtain $\boldsymbol{j}_h^{0,int} = 0$ from the boundary condition in (2.4) for $\boldsymbol{j}_h^{0,int}$ and have

$$\varphi^{0,int}(t,x,y) = -\Delta_h^{-1}\nabla_h^\perp \cdot \boldsymbol{u}_h^{0,int}. \tag{2.10}$$

On the basis of this analysis, it can be seen that the internal terms in (2.6) can be reduced to

$$\begin{cases} \partial_x p^{0,int} = \alpha u_2^{0,int}, \\ \partial_y p^{0,int} = -\alpha u_1^{0,int}. \end{cases}$$

By the incompressibility condition of $\boldsymbol{u}_h^{0,int}$, $p^{0,int}$ can be expressed as

$$p^{0,int}(t,x,y) = \alpha\Delta_h^{-1}\nabla_h^\perp \cdot \boldsymbol{u}_h^{0,int}. \tag{2.11}$$

Furthermore, the boundary terms (2.5) can be rewritten as

$$\begin{cases} \partial_\theta^2\boldsymbol{u}_h^{0,B} + \mathcal{A}\boldsymbol{u}_h^{0,B} = 0, \\ \boldsymbol{u}_h^{0,B}|_{\theta=0} = -\boldsymbol{u}_h^{0,int}, \quad \lim_{\theta\to\infty} \boldsymbol{u}_h^{0,B} = 0. \end{cases} \tag{2.12}$$

Equation (2.12) is a fourth-order ordinary differential system in $\boldsymbol{u}_h^{0,B}$. Solving this differential equation is straightforward, and we can solve it for

$$\boldsymbol{u}_h^{0,B}(t,x,y,\theta) = -\mathrm{e}^{-a\theta}\Big(\cos(b\theta)\boldsymbol{u}_h^{0,int} + \sin(b\theta)\boldsymbol{u}_{h,\perp}^{0,int}\Big), \tag{2.13}$$

where

$$a = (\alpha^2+\beta^2)^{\frac{1}{4}}\cos(\tfrac{\tau}{2}), \quad b = (\alpha^2+\beta^2)^{\frac{1}{4}}\sin(\tfrac{\tau}{2}).$$

Furthermore, from $(2.3)_2$ and $\varphi^{0,B} = 0$, we have

$$\boldsymbol{j}_h^{0,B}(t,x,y,\theta) = -\boldsymbol{u}_{h,\perp}^{0,B}. \tag{2.14}$$

Similar to the analysis above, we can also obtain the expressions for the top boundary terms as $\varphi^{0,T} = u_3^{0,T} = 0$ and

$$\boldsymbol{u}_h^{0,T}(t,x,y,\lambda) = -\mathrm{e}^{-a\lambda}\Big(\cos(b\lambda)\boldsymbol{u}_h^{0,int} + \sin(b\lambda)\boldsymbol{u}_{h,\perp}^{0,int}\Big). \tag{2.15}$$

2.1.3. The $O(1)$-order part in the system.

From the $O(1)$-order bottom boundary term, we get

$$\begin{cases}\partial_\theta^2\boldsymbol{u}_h^{1,B} + \mathcal{A}\boldsymbol{u}_h^{1,B} = \nabla_h p^{1,B} + \beta\nabla_h^{\perp}\varphi^{1,B},\\ \partial_\theta p^{2,B} = \partial_\theta^2 u_3^{1,B},\\ \partial_\theta u_3^{1,B} = -\nabla_h\cdot\boldsymbol{u}_h^{0,B},\\ \partial_\theta^2\varphi^{2,B} = -\nabla_h^{\perp}\cdot\boldsymbol{u}_h^{0,B}.\end{cases} \tag{2.16}$$

Firstly, from $(2.16)_3$ and the expression of (2.13) for $\boldsymbol{u}_h^{0,B}$, we have

$$\partial_\theta u_3^{1,B} = \nabla_h^{\perp}\cdot\Big(\mathrm{e}^{-a\theta}\Big(\cos(b\theta)\boldsymbol{u}_h^{0,int} + \sin(b\theta)\boldsymbol{u}_{h,\perp}^{0,int}\Big)\Big). \tag{2.17}$$

If we integrate Equation (2.17) concerning θ, we get

$$\begin{aligned}u_3^{1,B}(t,x,y,\theta) = &-(a^2+b^2)^{-1}\mathrm{e}^{-a\theta}(a\sin(b\theta)+b\cos(b\theta))\nabla_h^{\perp}\cdot\boldsymbol{u}_h^{0,int}\\ &-(a^2+b^2)^{-1}\mathrm{e}^{-a\theta}(a\cos(b\theta)-b\sin(b\theta))\nabla_h\cdot\boldsymbol{u}_h^{0,int}.\end{aligned} \tag{2.18}$$

From the boundary condition in (2.4), we can deduce that

$$u_3^{1,int}|_{z=0} = -u_3^{1,B}|_{\theta=0} = (a^2+b^2)^{-1}(b\nabla_h^{\perp}\cdot\boldsymbol{u}_h^{0,int} + a\nabla_h\cdot\boldsymbol{u}_h^{0,int}). \tag{2.19}$$

According to the boundary expression in (2.19), we take $u_3^{1,int}$ to be

$$u_3^{1,int}(t,x,y,z) = (1-2z)(a^2+b^2)^{-1}(b\nabla_h^{\perp}\cdot\boldsymbol{u}_h^{0,int} + a\nabla_h\cdot\boldsymbol{u}_h^{0,int}). \tag{2.20}$$

We then combine this with the incompressible condition of $\boldsymbol{u}^{1,int}$ that

$$\nabla_h\cdot\boldsymbol{u}_h^{1,int} = -\partial_z u_3^{1,int} = 2(a^2+b^2)^{-1}(b\nabla_h^{\perp}\cdot\boldsymbol{u}_h^{0,int} + a\nabla_h\cdot\boldsymbol{u}_h^{0,int}). \tag{2.21}$$

In this case, $\boldsymbol{u}_h^{1,int}$ can be expressed as

$$\boldsymbol{u}_h^{1,int} = 2(a^2+b^2)^{-1}(a\boldsymbol{u}_h^{0,int} - b\boldsymbol{u}_{h,\perp}^{0,int}) + \boldsymbol{g}^1(t,z),$$

where the expression for $\boldsymbol{g}^1(t,z)$ is determined below.

Remark 2.1. *It is worth noting that $\nabla_h \cdot \boldsymbol{u}_h^{0,int}$ in (2.18)–(2.21) practically vanishes. Since this term affects the construction of $\boldsymbol{u}_h^{1,int}$ and hence the limit equations, we keep it in this form.*

Below, we analyze the forms of $\boldsymbol{u}_h^{1,int}$ and $\boldsymbol{g}^{1,int}$. First of all, we know that the $O(1)$-order interior part in the approximate system is:

$$\begin{cases} \partial_t \boldsymbol{u}_h^{0,int} - \mathcal{A}\boldsymbol{u}_h^{1,int} + \nabla_h p^{1,int} + \beta \nabla_h^{\perp} \varphi^{1,int} = 0, \\ \partial_t u_3^{0,int} + \partial_z p^{1,int} = 0, \\ \nabla_h \cdot \boldsymbol{u}_h^{0,int} = 0, \\ \Delta \varphi^{1,int} = -\nabla_h^{\perp} \cdot \boldsymbol{u}_h^{1,int}. \end{cases} \tag{2.22}$$

Given $u_3^{0,int} = 0$ and $(2.22)_2$, it follows that $p^{1,int}(t,x,y)$ is independent of z. At this point, the expression for $\nabla_h^{\perp}\varphi^{1,int}$ is not determined, so we can assume that $\nabla_h^{\perp}\varphi^{1,int} = \boldsymbol{g}^2(t,x,y) + \boldsymbol{g}^3(t,z)$. Consequently, Eq $(2.22)_1$ can be decomposed into the parts related to (x,y) and the parts related to z, i.e.,

$$\partial_t \boldsymbol{u}_h^{0,int} - 2(a^2+b^2)^{-1}\mathcal{A}(a\boldsymbol{u}_h^{0,int} - b\boldsymbol{u}_{h,\perp}^{0,int}) + \nabla_h p^{1,int} + \boldsymbol{g}^2(t,x,y) = 0,$$

and

$$\boldsymbol{g}^1(t,z) + \boldsymbol{g}^3(t,z) = 0.$$

Furthermore, we can set $\boldsymbol{g}^1(t,z) = \boldsymbol{g}^3(t,z) = 0$, as this assumption does not affect the subsequent analysis. Therefore, both $\boldsymbol{u}_h^{1,int}$ and $\nabla_h^{\perp}\varphi^{1,int}$ are independent of z, and $\boldsymbol{u}_h^{1,int}$ can be expressed as

$$\boldsymbol{u}_h^{1,int}(t,x,y) = 2(a^2+b^2)^{-1}(a\boldsymbol{u}_h^{0,int} - b\boldsymbol{u}_{h,\perp}^{0,int}). \tag{2.23}$$

Next, we analyze $\varphi^{1,int}$. Assuming $\varphi^{1,int} = g^4(x,y) + g^5(z)$, then with the boundary condition in (2.4), we have

$$\partial_z \varphi^{1,int}|_{z=0,1} = \partial_z g^5(z)|_{z=0,1} = 0,$$

which gives

$$g^5(z) = \sum_{n=0}^{\infty} a_n \cos(nz),$$

where a_n is a family of constants. Thus

$$\Delta\varphi^{1,int} = \Delta_h g^4(x,y) - \sum_{n=0}^{\infty} n^2 a_n \cos(nz); \tag{2.24}$$

however, by $(2.22)_3$ and because $\boldsymbol{u}_h^{1,int}$ is independent of z, it follows that $\Delta\varphi^{1,int}$ is independent of z. This contradicts (2.24), and thus $a_n = 0$, i.e., $\varphi^{1,int}(t,x,y)$ is independent of z.

With the above analysis and the expression in (2.23) for $\boldsymbol{u}_h^{1,int}$, (2.22) can be rewritten as

$$\begin{cases} \partial_t \boldsymbol{u}_h^{0,int} + \gamma \boldsymbol{u}_h^{0,int} + \eta \boldsymbol{u}_{h,\perp}^{0,int} + \nabla_h p^{1,int} + \beta \nabla_h^{\perp}\varphi^{1,int} = 0, \\ \nabla_h \cdot \boldsymbol{u}_h^{0,int} = 0, \\ \Delta_h \varphi^{1,int} = -\frac{2\cos\frac{\tau}{2}}{(\alpha^2+\beta^2)^{\frac{1}{4}}} \nabla_h^{\perp} \cdot \boldsymbol{u}_h^{0,int}, \end{cases} \tag{2.25}$$

where

$$\gamma = \frac{2}{(\alpha^2+\beta^2)^{\frac{1}{4}}}(\alpha\sin\tfrac{\tau}{2} + \beta\cos\tfrac{\tau}{2}), \quad \eta = \frac{2}{(\alpha^2+\beta^2)^{\frac{1}{4}}}(\alpha\cos\tfrac{\tau}{2} - \beta\sin\tfrac{\tau}{2}).$$

On the basis of the expressions for $u_3^{1,B}$ and $\boldsymbol{u}_h^{0,B}$, we integrate $(2.16)_{2,4}$ to get

$$p^{2,B}(t,x,y,\theta) = \mathrm{e}^{-a\theta}\sin(b\theta)\nabla_h^{\perp}\cdot\boldsymbol{u}_h^{0,int}, \tag{2.26}$$

and

$$\partial_\theta\varphi^{2,B} = (a^2+b^2)^{-1}\mathrm{e}^{-a\theta}(b\sin(b\theta) - a\cos(b\theta))\nabla_h^{\perp}\cdot\boldsymbol{u}_h^{0,int} + g^6(t,x,y), \tag{2.27}$$

where the form of $g^6(t,x,y)$ is determined subsequently.

On the basis of the facts that the $O(1)$-order term $\partial_\theta\varphi^{1,B}|_{\theta=0} = -\partial_z\varphi^{0,int}|_{z=0} = 0$ in the boundary conditions in (2.4) and that $\partial_\theta\varphi^{1,B}$ is independent of θ, we can determine that $\varphi^{1,B}(t,x,y)$ is also independent of θ. Combining the boundary condition (2.2) with the boundary terms $p^{1,B}$ and $\varphi^{1,B}(t,x,y)$, independent of θ, and taking the limit ε to zero at both ends of $(2.16)_1$, we get

$$\nabla_h p^{1,B} = -\beta\nabla_h^{\perp}\varphi^{1,B}. \tag{2.28}$$

Thus $\boldsymbol{u}_h^{1,B}$ satisfies the following equations and boundary conditions, and the right-hand side of the system are all known terms:

$$\begin{cases} \partial_\theta^2\boldsymbol{u}_h^{1,B} + \mathcal{A}\boldsymbol{u}_h^{1,B} = 0, \\ \boldsymbol{u}_h^{1,B}|_{\theta=0} = -\boldsymbol{u}_h^{1,int}, \quad \lim\limits_{\varepsilon\to\infty}\boldsymbol{u}_h^{1,B} = 0. \end{cases}$$

Duhamel's principle leads to

$$\boldsymbol{u}_h^{1,B}(t,x,y,\theta) = -\mathrm{e}^{-a\theta}(\cos(b\theta)\boldsymbol{u}_h^{1,int} - \sin(b\theta)\boldsymbol{u}_{h,\perp}^{1,int}). \tag{2.29}$$

Remark 2.2. *Notably, the coefficient γ of the damping term of the linear limit system remains consistent with the results in [13]. Meanwhile, $\eta\boldsymbol{u}_{h,\perp}^{0,int}$ is due to the retention of $\nabla_h\cdot\boldsymbol{u}_h^{0,int}$ in* (2.18)–(2.21)*, reacting to the continuous effect of rotation on the fluid.*

Similarly, on the basis of the analysis above, we can get

$$\boldsymbol{u}^{1,T}(t,x,y,\lambda) = \begin{pmatrix} -\mathrm{e}^{-a\lambda}(\cos(b\lambda)\boldsymbol{u}_h^{1,int} - \sin(b\lambda)\boldsymbol{u}_{h,\perp}^{1,int}) \\ (a^2+b^2)^{-1}\mathrm{e}^{-a\lambda}(a\sin(b\lambda) + b\cos(b\lambda))\nabla_h^{\perp}\cdot\boldsymbol{u}_h^{0,int} \end{pmatrix}, \tag{2.30}$$

$$p^{2,T}(t,x,y,\lambda) = \mathrm{e}^{-a\lambda}\sin(b\lambda)\nabla_h^{\perp}\cdot\boldsymbol{u}_h^{0,int}, \tag{2.31}$$

$$\partial_\lambda\varphi^{2,T}(t,x,y,\lambda) = (a^2+b^2)^{-1}\mathrm{e}^{-a\lambda}(b\sin(b\lambda) - a\cos(b\lambda))\nabla_h^{\perp}\cdot\boldsymbol{u}_h^{0,int} + g^7(x,y), \tag{2.32}$$

and $\nabla_h p^{1,T} = -\beta\nabla_h^{\perp}\varphi^{1,T}$ and $g^7(x,y)$ are determined subsequently.

2.1.4. The $O(\varepsilon)$-order part in the system.

The boundary $O(\varepsilon)$-order term in the incompressibility condition is $\partial_\theta u_3^{2,B} = -\nabla_h\cdot\boldsymbol{u}_h^{1,B}$. It can then be found in the case where $\boldsymbol{u}_h^{1,B}$ is known that

$$u_3^{2,B}(t,x,y,\theta) = \int_\theta^{+\infty}\nabla_h\cdot\boldsymbol{u}_h^{1,B}(t,x,y,s)\,ds. \tag{2.33}$$

Similarly, according to the incompressibility condition, the upper boundary term $u_3^{2,T}$ is

$$u_3^{2,T}(t,x,y,\lambda) = -\int_\lambda^{+\infty} \nabla_h \cdot \boldsymbol{u}_h^{1,T}(t,x,y,s)\,ds. \tag{2.34}$$

Since the internal higher-order terms do not introduce singularities, they do not affect the subsequent analysis. Therefore, we take $\boldsymbol{u}^{2,int} = 0$, then $\boldsymbol{u}_h^{2,B} = \boldsymbol{u}_h^{2,T} = 0$. We will correct the boundary conditions for $u_3^{2,B}$ and $u_3^{2,T}$ subsequently.

On the basis of the facts that the $O(\varepsilon)$-order terms $\partial_\theta\varphi^{2,B}|_{\theta=0} = -\partial_z\varphi^{1,int}|_{z=0}$ and $\partial_\lambda\varphi^{2,T}|_{\lambda=0} = \partial_z\varphi^{1,int}|_{z=1}$ in the boundary conditions in (2.4) and that $\varphi^{1,int}$ is independent of z, we can get $\partial_\lambda\varphi^{2,T}|_{\lambda=0} = \partial_\theta\varphi^{2,B}|_{\theta=0} = 0$. Thus there is

$$\partial_\theta\varphi^{2,B} = (a^2+b^2)^{-1}\mathrm{e}^{-a\theta}(b\sin(b\theta) - a\cos(b\theta))\nabla_h^\perp \cdot \boldsymbol{u}_h^{0,int} + (a^2+b^2)^{-1}a\nabla_h^\perp \cdot \boldsymbol{u}_h^{0,int},$$

and

$$\partial_\lambda\varphi^{2,T} = -(a^2+b^2)^{-1}\mathrm{e}^{-a\lambda}(b\sin(b\lambda) - a\cos(b\lambda))\nabla_h^\perp \cdot \boldsymbol{u}_h^{0,int} - (a^2+b^2)^{-1}a\nabla_h^\perp \cdot \boldsymbol{u}_h^{0,int}.$$

In this subsection, we construct the top and bottom boundaries as well as the internal terms (see Figure 1),with the approximate solution $(\boldsymbol{u}_L^{1,app}, p_L^{1,app}, \varphi_L^{1,app}, \boldsymbol{j}_L^{1,app})$ given by

$$\begin{cases} \boldsymbol{u}_L^{1,app} = \begin{pmatrix} \boldsymbol{u}_h^{0,int} + \boldsymbol{u}_h^{0,B} + \boldsymbol{u}_h^{0,T} \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} \boldsymbol{u}_h^{1,int} + \boldsymbol{u}_h^{1,B} + \boldsymbol{u}_h^{1,T} \\ u_3^{1,int} + u_3^{1,B} + u_3^{1,T} \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 \\ u_3^{2,B} + u_3^{2,T} \end{pmatrix}, \\ p_L^{1,app} = p^{0,int} + \varepsilon(p^{1,int} + p^{1,B} + p^{1,T}) + \varepsilon^2(p^{2,B} + p^{2,T}), \\ \varphi_L^{1,app} = \varphi^{0,int} + \varepsilon(\varphi^{1,int} + \varphi^{1,B} + \varphi^{1,T}) + \varepsilon^2(\varphi^{2,B} + \varphi^{2,T}), \\ \boldsymbol{j}_L^{1,app} = \nabla\varphi_L^{1,app} - e^3 \wedge \boldsymbol{u}_L^{1,app}, \end{cases} \tag{2.35}$$

where the approximate solution $(\boldsymbol{u}_L^{1,app}, \boldsymbol{j}_L^{1,app})$ satisfies

$$\nabla \cdot \boldsymbol{u}_L^{1,app} = 0, \tag{2.36}$$

$$\nabla \cdot \boldsymbol{j}_L^{1,app} = \varepsilon(\Delta_h\varphi^{1,B} + \nabla_h^\perp \cdot \boldsymbol{u}_h^{1,B} + \Delta_h\varphi^{1,T} + \nabla_h^\perp \cdot \boldsymbol{u}_h^{1,T}) + \varepsilon^2(\Delta_h\varphi^{2,B} + \Delta_h\varphi^{2,T}), \tag{2.37}$$

and

$$\boldsymbol{u}_L^{1,app}|_{z=0,1} = \begin{pmatrix} \boldsymbol{u}_h^{0,B}|_{\theta=\frac{1}{\varepsilon}} \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} \boldsymbol{u}_h^{1,B}|_{\theta=\frac{1}{\varepsilon}} \\ (-1)^{1-z}_{|z=0,1} u_3^{1,B}|_{\theta=\frac{1}{\varepsilon}} \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 \\ (-1)^{z}_{|z=0,1} u_3^{2,B}|_{\theta=\frac{1}{\varepsilon}} \end{pmatrix}, \tag{2.38}$$

$$\boldsymbol{j}_{L,3}^{1,app}|_{z=0,1} = 0, \tag{2.39}$$

$$\boldsymbol{j}_{L,h}^{1,app} \cdot \boldsymbol{n}_s|_{\partial S} \neq 0, \quad \boldsymbol{u}_L^{1,app}|_{\partial S} \neq 0. \tag{2.40}$$

The next goal is to correct these incompressibility conditions and boundary conditions one by one.

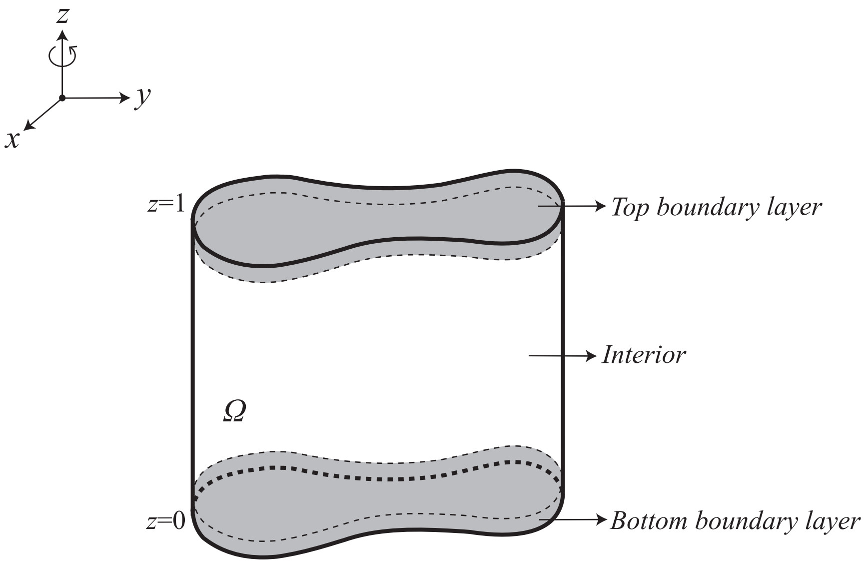

Figure 1. Schematic of the top and bottom boundary layers in a cylindrical domain Ω.

2.2. *Modification of the top and bottom boundary conditions of* $\boldsymbol{u}_L^{1,app}$

This subsection aims to correct the top and bottom boundary conditions in (2.40), and we establish the correction term $\boldsymbol{u}^c$, namely

$$\boldsymbol{u}^c = \boldsymbol{u}^{0,c} + \varepsilon \boldsymbol{u}^{1,c} + \varepsilon^2 \boldsymbol{u}^{2,c}.$$

Note that we can now construct the correction term $\boldsymbol{u}^c$ in such a way as to ensure that $\boldsymbol{u}^c$ satisfies the incompressibility condition. We therefore make $\boldsymbol{u}^{i,c}$ $(i = 0, 1, 2)$ satisfy

$$\begin{aligned}
\boldsymbol{u}^{0,c} =& (-\cos(2\pi z)\boldsymbol{u}_h^{0,B}|_{\theta=\frac{1}{\varepsilon}}, \frac{\sin(2\pi z)}{2\pi}\nabla_h \cdot \boldsymbol{u}_h^{0,B}|_{\theta=\frac{1}{\varepsilon}}), && (2.41)\\
\boldsymbol{u}^{1,c} =& (-\cos(2\pi z)\boldsymbol{u}_h^{1,B}|_{\theta=\frac{1}{\varepsilon}}, \frac{\sin(2\pi z)}{2\pi}\nabla_h \cdot \boldsymbol{u}_h^{1,B}|_{\theta=\frac{1}{\varepsilon}}) \\
&+ (\pi \sin(\pi z)\int_{\frac{1}{\varepsilon}}^{+\infty} \boldsymbol{u}_h^{0,B} d\theta, \cos(\pi z) u_3^{1,B}|_{\theta=\frac{1}{\varepsilon}}), && (2.42)\\
\boldsymbol{u}^{2,c} =& (-\pi \sin(\pi z)\int_{\frac{1}{\varepsilon}}^{+\infty} \boldsymbol{u}_h^{1,B} d\theta, -\cos(\pi z) u_3^{2,B}|_{\theta=\frac{1}{\varepsilon}}). && (2.43)
\end{aligned}$$

It is clear from the expression (2.41)–(2.43) above that

$$\|\boldsymbol{u}^c\|_{W^{1,\infty}(0,T;H^1(\Omega))} = O(\varepsilon^{\frac{1}{2}}). \tag{2.44}$$

At this point, an approximate solution $(\boldsymbol{u}_L^{2,app}, \boldsymbol{j}_L^{2,app})$ is obtained, i.e.

$$\begin{cases}
\boldsymbol{u}_L^{2,app} = \boldsymbol{u}_L^{1,app} + \boldsymbol{u}^c, \\
\boldsymbol{j}_L^{2,app} = \nabla \varphi_L^{1,app} - e^3 \wedge (\boldsymbol{u}_L^{1,app} + \boldsymbol{u}^c),
\end{cases}$$

and $(\boldsymbol{u}_L^{2,app}, \boldsymbol{j}_L^{2,app})$ satisfies

$$\nabla \cdot \boldsymbol{u}_L^{2,app} = 0, \tag{2.45}$$

$$\nabla \cdot \boldsymbol{j}_L^{2,app} = \varepsilon(\Delta_h \varphi^{1,B} + \nabla_h^\perp \cdot \boldsymbol{u}_h^{1,B} + \Delta_h \varphi^{1,T} + \nabla_h^\perp \cdot \boldsymbol{u}_h^{1,T}) + \varepsilon^2(\Delta_h \varphi^{2,B} + \Delta_h \varphi^{2,T}) \\ + \nabla_h^\perp \cdot \boldsymbol{u}_h^{0,c} + \varepsilon \nabla_h^\perp \cdot \boldsymbol{u}_h^{1,c} + \varepsilon^2 \nabla_h^\perp \cdot \boldsymbol{u}_h^{2,c}, \tag{2.46}$$

and

$$j_{L,3}^{2,app} \,|_{z=0,1} = 0, \quad \boldsymbol{u}_L^{2,app} \,|_{z=0,1} = 0, \tag{2.47}$$

$$\boldsymbol{j}_{L,h}^{2,app} \cdot \boldsymbol{n}_s \,|_{\partial S} \neq 0, \quad \boldsymbol{u}_L^{2,app} \,|_{\partial S} \neq 0. \tag{2.48}$$

In the analysis above, we corrected the top and bottom boundary conditions for the approximate solution of the velocity field. Below, we correct the lateral boundary conditions.

2.3. *Correction of the lateral boundary conditions for* $\boldsymbol{u}_L^{2,app}$

The purpose of this subsection is to correct the lateral boundary conditions in (2.48) for $\boldsymbol{u}_L^{2,app}$. The horizontal component of the approximate solution $\boldsymbol{u}^{2,app}$ consists of $\boldsymbol{u}_h^{0,int}$. It is therefore natural to impose a Dirichlet boundary condition on the velocity field $\boldsymbol{u}_h^{0,int}(t,x,y)$ in the bounded domain S in $\mathbb{R}^2$:

$$\boldsymbol{u}_h^{0,int} \,|_{\partial S} = 0. \tag{2.49}$$

Thus, we have

$$\boldsymbol{u}_h^{2,app} \,|_{\partial S} = 0. \tag{2.50}$$

Below, we correct the vertical component of the approximate solution $\boldsymbol{u}_L^{2,app}$. Referring to [3], we introduce $d : S \mapsto \mathbb{R}$ as a distance to the side S, and construct the lateral correction terms in the region of size ε^σ ($\frac{1}{2} < \sigma < 1$) near the lateral boundary (see Figure 2). The value of σ here will be determined later.

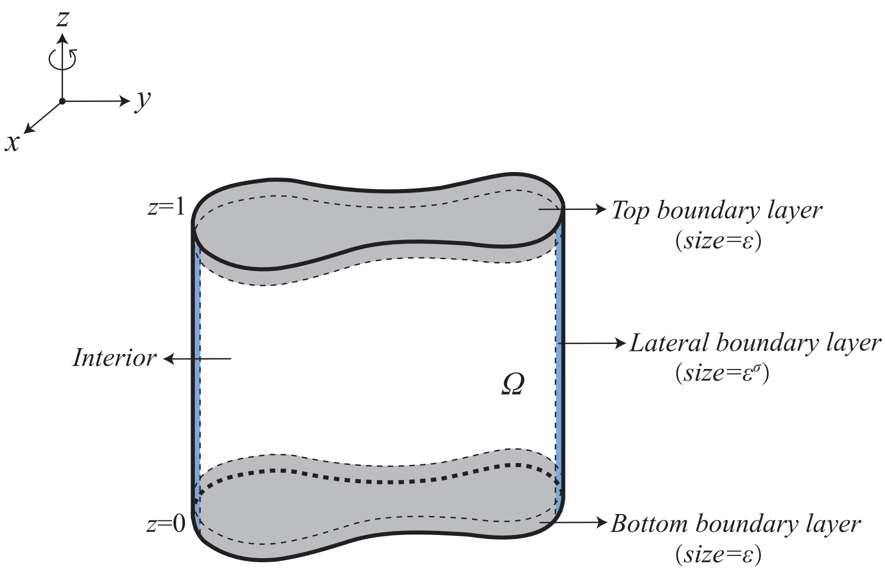

Figure 2. Schematic of the boundary layers in a cylindrical domain Ω.

First, using $\boldsymbol{w}^{0,c} = (\boldsymbol{w}_h^{0,c}, w_3^{0,c})$ to correct the ε^0-order term, we write

$$w_3^{0,c} = -\mathrm{e}^{-\frac{d(x,y)}{\varepsilon^\sigma}} u_3^{0,c} = -\mathrm{e}^{-\frac{d(x,y)}{\varepsilon^\sigma}} \Big(\frac{\sin(2\pi z)}{2\pi} \nabla_h \cdot \boldsymbol{u}_h^{0,B}|_{\theta=\frac{1}{\varepsilon}} \Big), \tag{2.51}$$

and $\boldsymbol{w}_h^{0,c} = 0$. $w_3^{0,c}$ vanishes at the top and bottom boundaries. Furthermore, the presence of $\mathrm{e}^{-\frac{d(x,y)}{\varepsilon^\sigma}}$ in $w_3^{0,c}$ causes $w_3^{0,c}$ to vanish when ε is sufficiently small, as well as away from the region where the size of the side edges is ε^σ. At the same time, $\boldsymbol{w}^{0,c}$ does not satisfy the incompressibility condition and has

$$\nabla \cdot \boldsymbol{w}^{0,c} = -\mathrm{e}^{-\frac{d(x,y)}{\varepsilon^\sigma}} \cos(2\pi z) \nabla_h \cdot \boldsymbol{u}_h^{0,B} \mid_{\theta=\frac{1}{\varepsilon}} .$$

Concerning $\boldsymbol{w}^{0,c}$, we have the following estimates:

$$\begin{cases} \|\nabla \cdot \boldsymbol{w}^{0,c}\|_{W^{1,\infty}(\mathbb{R}_+,L^2(\Omega))} = O(\varepsilon^{\frac{\sigma+1}{2}}), \\ \|\boldsymbol{w}^{0,c}\|_{W^{1,\infty}(\mathbb{R}_+,L^2(\Omega))} = O(\varepsilon^{\frac{\sigma+1}{2}}), \\ \|\boldsymbol{w}^{0,c}\|_{W^{1,\infty}(\mathbb{R}_+,H^1(\Omega))} = O(\varepsilon^{\frac{1}{2}}). \end{cases}$$

Secondly, using $\boldsymbol{w}^{1,c} = (\boldsymbol{w}_h^{1,c}, w_3^{1,c})$ to correct for the ε^1-order side boundary term, we write

$$w_3^{1,c} = -\mathrm{e}^{-\frac{d(x,y)}{\varepsilon^\sigma}} \varepsilon(u_3^{1,int} + u_3^{1,B} + u_3^{1,T} + u_3^{1,c}), \tag{2.52}$$

and

$$\begin{aligned} \boldsymbol{w}_h^{1,c} =& \varepsilon^\sigma(-1 + \mathrm{e}^{-\frac{d(x,y)}{\varepsilon^\sigma}}) \frac{\nabla d(x,y)}{|\nabla d(x,y)|^2} (\partial_\theta u_3^{1,B} - \partial_\lambda u_3^{1,T}) \\ &- \cos(2\pi z) \varepsilon^\sigma(-1 + \mathrm{e}^{-\frac{d(x,y)}{\varepsilon^\sigma}}) \frac{\nabla d(x,y)}{|\nabla d(x,y)|^2} \partial_\theta u_3^{1,B} \mid_{\theta=\frac{1}{\varepsilon}} . \end{aligned} \tag{2.53}$$

Clearly, from the definitions of $u_3^{1,int}$, $u_3^{1,B}$, $u_3^{1,T}$, and $u_3^{1,c}$, as well as the analysis above, it follows that $\boldsymbol{w}^{1,c}$ satisfies all boundary conditions. Nevertheless, it does not satisfy the incompressibility condition:

$$\begin{aligned} \nabla \cdot \boldsymbol{w}^{1,c} =& \varepsilon^\sigma(-1 + \mathrm{e}^{-\frac{d(x,y)}{\varepsilon^\sigma}}) \nabla_h \cdot \Big(\frac{\nabla d(x,y)}{|\nabla d(x,y)|^2} (\partial_\theta u_3^{1,B} - \partial_\lambda u_3^{1,T}) \Big) \\ &- \cos(2\pi z) \mathrm{e}^{-\frac{d(x,y)}{\varepsilon^\sigma}} \partial_\theta u_3^{1,B} \mid_{\theta=\frac{1}{\varepsilon}} - \mathrm{e}^{\frac{d(x,y)}{\varepsilon^\sigma}} \varepsilon(\partial_z u_3^{1,int} + \partial_z u_3^{1,c}) \\ &- \cos(2\pi z) \varepsilon^\sigma(-1 + \mathrm{e}^{-\frac{d(x,y)}{\varepsilon^\sigma}}) \nabla_h \cdot \Big(\frac{\nabla d(x,y)}{|\nabla d(x,y)|^2} \partial_\theta u_3^{1,B}|_{\theta=\frac{1}{\varepsilon}} \Big). \end{aligned} \tag{2.54}$$

We have the following estimates for $\boldsymbol{w}^{1,c}$:

$$\begin{cases} \|\nabla \cdot \boldsymbol{w}^{1,c}\|_{W^{1,\infty}(\mathbb{R}_+,L^2(\Omega))} = O(\varepsilon^\sigma), \\ \|\boldsymbol{w}^{1,c}\|_{W^{1,\infty}(\mathbb{R}_+,L^2(\Omega))} = O(\varepsilon^\sigma), \\ \|\boldsymbol{w}^{1,c}\|_{W^{1,\infty}(\mathbb{R}_+,L^2(\Omega))} = O(\varepsilon^{\sigma-\frac{1}{2}}). \end{cases}$$

Finally, utilizing $\boldsymbol{w}^{2,c} = (\boldsymbol{w}_h^{2,c}, w_3^{2,c})$ to correct the ε^2-order side boundary term, we write

$$w_3^{2,c} = -\mathrm{e}^{-\frac{d(x,y)}{\varepsilon^\sigma}} \varepsilon^2(u_3^{2,int} + u_3^{2,B} + u_3^{2,T} + u_3^{2,c}).$$

Since the higher-order correction term does not affect the subsequent analysis, we take $\boldsymbol{w}_h^{2,c} = 0$. Then $\boldsymbol{w}^{2,c}$ satisfies the boundary conditions, and

$$\nabla \cdot \boldsymbol{w}^{2,c} = -\mathrm{e}^{-\frac{d(x,y)}{\varepsilon^\sigma}} \varepsilon(\partial_\theta u_3^{2,B} - \partial_\lambda u_3^{2,T}) - \mathrm{e}^{-\frac{d(x,y)}{\varepsilon^\sigma}} \varepsilon^2(\partial_z u_3^{2,int} + \partial_z u_3^{2,c}),$$

as well as

$$\begin{cases}\|\nabla\cdot\boldsymbol{w}^{2,c}\|_{W^{1,\infty}(\mathbb{R}_+,L^2(\Omega))}=O(\varepsilon^{\frac{\sigma+1}{2}}),\\ \|\boldsymbol{w}^{2,c}\|_{W^{1,\infty}(\mathbb{R}_+,L^2(\Omega))}=O(\varepsilon^{\frac{\sigma+1}{2}}),\\ \|\boldsymbol{w}^{2,c}\|_{W^{1,\infty}(\mathbb{R}_+,H^1(\Omega))}=O(\varepsilon^{\frac{\sigma+1}{2}}).\end{cases}$$

Next, we take $\sigma=\frac{3}{4}$ and

$$\boldsymbol{w}^c=\boldsymbol{w}^{0,c}+\boldsymbol{w}^{1,c}+\boldsymbol{w}^{2,c}.$$

Moreover, $\boldsymbol{w}^c$ satisfies

$$\begin{cases}\|\nabla\cdot\boldsymbol{w}^{c}\|_{W^{1,\infty}(\mathbb{R}_+,L^2(\Omega))}=O(\varepsilon^{\frac{3}{4}}),\\ \|\boldsymbol{w}^{c}\|_{W^{1,\infty}(\mathbb{R}_+,L^2(\Omega))}=O(\varepsilon^{\frac{3}{4}}),\\ \|\boldsymbol{w}^{c}\|_{W^{1,\infty}(\mathbb{R}_+,H^1(\Omega))}=O(\varepsilon^{\frac{1}{4}}).\end{cases}\tag{2.55}$$

It is worth noting that while constructing $\boldsymbol{w}^c$, we need it to satisfy the incompressibility condition. According to [12, 21], $\boldsymbol{u}^w\in W^{1,\infty}(\mathbb{R}_+,H^1(\Omega))$ exists such that the following equations hold:

$$\begin{cases}\nabla\cdot\boldsymbol{u}^w=-\nabla\cdot\boldsymbol{w}^c,\\ \boldsymbol{u}^w|_{\partial\Omega}=0,\end{cases}$$

and

$$\|\boldsymbol{u}^w\|_{W^{1,\infty}(0,T;H^1(\Omega))}\lesssim\|\nabla\cdot\boldsymbol{w}^c\|_{W^{1,\infty}(0,T;L^2(\Omega))}=O(\varepsilon^{\frac{3}{4}}).\tag{2.56}$$

In the analysis above, we corrected the boundary and incompressibility conditions for the approximate solution of the velocity field. Moreover, we denote this new approximate solution $\boldsymbol{u}^{3,app}$ as

$$\boldsymbol{u}_L^{3,app}=\boldsymbol{u}_L^{1,app}+\boldsymbol{u}^c+\boldsymbol{w}^c+\boldsymbol{u}^w.$$

Moreover, let

$$\boldsymbol{j}_L^{3,app}=\nabla\varphi_L^{1,app}-e^3\wedge(\boldsymbol{u}_L^{1,app}+\boldsymbol{u}^c+\boldsymbol{w}^c+\boldsymbol{u}^w).$$

Due to we construct the correction term $\boldsymbol{u}^c+\boldsymbol{w}^c+\boldsymbol{u}^w$, relative to which we also correct the magnetic potential.

2.4. Correction of the incompressibility and boundary conditions for $\boldsymbol{j}_L^{3,app}$

In this subsection, we correct the incompressibility condition of $\boldsymbol{j}_L^{3,app}$ by constructing a correction term φ^c for the magnetic potential. By the order of ε in (2.46), we write φ^c as

$$\varphi^c=\varphi^{0,c}+\varphi^{1,c}+\varphi^{2,c}.$$

Next, according to [12,21], $\nabla\varphi^{0,c},\nabla\varphi^{1,c},\nabla\varphi^{2,c}\in L^\infty(\mathbb{R}_+,H^1(\Omega))$ exists such that the following equations hold:

$$\begin{cases}\nabla\cdot(\nabla\varphi^{0,c})=-\nabla_h^\perp\cdot\boldsymbol{u}_h^{0,c},\\ \nabla\varphi^{0,c}\ |_{\partial\Omega}=0,\end{cases}\tag{2.57}$$

$$\begin{cases}\nabla\cdot(\nabla\varphi^{1,c}) = -\nabla_h^{\perp}\cdot(\varepsilon\boldsymbol{u}_h^{1,c}+\varepsilon\boldsymbol{u}_h^{1,B}+\varepsilon\boldsymbol{u}_h^{1,T}+\boldsymbol{w}_h^c+\boldsymbol{u}_h^w)-\varepsilon\Delta_h(\varphi^{1,B}+\varphi^{1,T}),\\ \nabla\varphi^{1,c}\mid_{\partial\Omega}=0,\end{cases}\tag{2.58}$$

$$\begin{cases}\nabla\cdot(\nabla\varphi^{2,c}) = -\varepsilon^2\nabla_h^{\perp}\cdot\boldsymbol{u}_h^{2,c}-\varepsilon^2\Delta_h(\varphi^{2,B}+\varphi^{2,T}),\\ \nabla\varphi^{2,c}\mid_{\partial\Omega}=0.\end{cases}\tag{2.59}$$

Thus, we obtain a new approximate solution for the magnetic potential $\boldsymbol{j}_L^{4,app}$, i.e.,

$$\boldsymbol{j}_L^{4,app} = \nabla(\varphi_L^{1,app}+\varphi^c)-e^3\wedge(\boldsymbol{u}_L^{1,app}+\boldsymbol{u}^c+\boldsymbol{w}^c+\boldsymbol{u}^w),\tag{2.60}$$

which satisfies

$$\nabla\cdot\boldsymbol{j}_L^{4,app}=0,\quad j_{L,3}^{4,app}\mid_{z=0,1}=0.\tag{2.61}$$

In the following, correcting only the lateral boundary conditions of $\boldsymbol{j}^{4,app}$ is necessary. Through the analysis and construction process above, we can get

$$\boldsymbol{j}_{L,H}^{4,app}\cdot\boldsymbol{n}_s\mid_{\partial S}=(\varepsilon\nabla_h\varphi^{1,int}+\varepsilon\nabla_h\varphi^{1,B}+\varepsilon\nabla_h\varphi^{1,T}+\varepsilon^2\nabla_h\varphi^{2,B}+\varepsilon^2\nabla_h\varphi^{2,T})\cdot\boldsymbol{n}_s\mid_{\partial S}\ .$$

First, we take

$$\nabla_h\varphi^{1,int}\cdot\boldsymbol{n}_s\mid_{\partial S}=0.\tag{2.62}$$

Secondly, according to [12,21], $\nabla_h\varphi^w\in L^\infty(\mathbb{R}_+,H^1(\Omega))$ exists such that the following equations hold:

$$\begin{cases}\nabla_h\cdot(\nabla_h\varphi^w)=0,\\ \nabla_h\varphi^w\mid_{\partial\Omega}=-(\varepsilon\nabla_h\varphi^{1,B}+\varepsilon\nabla_h\varphi^{1,T}+\varepsilon^2\nabla_h\varphi^{2,B}+\varepsilon^2\nabla_h\varphi^{2,T})\mid_{\partial\Omega}\ .\end{cases}$$

In summary, we have completed the construction of the approximate solution and satisfied all its incompressibility and boundary conditions.

2.5. Construction of a nonlinear approximate solution

The previous subsections considered the approximate system under the linear system in (2.3). On the basis of the analysis above, we construct the approximate solution to the following system:

$$\begin{cases}\partial_t\boldsymbol{u}^{app}-\varepsilon\Delta\boldsymbol{u}^{app}+(\boldsymbol{u}^{app}\cdot\nabla)\boldsymbol{u}^{app}+\frac{\alpha}{\varepsilon}e^3\wedge\boldsymbol{u}^{app}+\frac{\beta}{\varepsilon}e^3\wedge\boldsymbol{j}^{app}+\frac{1}{\varepsilon}\nabla p^{app}=\boldsymbol{R}^{app},\\ \boldsymbol{j}^{app}-\nabla\varphi^{app}+e^3\wedge\boldsymbol{u}^{app}=0,\\ \nabla\cdot\boldsymbol{u}^{app}=\nabla\cdot\boldsymbol{j}^{app}=0,\end{cases}\tag{2.63}$$

where $\boldsymbol{R}^{app}$ represents the residual term obtained by substituting the corrected approximate solution into the original system, with the boundary conditions

$$\boldsymbol{u}^{app}|_{\partial\Omega}=0,\quad j_3^{app}|_{z=0,1}=0,\quad \boldsymbol{j}_h^{app}\cdot\boldsymbol{n}_s|_{\partial S}=0.\tag{2.64}$$

First, we consider the principal part of the approximate solution $\boldsymbol{u}^{app}$ and let it be $\bar{\boldsymbol{u}}(t,x,y)$. By analyzing the linear part above and combining (2.25), (2.49), and (2.62), it is natural to set $\bar{\boldsymbol{u}}(t,x,y)=(\bar{\boldsymbol{u}}_h(t,x,y),0)$ as

$$\begin{cases}\partial_t\bar{\boldsymbol{u}}_h+(\bar{\boldsymbol{u}}_h\cdot\nabla_h)\bar{\boldsymbol{u}}_h+\gamma\bar{\boldsymbol{u}}_h+\eta\bar{\boldsymbol{u}}_{h,\perp}+\nabla_h\bar{p}+\beta\nabla_h^{\perp}\bar{\varphi}=0,\\ \nabla_h\cdot\bar{\boldsymbol{u}}_h=0,\\ \Delta_h\bar{\varphi}=-\frac{2\cos\frac{\tau}{2}}{(\alpha^2+\beta^2)^{\frac{1}{4}}}\nabla_h^{\perp}\cdot\bar{\boldsymbol{u}}_h,\end{cases}\tag{2.65}$$

with the boundary conditions

$$\bar{\boldsymbol{u}}_h \mid_{\partial S} = 0, \quad \nabla_h \bar{\varphi} \cdot \boldsymbol{n}_s \mid_{\partial S} = 0. \tag{2.66}$$

The remaining terms all consist of the central part $\bar{\boldsymbol{u}}_h(t, x, y)$. It may be helpful to use the original notation so that the approximate solution $(\boldsymbol{u}^{app}, p^{app}, \varphi^{app}, \boldsymbol{j}^{app})$ is

$$\begin{cases} \boldsymbol{u}^{app} = \bar{\boldsymbol{u}} + \boldsymbol{u}^{0,B} + \boldsymbol{u}^{0,T} + \varepsilon(\boldsymbol{u}^{1,int} + \boldsymbol{u}^{1,B} + \boldsymbol{u}^{1,T}) + \varepsilon^2(\boldsymbol{u}^{2,B} + \boldsymbol{u}^{2,T}) + \boldsymbol{u}^c + \boldsymbol{w}^c + \boldsymbol{u}^w, \\ p^{app} = p^{0,int} + \varepsilon(\bar{p} + p^{1,B} + p^{1,T}) + \varepsilon^2(p^{2,B} + p^{2,T}), \\ \varphi^{app} = \varphi^{0,int} + \varepsilon(\bar{\varphi} + \varphi^{1,B} + \varphi^{1,T}) + \varepsilon^2(\varphi^{2,B} + \varphi^{2,T}) + \varphi^c + \varphi^w, \\ \boldsymbol{j}^{app} = \nabla\varphi^{app} - e^3 \wedge \boldsymbol{u}^{app}, \end{cases} \tag{2.67}$$

where $\boldsymbol{u}_h^{0,int}$ in the original forms is substituted for $\bar{\boldsymbol{u}}_h$ in all but the main part $(\bar{\boldsymbol{u}}, \bar{p}, \bar{\varphi})$.

According to the construction, the following asymptotic behavior holds.

Proposition 2.1. *For the approximate solution $\boldsymbol{u}^{app}$ given above, if $\bar{\boldsymbol{u}}_h \in L^2(\mathbb{R}^2)$, it satisfies*

$$\lim_{\varepsilon \to 0^+} \|\boldsymbol{u}^{app} - \bar{\boldsymbol{u}}\|_{L^2(\Omega)} = 0.$$

Proof. With the expression $(2.67)_1$ for $\boldsymbol{u}^{app}$, it can be shown that

$$\begin{aligned} &\|\boldsymbol{u}^{app} - \bar{\boldsymbol{u}}\|_{L^2(\Omega)} \\ &\le \sum_{i=0}^{2} \varepsilon^i(\|\boldsymbol{u}^{i,B}\|_{L^2(\Omega)} + \|\boldsymbol{u}^{i,T}\|_{L^2(\Omega)}) + \varepsilon\|\boldsymbol{u}^{1,\text{int}}\|_{L^2(\Omega)} + \|\boldsymbol{u}^c\|_{L^2(\Omega)} + \|\boldsymbol{w}^c\|_{L^2(\Omega)} + \|\boldsymbol{u}^w\|_{L^2(\Omega)}. \end{aligned}$$

□

The presence of the exponential factors $\mathrm{e}^{-a\theta}$and $\mathrm{e}^{-a\lambda}$ in the boundary layer terms results in the subsequent estimates being small. As an illustration, consider the example of the bottom boundary term $\|\boldsymbol{u}^{0,B}\|_{L^2(\Omega)}$. From (2.13), we have

$$\begin{aligned} &\|\boldsymbol{u}^{0,B}\|_{L^2(\Omega)}^2 \\ &= \int_0^1 \int_S \left|\mathrm{e}^{-a\theta}\Big(\cos(b\theta)\bar{\boldsymbol{u}}_h + \sin(b\theta)\bar{\boldsymbol{u}}_{h,\perp}\Big)\right|^2 dxdydz \\ &\le \int_0^1 \int_S |\mathrm{e}^{-\frac{az}{\varepsilon}}(\bar{\boldsymbol{u}}_h + \bar{\boldsymbol{u}}_{h,\perp})|^2 \, dxdydz \\ &= \int_0^{\frac{a}{\varepsilon}} \int_S \frac{\varepsilon}{a}|\mathrm{e}^{-\frac{az}{\varepsilon}}(\bar{\boldsymbol{u}}_h + \bar{\boldsymbol{u}}_{h,\perp})|^2 \, dxdyd\tfrac{az}{\varepsilon} \lesssim \varepsilon\|\bar{\boldsymbol{u}}_h\|_{L^2(S)}^2. \end{aligned} \tag{2.68}$$

Similarly, to estimate other top and bottom boundary terms, by combining (2.68) with the expression (2.23) for the interior term $\boldsymbol{u}^{1,\text{int}}$, we get

$$\sum_{i=0}^{2} \varepsilon^i(\|\boldsymbol{u}^{i,B}\|_{L^2(\Omega)} + \|\boldsymbol{u}^{i,T}\|_{L^2(\Omega)}) + \varepsilon\|\boldsymbol{u}^{1,\text{int}}\|_{L^2(\Omega)} \lesssim \varepsilon^{\frac{1}{2}}\|\bar{\boldsymbol{u}}_h\|_{L^2(S)}. \tag{2.69}$$

Recalling (2.44), (2.55), and (2.56), one has

$$\|\boldsymbol{u}^c\|_{L^2(\Omega)} + \|\boldsymbol{w}^c\|_{L^2(\Omega)} + \|\boldsymbol{u}^w\|_{L^2(\Omega)} \lesssim \varepsilon^{\frac{1}{2}}\|\bar{\boldsymbol{u}}_h\|_{L^2(S)}. \tag{2.70}$$

Combining (2.68)–(2.70) gives

$$\lim_{\varepsilon \to 0^+} \|\boldsymbol{u}^{app} - \bar{\boldsymbol{u}}\|_{L^2(\Omega)} = 0.$$

3. Properties of the 2D limiting system

This section investigates the following properties of 2D limit system:

$$\begin{cases} \partial_t \bar{\boldsymbol{u}}_h + (\bar{\boldsymbol{u}}_h \cdot \nabla_h)\bar{\boldsymbol{u}}_h + \gamma \bar{\boldsymbol{u}}_h + \eta \bar{\boldsymbol{u}}_{h,\perp} + \nabla_h \bar{p} + \beta \nabla_h^{\perp} \bar{\varphi} = 0, \\ \nabla_h \cdot \bar{\boldsymbol{u}}_h = 0, \\ \Delta_h \bar{\varphi} = -\frac{2\cos\frac{\tau}{2}}{(\alpha^2+\beta^2)^{\frac{1}{4}}} \nabla_h^{\perp} \cdot \bar{\boldsymbol{u}}_h, \end{cases} \tag{3.1}$$

with the boundary conditions

$$\bar{\boldsymbol{u}}_h \mid_{\partial S} = 0, \quad \nabla_h \bar{\varphi} \cdot \boldsymbol{n}_s \mid_{\partial S} = 0. \tag{3.2}$$

By applying $\nabla_h^{\perp} \cdot$ to (3.1), and writing $\bar{\omega} = \nabla_h^{\perp} \cdot \bar{\boldsymbol{u}}_h$, we can obtain the vorticity system:

$$\partial_t \bar{\omega} + (\bar{\boldsymbol{u}}_h \cdot \nabla_h)\bar{\omega} + \gamma \bar{\omega} + \beta \Delta_h \bar{\varphi} = 0, \tag{3.3}$$

where

$$\Delta_h \bar{\varphi} = -\frac{2\cos\frac{\tau}{2}}{(\alpha^2+\beta^2)^{\frac{1}{4}}} \bar{\omega}.$$

Therefore, combined with the definition of γ, (3.3) can be rewritten as

$$\partial_t \bar{\omega} + (\bar{\boldsymbol{u}}_h \cdot \nabla_h)\bar{\omega} + \frac{2\alpha \sin\frac{\tau}{2}}{(\alpha^2+\beta^2)^{\frac{1}{4}}} \bar{\omega} = 0. \tag{3.4}$$

As the flow is divergence-free, with $\nabla_h \cdot \bar{\boldsymbol{u}}_h = 0$, we have

$$\bar{\omega} = \nabla_h^{\perp} \cdot \bar{\boldsymbol{u}}_h, \quad \bar{\boldsymbol{u}}_h = -\nabla_h^{\perp} (-\Delta_h)^{-1} \bar{\omega}. \tag{3.5}$$

Proposition 3.1. *Let $\bar{\boldsymbol{u}}_{0,h}(x,y) \in H^1(S)$ be a divergence-free vector field, $\bar{\omega}_0 = \nabla_h^{\perp} \cdot \bar{\boldsymbol{u}}_{0,h}$ be the initial vorticity, and $(\bar{\boldsymbol{u}}, \bar{p}, \bar{\varphi})$ be a pair of solution to the systems in* (3.1) *and* (3.2) *with the initial data $\bar{\boldsymbol{u}}_0 = (\bar{\boldsymbol{u}}_{0,h}, 0)$. Then the following estimations are valid:*

$$\|\bar{\boldsymbol{u}}_h\|^2_{L^2(S)}, \|\bar{\omega}\|^2_{L^2(S)}, \|\nabla_h \bar{\boldsymbol{u}}_h\|^2_{L^2(S)} \lesssim \mathrm{e}^{-\nu t}, \tag{3.6}$$

where $\nu = \frac{2\alpha \sin\frac{\tau}{2}}{(\alpha^2+\beta^2)^{\frac{1}{4}}}$.

Proof. Given the divergence-free condition, we derive the L^2 estimate for $\bar{\boldsymbol{u}}_h$ as follows:

$$\frac{1}{2}\frac{d}{dt}\|\bar{\boldsymbol{u}}_h\|^2_{L^2(S)} + \gamma \|\bar{\boldsymbol{u}}_h\|^2_{L^2(S)} + \beta \langle \nabla_h^{\perp} \bar{\varphi}, \bar{\boldsymbol{u}}_h \rangle = 0.$$

From $(2.67)_4$, it follows that $\nabla \bar{\varphi} = \boldsymbol{j}^{1,int} + e^3 \wedge \boldsymbol{u}^{1,int}$, and from $\bar{\boldsymbol{u}} = (\bar{\boldsymbol{u}}_h, 0)$, we have

$$\begin{aligned} \langle \nabla_h^{\perp} \bar{\varphi}, \bar{\boldsymbol{u}}_h \rangle &= \left\langle e^3 \wedge \boldsymbol{j}^{1,int}, \bar{\boldsymbol{u}} \right\rangle + \left\langle e^3 \wedge (e^3 \wedge \boldsymbol{u}^{1,int}), \bar{\boldsymbol{u}} \right\rangle \\ &= \int_S e^3 \wedge \boldsymbol{j}^{1,int} \cdot \bar{\boldsymbol{u}} - \int_S \boldsymbol{u}^{1,int} \cdot \bar{\boldsymbol{u}} \\ &= -\int_S \boldsymbol{j}^{1,int} \cdot (e^3 \wedge \bar{\boldsymbol{u}}) + \int_S \Big(\frac{2\sin\frac{\tau}{2}}{(\alpha^2+\beta^2)^{\frac{1}{4}}} \bar{\boldsymbol{u}}_{h,\perp} - \frac{2\cos\frac{\tau}{2}}{(\alpha^2+\beta^2)^{\frac{1}{4}}} \bar{\boldsymbol{u}}_h \Big) \cdot \bar{\boldsymbol{u}}_h \\ &= -\int_S \boldsymbol{j}^{1,int} \cdot \nabla \varphi^{0,int} - \int_S \frac{2\cos\frac{\tau}{2}}{(\alpha^2+\beta^2)^{\frac{1}{4}}} |\bar{\boldsymbol{u}}_h|^2 \\ &= -\frac{2\cos\frac{\tau}{2}}{(\alpha^2+\beta^2)^{\frac{1}{4}}} \|\bar{\boldsymbol{u}}_h\|^2_{L^2(S)}. \end{aligned}$$

A simple derivation gives

$$\frac{1}{2}\frac{d}{dt}\|\bar{\boldsymbol{u}}_h\|_{L^2(S)}^2+\frac{2\alpha\sin\frac{\tau}{2}}{(\alpha^2+\beta^2)^{\frac{1}{4}}}\|\bar{\boldsymbol{u}}_h\|_{L^2(S)}^2=0.$$

We write $\nu=\frac{2\alpha\sin\frac{\tau}{2}}{(\alpha^2+\beta^2)^{\frac{1}{4}}}$, and thus

$$\|\bar{\boldsymbol{u}}_h\|_{L^2(S)}^2\le \mathrm{e}^{-\nu t}\|\bar{\boldsymbol{u}}_{0,h}\|_{L^2(S)}^2.$$

From (3.5), it is clear that

$$\frac{1}{2}\frac{d}{dt}\|\bar{\omega}\|_{L^2(S)}^2+\nu\|\bar{\omega}\|_{L^2(S)}^2=0,$$

and thus

$$\|\bar{\omega}\|_{L^2(S)}^2\le \mathrm{e}^{-\nu t}\|\bar{\omega}_0\|_{L^2(S)}^2.$$

as well as

$$\|\nabla_h\bar{\boldsymbol{u}}_h\|_{L^2(S)}^2\lesssim\|\bar{\omega}\|_{L^2(S)}^2\le \mathrm{e}^{-\nu t}\|\bar{\omega}_0\|_{L^2(S)}^2.$$ □

Remark 3.1. *Furthermore, combining this with* (3.6) *yields an estimate of* (3.6) *for* $\|\bar{\boldsymbol{u}}\|_{L^\infty(S)}$.

In our proof, we require the bound for $\|\nabla_h\bar{\boldsymbol{u}}_h\|_{L^\infty(S)}$. This necessity arises from the Calderón-Zygmund theory of singular integral operators, which asserts that the mapping $\bar{\omega}\to\nabla_h\bar{\boldsymbol{u}}_h$ is continuous within the space $L^s(S)$ for $1<s<\infty$. However, the case when $s=\infty$ presents additional complexities. To address this, we will establish the desired bound by employing the Littlewood-Paley decomposition in the subsequent steps.

First, let $C=\{\xi\in\mathbb{R}^2|\frac{3}{4}\le|\xi|\le\frac{4}{3}\}$. The radial functions ψ_{-1} and ψ take values in $[0,1]$ and have support in $B(0,\frac{4}{3})$ and C, respectively, such that

$$\forall\xi\in\mathbb{R}^2,\quad \psi_{-1}(\xi)+\sum_{j\ge0}\psi(2^{-j}\xi)=1.$$

We then take $\psi_j(\xi)=\psi(2^{-j}\xi)$. Obviously, $\psi_j(j>-1)$ is supported in $2^{j-1}<|\xi|<2^{j+2}$. We write

$$f_j(x)=\mathcal{F}^{-1}[\psi_j(\xi)\mathcal{F}(f)],\quad j\in\mathbb{Z},\tag{3.7}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier and inverse Fourier transforms, respectively. Recalling (3.1), we see that any function $f\in L^1(S)$ holds:

$$f=\sum_{j\ge-1}f_j(x).\tag{3.8}$$

Proposition 3.2. *Let* $\bar{\boldsymbol{u}}_{0,h}(x,y)\in H^{a+1}(S)(a>1)$ *be a divergence-free vector field,* $\bar{\omega}_0=\nabla_h^\perp\cdot\bar{\boldsymbol{u}}_{0,h}$ *be the initial vorticity, and* $(\bar{\boldsymbol{u}},\bar{p},\bar{\varphi})$ *be a pair of solutions to the system in* (3.1) *with the initial data* $\bar{\boldsymbol{u}}_0=(\bar{\boldsymbol{u}}_{0,h},0)$. *Then there holds*

$$\|\nabla_h\bar{\boldsymbol{u}}_h\|_{L^\infty(S)}\lesssim \mathrm{e}^{-\nu t}.\tag{3.9}$$

Proof. From (3.8), one has

$$\bar{\boldsymbol{u}}_h=\sum_{j\ge-1}\mathcal{F}^{-1}\Big(\psi_j\mathcal{F}(\bar{\boldsymbol{u}}_h)\Big)=\sum_{j\ge-1}\bar{\boldsymbol{u}}_{h,j},$$

and

$$\|\nabla_h\bar{\boldsymbol{u}}_h\|_{L^\infty(S)}\le\sum_{j\ge-1}\|\nabla_h\bar{\boldsymbol{u}}_{h,j}\|_{L^\infty(S)}\le\|\nabla_h\bar{\boldsymbol{u}}_{h,-1}\|_{L^\infty(S)}+\sum_{j>-1}\|\nabla_h\bar{\boldsymbol{u}}_{h,j}\|_{L^\infty(S)}.\tag{3.10}$$

The first term on the right-hand side can easily be bounded using the Bernstein inequality (for a more specific elaboration of the inequality, see [5])

$$\|\nabla_h \bar{\boldsymbol{u}}_{h,-1}\|_{L^\infty(S)} \lesssim \|\bar{\boldsymbol{u}}_h\|_{L^\infty(S)}. \tag{3.11}$$

Combining this with (3.5), we have

$$\sum_{j>-1} \|\nabla_h \bar{\boldsymbol{u}}_{h,j}\|_{L^\infty(S)} = \sum_{j>-1} \|\nabla_h \nabla_h^\perp (-\Delta_h)^{-1} \bar{\omega}_j\|_{L^\infty(S)} \le \sum_{j>-1} \|\bar{\omega}_j\|_{L^\infty(S)}. \tag{3.12}$$

Thus, from (3.10)–(3.12) and the results of Proposition 3.1, we have

$$\begin{aligned} &\|\nabla_h \bar{\boldsymbol{u}}_h\|_{L^\infty(S)} \\ &\lesssim \|\bar{\boldsymbol{u}}_h\|_{L^\infty(S)} + \sum_{j>-1} \|\bar{\omega}_j\|_{L^\infty(S)} \lesssim \|\bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)} \mathrm{e}^{-\nu t} + \sum_{j>-1} \|\bar{\omega}_j\|_{L^\infty(S)}. \end{aligned} \tag{3.13}$$

Now, we turn to the term $\sum_{j>-1} \|\bar{\omega}_j\|_{L^\infty(S)}$. Applying δ_j to (3.4), we have

$$\begin{cases} \partial_t \bar{\omega}_j + (\bar{\boldsymbol{u}}_h \cdot \nabla_h) \bar{\omega}_j + \nu \bar{\omega}_j = -[\delta_j, (\bar{\boldsymbol{u}}_h \cdot \nabla_h)] \bar{\omega}, \\ \bar{\omega}_j \mid_{t=0} = \bar{\omega}_{j0}, \end{cases} \tag{3.14}$$

where [,] stands for the commutator. Let

$$\bar{\omega} = \sum_{j \ge -1} \bar{\omega}_j,$$

with N to be determined later. One then has

$$\sum_{j>-1} \|\bar{\omega}_j\|_{L^\infty(S)} = \sum_{-1<j<N} \|\bar{\omega}_j\|_{L^\infty(S)} + \sum_{j \ge N} \|\bar{\omega}_j\|_{L^\infty(S)}. \tag{3.15}$$

From (3.6)–(3.8) and (3.14), for $j < N$, we get

$$\begin{aligned} &\|\bar{\omega}_j\|_{L^\infty(S)} \\ &\lesssim \mathrm{e}^{-\nu t} \|\bar{\omega}_{j0}\|_{L^\infty(S)} + \int_0^t \mathrm{e}^{-\nu(t-s)} \|[\delta_j, (\bar{\boldsymbol{u}}_h \cdot \nabla_h)] \bar{\omega}\|_{L^\infty(S)} \, ds \\ &\lesssim \mathrm{e}^{-\nu t} \|\bar{\omega}_{j0}\|_{L^\infty(S)} + \int_0^t \mathrm{e}^{-\nu(t-s)} \|\psi_j \mathcal{F}((\bar{\boldsymbol{u}}_h \cdot \nabla_h) \bar{\omega})\|_{L^1(S)} \, ds \\ &\lesssim \mathrm{e}^{-\nu t} \|\bar{\omega}_{j0}\|_{L^\infty(S)} + \mathrm{e}^{-\nu t} (\|\bar{\boldsymbol{u}}_{0,h}\|_{L^2(S)} + \|\bar{\omega}_0\|_{H^1(S)}), \end{aligned} \tag{3.16}$$

where we used the results of Proposition 3.2. Thus we get

$$\sum_{j \le N} \|\bar{\omega}_j\|_{L^\infty(S)} \le \mathrm{e}^{-\nu t} N \sum_{j \le N} \|\bar{\omega}_{j0}\|_{L^\infty(S)} + \mathrm{e}^{-\nu t} N (\|\bar{\boldsymbol{u}}_{0,h}\|_{L^2(S)} + \|\bar{\omega}_0\|_{H^1(S)}). \tag{3.17}$$

Furthermore, to deal with the case $j > N$, similar to (3.16), we get

$$\sum_{j>N} \|\bar{\omega}_j\|_{L^\infty(S)} \tag{3.18}$$

$$\leq \mathrm{e}^{-\nu t}2^{-\frac{(a-1)N}{2}}\sum_{j>N}\|\nabla_h^{\frac{a-1}{2}}\bar{\omega}_{j0}\|_{L^\infty(S)}+\mathrm{e}^{-\nu t}2^{-(a-1)N}(\|\bar{\boldsymbol{u}}_{0,h}\|_{L^2(S)}+\|\bar{\omega}_0\|_{H^1(S)}).$$

If we combine (3.17) and (3.18), taking $N=\Big[\log_2\frac{2}{a-1}\Big(1+\frac{1+\sum_{j\geq-1}\|\nabla_h^{\frac{a-1}{2}}\bar{\omega}_{j0}\|_{L^\infty(S)}}{1+\sum_{j\geq-1}\|\bar{\omega}_{j0}\|_{L^\infty(S)}}\Big)\Big]$, the following holds:

$$\begin{aligned}&\sum_{j\geq-1}\|\bar{\omega}_j\|_{L^\infty(S)} \qquad (3.19)\\ &\lesssim \mathrm{e}^{-\nu t}\Big(\ln(1+\sum_{j\geq-1}\|\nabla_h^{\frac{a-1}{2}}\bar{\omega}_{j0}\|_{L^\infty(S)})+\|\bar{\boldsymbol{u}}_{0,h}\|_{L^2(S)}+\|\bar{\omega}_0\|_{H^1(S)}\Big)\\ &\quad\cdot\Big(\sum_{j\geq-1}\|\bar{\omega}_{j0}\|_{L^\infty(S)}+\|\bar{\boldsymbol{u}}_{0,h}\|_{L^2(S)}+\|\bar{\omega}_0\|_{H^1(S)}\Big)\\ &\lesssim \mathrm{e}^{-\nu t}\Big(\ln(1+\|\bar{\omega}_0\|_{H^a(S)})+\|\bar{\omega}_0\|_{H^a(S)}\Big)\|\bar{\omega}_0\|_{H^a(S)}.\end{aligned}$$

Therefore, the result is derived from from (3.13) and (3.19).

□

4. Proof of the Theorem

In this section, we aim to demonstrate that as $\varepsilon\to0^+$, the weak solution $\boldsymbol{u}^\varepsilon$ of the system given by (1.1)–(1.4) converges in the $L^2(\Omega)$ norm to $\bar{\boldsymbol{u}}$. Specifically, we show that $\|\boldsymbol{u}^\varepsilon-\bar{\boldsymbol{u}}\|_{L^2(\Omega)}$ tends to zero. Given Proposition 2.1, it suffices to establish that $\|\boldsymbol{u}^\varepsilon-\boldsymbol{u}^{app}\|_{L^2(\Omega)}$ also approaches zero.

Note that $\boldsymbol{u}^\varepsilon$ and $\boldsymbol{u}^{app}$ satisfy the following systems:

$$\begin{cases}\partial_t\boldsymbol{u}^\varepsilon+(\boldsymbol{u}^\varepsilon\cdot\nabla)\boldsymbol{u}^\varepsilon-\varepsilon\Delta\boldsymbol{u}^\varepsilon+\frac{\alpha}{\varepsilon}e^3\wedge\boldsymbol{u}^\varepsilon+\frac{\beta}{\varepsilon}e^3\wedge\boldsymbol{j}^\varepsilon+\frac{1}{\varepsilon}\nabla p^\varepsilon=0,\\ \boldsymbol{j}^\varepsilon-\nabla\varphi^\varepsilon+e^3\wedge\boldsymbol{u}^\varepsilon=0,\\ \nabla\cdot\boldsymbol{u}^\varepsilon=\nabla\cdot\boldsymbol{j}^\varepsilon=0,\\ \boldsymbol{u}^\varepsilon(t,\boldsymbol{x})|_{t=0}=\boldsymbol{u}_0^\varepsilon(\boldsymbol{x}),\\ \boldsymbol{u}^\varepsilon(t,\boldsymbol{x})|_{\partial\Omega}=0,\quad j_3^\varepsilon(t,\boldsymbol{x})|_{z=0,1}=0,\quad \boldsymbol{j}_h^\varepsilon\cdot\boldsymbol{n}_s|_{\partial S}=0,\end{cases}$$

and

$$\begin{cases}\partial_t\boldsymbol{u}^{app}+(\boldsymbol{u}^{app}\cdot\nabla)\boldsymbol{u}^{app}-\varepsilon\Delta\boldsymbol{u}^{app}+\frac{\alpha}{\varepsilon}e^3\wedge\boldsymbol{u}^{app}+\frac{\beta}{\varepsilon}e^3\wedge\boldsymbol{j}^{app}+\frac{1}{\varepsilon}\nabla p^{app}=\boldsymbol{R}^{app},\\ \boldsymbol{j}^{app}-\nabla\varphi^{app}+e^3\wedge\boldsymbol{u}^{app}=0,\\ \nabla\cdot\boldsymbol{u}^{app}=\nabla\cdot\boldsymbol{j}^{app}=0,\\ \boldsymbol{u}^{app}|_{t=0}=(\bar{\boldsymbol{u}}+\boldsymbol{u}^{0,B}+\boldsymbol{u}^{0,T}+\varepsilon(\boldsymbol{u}^{1,int}+\boldsymbol{u}^{1,B}+\boldsymbol{u}^{1,T})+\varepsilon^2(\boldsymbol{u}^{2,B}+\boldsymbol{u}^{2,T})+\boldsymbol{u}^c+\boldsymbol{w}^c+\boldsymbol{u}^w)|_{t=0},\\ \boldsymbol{u}^{app}(t,\boldsymbol{x})|_{\partial\Omega}=0,\quad j_3^{app}(t,\boldsymbol{x})|_{z=0,1}=0,\quad \boldsymbol{j}_h^{app}\cdot\boldsymbol{n}_s|_{\partial S}=0,\end{cases}$$

where

$$\begin{aligned}\boldsymbol{R}^{app}=&\partial_t(\boldsymbol{u}^{0,B}+\boldsymbol{u}^{0,T}+\varepsilon(\boldsymbol{u}^{1,int}+\boldsymbol{u}^{1,B}+\boldsymbol{u}^{1,T})+\varepsilon^2(\boldsymbol{u}^{2,B}+\boldsymbol{u}^{2,T})+\boldsymbol{u}^c+\boldsymbol{w}^c+\boldsymbol{u}^w) \qquad (4.1)\\ &-\varepsilon\Delta(\bar{\boldsymbol{u}}+\varepsilon\boldsymbol{u}^{1,int}+\varepsilon^2(\boldsymbol{u}^{2,B}+\boldsymbol{u}^{2,T})+\boldsymbol{u}^c+\boldsymbol{w}^c+\boldsymbol{u}^w)\\ &-\varepsilon\Delta_h(\boldsymbol{u}^{0,B}+\boldsymbol{u}^{0,T}+\varepsilon(\boldsymbol{u}^{1,B}+\boldsymbol{u}^{1,T}))+(\boldsymbol{u}^{app}\cdot\nabla)(\varepsilon\boldsymbol{u}^{1,int}+\boldsymbol{u}^c+\boldsymbol{w}^c+\boldsymbol{u}^w)\end{aligned}$$

$$+ \frac{\alpha}{\varepsilon}e^3 \wedge (\boldsymbol{u}^c + \boldsymbol{w}^c + \boldsymbol{u}^w) + \varepsilon\begin{pmatrix}\nabla_h p^{2,B} + \nabla_h p^{2,T}\\ 0\end{pmatrix} + \beta\varepsilon\begin{pmatrix}\nabla_h^{\perp}\varphi^{2,B} + \nabla_h^{\perp}\varphi^{2,T}\\ 0\end{pmatrix}$$
$$+ \sum_{i=0}^{2}\varepsilon^i\Big((\boldsymbol{u}_h^{0,B} + \boldsymbol{u}_h^{0,T} + \varepsilon(\boldsymbol{u}_h^{1,int} + \boldsymbol{u}_h^{1,B} + \boldsymbol{u}_h^{1,T})) \cdot \nabla_h(\boldsymbol{u}^{i,B} + \boldsymbol{u}^{i,T})$$
$$+ (\bar{\boldsymbol{u}} + \boldsymbol{u}^c + \boldsymbol{w}^c + \boldsymbol{u}^w) \cdot \nabla(\boldsymbol{u}^{i,B} + \boldsymbol{u}^{i,T})$$
$$+ (\varepsilon(u_3^{1,int} + u_3^{1,B} + u_3^{1,T}) + \varepsilon^2(u_3^{2,B} + u_3^{2,T}))\partial_z(\boldsymbol{u}^{i,B} + \boldsymbol{u}^{i,T})\Big).$$

Below, we compute the error estimate between $\boldsymbol{u}^\varepsilon$ and $\boldsymbol{u}^{app}$. Let $\boldsymbol{v} = \boldsymbol{u}^\varepsilon - \boldsymbol{u}^{app}$, $\boldsymbol{j}^v = \boldsymbol{j}^\varepsilon - \boldsymbol{j}^{app}$, $\varphi^v = \varphi^\varepsilon - \varphi^{app}$, and $p^v = p^\varepsilon - p^{app}$. We then have

$$\begin{cases}\partial_t \boldsymbol{v} + (\boldsymbol{u}^\varepsilon \cdot \nabla)\boldsymbol{v} - \varepsilon\Delta\boldsymbol{v} + \frac{\alpha}{\varepsilon}e^3 \wedge \boldsymbol{v} + \frac{\beta}{\varepsilon}e^3 \wedge \boldsymbol{j}^v + \frac{1}{\varepsilon}p^v + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u}^{app} + \boldsymbol{R}^{app} = 0,\\ \boldsymbol{j}^v - \nabla\varphi^v + e^3 \wedge \boldsymbol{v} = 0,\\ \nabla \cdot \boldsymbol{v} = \nabla \cdot \boldsymbol{j}^v = 0,\\ \boldsymbol{v}(t,\boldsymbol{x})|_{\partial\Omega} = 0, \quad j_3^v(t,\boldsymbol{x})|_{z=0,1} = 0, \quad \boldsymbol{j}_h^v \cdot \boldsymbol{n}_s|_{\partial S} = 0.\end{cases} \tag{4.2}$$

Estimating $\|\boldsymbol{v}\|_{L^2}^2$ using Eq (4.2) naturally yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}\|_{L^2}^2 + \varepsilon\|\nabla\boldsymbol{v}\|_{L^2}^2 + \langle\frac{\alpha}{\varepsilon}e^3 \wedge \boldsymbol{v} + \frac{1}{\varepsilon}\nabla p^v + (\boldsymbol{u}^\varepsilon \cdot \nabla)\boldsymbol{v}, \boldsymbol{v}\rangle + \langle\frac{\beta}{\varepsilon}e^3 \wedge \boldsymbol{j}^v, \boldsymbol{v}\rangle\\ &= -\langle(\boldsymbol{v} \cdot \nabla)\boldsymbol{u}^{app}, \boldsymbol{v}\rangle - \langle\boldsymbol{R}^{app}, \boldsymbol{v}\rangle.\end{aligned} \tag{4.3}$$

Using the incompressibility condition for $\boldsymbol{v}$ and the structure of $e^3 \wedge \boldsymbol{v}$, the third term on the left-hand side of (4.3) is

$$\langle\frac{\alpha}{\varepsilon}e^3 \wedge \boldsymbol{v} + \frac{1}{\varepsilon}\nabla p^v + (\boldsymbol{u}^\varepsilon \cdot \nabla)\boldsymbol{v}, \boldsymbol{v}\rangle = 0.$$

By definition and the boundary conditions in $(4.2)_2 - (4.2)_4$ of $\boldsymbol{j}_v$, the fourth term reduces to

$$\begin{aligned}\langle\frac{\beta}{\varepsilon}e^3 \wedge \boldsymbol{j}^v, \boldsymbol{v}\rangle &= -\frac{\beta}{\varepsilon}\langle\boldsymbol{j}^v, e^3 \times \boldsymbol{v}\rangle = \frac{\beta}{\varepsilon}\langle\boldsymbol{j}^v, \boldsymbol{j}^v - \nabla\varphi^v\rangle\\ =&\frac{\beta}{\varepsilon}\int_\Omega |\boldsymbol{j}^v|^2\,dx - \frac{\beta}{\varepsilon}\int_\Omega \boldsymbol{j}^v \cdot \nabla\varphi^v\,dx = \frac{\beta}{\varepsilon}\|\boldsymbol{j}^v\|_{L^2}^2 \geq 0.\end{aligned}$$

Next, we estimate the right-hand side of (4.3). The first term can be expanded to

$$\begin{aligned}&\langle(\boldsymbol{v} \cdot \nabla)\boldsymbol{u}^{app}, \boldsymbol{v}\rangle\\ &= \langle\boldsymbol{v} \cdot \nabla(\bar{\boldsymbol{u}} + \boldsymbol{u}^{0,B} + \boldsymbol{u}^{0,T} + \varepsilon(\boldsymbol{u}^{1,int} + \boldsymbol{u}^{1,B} + \boldsymbol{u}^{1,T}) + \varepsilon^2(\boldsymbol{u}^{2,B} + \boldsymbol{u}^{2,T}) + \boldsymbol{u}^c + \boldsymbol{w}^c + \boldsymbol{u}^w), \boldsymbol{v}\rangle.\end{aligned}$$

First, by Hölder's inequality and Proposition 3.2, one obtains

$$\begin{aligned}&|\langle\boldsymbol{v} \cdot \nabla\bar{\boldsymbol{u}}, \boldsymbol{v}\rangle| = |\langle\boldsymbol{v}_h \cdot \nabla_h\bar{\boldsymbol{u}}_h, \boldsymbol{v}_h\rangle|\\ \leq&\|\nabla_h\bar{\boldsymbol{u}}_h\|_{L^\infty(S)}\|\boldsymbol{v}_h\|_{L^2(\Omega)}^2 \leq \|\nabla_h\bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)}\|\boldsymbol{v}\|_{L^2(\Omega)}^2 \mathrm{e}^{-\nu t}.\end{aligned} \tag{4.4}$$

Second, for the ε^0-order boundary term, in the case of $\boldsymbol{u}^{0,B}$, we utilize the integration by parts, which is computed as

$$\begin{aligned}&|\langle\boldsymbol{v} \cdot \nabla(\boldsymbol{u}^{0,B} + \boldsymbol{u}^{0,T}), \boldsymbol{v}\rangle| = |\langle\boldsymbol{v} \cdot \nabla\boldsymbol{v}, \boldsymbol{u}^{0,B} + \boldsymbol{u}^{0,T}\rangle|\\ \leq&\int_{S\times[0,\frac{1}{2}]} |\boldsymbol{v}||\nabla\boldsymbol{v}||\boldsymbol{u}^{0,B} + \boldsymbol{u}^{0,T}|\,dx + \int_{S\times[\frac{1}{2},1]} |\boldsymbol{v}||\nabla\boldsymbol{v}| \cdot |\boldsymbol{u}^{0,B} + \boldsymbol{u}^{0,T}|\,dx.\end{aligned}$$

Due to the boundary conditions on $\boldsymbol{v}$, we can deduce that

$$|\boldsymbol{v}| = \Big| \int_0^z \partial_{z'} \boldsymbol{v} dz' \Big| \leq d(z)^{\frac{1}{2}} \|\partial_z \boldsymbol{v}\|_{L^2(0,1)},$$

where $d(z)$ is the distance to the bottom boundary. Then

$$\begin{aligned}
&\int_{S\times[0,\frac{1}{2}]} |\boldsymbol{v}||\nabla \boldsymbol{v}| \cdot |\boldsymbol{u}^{0,B} + \boldsymbol{u}^{0,T}|\, dx \\
&\leq \int_\Omega \|\partial_z \boldsymbol{v}\|_{L^2(0,1)} \cdot |\nabla \boldsymbol{v}| \cdot d(z)^{\frac{1}{2}} |\boldsymbol{u}^{0,B} + \boldsymbol{u}^{0,T}|\, dx \\
&\leq \|\nabla \boldsymbol{v}\|^2_{L^2(\Omega)} \|d(z)^{\frac{1}{2}} |\boldsymbol{u}^{0,B} + \boldsymbol{u}^{0,T}|\|_{L^2(0,1;L^\infty(S))},
\end{aligned}$$

where

$$\begin{aligned}
&\|d(z)^{\frac{1}{2}} |\boldsymbol{u}^{0,B} + \boldsymbol{u}^{0,T}|\|_{L^2(0,1;L^\infty(S))} \\
\lesssim&\frac{\varepsilon}{a} \|\bar{\boldsymbol{u}}_h\|_{L^\infty(S)} \int_{[0,\frac{a}{\varepsilon}]} \frac{az}{\varepsilon} \mathrm{e}^{-\frac{az}{\varepsilon}} d\frac{az}{\varepsilon} \\
&+ \frac{\varepsilon}{a} \|\bar{\boldsymbol{u}}_h\|_{L^\infty(S)} \int_{[0,\frac{a}{\varepsilon}]} \frac{a(1-z)}{\varepsilon} \mathrm{e}^{-\frac{a(1-z)}{\varepsilon}} d\frac{a(1-z)}{\varepsilon} \\
\lesssim&\varepsilon \|\bar{\boldsymbol{u}}_h\|_{L^\infty(S)}.
\end{aligned}$$

In summary, this gives

$$|\langle \boldsymbol{v} \cdot \nabla(\boldsymbol{u}^{0,B} + \boldsymbol{u}^{0,T}), \boldsymbol{v}\rangle| \lesssim \varepsilon \|\bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)} \|\nabla \boldsymbol{v}\|^2_{L^2(\Omega)} \mathrm{e}^{-\nu t}. \tag{4.5}$$

Next, for higher-order terms, it is easy to obtain

$$\begin{aligned}
&\Big|\langle \boldsymbol{v} \cdot \nabla(\varepsilon(\boldsymbol{u}^{1,int} + \boldsymbol{u}^{1,B} + \boldsymbol{u}^{1,T}) + \varepsilon^2(\boldsymbol{u}^{2,B} + \boldsymbol{u}^{2,T})), \boldsymbol{v}\rangle\Big| \\
=&\varepsilon\Big|\langle \boldsymbol{v} \cdot \nabla \boldsymbol{v}, (\boldsymbol{u}^{1,int} + \boldsymbol{u}^{1,B} + \boldsymbol{u}^{1,T}) + \varepsilon(\boldsymbol{u}^{2,B} + \boldsymbol{u}^{2,T})\rangle\Big| \\
\leq&\varepsilon \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\boldsymbol{v}\|_{L^2(\Omega)} \|\bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)} \mathrm{e}^{-\nu t} \\
\leq&\frac{\varepsilon}{4} \|\nabla \boldsymbol{v}\|^2_{L^2(\Omega)} + 4\varepsilon \|\bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)} \|\boldsymbol{v}\|^2_{L^2(\Omega)} \mathrm{e}^{-\nu t}.
\end{aligned} \tag{4.6}$$

Finally, for the correction term, according to (2.44), (2.55), and (2.56), it follows that

$$|\langle \boldsymbol{v} \cdot \nabla(\boldsymbol{u}^c + \boldsymbol{w}^c + \boldsymbol{u}^w), \boldsymbol{v}\rangle| \leq \varepsilon^{\frac{1}{4}} \|\boldsymbol{v}\|^2_{L^2(\Omega)} \|\nabla_h \bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)} \mathrm{e}^{-\nu t}. \tag{4.7}$$

Combining (4.4)–(4.7), we get

$$\begin{aligned}
\langle (\boldsymbol{v} \cdot \nabla) \boldsymbol{u}^{app}, \boldsymbol{v}\rangle \lesssim&\varepsilon(\tfrac{1}{4} + \|\bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)} \mathrm{e}^{-\nu t}) \|\nabla \boldsymbol{v}\|^2_{L^2(\Omega)} \\
&+ (\|\nabla_h \bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)} + 4\varepsilon \|\bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)} + \varepsilon^{\frac{1}{4}} \|\nabla_h \bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)}) \|\boldsymbol{v}\|^2_{L^2(\Omega)} \mathrm{e}^{-\nu t}.
\end{aligned} \tag{4.8}$$

The second term on the right-hand side of (4.3), from the expression for $\boldsymbol{R}^{app}$, can be easily obtained as

$$|\langle \boldsymbol{v}, \boldsymbol{R}^{app}\rangle| = \|\boldsymbol{R}^{app}\|_{L^2(\Omega)} \|\boldsymbol{v}\|_{L^2(\Omega)} \tag{4.9}$$

$$
\begin{aligned}
&\leq \varepsilon \|\bar{\boldsymbol{u}}_{0,h}\|_{L^2(S)} \|\boldsymbol{v}\|_{L^2(\Omega)} \mathrm{e}^{-\nu t} \\
&\leq \varepsilon \|\bar{\boldsymbol{u}}_{0,h}\|_{L^2(S)}^2 \mathrm{e}^{-\nu t} + \varepsilon \|\boldsymbol{v}\|_{L^2(\Omega)}^2 \mathrm{e}^{-\nu t}.
\end{aligned}
$$

Thus, on the basis of (4.3), (4.8) and (4.9), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}\|_{L^2(\Omega)}^2 + \frac{3\varepsilon}{4}\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}^2 \\
\leq& \varepsilon \|\bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)} \mathrm{e}^{-\nu t} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)}^2 + \varepsilon \|\bar{\boldsymbol{u}}_{0,h}\|_{L^2(S)}^2 \mathrm{e}^{-\nu t} \\
&+ (\|\nabla_h \bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)} + 4\varepsilon \|\bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)} + \varepsilon^{\frac{1}{4}} \|\nabla_h \bar{\boldsymbol{u}}_{0,h}\|_{L^\infty(S)} + \varepsilon) \|\boldsymbol{v}\|_{L^2(\Omega)}^2 \mathrm{e}^{-\nu t}.
\end{aligned}
\tag{4.10}
$$

Due to the initial conditions in (1.5) and (1.6), and by integrating the inequality (4.10) with respect to the variable t, we can complete the proof of the theorem.

5. Conclusions

This paper employs a multiscale analysis approach to investigate the impact of the Ekman-Hartmann boundary layer within rotating MHD flows confined to cylindrical domains and develops the corresponding approximate solutions. These solutions are valuable for numerical computations in geophysics and metal engineering industries, aiding in more accurate simulations of fluid dynamic behaviors. Although our model has achieved innovation in handling constant magnetic fields and rotation axes, it has limitations in modeling variations in the magnetic fields and rotation axes over time and space, and in adapting to more complex geometrical shapes. Future research will explore the effects of complex variations in the magnetic fields and rotation axes on the boundary layer. It may extend the model to accommodate various geometries, including spherical and nonplanar, to solve more practical problems.

Author contributions

Yifei Jia: Writing-original and draft; Guanglei Zhang and Kexue Chen: Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this article.

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