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Research article

Further norm and numerical radii inequalities for operators involving a positive operator

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Abstract: The article examines inequalities for norms and numerical radii of bounded linear operators on complex Hilbert spaces. It focuses on scenarios where three operators are involved, with one being positive, and investigates their sums or products. Some of our findings extend existing inequalities established in the literature.

Keywords: positive operator; Hilbert space; numerical radius; operator norm; inequalities
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1. Introduction

Mathematical inequalities are powerful tools that establish relationships and bounds between mathematical quantities. They have gained increasing importance in recent years, driving research and advancements in various fields of study. For investigations related to the theory of mathematical inequalities, we refer the reader to several key works. Foundational results on operator inequalities and numerical radius inequalities can be found in [1, 2]. Classical refinements of the Cauchy-Schwarz inequality and discrete inequalities are discussed in [3–5]. Comprehensive studies on numerical radius inequalities and matrix exponential inequalities are presented in [6, 7]. Recent developments involving preinvexity and stochastic harmonically convexity are explored in [8, 9]. Further insights and related results are available in [10] and the references therein. Additionally, for inequalities in different spaces, we refer to the works of S. Shi et al. [11], G. Wang et al. [5], and Y. Wu et al. [12].

Recently, the authors of this paper have previously investigated the Selberg inequality and the Selberg operator in [13, 14], focusing specifically on norm and numerical radius inequalities related to any positive operator, since every Selberg operator is a positive contraction. This work builds upon their previous research in this area.

Before delving into these results, we let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$, with the identity operator denoted by I. If $S \in \mathcal{B}(\mathcal{H})$, then S^* denotes the adjoint of S. We define a positive operator, denoted $S \geq 0$, as one for which $\langle Sy, y\rangle \geq 0$ holds for all $y \in \mathcal{H}$. This notion of positivity induces an ordering $A \geq B$ for self-adjoint operators if and only if $A - B \geq 0$.

The numerical radius and operator norm of $S \in \mathcal{B}(\mathcal{H})$ are respectively given by:

$$\omega(S) = \sup\{|\lambda| : \lambda \in W(S)\} = \sup_{\|y\|=1} |\langle Sy, y\rangle| \quad \text{and} \quad \|S\| = \sup_{\|y\|=1} \|Sy\|,$$

where $W(S) = \{\langle Sy, y\rangle \ : \ y \in \mathcal{H},\ \|y\| = 1\}$ is the numerical range of S. Recent developments and research on numerical range, norm, and numerical radius inequalities of operators are discussed in several key works. For refinements and improvements of generalized numerical radius inequalities, see [15–17]. Numerical radius inequalities for operator matrices and certain 2×2 operator matrices are explored in [18, 19]. Additionally, norm and numerical radius inequalities for sums of operators are presented in [20]. Further insights and related results can be found in these works and the references cited therein.

It is well known that the numerical radius is not submultiplicative, meaning that we cannot assert $\omega(AB) \leq \omega(A)\omega(B)$ for operators A and B, even when A and B commute. Due to this fact, it is essential to find upper bounds for the numerical radius of the product of operators. For this reason, several authors have explored various bounds concerning the norms and numerical radii of products and sums of Hilbert space operators. This line of research is documented in works such as [21, 22], along with their respective references. In a recent study by Sababheh et al. [23], the following result was established: for $A, B, P \in \mathcal{B}(\mathcal{H})$ with P a positive contraction, i.e., $0 \leq P \leq I$, then the following inequality holds:

$$\omega(BPA) \leq \frac{1}{2}\left(\frac{1}{2}\left\||B^*|^2 + |A|^2\right\| + \omega(BA)\right). \tag{1.1}$$

We are motivated by the inequality (1.1), which prompts us to extend this result to a broader context involving a positive non-zero operator P on $\mathcal{H}$. We also explore norm and numerical radius inequalities for bounded linear operators on $\mathcal{H}$, focusing on scenarios involving the sum or product of three operators, one of which is a positive non-zero operator P acting on $\mathcal{H}$. Our results contribute to extending various inequalities established by other mathematicians in recent years.

2. Main results

In this section, we will present the proofs of our main results. In order to achieve this, we will make use of the following lemma, which draws its inspiration from the research conducted by Bottazzi and Conde in [24].

Lemma 2.1. *Let $P \in \mathcal{B}(\mathcal{H})$ be a non-zero positive operator. Then for any $x, y \in \mathcal{H}$ the following inequality holds:*

$$\left|\left\langle \left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)x, y\right\rangle\right| \le \frac{1}{2}\|x\|\|y\|. \tag{2.1}$$

Proof. By the positivity of P and [10, Lemma 3.2], we deduce that

$$\left\|\frac{2}{\|P\|}P - I\right\| \le 1,$$

or equivalently $\left\|\frac{1}{\|P\|}P - \frac{1}{2}I\right\| \le \frac{1}{2}$.

Then, by the Cauchy–Bunyakovsky–Schwarz inequality, we note that

$$\begin{aligned}\left|\left\langle \left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)x, y\right\rangle\right| &\le \left\|\left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)x\right\|\|y\| \\ &\le \left\|\frac{1}{\|P\|}P - \frac{1}{2}I\right\|\|x\|\|y\| \\ &\le \frac{1}{2}\|x\|\|y\|.\end{aligned}$$

This proves (2.1) as requested. □

Based on Lemma 2.1 and recent results obtained for the Selberg operator in [13], we can derive the following inequalities for operator norms.

Theorem 2.1. *Let $A, B, P \in \mathcal{B}(\mathcal{H})$, with P being a non-zero positive operator. Then, for any $z_k \in \mathbb{C}$ with $k = 1, \ldots, n$, we determine the following norm inequalities:*

$$\left\|\sum_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)A\right\| \le \frac{\sum_{k=1}^{n}|z_k|}{2}\|A\|\|B\|, \tag{2.2}$$

and

$$\left\|\prod_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)A\right\| \le \frac{\prod_{k=1}^{n}|z_k|}{2^n}\|A\|\|B\|. \tag{2.3}$$

Proof. From Lemma 2.1, we see that for Ax instead of x and B^*y instead of y, that

$$\left|\left\langle \sum_{k=1}^{n} z_k \left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)Ax, B^*y\right\rangle\right| \le \frac{\sum_{k=1}^{n}|z_k|}{2}\|Ax\|\|B^*y\|,$$

and

$$\left|\left\langle \prod_{k=1}^{n} z_k \left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)Ax, B^*y\right\rangle\right| \le \frac{\prod_{k=1}^{n}|z_k|}{2^n}\|Ax\|\|B^*y\|$$

for all $x, y \in \mathcal{H}$. This is equivalent to

$$\left|\left\langle \sum_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)Ax, y\right\rangle\right| \le \frac{\sum_{k=1}^{n}|z_k|}{2}\|Ax\|\|B^*y\|,$$

and

$$\left|\left\langle \prod_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)Ax, y\right\rangle\right| \leq \frac{\prod_{k=1}^{n}|z_k|}{2^n}\|Ax\|\|B^*y\| \tag{2.4}$$

for all $x, y \in \mathcal{H}$. If we take the supremum over $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$, then we get the norm inequalities (2.2) and (2.3). □

To derive the following power inequalities, we recall McCarthy's inequality [25, Theorem 1.2], which asserts that if $Q \geq 0$, then the following inequality holds for all $s \geq 1$ and for all $x \in \mathcal{H}$ with $\|x\| = 1$:

$$\langle Qx, x\rangle^s \leq \langle Q^s x, x\rangle. \tag{2.5}$$

Theorem 2.2. *Let $A, B, P \in \mathcal{B}(\mathcal{H})$ with P being a non-zero positive operator; then for any $z_k \in \mathbb{C}$, we have the numerical radius inequalities*

$$\omega\left(\sum_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)A\right) \leq \frac{\sum_{k=1}^{n}|z_k|}{2}\left\|\frac{1}{p}|A|^{rp} + \frac{1}{q}|B^*|^{rq}\right\|^{\frac{1}{r}}, \tag{2.6}$$

and

$$\omega\left(\prod_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)A\right) \leq \frac{\prod_{k=1}^{n}|z_k|}{2^n}\left\|\frac{1}{p}|A|^{rp} + \frac{1}{q}|B^*|^{rq}\right\|^{\frac{1}{r}}, \tag{2.7}$$

where $r \geq 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $rp \geq 2$, $rq \geq 2$.

Proof. We will only demonstrate inequality (2.7). The proof of (2.6) follows a similar approach.

From Young's inequality

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q,\ a,\ b \geq 0,\ p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \tag{2.8}$$

we infer that

$$\begin{aligned}\|Ax\|^r\|B^*x\|^r &\leq \frac{1}{p}\|Ax\|^{rp} + \frac{1}{q}\|B^*x\|^{rq}\\ &= \frac{1}{p}\|Ax\|^{2\frac{rp}{2}} + \frac{1}{q}\|B^*x\|^{2\frac{rq}{2}}\\ &= \frac{1}{p}\langle |A|^2x, x\rangle^{\frac{rp}{2}} + \frac{1}{q}\langle |B^*|^2x, x\rangle^{\frac{rq}{2}}\end{aligned}$$

for all $x \in \mathcal{H}$, where $|T|^2 = T^*T$ for any $T \in \mathcal{B}(\mathcal{H})$.
Since $rp \geq 2$ and $rq \geq 2$, we can apply McCarthy's inequality (2.5) to obtain:

$$\begin{aligned}\frac{1}{p}\langle |A|^2x, x\rangle^{\frac{rp}{2}} + \frac{1}{q}\langle |B^*|^2x, x\rangle^{\frac{rq}{2}} &\leq \frac{1}{p}\langle |A|^{rp}x, x\rangle + \frac{1}{q}\langle |B^*|^{rq}x, x\rangle\\ &= \left\langle\left(\frac{1}{p}|A|^{rp} + \frac{1}{q}|B^*|^{rq}\right)x, x\right\rangle\end{aligned}$$

for $x \in \mathcal{H}$, $\|x\| = 1$.

By (2.4)

$$\left|\left\langle \prod_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P-\frac{1}{2}I\right)Ax,x\right\rangle\right| \le \frac{\prod_{k=1}^{n}|z_k|}{2^n}\left\langle \left(\frac{1}{p}|A|^{rp}+\frac{1}{q}|B^*|^{rq}\right)x,x\right\rangle^{\frac{1}{r}}$$

for all $x \in \mathcal{H}$.

By taking the supremum over $x \in \mathcal{H}$, $\|x\| = 1$, we deduce (2.7). This concludes the proof of our result. □

By considering the particular values $r = 1$ and $p = q = 2$ in the Theorem 2.2, we derive the following inequalities

$$\omega\left(\sum_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P-\frac{1}{2}I\right)A\right) \le \frac{\sum_{k=1}^{n}|z_k|}{4}\left\||A|^2+|B^*|^2\right\|, \tag{2.9}$$

and

$$\omega\left(\prod_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P-\frac{1}{2}I\right)A\right) \le \frac{\prod_{k=1}^{n}|z_k|}{2^{n+1}}\left\||A|^2+|B^*|^2\right\|. \tag{2.10}$$

Corollary 2.1. *Let $A, B, P \in \mathcal{B}(\mathcal{H})$ with P being a positive operator; then for any $z_k \in \mathbb{C}$, we have the following inequality:*

$$\omega\left(\sum_{k=1}^{n} z_k BPA\right) \le \frac{\|P\|\sum_{k=1}^{n}|z_k|}{2}\left(\left\|\frac{1}{p}|A|^{rp}+\frac{1}{q}|B^*|^{rq}\right\|^{\frac{1}{r}}+\omega(BA)\right),$$

where $r \ge 1$, $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $rp \ge 2$, $rq \ge 2$,

Proof. We will only consider the case $P \neq 0$, as the other case is trivial. Using the well-known fact that the numerical radius is a norm and (2.6), we obtain that

$$\begin{aligned}\left|\omega\left(\sum_{k=1}^{n} z_k B\frac{1}{\|P\|}PA\right)-\omega\left(\frac{\sum_{k=1}^{n} z_k}{2}BA\right)\right| &\le \omega\left(\sum_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P-\frac{1}{2}I\right)A\right).\\ &\le \frac{\sum_{k=1}^{n}|z_k|}{2}\left\|\frac{1}{p}|A|^{rp}+\frac{1}{q}|B^*|^{rq}\right\|^{\frac{1}{r}}.\end{aligned}$$

Then, for the triangle inequality for the modulus of a complex number, we conclude that

$$\begin{aligned}\omega\left(\sum_{k=1}^{n} z_k B\frac{1}{\|P\|}PA\right) &\le \left|\omega\left(\sum_{k=1}^{n} z_k B\frac{1}{\|P\|}PA\right)-\omega\left(\frac{\sum_{k=1}^{n} z_k}{2}BA\right)\right|+\omega\left(\frac{\sum_{k=1}^{n} z_k}{2}BA\right)\\ &\le \frac{\sum_{k=1}^{n}|z_k|}{2}\left\|\frac{1}{p}|A|^{rp}+\frac{1}{q}|B^*|^{rq}\right\|^{\frac{1}{r}}+\omega\left(\frac{\sum_{k=1}^{n} z_k}{2}BA\right)\\ &\le \frac{\sum_{k=1}^{n}|z_k|}{2}\left(\left\|\frac{1}{p}|A|^{rp}+\frac{1}{q}|B^*|^{rq}\right\|^{\frac{1}{r}}+\omega(BA)\right).\end{aligned}$$

Multiplying the last inequality by $\|P\|$, we derive the desired inequality. □

Remark 2.1. *Corollary 2.1 extends the inequality* (1.1) *by considering specific values:* $r = 1$, $p = q = 2$, $z_1 = 1$, *and* $z_k = 0$ *for any* $k = 2, \cdots, n$. *This particular choice of parameters allows us to recover the mentioned inequality.*

Now, we are able to derive new upper bounds for the sum or product of operators. In these bounds, one of the operators is a linear combination of a positive operator and the identity operator, incorporating convex combinations of the operators.

Theorem 2.3. *Let* $A, B, P \in \mathcal{B}(\mathcal{H})$ *with* P *being a positive operator; then for any* $z_k \in \mathbb{C}$, *we have the numerical radius inequalities*

$$\omega\left(\sum_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)A\right) \leq \frac{\sum_{k=1}^{n}|z_k|}{2}\left\|(1-\alpha)|A|^2 + \alpha|B^*|^2\right\|^{\frac{1}{2}}\|A\|^{\alpha}\|B\|^{1-\alpha}, \tag{2.11}$$

and

$$\omega\left(\prod_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)A\right) \leq \frac{\prod_{k=1}^{n}|z_k|}{2^n}\left\|(1-\alpha)|A|^2 + \alpha|B^*|^2\right\|^{\frac{1}{2}}\|A\|^{\alpha}\|B\|^{1-\alpha} \tag{2.12}$$

for all $\alpha \in [0,1]$.

Proof. Observe that

$$\begin{aligned}
\|Ax\|^2\|B^*x\|^2 &= \langle |A|^2x, x\rangle\langle |B^*|^2x, x\rangle \\
&= \langle |A|^2x, x\rangle^{1-\alpha}\langle |B^*|^2x, x\rangle^{\alpha}\langle |A|^2x, x\rangle^{\alpha}\langle |B^*|^2x, x\rangle^{1-\alpha} \\
&\leq \Big((1-\alpha)\langle |A|^2x, x\rangle + \alpha\langle |B^*|^2x, x\rangle\Big)\|Ax\|^{2\alpha}\|B^*x\|^{2(1-\alpha)} \\
&= \left\langle \left[(1-\alpha)|A|^2 + \alpha|B^*|^2\right]x, x\right\rangle\|Ax\|^{2\alpha}\|B^*x\|^{2(1-\alpha)},
\end{aligned} \tag{2.13}$$

for all $x \in \mathcal{H}$.
By Lemma 2.1, we then have

$$\left|\left\langle \sum_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)Ax, x\right\rangle\right| \leq \frac{\sum_{k=1}^{n}|z_k|}{2}\left\langle \left[(1-\alpha)|A|^2 + \alpha|B^*|^2\right]x, x\right\rangle^{\frac{1}{2}}\|Ax\|^{\alpha}\|B^*x\|^{1-\alpha}, \tag{2.14}$$

and

$$\left|\left\langle \prod_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)Ax, x\right\rangle\right| \leq \left(\frac{\prod_{k=1}^{n}|z_k|}{2^n}\right)\left\langle \left[(1-\alpha)|A|^2 + \alpha|B^*|^2\right]x, x\right\rangle^{\frac{1}{2}}\|Ax\|^{\alpha}\|B^*x\|^{1-\alpha}, \tag{2.15}$$

for all $x \in \mathcal{H}$.

Taking the supremum in (2.14) and (2.15) over $\|x\| = 1$, we derive (2.11) and (2.12). □

Remark 2.2. *We observe that for* $\alpha = \frac{1}{2}$ *in* (2.11) *and* (2.12), *we derive the following inequalities:*

$$\omega\left(\sum_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P-\frac{1}{2}I\right)A\right) \le \frac{\sum_{k=1}^{n}|z_k|}{2}\left\|\frac{|A|^2+|B^*|^2}{2}\right\|^{\frac{1}{2}} \|A\|^{\frac{1}{2}}\|B\|^{\frac{1}{2}},$$

and

$$\omega\left(\prod_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P-\frac{1}{2}I\right)A\right) \le \frac{\prod_{k=1}^{n}|z_k|}{2^n}\left\|\frac{|A|^2+|B^*|^2}{2}\right\|^{\frac{1}{2}} \|A\|^{\frac{1}{2}}\|B\|^{\frac{1}{2}}.$$

We now turn our attention to the bounds obtained above, and we attempt to make a comparison with the inequalities (2.9) and (2.10), respectively. We show that, in general, they are not directly comparable.

Consider $A = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ and $B = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. Then, it is straightforward to verify that $\|A\|\|B\| = 1$ and $\frac{1}{2}\left\||A|^2+|B^*|^2\right\| = \frac{1}{2}\left\|\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\| = \frac{1}{2}$. Thus,

$$\left\|\frac{|A|^2+|B^*|^2}{2}\right\| = \frac{1}{2} < \frac{1}{\sqrt{2}} = \left\|\frac{|A|^2+|B^*|^2}{2}\right\|^{\frac{1}{2}} \|A\|^{\frac{1}{2}}\|B\|^{\frac{1}{2}}.$$

Again, if we consider $A = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $B = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$, then $\|A\|\|B\| = 2$ and $\frac{1}{2}\left\||A|^2+|B^*|^2\right\| = \frac{1}{2}\left\|\begin{pmatrix}1 & 0\\ 0 & 5\end{pmatrix}\right\| = \frac{5}{2}$, and in particular, we obtain:

$$\left\|\frac{|A|^2+|B^*|^2}{2}\right\|^{\frac{1}{2}} \|A\|^{\frac{1}{2}}\|B\|^{\frac{1}{2}} = \sqrt{5} < \frac{5}{2} = \left\|\frac{|A|^2+|B^*|^2}{2}\right\|.$$

Moreover, the preceding examples demonstrate that, in general, the expressions $\|A\|\,\|B\|$ and $\left\|\frac{|A|^2+|B^*|^2}{2}\right\|$ are not directly comparable, even though Young's inequality (2.8) might suggest a potential relationship between them.

By mimicking the idea used in the proof of Corollary 2.1, we obtain the following result.

Corollary 2.2. *Let $A, B, P \in \mathcal{B}(\mathcal{H})$ with P being a positive operator; then for any $z_k \in \mathbb{C}$, we obtain the following inequality:*

$$\omega\left(\sum_{k=1}^{n} z_k BPA\right) \le \frac{\|P\|\sum_{k=1}^{n}|z_k|}{2}\left(\left\|(1-\alpha)|A|^2+\alpha|B^*|^2\right\|^{\frac{1}{2}} \|A\|^{\alpha}\|B\|^{1-\alpha}+\omega(BA)\right),$$

where $\alpha \in [0,1]$.

Next, it would be advantageous to consider various specific cases of interest by selecting appropriate values for z_k. We begin by presenting a generalization of Corollary 2.2.

Theorem 2.4. *Let $A, B, P \in \mathcal{B}(\mathcal{H})$ with P being a positive operator. Then, for any $z_k \in \mathbb{C}$ with $\sum_{k=1}^{n}|z_k| \le 1$ and $r \ge 1$, we obtain the following numerical radius inequality for $\alpha \in [0,1]$:*

$$\omega\left(\sum_{k=1}^{n} z_k BPA\right) \leq \frac{\|P\|}{2^{\frac{1}{r}}}\left[\omega^r(BA) + \left\|(1-\alpha)|A|^2 + \alpha|B^*|^2\right\|^{\frac{r}{2}} \|A\|^{r\alpha}\|B\|^{r(1-\alpha)}\right]^{\frac{1}{r}}, \tag{2.16}$$

and

$$\omega\left(\sum_{k=1}^{n} z_k BPA\right) \leq \frac{\|P\|}{2^{\frac{1}{r}}}\left[\omega^r(BA) + \left\|(1-\alpha)|A|^2 + \alpha|B^*|^2\right\|^{\frac{r}{2}} \left\|\alpha|A|^2 + (1-\alpha)|B^*|^2\right\|^{\frac{r}{2}}\right]^{\frac{1}{r}}. \tag{2.17}$$

Proof. Let us note that if $P = 0$, the inequality reduces trivially to an equality. Therefore, we shall assume that $P \neq 0$. From Lemma 2.1, we conclude the following inequality:

$$\left|\left\langle\left(\sum_{k=1}^{n} z_k B\frac{1}{\|P\|}PA\right)x, y\right\rangle\right| \leq \frac{|\langle BAx, y\rangle| + \|Ax\|\|B^*y\|}{2} \tag{2.18}$$

for all $x, y \in \mathcal{H}$.

If we take $y = x$ in (2.18), then we obtain

$$\left|\left\langle\left(\sum_{k=1}^{n} z_k B\frac{1}{\|P\|}PA\right)x, x\right\rangle\right| \leq \frac{|\langle BAx, x\rangle| + \|Ax\|\|B^*x\|}{2}, \tag{2.19}$$

for all $x \in \mathcal{H}$.

For $r \geq 1$ and (2.13), then we obtain

$$\|Ax\|^r\|B^*x\|^r \leq \left\langle\left[(1-\alpha)|A|^2 + \alpha|B^*|^2\right]x, x\right\rangle^{\frac{r}{2}} \|Ax\|^{r\alpha}\|B^*x\|^{r(1-\alpha)},$$

for all $x \in \mathcal{H}$.

If we take the power $r \geq 1$ in (2.19) and use the convexity of the function $g(t) = t^r$ with $t \in [0, \infty)$, then we obtain:

$$\begin{aligned}\left|\left\langle\left(\sum_{k=1}^{n} z_k B\frac{1}{\|P\|}PA\right)x, x\right\rangle\right|^r &\leq \left(\frac{|\langle BAx, x\rangle| + \|Ax\|\|B^*x\|}{2}\right)^r \\ &\leq \frac{|\langle BAx, x\rangle|^r + \|Ax\|^r\|B^*x\|^r}{2}.\end{aligned} \tag{2.20}$$

From (2.20), we then have

$$\left|\left\langle\left(\sum_{k=1}^{n} z_k B\frac{1}{\|P\|}PA\right)x, x\right\rangle\right|^r \leq \frac{|\langle BAx, x\rangle|^r + \left\langle\left[(1-\alpha)|A|^2 + \alpha|B^*|^2\right]x, x\right\rangle^{\frac{r}{2}}\|Ax\|^{r\alpha}\|B^*x\|^{r(1-\alpha)}}{2}$$

for all $x \in \mathcal{H}$. Taking the supremum over $\|x\| = 1$, we derive (2.16).

In a similar way, we obtain

$$\left|\left\langle\left(\sum_{k=1}^{n} z_k B\frac{1}{\|P\|}PA\right)x, x\right\rangle\right|^r \leq \frac{|\langle BAx, x\rangle|^r + \left\langle\left[(1-\alpha)|A|^2 + \alpha|B^*|^2\right]x, x\right\rangle^{\frac{r}{2}}\left\langle\left[\alpha|A|^2 + (1-\alpha)|B^*|^2\right]x, x\right\rangle^{\frac{r}{2}}}{2}$$

for all $x \in \mathcal{H}$, which proves (2.17). This marks the completion of our result's proof. □

Remark 2.3. *We note that inequality* (2.16) *can be deduced from Corollary 2.2, repeating the proof idea used in Theorem 2.5.*

We observe that for $\alpha = \frac{1}{2}$, in Theorem 2.4, we obtain:

$$\omega\left(\sum_{k=1}^{n} z_k BPA\right) \leq \frac{\|P\|}{2^{\frac{1}{r}}}\left[\omega^r(BA) + \left\|\frac{|A|^2 + |B^*|^2}{2}\right\|^{\frac{r}{2}} \|A\|^{\frac{r}{2}} \|B\|^{\frac{r}{2}}\right]^{\frac{1}{r}}$$

for $r \geq 1$.

In the case $r = 1$, we obtain

$$\omega\left(\sum_{k=1}^{n} z_k BPA\right) \leq \frac{\|P\|}{2}\left[\omega(BA) + \left\|\frac{|A|^2 + |B^*|^2}{2}\right\|^{\frac{1}{2}} \|A\|^{\frac{1}{2}} \|B\|^{\frac{1}{2}}\right],$$

while for $r = 2$,

$$\omega\left(\sum_{k=1}^{n} z_k BPA\right) \leq \frac{\|P\|\sqrt{2}}{2}\sqrt{\omega^2(BA) + \left\|\frac{|A|^2 + |B^*|^2}{2}\right\| \|A\| \|B\|}.$$

We also conclude that

Corollary 2.3. *Let $A, B, P \in \mathcal{B}(\mathcal{H})$ with P being a positive operator, then for any $z_k \in \mathbb{C}$ with $\sum_{k=1}^{n} |z_k| \leq 1$, we infer the norm inequality*

$$\left\|\sum_{k=1}^{n} z_k BPA\right\| \leq \frac{\|P\|}{2}\left(\|BA\| + \|A\| \|B\|\right),$$

and the numerical radius inequality

$$\omega\left(\sum_{k=1}^{n} z_k BPA\right) \leq \frac{\|P\|}{2}\left(\omega(BA) + \frac{1}{2}\left\||A|^2 + |B^*|^2\right\|\right). \tag{2.21}$$

Proof. It is sufficient to consider the case where $P \neq 0$. We only prove the inequality (2.21). The other one follows similarly.

By (2.9), we obtain:

$$\begin{aligned}\omega\left(\sum_{k=1}^{n} z_k B\frac{1}{\|P\|}PA\right) &\leq \omega\left(\sum_{k=1}^{n} z_k B\left(\frac{1}{\|P\|}P - \frac{1}{2}I\right)A\right) + \omega\left(\frac{1}{2}\sum_{k=1}^{n} z_k BA\right) \\ &\leq \frac{1}{4}\left\||A|^2 + |B^*|^2\right\| + \frac{1}{2}\omega(BA), \qquad (2.22)\end{aligned}$$

and the inequality (2.21) is proved. □

Corollary 2.4. *With the assumptions of Corollary 2.3, we establish the following norm inequality:*

$$\left\|\sum_{k=1}^{n} z_k APA\right\| \leq \frac{\|P\|}{2}\left(\left\|A^2\right\| + \|A\|^2\right),$$

and the numerical radius inequality

$$\omega\left(\sum_{k=1}^{n} z_k APA\right) \leq \frac{\|P\|}{2}\left(\omega\left(A^2\right) + \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|\right).$$

We also have:

Corollary 2.5. *Let $A, B, P \in \mathcal{B}(\mathcal{H})$ with P being a positive operator, then for any $z_k \in \mathbb{C}$ with $\sum_{k=1}^{n} |z_k| \leq 1$, we have the numerical radius inequality*

$$\omega\left(\sum_{k=1}^{n} z_k BPA\right) \leq \frac{\|P\|}{2^{\frac{1}{r}}}\left[\omega^r(BA) + \left\|\frac{1}{p}|A|^{rp} + \frac{1}{q}|B^*|^{rq}\right\|\right]^{\frac{1}{r}}, \tag{2.23}$$

for any $r \geq 1$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $rp \geq 2$, $rq \geq 2$.

Proof. Utilizing Corollary 2.1 and the fact that $f(t) = t^{\frac{1}{r}}$ is concave on $[0, \infty)$, we obtain:

$$\begin{aligned}
\omega\left(\sum_{k=1}^{n} z_k B\frac{1}{\|P\|}PA\right) &\leq \frac{1}{2}\left\|\frac{1}{p}|A|^{rp} + \frac{1}{q}|B^*|^{rq}\right\|^{\frac{1}{r}} + \frac{1}{2}[\omega^r(BA)]^{\frac{1}{r}} \\
&\leq \left[\frac{1}{2}\left\|\frac{1}{p}|A|^{rp} + \frac{1}{q}|B^*|^{rq}\right\| + \frac{1}{2}\omega^r(BA)\right]^{\frac{1}{r}} \\
&= \frac{1}{2^{\frac{1}{r}}}\left[\omega^r(BA) + \left\|\frac{1}{p}|A|^{rp} + \frac{1}{q}|B^*|^{rq}\right\|\right]^{\frac{1}{r}}.
\end{aligned}$$

This concludes the proof of our result. □

Let $A, B, P \in \mathcal{B}(H)$ with P being a non-zero, positive operator. Then, for any $z_k \in \mathbb{C}$ with $\sum_{k=1}^{n} |z_k| \leq 1$, setting $r = 1$ and $p = q = 2$ in (2.23), we obtain

$$\omega\left(\sum_{k=1}^{n} z_k BPA\right) \leq \frac{\|P\|}{2}\left[\omega(BA) + \left\|\frac{|A|^2 + |B^*|^2}{2}\right\|\right],$$

while for $r = 2$ and $p = q = 2$, we obtain:

$$\omega\left(\sum_{k=1}^{n} z_k BPA\right) \leq \frac{\|P\|\sqrt{2}}{2}\sqrt{\omega^2(BA) + \left\|\frac{|A|^4 + |B^*|^4}{2}\right\|}.$$

Finally, for $r = 2$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we infer

$$\omega\left(\sum_{k=1}^{n} z_k BPA\right) \leq \frac{\|P\|\sqrt{2}}{2}\sqrt{\omega^2(BA) + \left\|\frac{1}{p}|A|^{2p} + \frac{1}{q}|B^*|^{2q}\right\|}.$$

Corollary 2.6. *With the assumptions of Corollary 2.5, we deduce that for* $r \geq 1$*,*

$$\omega\left(\sum_{k=1}^{n} z_k APA\right) \leq \frac{\|P\|}{2^{\frac{1}{r}}}\left[\omega^r\left(A^2\right) + \left\|\frac{|A|^{2r} + |A^*|^{2r}}{2}\right\|\right]^{\frac{1}{r}}.$$

If $r \geq 1$, then for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $rp \geq 2$, $rq \geq 2$,

$$\omega\left(\sum_{k=1}^{n} z_k APA\right) \leq \frac{\|P\|}{2^{\frac{1}{r}}}\left[\omega^r\left(A^2\right) + \left\|\frac{1}{p}|A|^{rp} + \frac{1}{q}|A^*|^{rq}\right\|\right]^{\frac{1}{r}}.$$

3. Conclusions

In this paper, we have explored new inequalities for the norms and numerical radii of bounded linear operators on complex Hilbert spaces, specifically involving one positive operator. Our results contribute to extending existing inequalities in the literature, offering a deeper understanding of the relationships between these operators. This work establishes a foundation for further research and serves as a basis for future studies in this area. We hope that our findings will inspire further exploration and development of inequalities involving operators in complex Hilbert spaces, as well as their potential applications in various mathematical fields.

Author Contributions

Najla Altwaijry: Conceptualization, Visualization, Funding, Resources, Writing–review & editing, Formal analysis, Project administration, Validation, Investigation; Cristian Conde: Conceptualization, Visualization, Funding, Writing–review & editing, Formal analysis, Project administration, Validation, Investigation; Silvestru Sever Dragomir: Conceptualization, Visualization, Funding, Writing–review & editing, Formal analysis, Project administration, Validation, Investigation; Kais Feki: Conceptualization, Visualization, Funding, Writing–review & editing, Formal analysis, Project administration, Validation, Investigation. All authors declare that they have contributed equally to this paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. M. W. Alomari, C. Chesneau, A. Al-Khasawneh, Operator Jensen's inequality for operator superquadratic functions, *Axioms*, **11** (2022), 617. https://doi.org/10.3390/axioms11110617
2. A. Abu-Omar, F. Kittaneh, Numerical radius inequalities for products and commutators of operators, *Houston J. Math.*, **41** (2015), 1163–1173.
3. S. S. Dragomir, *Discrete inequalities of the Cauchy-Bunyakovsky-Schwarz type*, New York: Nova Science Publishers, 2004.
4. C. Lupu, D. Schwarz, Another look at some new Cauchy-Schwarz type inner product inequalities, *Appl. Math. Comput.*, **231** (2014), 463–477. https://doi.org/10.1016/j.amc.2013.11.047
5. G. L. Wang, Y. Wu, G. L. Li, Sharp Adams type inequalities in Lorentz-Sobolev space, *AIMS Mathematics*, **8** (2023), 22192–22206. https://doi.org/10.3934/math.20231131
6. P. Bhunia, S. S. Dragomir, M. S. Moslehian, K. Paul, *Lectures on numerical radius inequalities*, Cham: Springer, 2022. https://doi.org/10.1007/978-3-031-13670-2
7. L. Gan, X. H. Liu, T. Y. Tam, Inequalities for matrix exponentials and their extensions to Lie groups, In: *Matrix and operator equations and applications*, Cham: Springer, 2023, 373–411. https://doi.org/10.1007/16618_2023_51
8. T. Chiheb, B. Meftah, A. Moumen, M. B. Mesmouli, M. Bouye, Some Simpson-like inequalities involving the (s, m)-preinvexity, *Symmetry*, **15** (2023), 2178. https://doi.org/10.3390/sym15122178
9. M. A. Latif, On symmetrized stochastic harmonically convexity and Hermite-Hadamard type inequalities, *Axioms*, **11** (2022), 570. https://doi.org/10.3390/axioms11100570
10. O. Hirzallah, Commutator inequalities for Hilbert space operators, *Linear Algebra Appl.*, **431** (2009), 1571–1578. https://doi.org/10.1016/j.laa.2009.05.026
11. S. G. Shi, L. Zhang, Norm inequalities for higher-order commutators of one-sided oscillatory singular integrals, *J. Inequal. Appl.*, **2016** (2016), 88. https://doi.org/10.1186/s13660-016-1025-0
12. Y. Wu, G. L. Wang, Fractional Adams-Moser-Trudinger type inequality with singular term in Lorentz space and L_p space, *J. Appl. Anal. Comput.*, **14** (2024), 133–145. https://doi.org/10.11948/20230094
13. N. Altwaijry, C. Conde, S. S. Dragomir, K. Feki, Norm and numerical radius inequalities related to the Selberg operator, *Symmetry*, **15** (2023), 1860. https://doi.org/10.3390/sym15101860
14. N. Altwaijry, C. Conde, S. S. Dragomir, K. Feki, Some refinements of Selberg inequality and related results, *Symmetry*, **15** (2023), 1486. https://doi.org/10.3390/sym15081486
15. P. Bhunia, R. K. Nayak, K. Paul, Improvement of A-numerical radius inequalities of semi-Hilbertian space operators, *Results Math.*, **76** (2021), 120. https://doi.org/10.1007/s00025-021-01439-w

16. P. Bhunia, R. K. Nayak, K. Paul, Refinements of A-numerical radius inequalities and their applications, *Adv. Oper. Theory*, **5** (2020), 1498–1511. https://doi.org/10.1007/s43036-020-00056-8

17. A. Zamani, A-numerical radius inequalities for semi-Hilbertian space operators, *Linear Algebra Appl.*, **578** (2019), 159–183. https://doi.org/10.1016/j.laa.2019.05.012

18. S. Sahoo, N. Das, D. Mishra, Numerical radius inequalities for operator matrices, *Adv. Oper. Theory*, **4** (2019), 197–214. https://doi.org/10.15352/aot.1804-1359

19. K. Shebrawi, Numerical radius inequalities for certain 2×2 operator matrices II, *Linear Algebra Appl.*, **523** (2017), 1–12. https://doi.org/10.1016/j.laa.2017.02.019

20. A. Z. Vakili, A. Farokhinia, Norm and numerical radius inequalities for sum of operators, *Boll. Unione Mat. Ital.*, **14** (2021), 647–657. https://doi.org/10.1007/s40574-021-00289-2

21. M. H. M. Rashid, N. H. Altaweel, Some generalized numerical radius inequalities for Hilbert space operators, *J. Math. Inequal.*, **16** (2022), 541–560. https://doi.org/10.7153/jmi-2022-16-39

22. K. Shebrawi, H. Albadawi, Numerical radius and operator norm inequalities, *J. Inequal. Appl.*, **2009** (2009), 492154. https://doi.org/10.1155/2009/492154

23. M. Sababheh, H. R. Moradi, Z. Heydarbeygi, Buzano, Kreǐn and Cauchy-Schwarz inequalities, *Oper. Matrices*, **16** (2022), 239–250. https://doi.org/10.7153/oam-2022-16-19

24. T. Bottazzi, C. Conde, Generalized buzano inequality, *Filomat*, **37** (2023), 9377–9390. https://doi.org/10.2298/FIL2327377B

25. T. Furuta, J. M. Hot, J. Pečarić, Y. Seo, *Mond-Pečarić method in operator inequalities: inequalities for bounded self-adjoint operators on a Hilbert space*, Zagreb: Element, 2005.

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