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Research article

Exploring differential equations and fundamental properties of Generalized Hermite-Frobenius-Genocchi polynomials

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Abstract: This study introduces an innovative framework for generalized Hermite-Frobenius-Genocchi polynomials in two variables, parameterized by a single variable. The focus is on providing a comprehensive characterization of these polynomials through various mathematical tools, including generating functions, series expansions, and summation identities that uncover their essential properties. The work extends to the derivation of recurrence relations, the investigation of shift operators, and the formulation of multiple types of differential equations. In particular, the study delves into integro-differential and partial differential equations, employing a factorization technique to develop different forms and solutions. This multifaceted approach not only enhances our understanding of these polynomials, but also lays the groundwork for their further exploration in diverse areas of mathematical research.

Keywords: Frobenius-Genocchi polynomials; 1-parameter generalized Hermite-Frobenius-Genocchi polynomials; summation formulae; recurrence relations; differential equations
Mathematics Subject Classification: 33E20, 33C45, 33B10, 33E30, 39A14, 45J05, 11T23

1. Introduction and preliminaries

Frobenius-Genocchi polynomials have garnered significant attention due to their versatile applications in several key areas of mathematics, including combinatorial analysis, number theory, and differential equations. These polynomials are not merely abstract constructs; they serve as powerful tools in enumerating combinatorial structures, such as permutations, partitions, and other discrete objects. In combinatorial mathematics, their generating functions often simplify the complex task of counting arrangements or configurations that follow specific rules. In number theory, Frobenius-Genocchi polynomials assist in analyzing numerical sequences, particularly those related to special number classes, like Genocchi and Bernoulli numbers. Their ability to represent and generalize such sequences opens up new avenues for understanding the deeper properties of integers and their relationships. Additionally, these polynomials contribute to discovering new identities and relations between number-theoretic functions, further enriching the field. In the realm of differential equations, Frobenius-Genocchi polynomials provide solutions to various classes of equations, including both ordinary and partial differential equations. Their recurrence relations and generating functions are often used to formulate and solve complex integro-differential and integro-partial differential equations. This capability enables mathematicians and physicists to tackle problems in mathematical physics, fluid dynamics, and other applied fields. Therefore, Frobenius-Genocchi polynomials bridge the gap between pure and applied mathematics, offering critical insights that help address a wide range of mathematical challenges. Their ability to unify different branches of mathematics underscores their enduring relevance and utility.

Banu and Özarslan [29] conducted an in-depth study of the Frobenius-Genocchi polynomials, denoted as ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1|u)$, which adhere to a specific exponential generating function. Their research is significant because it expands the theoretical understanding of these polynomials, which are a generalization of classical Genocchi polynomials. The Frobenius-Genocchi polynomials play a crucial role in various branches of mathematical analysis and number theory, including combinatorics, special functions, and algebraic identities. By exploring the properties, recurrence relations, and applications of ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1|u)$, Banu and Özarslan have provided valuable insights that can be utilized in solving complex mathematical problems. Their findings also have potential implications for related fields such as mathematical physics and coding theory, where polynomial structures are often employed to model and solve intricate problems. Moreover, the detailed examination of the exponential generating function associated with these polynomials offers a deeper comprehension of their behavior and interrelations with other mathematical entities. This work not only contributes to the existing body of knowledge, but also paves the way for future research endeavors aimed at uncovering further applications and generalizations of Frobenius-Genocchi polynomials. This foundational work has allowed mathematicians to further explore the algebraic properties and relationships within polynomial sequences, leading to advancements in mathematical research and practical problem-solving. These polynomials possess the following generating relation:

$$\frac{(1-u)\tau}{e^{\tau}-u}\, e^{\rho_1\tau} = \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_n(\rho_1|u)\frac{\tau^n}{n!}, \quad \forall\ u \in \mathbb{C}; \quad u \neq 1. \tag{1.1}$$

The Frobenius-Genocchi polynomials, particularly ${}_{\mathcal{P}}\mathcal{J}_n(u) = {}_{\mathcal{P}}\mathcal{J}_n(0|u)$, known as the Frobenius-Genocchi numbers, are significant for their broad applications in combinatorial mathematics, number theory, and the resolution of differential equations. These numbers facilitate the enumeration of combinatorial structures, the examination of numerical sequences, and the solution of intricate differential equations, providing essential insights into diverse mathematical problems possessing the generating relation:

$$\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_n(0|u)\frac{\tau^n}{n!} = \frac{(1-u)\tau}{e^{\tau}-u}, \quad \forall\ u \in \mathbb{C}; \quad u \neq 1.$$

The "Frobenius-Genocchi polynomials" possess the series representation:

$$\sum_{k=0}^{n}\binom{n}{k}{}_{\mathcal{P}}\mathcal{J}_k(u)\rho_1^{n-k} = {}_{\mathcal{P}}\mathcal{J}_n(\rho_1|u), \quad n \geq 0, \quad {}_{\mathcal{P}}\mathcal{J}_0(u) = 1.$$

The "classical Genocchi polynomials" $\mathcal{J}_n(\rho_1)$ are analogues of the "Frobenius-Genocchi polynomials". The following generating relations define them:

$$\frac{2\tau}{e^{\tau}-1}\, e^{\rho_1\tau} = \sum_{n=0}^{\infty}\mathcal{J}_n(\rho_1)\frac{\tau^n}{n!}.$$

For, $\mathcal{J}_n = \mathcal{J}_n(0)$, they are called as the "classical Genocchi numbers". These numerical values and polynomial functions hold pivotal significance across various mathematical domains, encompassing number theory, combinatorics, special functions, and analysis. Notably, the "Frobenius-Genocchi polynomials yield the classical Genocchi polynomials" when $u = -1$ in Eq (1.1).

From the perspective of applications, the special polynomials of two variables are crucial. By using these polynomials, new families of special polynomials may be introduced, and valuable identities can be derived very straightforwardly. Using features of an iterated isomorphism connected to the Laguerre-type exponentials, Bretti et al. [7], for instance, established extended classes of the Appell polynomials of two variables. Various writers have examined the two variable versions of the Hermite, Laguerre, and truncated exponential polynomials, along with their generalizations in [4, 8, 12–14, 18, 19].

In recent years, there has been a remarkable surge in the development and application of special functions in mathematical physics, significantly broadening their scope and potential. These innovations have provided a robust and versatile analytical framework capable of addressing a diverse range of complex problems across the field of mathematical physics. As a result, these advancements have found extensive use in numerous practical domains, influencing not only theoretical research, but also real-world applications in various industries, see [1, 2, 6, 23–25].

The significance of generalized Hermite polynomials has been thoroughly examined in previous studies, including those by Datolli et al. [15, 16]. These polynomials have emerged as indispensable tools in solving fundamental challenges in areas such as quantum mechanics, where they are pivotal in modeling wave functions and particle dynamics. Furthermore, they play a central role in the analysis of optical beam propagation, facilitating the study of light interaction in nonlinear media. Additionally, generalized Hermite polynomials are instrumental in solving a variety of problems related to partial differential equations and have broad implications in the study of abstract group theory, providing profound insights into symmetry and algebraic structures.

The "2-variable Hermite Kampé de Feriet polynomials (2VHKdFP)", denoted as $\mathcal{P}_n(\rho_1, \rho_2)$ [4], can be represented by the following generating function:

$$e^{\rho_1\tau+\rho_2\tau^2} = \sum_{n=0}^{\infty}\mathcal{P}_n(\rho_1,\rho_2)\frac{\tau^n}{n!},$$

which, for $\rho_2 = 0$, gives

$$e^{\rho_1\tau} = \sum_{n=0}^{\infty}\mathcal{P}_n(\rho_1)\frac{\tau^n}{n!}.$$

Further, the "2-variable 1-parameter Hermite polynomials (2V1PHP)", represented as $\mathcal{P}_n(\rho_1, \rho_2, C)$, can be represented by the following generating function [29]:

$$C^{\rho_1\tau+\rho_2\tau^2} = \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_n(\rho_1, \rho_2, C)\frac{\tau^n}{n!}, \quad C > 1. \tag{1.2}$$

The benefits and rationale behind using $C^{\rho_1\tau+\rho_2\tau^2}$ over $e^{\rho_1\tau+\rho_2\tau^2}$ is due to the flexibility with the Base C. The introduction of the base C allows us to tailor the behavior of the function to better fit specific applications. For instance, selecting C appropriately can help in achieving a desired growth rate or in matching empirical data more accurately. Also, in many real-world scenarios, exponential growth might be too rapid. By using a base C smaller than e, we can model phenomena with slower growth more effectively. Conversely, a larger C can be used to model faster growth or decay. Further more, depending on the value of C and the computational context, evaluating $C^{\rho_1\tau+\rho_2\tau^2}$ might be more efficient or numerically stable compared to $e^{\rho_1\tau+\rho_2\tau^2}$. This is particularly relevant in high-performance computing or large dataset analyses.

In their two-variable formulation, these polynomials have become indispensable across a wide range of disciplines within both pure and applied mathematics and physics. Their versatility allows them to be applied to complex problems in diverse fields as a foundational tool in various mathematical models and physical theories.

For instance, in mathematical physics, these polynomials play a crucial role in solving Laplace's equation when expressed in parabolic coordinates. This equation, which is fundamental in studying potential theory, electrostatics, and fluid dynamics, often requires special polynomials to simplify and solve problems involving boundary conditions or specific geometries. Moreover, in quantum mechanics, these polynomials are instrumental in addressing scenarios where wave functions or quantum states need to be described in parabolic coordinates, particularly in systems with cylindrical or parabolic symmetry. Their application extends to solving the Schrödinger equation in such contexts, providing exact solutions that describe the behavior of quantum particles under specific potential fields.

In probability theory, these polynomials are equally important. They are used to model distributions and stochastic processes, particularly when the underlying processes have symmetries or constraints that can be described using parabolic coordinates. This includes applications in financial mathematics, where they help analyze random walks or diffusion processes.

One of the most notable features of these polynomials is their ability to provide specific solutions to the heat equation or generalized heat problems for any integral value of n. The heat equation, which describes the distribution of heat (or variation in temperature) in a given region over time, is a critical partial differential equation in theoretical and applied contexts. The corresponding Gauss-Weierstrass transforms, integral transforms used to smooth or regularize functions, facilitate these solutions by linking the polynomials to the broader context of heat diffusion and propagation.

Consider the sequence of polynomials $\{\mathcal{P}_n(\rho_1)\}_{n=0}^{\infty}$, where each $\mathcal{P}_n(\rho_1)$ represents a polynomial. It can be observed that the degree of $\mathcal{P}_n(\rho_1)$ is n, for all $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$. The differential operators Ψ_n^- and Ψ_n^+ meeting the criteria

$$\Psi_n^-\{\mathcal{P}_n(\rho_1)\} = \mathcal{P}_{n-1}(\rho_1), \tag{1.3}$$

$$\Psi_n^+\{\mathcal{P}_n(\rho_1)\} = \mathcal{P}_{n+1}(\rho_1) \tag{1.4}$$

are referred to as the multiplicative and derivative operators, respectively. A sequence of polynomials $\{\mathcal{P}_n(\rho_1)\}_{n=0}^{\infty}$ is classified as quasi-monomial if, and only if, it satisfies the conditions given by Eqs (1.3) and (1.4). This particular differential equation can be derived by identifying the corresponding derivative and multiplicative operators for a specific family of polynomials, as outlined below:

$$(\Psi_{n+1}^{-}\Psi_n^{+})\{\mathcal{P}_n(\rho_1)\} = \mathcal{P}_n(\rho_1).$$

This process is known as the factorization technique. The core of this approach lies in determining the multiplicative operator Ψ_n^{+} and the derivative operator Ψ_n^{-}, as outlined in references [3, 5, 9, 10, 17, 22, 26, 27]. Another perspective on this method is provided by the monomiality principle. When applied to multivariable special functions, the factorization technique introduces new analytical tools for solving a wide range of partial differential equations commonly encountered in practical applications.

Differential equations are integral to various fields such as physics, engineering, and both pure and applied mathematics. Many problems in scientific and technical disciplines are often modeled by differential equations, typically solved using specialized functions. Over the past three decades, there has been a resurgence of interest in differential equation theory, driven by advances in nonlinear analysis, dynamical systems, and their practical applications in science and engineering.

Numerous studies have systematically explored and analyzed hybrid families of special polynomials using various generating function approaches and analytical methods [11, 20, 21, 28]. Key features of these multi-variable hybrid special polynomials include "recurrence relations, explicit formulas, functional and differential equations, summation formulas, symmetric and convolution identities, and determinant techniques," all of which contribute to their significance. These polynomials are valuable tools in diverse fields, such as number theory, combinatorics, classical and numerical analysis, theoretical physics, and approximation theory. The unique properties of hybrid special polynomials make them particularly useful for addressing new challenges across various scientific disciplines.

In this work, we introduce a novel technique for analyzing generalized Hermite-Frobenius-Genocchi polynomials, which offers distinct advantages over traditional methods. This approach provides a more efficient framework for deriving recurrence relations, differential equations, and summation identities, allowing for easier manipulation and application in various mathematical contexts. Compared to previous studies, our technique simplifies the formulation of complex relationships and enhances computational tractability. A thorough review of the existing literature reveals that while significant progress has been made in the study of special polynomials, many approaches are limited by computational complexity or lack of generalizability. Our technique addresses these limitations by offering a versatile and scalable method for dealing with two-variable polynomials. Furthermore, it extends the scope of previous research by introducing factorization methods and providing insights into the broader algebraic structures. This paper fills a gap in the current literature and opens up new possibilities for future research in both theoretical and applied mathematics. The article is structured as follows:

Section 2 introduces the generalized Hermite-Frobenius-Genocchi polynomials with one parameter and two variables (1P2VGHFGP), presenting their series representations, generating functions, and operational formalism. Section 3 applies the factorization method to derive various differential equations, including traditional, integro-differential, and partial differential equations, highlighting the polynomials' versatility in solving complex problems. Section 4 provides summation formulae derived from the polynomials' series representations, offering insights into their summing behavior and practical applications. The final section summarizes the findings and discusses potential future research directions, focusing on the further exploration of these polynomials and their broader applications in mathematical and scientific fields.

2. 1-parameter 2-variable generalized Hermite-Frobenius-Genocchi polynomials

This section presents a novel hybrid family of polynomials, referred to as the "1-parameter 2-variable generalized Hermite-Frobenius-Genocchi polynomials" (1P2VGHFGP). This family is defined through a combination of key mathematical concepts, offering a unique structure for further exploration. In this regard, we derive and establish several significant properties of these polynomials, highlighting their importance and utility in various mathematical contexts. To facilitate the understanding and application of these polynomials, we proceed to derive their generating function. A pivotal result is introduced in the following, which serves as the foundation for deriving the generating function of the 1P2VGHFGP.

Theorem 2.1. *For the "1-parameter 2-variable generalized Hermite-Frobenius-Genocchi polynomials" ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)$, the following generating relation is demonstrated:*

$$\frac{(1-u)\tau}{e^{\tau}-u}C^{\rho_1\tau+\rho_2\tau^2}=\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\frac{\tau^n}{n!},\quad C>1, \tag{2.1}$$

or, equivalently,

$$\frac{(1-u)\tau}{e^{\tau}-u}e^{(\ln(\rho_1\tau+\rho_2\tau^2)C)}=\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\frac{\tau^n}{n!},\quad C>1. \tag{2.2}$$

Proof. Substituting the exponents of τ, *i.e.* $\rho_1^0,\ \rho_1^1,\ \rho_1^2,\cdots,\rho_1^n$, in the expansion of $e^{\rho_1\tau}$ by the polynomials ${}_{\mathcal{P}}\mathcal{J}_0(\rho_1,\rho_2;C|u)$, ${}_{\mathcal{P}}\mathcal{J}_2(\rho_1,\rho_2;C|u)$, $\cdots,{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)$ in the left hand part and ρ_1 by ${}_{\mathcal{P}}\mathcal{J}_1(\rho_1,\rho_2;C|u)$ in right-hand part of the expression (1.1), further adding up the expressions in the left-hand part of the resultant expression, we have

$$\frac{(1-u)\tau}{e^{\tau}-u}\sum_{n=0}^{\infty}\mathcal{P}_n(\rho_1,\rho_2;C|u)\frac{\tau^n}{n!}=\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{J}_n(\mathcal{P}_1(\rho_1,\rho_2;C|u))\frac{\tau^n}{n!},$$

which indicates the resulting 1P2VGHFGP on the r.h.s., that is, ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u):={}_{\mathcal{P}}\mathcal{J}_n\{\mathcal{P}_1(\rho_1,\rho_2;C|u)\}$, leading to (2.1).

Further, the generating function (2.2) is obtained by simplifying the l.h.s. of Eq (2.1). □

Remark 2.1. *For $\rho_2=0$, the "1-parameter 2-variable generalized Hermite-Frobenius-Genocchi polynomials" ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)$ reduces to the generalized 1-parameter Frobenius-Genocchi polynomials ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1C)$, possessing generating relation:*

$$\frac{(1-u)\tau}{e^{\tau}-u}C^{\rho_1\tau}=\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1;C|u)\frac{\tau^n}{n!},\quad C>1,$$

or, equivalently,

$$\frac{(1-u)\tau}{e^{\tau}-u}e^{(\rho_1\tau)\ln C}=\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1;C|u)\frac{\tau^n}{n!},\quad C>1.$$

Remark 2.2. *For $\rho_2=0=\rho_1$, the "1-parameter 2-variable generalized Hermite-Frobenius-Genocchi polynomials" ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)$ reduces to the Frobenius-Genocchi numbers ${}_{\mathcal{P}}\mathcal{J}_n$, possessing the generating relation:*

$$\frac{(1-u)\tau}{e^{\tau}-u}=\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1;C|u)\frac{\tau^n}{n!}.$$

The following theorem gives the series definition for the 1P2VGHFGP ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)$:

Operational techniques for special polynomials encompass a diverse array of algebraic and analytical approaches that enable their effective manipulation and application across various mathematical and practical domains. Central to these techniques are generating functions, which consolidate entire polynomial sequences into a single, compact function. This representation not only simplifies the derivation of relationships and identities among the polynomials, but also facilitates their deeper analysis.

Another cornerstone is the use of differential operators, which serve as powerful tools for expressing recurrence relations, performing transformations, and streamlining the resolution of complex differential equations. By leveraging these operators, one can gain insights into the structural properties and dynamic behavior of the polynomials.

Furthermore, integral transforms, including the Laplace and Fourier transforms, significantly extend the scope of special polynomials, enabling their application in practical contexts such as quantum mechanics, signal processing, and control theory. These transforms provide a bridge between abstract polynomial theory and real-world problem-solving, enhancing both their versatility and utility.

Altogether, these operational techniques not only simplify intricate mathematical computations, but also underscore the critical role of special polynomials in advancing both pure and applied mathematics. Their efficiency and adaptability make them indispensable in tackling theoretical challenges and modeling complex systems across scientific disciplines.

Differentiating (2.1) or (2.2) w.r.t. ρ_1 successively, we find

$$\frac{\partial}{\partial\rho_1}\left(\frac{(1-u)\tau}{e^{\tau}-u}C^{\rho_1\tau+\rho_2\tau^2}\right)=(\ln C)\tau\left(\frac{(1-u)\tau}{e^{\tau}-u}C^{\rho_1\tau+\rho_2\tau^2}\right),$$

which can be further expressed as

$$\frac{\partial}{\partial\rho_1}\left(\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\frac{\tau^n}{n!}\right)=(\ln C)\left(\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\frac{\tau^{n+1}}{n!}\right).$$

Then, replacing $n\to n-1$ in r.h.s. of the preceding expression and then comparing the coefficients of the same exponents on both sides of the resultant expression, we find

$$\frac{\partial}{\partial\rho_1}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)=n\ln C\,{}_{\mathcal{P}}\mathcal{J}_{n-1}(\rho_1,\rho_2;C|u).\tag{2.3}$$

Continuing in similar fashion, we have

$$\frac{\partial^2}{\partial\rho_1^2}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u) = n(n-1)(\ln C)^2{}_{\mathcal{P}}\mathcal{J}_{n-2}(\rho_1,\rho_2;C|u),$$

$$\frac{\partial^3}{\partial\rho_1^3}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u) = n(n-1)(n-2)(\ln C)^3{}_{\mathcal{P}}\mathcal{J}_{n-3}(\rho_1,\rho_2;C|u),$$

$$\vdots \qquad \vdots$$

$$\frac{\partial^m}{\partial\rho_1^m}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u) = n(n-1)\cdots(n-m+1)(\ln C)^m{}_{\mathcal{P}}\mathcal{J}_{n-m}(\rho_1,\rho_2;C|u).$$

Further, differentiating (2.1) or (2.2) w.r.t. ρ_2 successively, we find

$$\frac{\partial}{\partial\rho_2}\left(\frac{(1-u)\tau}{e^\tau-u}C^{\rho_1\tau+\rho_2\tau^2}\right) = (\ln C)\tau^2\left(\frac{(1-u)\tau}{e^\tau-u}C^{\rho_1\tau+\rho_2\tau^2}\right),$$

which can be further expressed as

$$\frac{\partial}{\partial\rho_2}\left(\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\frac{\tau^n}{n!}\right) = (\ln C)\left(\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\frac{\tau^{n+2}}{n!}\right),$$

Then, replacing $n \to n-2$, on the r.h.s. of the preceding expression, and then comparing the coefficients of the same exponents on both sides of the resultant expression, we find

$$\frac{\partial}{\partial\rho_2}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u) = n(n-1)\ln C{}_{\mathcal{P}}\mathcal{J}_{n-2}(\rho_1,\rho_2;C|u). \tag{2.4}$$

Thus, the expressions (2.3) and (2.4) satisfy the relation:

$$\frac{\partial}{\partial\rho_2}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u) = \frac{1}{\ln C}\frac{\partial^2}{\partial\rho_1^2}{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u) \tag{2.5}$$

which, in consideration of the initial condition:

$${}_{\mathcal{P}}\mathcal{J}_n(\rho_1,0;C|u) = {}_{\mathcal{P}}\mathcal{J}_n(\rho_1;C|u) \tag{2.6}$$

provides the operational representation for ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)$ via the result.

Theorem 2.2. *For the 1P2VGHFGP ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)$, the following operational representation is given:*

$${}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u) = \exp\left(\frac{\rho_2}{\ln C}\frac{\partial^2}{\partial\rho_1^2}\right)\left\{{}_{\mathcal{P}}\mathcal{J}_n(\rho_1;C|u)\right\}. \tag{2.7}$$

Proof. In view of the expressions (2.5) and (2.6), the assertion (2.7) is established. □

3. Recurrence relations, shift operators, and families of differential equations

Recurrence relations are pivotal in deriving families of differential equations as they systematically link solutions of different orders, enabling the reduction of complex problems to simpler ones. They facilitate the derivation and simplification of differential equations by expressing higher-order solutions in terms of lower-order ones, thus making the problem more manageable. Moreover, they offer analytical insights into the structure and properties of solutions, such as orthogonality and asymptotic behavior, and provide computational efficiency by enabling algorithms for practical applications. Overall, recurrence relations not only unify the approach to solving differential equations, but also enhance our ability to apply these solutions in various scientific and engineering fields.

Theorem 3.1. *The 1P2VGHFGP* ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1, \rho_2; C|u)$ *adhere, to the following recurrence relation:*

$$\begin{aligned}{}_{\mathcal{P}}\mathcal{J}_{n+1}(\rho_1, \rho_2; C|u) &= (\rho_1(\ln C) - \tfrac{n+1}{2(1-u)})\, {}_{\mathcal{P}}\mathcal{J}_n(\rho_1, \rho_2; C|u) + 2n\rho_2(\ln C){}_{\mathcal{P}}\mathcal{J}_{n-1}(\rho_1, \rho_2; C|u)\\ &-\tfrac{1}{(1-u)} \sum_{k=2}^{n+1} \tbinom{n+1}{k} {}_{\mathcal{P}}\mathcal{J}_{n-k+1}(\rho_1, \rho_2; C|u)\mathfrak{G}_k(u),\end{aligned} \tag{3.1}$$

where the expression:

$$\mathfrak{G}_k(u) := -\sum_{i=0}^{k} \frac{1}{2^i}\binom{k}{i}\mathfrak{G}_{k-i}\left(\frac{1}{2}|u\right), \ \mathfrak{G}_0 = -1, \ \ \mathfrak{G}_1 = \frac{1}{2}$$

is expressed using the numerical coefficients $\mathfrak{G}_n(u)$*, which are connected to the Frobenius-Genocchi polynomials* ${}_{\mathcal{P}}\mathcal{J}_k(\rho_1|u)$ *and*

$${}_{\mathcal{P}}\mathcal{J}_{-k}(\rho_1, \rho_2; C|u) := 0, \quad k = 1, 2.$$

Proof. After incorporating τ and differentiating both sides of the generating function (2.1), we obtain:

$$\frac{\partial}{\partial \tau}\left\{\frac{(1-u)\tau}{e^\tau - u} C^{\rho_1\tau+\rho_2\tau^2}\right\} = \frac{\partial}{\partial \tau}\left\{\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_n(\rho_1, \rho_2; C|u)\frac{\tau^n}{n!}\right\}$$

which can be simplified as

$$\begin{aligned}&\left\{\rho_1 \ln(C) + 2\rho_2 \ln(C)\tau\right\} \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_n(\rho_1, \rho_2; C|u)\frac{\tau^n}{n!} - \frac{1}{(1-u)} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_n(\rho_1, \rho_2; C|u)\\ &\times\ \mathfrak{G}_k(u)\frac{\tau^{n+k}}{n!\,k!} = \sum_{n=0}^{\infty} n\, {}_{\mathcal{P}}\mathcal{J}_n(\rho_1, \rho_2; C|u)\frac{\tau^{n-1}}{n!}.\end{aligned}$$

Furthermore, the previous expression can be rewritten using the Cauchy product formula as:

$$\begin{aligned}&\sum_{n=0}^{\infty}\Bigg[\Big(\rho_1 \ln(C) - \frac{n+1}{2(1-u)}\Big){}_{\mathcal{P}}\mathcal{J}_n(\rho_1, \rho_2; C|u) + 2n\rho_2 \ln(C){}_{\mathcal{P}}\mathcal{J}_{n-1}(\rho_1, \rho_2; C|u)\\ &- \frac{1}{(1-u)} \sum_{k=2}^{n+1} \binom{n+1}{k} {}_{\mathcal{P}}\mathcal{J}_{n-k+1}(\rho_1, \rho_2; C|u)\mathfrak{G}_k(u)\Bigg]\frac{\tau^n}{n!} = \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_{n+1}(\rho_1, \rho_2; C|u)\frac{\tau^n}{n!}.\end{aligned}$$

Statement (3.1) is derived by equating the coefficients of corresponding powers of τ on both sides of the given equation. □

Theorem 3.2. *The 1P2VGHFGP ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)$ adhere, to the following shift operators:*

$$ {}_{\rho_1}\mathcal{L}_n^- := \frac{1}{n(\ln C)} D_{\rho_1}, \tag{3.2}$$

$$ {}_{\rho_2}\mathcal{L}_n^- := \frac{1}{n} D_{\rho_1}^{-1} D_{\rho_2}, \tag{3.3}$$

$$ {}_{\rho_1}\mathcal{L}_n^+ := (\rho_1 \ln C - \frac{n+1}{2(1-u)}) + 2\rho_2 D_{\rho_1} + 3\rho_3 (\ln C)^{-1} D_{\rho_1}^2 - \frac{n+1}{1-u}\sum_{k=2}^{n+1} (\ln C)^{-k} D_{\rho_1}^{k-1} \frac{\mathfrak{G}_k(u)}{k!} \tag{3.4}$$

and

$$ {}_{\rho_2}\mathcal{L}_n^+ := (\rho_1 \ln C - \frac{n+1}{2(1-u)}) + 2\rho_2 D_{\rho_1}^{-1} D_{\rho_2} + 3\rho_3 (\ln C)^{-1} D_{\rho_1}^{-2} D_{\rho_2}^2 - \frac{n+1}{1-u}\sum_{k=2}^{n+1} (\ln C)^{-k} D_{\rho_1}^{-(k-1)} D_{\rho_2}^{k-1} \frac{\mathfrak{G}_k(u)}{k!}, \tag{3.5}$$

respectively, where

$$ D_{\rho_1} := \frac{\partial}{\partial_{\rho_1}}, \;\; D_{\rho_2} := \frac{\partial}{\partial_{\rho_2}}; \;\; D_{\rho_1}^{-1} := \int_0^{\rho_1} g(\tau) d\tau.$$

Proof. By reorganizing the terms according to their powers and differentiating both sides of Eq (2.1) with respect to ρ_1, we equate the coefficients of the corresponding powers of τ from both sides of the resulting expression, as shown below:

$$ D_{\rho_1}\{{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\} = n(\ln C)\, {}_{\mathcal{P}}\mathcal{J}_{n-1}(\rho_1,\rho_2;C|u).$$

As a consequence, the operator defined by Eq (3.2) fulfills the requirements of the equation

$$ {}_{\rho_1}\mathcal{L}_n^-\{{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\} = {}_{\mathcal{P}}\mathcal{J}_{n-1}(\rho_1,\rho_2;C|u).$$

Subsequently, differentiating both sides of Eq (2.1) with respect to ρ_2, rearranging the powers, and then calculating the coefficients of the identical powers of τ on both sides of the resulting equation gives:

$$ D_{\rho_2}\{{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\} = (\ln C) n(n-1)\, {}_{\mathcal{P}}\mathcal{J}_{n-2}(\rho_1,\rho_2;C|u),$$

which can be further stated as

$$ D_{\rho_2}\{{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\} = n(\ln C) D_{\rho_1}\, {}_{\mathcal{P}}\mathcal{J}_{n-1}(\rho_1,\rho_2;C|u).$$

Thus, it follows that

$$ \frac{1}{n} D_{\rho_2} D_{\rho_1}^{-1}\{{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\} = {}_{\mathcal{P}}\mathcal{J}_{n-1}(\rho_1,\rho_2;C|u).$$

Thus, the operator given by Eq (3.3) satisfies the above equation.

The raising operator given by (3.4) can be determined using the following relation:

$$ {}_{\mathcal{P}}\mathcal{J}_{n-k}(\rho_1,\rho_2;C|u) = \left({}_{\rho_1}\mathcal{L}_{n-k+1}^- \; {}_{\rho_1}\mathcal{L}_{n-k+2}^- \cdots {}_{\rho_1}\mathcal{L}_{n-1}^- \; {}_{\rho_1}\mathcal{L}_n^-\right)\{{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\}. \tag{3.6}$$

By combining Eq (3.2) with Eq (3.6), we obtain:

$$_{\mathcal{P}}\mathcal{J}_{n-k}(\rho_1,\rho_2;C|u) = \left(\frac{1}{(n-k+1)(\ln C)}D_{\rho_1}\cdots\frac{1}{(n-1)(\ln C)}D_{\rho_1}\ \frac{1}{n(\ln C)}D_{\rho_1}\right)\{_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\}$$

which can be further casted as

$$_{\mathcal{P}}\mathcal{J}_{n-k}(\rho_1,\rho_2;C|u) = \frac{(n-k)!}{n!}(\ln C)^{-k}D^k_{\rho_1}\{_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\}. \tag{3.7}$$

Further, we have

$$_{\mathcal{P}}\mathcal{J}_{n-1}(\rho_1,\rho_2;C|u) = \frac{1}{n}(\ln C)^{-1}D_{\rho_1}\{_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\}.$$

Thus, inserting expressions (3.6) and (3.7) into Eq (3.1), we find

$$\begin{aligned}_{\mathcal{P}}\mathcal{J}_{n+1}(\rho_1,\rho_2;C|u) = \Bigg(\Big(\rho_1\ln C-\frac{n+1}{2(1-u)}\Big)+2\rho_2 D^{-1}_{\rho_1}D_{\rho_2}\\ -\frac{n+1}{1-u}\sum_{k=2}^{n+1}(\ln C)^{-k}D^{k-1}_{\rho_1}\frac{\mathfrak{G}_k(u)}{k!}\Bigg)\{_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\}.\end{aligned}$$

Thus, we derive the expression (3.4) for the raising operator $_{\rho_1}\mathcal{L}^+_n$.

Next, we use the following relation to determine the raising operator given by (3.5):

$$_{\mathcal{P}}\mathcal{J}_{n-k}(\rho_1,\rho_2;C|u) = \left({}_{\rho_2}\mathcal{L}^-_{n-k+1}\ {}_{\rho_2}\mathcal{L}^-_{n-k+2}\cdots{}_{\rho_2}\mathcal{L}^-_{n-1}\ {}_{\rho_2}\mathcal{L}^-_{n}\right)\{_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\}. \tag{3.8}$$

By applying Eq (3.3) to Eq (3.8) and simplifying, we obtain:

$$_{\mathcal{P}}\mathcal{J}_{n-k}(\rho_1,\rho_2;C|u) = \frac{(n-k)!}{n!}(\ln C)^{-k}D^{-k}_{\rho_1}D^k_{\rho_2}\{_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\}. \tag{3.9}$$

Further, we find

$$_{\mathcal{P}}\mathcal{J}_{n-1}(\rho_1,\rho_2;C|u) = \frac{1}{n}(\ln C)^{-1}D^{-1}_{\rho_1}D_{\rho_2}\{_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\}. \tag{3.10}$$

By substituting Eqs (3.9) and (3.10) into Eq (3.1), we find:

$$\begin{aligned}_{\mathcal{P}}\mathcal{J}_{n+1}(\rho_1,\rho_2;C|u) = &\Big(\Big(\rho_1\ln C-\tfrac{n+1}{2(1-u)}\Big)+2\rho_2 D^{-1}_{\rho_1}D_{\rho_2}\\ &-\tfrac{n+1}{1-u}\textstyle\sum_{k=2}^{n+1}(\ln C)^{-k}D^{-(k-1)}_{\rho_1}D^{k-1}_{\rho_2}\tfrac{\mathfrak{G}_k(u)}{k!}\Big)\{_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\},\end{aligned}$$

Thus, we obtain the expression (3.5) for the raising operator $_{\rho_2}\mathcal{L}^+_n$. □

Next, we derive the "differential, integro-differential, and partial differential equations" for the generalized one-parameter Hermite-Frobenius-Genocchi polynomials $_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)$. To achieve this, we consider the following results.

Theorem 3.3. *The generalized one-parameter Hermite-Frobenius-Genocchi polynomials $_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)$ satisfy the following differential equation:*

$$\begin{aligned}\Bigg(\Big(\rho_1-(\ln C)^{-1}\frac{n+1}{2(1-u)}\Big)D_{\rho_1}+2\rho_2(\ln C)^{-1}D^2_{\rho_1}-\frac{n+1}{1-u}\sum_{k=2}^{n+1}(\ln C)^{-k-1}D^k_{\rho_1}\frac{\mathfrak{G}_k(u)}{k!}-n\Bigg)\\ \times\,{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)=0.\end{aligned} \tag{3.11}$$

Proof. By utilizing expressions (3.2) and (3.4) for the shift operators ${}_{\rho_1}\mathcal{L}_n^-$ and ${}_{\rho_1}\mathcal{L}_n^+$ in the factorization equation ${}_{\rho_1}\mathcal{L}_{n+1}^-\,{}_{\rho_1}\mathcal{L}_n^+\{{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\} = {}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)$, we arrive at expression (3.11). □

Theorem 3.4. *The generalized one-parameter Hermite-Frobenius-Genocchi polynomials ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)$ satisfy the following integro-differential equation:*

$$\Bigg(\Big(\rho_1 - (\ln C)^{-1}\frac{n+1}{2(1-u)}\Big)D_{\rho_2} + 2\rho_2(\ln C)^{-1}D_{\rho_1}^{-1}D_{\rho_2}^2 - \frac{n+1}{1-u}\sum_{k=2}^{n+1}(\ln C)^{-k-1} \times D_{\rho_1}^{-(k-1)}D_{\rho_2}^k\frac{\mathfrak{G}_k(u)}{k!} - (n+1)D_{\rho_1}\Bigg){}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u) = 0. \tag{3.12}$$

Proof. By utilizing expressions (3.3) and (3.5) for the shift operators $\mathcal{L}_n^-$ and $\mathcal{L}_n^+$ in the factorization equation

$$\mathcal{L}_{n+1}^-\,\mathcal{L}_n^+\{{}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)\} = {}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u),$$

we derive expressions (3.12). □

Theorem 3.5. *The generalized one-parameter Hermite-Frobenius-Genocchi polynomials, denoted as ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u)$, satisfy the following partial differential equation:*

$$\Bigg(\Big(\rho_1 - \frac{(\ln C)^{-1}(n+1)}{2(1-u)}\Big)D_{\rho_1}^{2n}D_{\rho_2} + 2nD_{\rho_1}^{2n-1}D_{\rho_2} + 2\rho_2(\ln C)^{-1}D_{\rho_1}^{2n-1}D_{\rho_2}^2 + 2(\ln C)^{-1}D_{\rho_1}^{2n-1}D_{\rho_2} + 3\rho_3(\ln C)^{-2} \times D_{\rho_1}^{2n-2}D_{\rho_2}^3 - \frac{n+1}{1-u}\sum_{k=2}^{n+1}(\ln C)^{-k-1}D_{\rho_1}^{2n-k+1}D_{\rho_2}^{2n+k}\frac{\mathfrak{G}_k(u)}{k!} - (n+1)D_{\rho_1}^{2n+1}\Bigg){}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C|u) = 0. \tag{3.13}$$

Proof. By differentiating expressions (3.12) concerning D_{ρ_1} a total of $2n$ times, we obtain the partial differential equations (3.13). □

4. Summation formulae

Summation formulas hold a pivotal place in mathematics, underpinning numerous applications across diverse fields such as probability theory, combinatorics, and algebra. These formulas facilitate key computations, including the determination of expected values in probabilistic models, efficient methods for polynomial interpolation, and streamlined approaches to solving combinatorial counting problems.

In number theory, summation formulas are instrumental in the exploration and analysis of arithmetic functions, while in applied mathematics, they find practical utility in areas like signal processing, optimization, and numerical analysis. By providing systematic techniques for evaluating sequences and series, summation formulas become indispensable tools for addressing complex challenges in both theoretical and applied contexts.

The summation formulas associated with the generalized 1-parameter Hermite-Frobenius-Genocchi polynomials, denoted as ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1,\rho_2;C \mid u)$, extend these capabilities by offering robust solutions to intricate problems across various mathematical disciplines. The explicit expressions for these summation formulas are as follows.

Theorem 4.1. *For the generalized 1-parameter Hermite-Frobenius-Genocchi polynomials* ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1, \rho_2; C|u)$*, the following summation formula holds true:*

$$ {}_{\mathcal{P}}\mathcal{J}_n(\rho_1 + \upsilon, \rho_2; C|u) = \sum_{k=0}^{n} \binom{n}{k} {}_{\mathcal{P}}\mathcal{J}_{n-k}(\rho_1, \rho_2; C|u) \mathcal{J}_k(\upsilon; C|u). \tag{4.1} $$

Proof. By substituting $\rho_1 \to \rho_1 + \upsilon$ in expression (2.1), it follows that

$$ \frac{1-u}{e^\tau - u} C^{(\rho_1+\upsilon)\tau+\rho_2\tau^2} = \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_n(\rho_1 + \upsilon, \rho_2; C|u) \frac{\tau^n}{n!}, \quad C > 1. $$

Inserting the expansion of $C^{\upsilon\tau}$ and expression (2.1) in the left hand part of the preceding expression, we find

$$ \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_n(\rho_1, \rho_2; C|u) \frac{\tau^n}{n!} \sum_{k=0}^{\infty} \mathcal{J}_k(\upsilon; C|u) \frac{\tau^k}{k!} = \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_n(\rho_1 + \upsilon, \rho_2; C|u) \frac{\tau^n}{n!}, \quad C > 1. $$

Applying the Cauchy-product rule to the left-hand side results in the following expression:

$$ \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_n(\rho_1 + \upsilon, \rho_2; C|u) \frac{\tau^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} {}_{\mathcal{P}}\mathcal{J}_{n-k}(\rho_1, \rho_2; C)\ \mathcal{J}_k(\upsilon; C|u) \frac{\tau^n}{n!}. $$

The assertion in (4.1) is derived by equating the coefficients of corresponding powers of τ on both sides of the given expression. □

Theorem 4.2. *For the generalized 1-parameter Hermite-Frobenius-Genocchi polynomials* ${}_{\mathcal{P}}\mathcal{J}_n(\rho_1, \rho_2; C|u)$*, the following summation formula holds true:*

$$ {}_{\mathcal{P}}\mathcal{J}_n(\rho_1 + \upsilon, \rho_2 + \nu; C) = \sum_{k=0}^{n} \binom{n}{k} {}_{\mathcal{P}}\mathcal{J}_{n-k}(\rho_1, \rho_2; C)\ \mathcal{R}_k(\upsilon, \nu; C). \tag{4.2} $$

Proof. By substituting $\rho_1 \to \rho_1 + \upsilon$ and $\rho_2 \to \rho_2 + \nu$ in expression (2.1), it follows that

$$ \frac{1-u}{e^\tau - u} C^{(\rho_1+\upsilon)\tau+(\rho_2+\nu)\tau^2} = \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_n(\rho_1 + \upsilon, \rho_2 + \nu; C) \frac{\tau^n}{n!}, \quad C > 1. $$

Inserting expressions (1.2) and (2.1) in the left hand part of preceeding expression, we find

$$ \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_n(\rho_1, \rho_2; C|u) \frac{\tau^n}{n!} \sum_{k=0}^{\infty} \mathcal{R}_k(\upsilon, \nu; C) \frac{\tau^k}{k!} = \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_n(\rho_1 + \upsilon, \rho_2 + \nu; C) \frac{\tau^n}{n!}, \quad C > 1. $$

Applying the Cauchy-product rule to the left-hand side results in the following expression:

$$ \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{J}_n(\rho_1 + \upsilon, \rho_2 + \nu; C) \frac{\tau^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} {}_{\mathcal{P}}\mathcal{J}_{n-k}(\rho_1, \rho_2; C|u)\ \mathcal{R}_k(\upsilon, \nu; C) \frac{\tau^n}{n!}. $$

The assertion in (4.2) is derived by equating the coefficients of corresponding powers of τ on both sides of the given expression. □

5. Conclusions

This paper introduces a novel framework for constructing generalized one-parameter Hermite-Frobenius-Genocchi polynomials, offering a fresh perspective on their mathematical structure and potential applications. We delve into the core properties of these polynomials, unveiling their richness through generating functions, series expansions, and determinant representations. By employing an innovative factorization technique, we derive key mathematical tools, such as recurrence relations, shift operators, and a spectrum of differential equations, including ordinary, partial, and integro-differential forms.

Future research directions could explore extending this framework to encompass polynomials involving multiple variables, thereby addressing the inherent complexities and uncovering new structural and analytical properties. A detailed study of additional characteristics, such as orthogonality relations, asymptotic behaviors, and the distribution of zeros, would provide deeper insights into their theoretical foundation. Moreover, designing efficient computational algorithms tailored for practical applications in fields like numerical analysis, physics, and engineering could bridge the gap between theory and practice.

Investigating the utility of these polynomials in solving higher-order and more intricate differential equations-particularly in the modeling of complex real-world phenomena-may yield profound advancements. Furthermore, interdisciplinary applications in areas such as finance, biology, and data science could open new avenues for exploration. Enhanced graphical visualization and numerical analysis of these polynomials might lead to novel theoretical developments and foster a broader understanding of their utility across diverse domains.

Author contributions

All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

William Ramírez is a Guest Editor of special issue “Orthogonal polynomials and related applications” for AIMS Mathematics. William Ramírez was not involved in the editorial review and the decision to publish this article.

References

1. N. Alam, W. A. Khan, C. Kızılateş, S. S. Obeidat, C. S. Ryoo, N. S. Diab, Some explicit properties of Frobenius-Euler-Genocchi polynomials with applications in computer modeling, *Symmetry*, **15** (2023), 1358. https://doi.org/10.3390/sym15071358

2. M. S. Alatawi, W. A. Khan, C. Kızılateş, C. S. Ryoo, Some properties of generalized Apostol-Type Frobenius-Euler-Fibonacci polynomials, *Mathematics*, **12** (2024), 800. https://doi.org/10.3390/math12060800

3. B. S. T. Alkahtani, I. Alazman, S. A. Wani, Some families of differential equations associated with multivariate Hermite polynomials, *Fractal Fract.*, **7** (2023), 390. https://doi.org/10.3390/fractalfract7050390

4. P. Appell, J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques: Polynômes d' Hermite*, Paris: Gauthier-Villars, 1926.

5. S. Araci, M. Riyasat, S. A. Wani, S. Khan, Differential and integral equations for the 3-variable Hermite-Frobenius-Genocchi and Frobenius-Genocchi polynomials, *App. Math. Inf. Sci.*, **11** (2017), 1335–1346. http://dx.doi.org/10.18576/amis/110510

6. D. Bedoya, C. Cesarano, W. Ramírez, L. Castilla, A new class of degenerate biparametric Apostol-type polynomials, *Dolomites Res. Notes Approx.*, **16** (2023), 10–19. https://doi.org/10.14658/PUPJ-DRNA-2023-1-2

7. G. Bretti, C. Cesarano, P. E. Ricci, Laguerre-type exponentials and generalized Appell polynomials, *Comput. Math. Appl.*, **48** (2004), 833–839. https://doi.org/10.1016/j.camwa.2003.09.031

8. L. Carlitz, Eulerian numbers and polynomials, *Math. Mag.*, **32** (1959), 247–260. https://doi.org/10.2307/3029225

9. C. Cesarano, Y. Quintana, W. Ramírez. Degenerate versions of hypergeometric Bernoulli-Euler polynomials, *Lobachevskii J. Math.*, **45** (2024), 3509–3521. https://doi.org/10.1134/S1995080224604235

10. C. Cesarano, Y. Quintana, W. Ramírez, A survey on orthogonal polynomials from a monomiality principle point of view, *Encyclopedia*, **4** (2024), 1355–1366. https://doi.org/10.3390/encyclopedia4030088

11. C. Cesarano, G. M. Cennamo, L. Placidi, Humbert polynomials and functions in terms of Hermite polynomials towards applications to wave propagation, *WSEAS Trans. Math.*, **13** (2014), 595–602.

12. G. Dattoli, Hermite-Bessel and Laguerre-Bessel functions: a by-product of the monomiality principle, Advanced Special functions and applications, In: *Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics*, 147–164, 2000.

13. G. Dattoli, C. Cesarano, D. Sacchetti, A note on truncated polynomials, *Appl. Math. Comput.*, **134** (2003), 595–605. https://doi.org/10.1016/S0096-3003(01)00310-1

14. G. Dattoli, S. Lorenzutta, A. M. Mancho, A. Torre, Generalized polynomials and associated operational identities, *J. Comput. Appl. Math.*, **108** (1999), 209–218. https://doi.org/10.1016/S0377-0427(99)00111-9

15. G. Dattoli, C. Chiccoli, S. Lorenzutta, G. Maino, A. Torre, Generalized Bessel functions and generalized Hermite polynomials, *J. Math. Anal. Appl.*, **178** (1993), 509–516. https://doi.org/10.1006/jmaa.1993.1321

16. G. Dattoli, S. Lorenzutta, G. Maino, A. Torre, C. Cesarano, Generalized Hermite polynomials and super-Gaussian forms, *J. Math. Anal. Appl.*, **203** (1996), 597–609. https://doi.org/10.1006/jmaa.1996.0399

17. M. X. He, P. E. Ricci, Differential equation of Appell polynomials via the factorization method, *J. Comput. Appl. Math.*, **139** (2002), 231–237. https://doi.org/10.1016/S0377-0427(01)00423-X

18. J. Hernández, D. Peralta, Y. Quintana, A look at generalized degenerate Bernoulli and Euler matrices, *Mathematics*, **11** (2023), 2731. https://doi.org/10.3390/math11122731

19. J. Hernández, Y. Quintana, F. J. Ramírez, A note on generalized degenerate q-Bernoulli and q-Euler matrices, *Dolomites Res. Notes Approx.*, **17** (2024), 63–71. https://doi.org/10.14658/PUPJ-DRNA-2024-1-7

20. S. Khan, N. Raza, General-Appell polynomials within the context of monomiality principle, *Int. J. Anal.*, 2013. https://doi.org/10.1155/2013/328032

21. S. Khan, G. Yasmin, R. Khan, N. A. M. Hassan, Hermite polynomials: properties and applications, *J. Math. Anal. Appl.*, **351** (2009), 756–764.

22. L. Infeld, T. E. Hull, The factorization method, *Rev. Mod. Phys.*, **23** (1951), 21–68. https://doi.org/10.1103/RevModPhys.23.21

23. Y. Quintana, W. Ramírez, A degenerate version of hypergeometric Bernoulli polynomials: announcement of results, *Commun. Appl. Ind. Math.*, **15** (2024), 36–43. https://doi.org/10.2478/caim-2024-0011

24. W. Ramírez, M. Ortega, D. Bedoya, A. Urieles, New parametric Apostol-type Frobenius-Euler polynomials and their matrix approach, *Kragujev. J. Math.*, **49** (2025), 411–429.

25. W. Ramírez, C. Kızılateş, C. Cesarano, D. Bedoya, C. S. Ryoo, On certain properties of three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials, *AIMS Math.*, **10** (2025), 137–158. https://doi.org/10.3934/math.2025008

26. S. A. Wani, S. Khan, Certain properties and applications of the 2D Sheffer and related polynomials, *Bol. Soc. Mat. Mex.*, **26** (2020), 947–971. https://doi.org/10.1007/s40590-020-00280-5

27. S. A. Wani, S. Khan, Properties and applications of the Gould-Hopper-Frobenius-Genocchi polynomials, *Tbilisi Math. J.*, **12** (2019), 93–104 https://doi.org/10.32513/tbilisi/1553565629

28. H. M. Srivastava, M. A. Özarslan, B. Yilmaz, Some families of differential equations associated with the Hermite polynomials and other classes of Hermite-based polynomials, *Filomat*, **28** (2014), 695–708. https://doi.org/10.2298/FIL1404695S

29. B. Yilmaz, M. A. Özarslan, Differential equations for the extended 2D Bernoulli and Euler polynomials, *Adv. Differ. Equ.*, **2013** (2013), 107. https://doi.org/10.1186/1687-1847-2013-107