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Research article

An age-dependent hybrid system for optimal contraception control of vermin

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Abstract: This paper discusses the optimal contraception control problem for vermin. The novel model consists of a first-order partial differential equation for the age-dependent density of vermin and two ordinary differential equations for the amounts of female sterilant in the environment and in an individual. We first show that the hybrid system is well-posed by applying the fixed-point theorem. Then the structure of an optimal contraception policy is established by considering the normal cone and adjoint system. Moreover, there is a unique optimal policy by employing Ekeland's variational principle and fixed-point theory. The optimal policy that we have derived offers a rational deployment strategy for the use of sterilants as a means of efficacious pest control. These criteria guarantee that during the application of sterilants, the predetermined objectives are attained while simultaneously minimizing expenditure and environmental implications. Utilizing these optimality criteria facilitates the development of streamlined and economically viable pest management protocols.

Keywords: age structure; hybrid system; contraception control; vermin
Mathematics Subject Classification: 35F50, 49K15, 49K20, 92D25

1. Description of the problem

Zhang and Liu [1] introduced vermin, particularly small rodents with relatively high densities. These vermin demonstrate substantial population sizes and expansive geographical proliferation in the ecological environment. This exerts a profound influence on the global agricultural framework and ecological balance. The ramifications of their presence are evidenced by diminished crop yields, alterations in the geographical distribution patterns of pest populations, and the facilitation of transboundary dissemination through international trade activities. In addition, Jacob et al. [2] further stated vermin can transmit several zoonotic viruses, bacteria, and parasites that endanger human and livestock health. Thus, it is necessary to control them. Jacob et al. [3] contributed by using female sterilants to achieve this purpose. Liu et al. [4] proposed that this is because female sterilants have the dual functions of causing sterility and death of vermin. Further, research shows that the reproductive ability of vermin is related to the ages of individuals.

Mathematical models can be used to study the dynamics of infectious diseases and biological population dynamics. In the field of epidemiology, reference literature includes [5–9]. We will focus on introducing the related work in the area of biological population dynamics. There is some work on the theoretical analysis and optimal control of population models with age and size structures. Liu et al. [10], Golubtsov and Steinshamn [11], and Skritek and Veliov [12] considered separately discrete-time, continuous-time, and infinite-horizon optimal harvesting problems for populations with age structure. Liu et al. [13] and Zhang et al. [14] studied periodic optimal harvesting problems for the food chain model. Li et al. [15] studied the optimal control of an age-structured problem modeling mosquito plasticity. He and Liu [16], and Li et al. [17] discussed separately the optimal birth control and the optimal harvesting problem for a size-stage-structured population model. Osmolovskii and Veliov [18] proposed the optimal control strategy for the age-structured system with mixed state-control constraints. Bandyopadhyay and Chattopadhyay [19] investigated the impact of harvesting on the evolutionary dynamics of the prey species. However, so far, there are few theoretical studies on contraception control of vermin. As far as we know, only Liu and Liu [20–22] have studied contraception control problems of vermin with structural differences.

Aniţa [23] considered the linear model describing the evolution of an age-structured population:

$$
\begin{cases}
Dp(a,t) + \mu(a,t)p(a,t) = f(a,t), & (a,t) \in Q_T, \\
p(0,t) = \int_0^{a_\dagger} \beta(a,t)p(a,t)\mathrm{d}a, & t \in (0,T), \\
p(a,0) = p_0(a), & a \in (0,a_\dagger),
\end{cases}
$$

where $p(a,t)$ is the density of individuals; $T \in (0,+\infty)$, $Q_T = (0,a_\dagger) \times (0,T)$, and $Dp(a,t) = \lim_{\varepsilon\to 0} \frac{p(a+\varepsilon,t+\varepsilon)-p(a,t)}{\varepsilon}$; $\mu(a)$ is the mortality rate; $p(0,t)$ is the number of newborn population at the moment t; $\beta(a)$ is the fertility rate. This paper is devoted to the basic properties of the models of age-dependent population dynamics without diffusion. Luo and He [24] proposed a toxicant-population model for modeling an age-dependent population dynamic system in a polluted environment:

$$
\begin{cases}
\dfrac{\partial p(a,t)}{\partial a} + \dfrac{\partial p(a,t)}{\partial t} = -\mu(a,c_0(t))p(a,t), \\
\dfrac{\mathrm{d}c_0(t)}{\mathrm{d}t} = kc_e(t) - gc_0(t) - mc_0(t), \\
\dfrac{\mathrm{d}c_e(t)}{\mathrm{d}t} = -k_1c_e(t)P(t) + g_1c_0(t)P(t) - hc_e(t) + v(t), \\
p(0,t) = \displaystyle\int_0^{a_+} \beta(a,c_0(t))p(a,t)\mathrm{d}a, \qquad p(a,0) = p_0(a), \\
0 \le c_0(t) \le 1, \quad 0 \le c_e(t) \le 1, \\
P(t) = \displaystyle\int_0^{a_+} p(a,t)\mathrm{d}a, \qquad (a,t) \in Q,
\end{cases}
$$

where $p(a,t)$ is the density of individuals; $c_0(t)$ is the concentration of the toxicant in an organism; $c_e(t)$ is the concentration of the toxicant in the environment; $\mu(a, c_0(t))$ is the vital rates; and $v(t)$ is the exogenous toxicant input rate. This paper studied the existence and uniqueness of a nonnegative solution, deduct the optimality conditions for the control problem.

Motivated by the idea of Aniţa [23], Luo and He [24], we propose the following age-dependent hybrid system for addressing the contraception problem of vermin

$$\begin{cases} \dfrac{\partial p(a,t)}{\partial t} + \dfrac{\partial p(a,t)}{\partial a} = -\mu(a,t)p(a,t) - m(a, c_0(t))p(a,t), & (a,t) \in D = (0, a_\dagger) \times (0, T), \\ \dfrac{\mathrm{d}c_e(t)}{\mathrm{d}t} = u(t) - k_1 c_e(t)P(t) - h_1 c_e(t), & t \in (0, T), \\ \dfrac{\mathrm{d}c_0(t)}{\mathrm{d}t} = k_2 c_e(t) - h_2 c_0(t), & t \in (0, T), \\ p(0,t) = \displaystyle\int_0^{a_\dagger} \beta(a,t)[1 - b(a, c_0(t))]\omega(a)p(a,t)\,\mathrm{d}a, & t \in (0, T), \\ p(a,0) = p_0(a), \quad c_e(0) = c_0(0) = 0, & a \in [0, a_\dagger), \end{cases} \tag{1.1}$$

where $p(a,t)$ is the density of vermin with age a at time t, and $c_e(t)$ and $c_0(t)$ are respectively the amounts of female sterilant in the environment and an individual. $a_\dagger$ is the maximum age of survival of vermin, $T \in (0, +\infty)$ is the control horizon, and $\int_0^{a_\dagger} p(a,t)\,\mathrm{d}a$ is the total size of vermin at time t. Here, $\beta(a,t)$ and $\mu(a,t)$ are the natural fertility and mortality of vermin, respectively; the mortality and sterility of vermin with age a due to female sterilant are $m(a, c_0(t))$ and $b(a, c_0(t))$, respectively; $u(t)$ stands for the rate of female sterilant administered to the environment, which is also the control variable; $-k_1 c_e(t)P(t)$ and $-h_1 c_e(t)$ are the sterilant loss from the environment due to eating by vermin and eating by other animals or volatilization, respectively; $k_2 c_e(t)$ means the amount of sterilant absorbed by an individual from the environment; $-h_2 c_0(t)$ represents the egestion or depuration amount of sterilant in an individual; $\omega(a)$ represents the fraction of females that are of age a; and $p_0(a)$ is the initial distribution. All the constant parameters are positive, $u \in \mathcal{U} = \{u \in L^\infty[0,T] : 0 \le u(t) \le L,\ \text{a.e. } t \in [0,T]\}$, and the parameter functions are assumed to satisfy

(A$_1$) $\mu \in L^1_{loc}(D)$ and $\mu(a,t) \ge 0$ a.e. $(a,t) \in D$; $\int_0^{a_\dagger} \mu(a, t + a - a_\dagger)\,\mathrm{d}a = +\infty$, where $\mu(a,t) = 0$ for $(a,t) \in (0, a_\dagger) \times (-\infty, 0)$; $\beta : D \to R_+$ is measurable, and $0 \le \beta(x,t) \le \bar{\beta}$ a.e. $(a,t) \in D$ for some $\bar{\beta} > 0$, where $R_+ = [0, +\infty)$.

(A$_2$) $m(\cdot,\cdot)$ is measurable, and there is a positive constant $\bar{m}$ such that $m(a,s) \le \bar{m}$ a.e $(a,s) \in (0, a_\dagger) \times R_+$. Moreover, there exists an increasing function $C_m : R_+ \to R_+$ such that for $a \in (0, a_\dagger)$

$$|m(a, s_1) - m(a, s_2)| \le C_m(r)|s_1 - s_2|, \quad 0 \le s_1, s_2 \le r.$$

(A$_3$) $b(\cdot,\cdot)$ is measurable, and $b(a,s) < 1$ a.e $(a,s) \in (0, a_\dagger) \times R_+$. Furthermore, there exists an increasing function $C_b : R_+ \to R_+$ such that for $a \in (0, a_\dagger)$

$$|b(a, s_1) - b(a, s_2)| \le C_b(r)|s_1 - s_2|, \quad 0 \le s_1, s_2 \le r.$$

(A$_4$) $p_0 \in L^1_+ \doteq L^1(0, m; R_+)$; $\omega \in L^\infty_+ \doteq L^\infty(0, a_\dagger; R_+)$ and $0 \le \omega(a) \le 1$ a.e. $a \in (0, a_\dagger)$.

The current work incorporates two novel features:

- To our knowledge, only the literature [17,18] has studied the contraception control problems of vermin with structural differences, where it is assumed that any female sterilant administered will be completely ingested by the vermin at any time. However, this assumption deviates from reality. Furthermore, the unused sterilants may have a negative impact on the environment. To address this issue more accurately, this study employs an infinite-dimensional hybrid system (1.1) to investigate the dynamic changes in the pest rodent population and the sterilant stock.
- Kato [25] does not give existence or uniqueness results for optimal strategies, while the only result for the harvesting problem is the existence of an optimal solution in [26]. The paper states that the contraception control problem (3.1) has a unique optimal solution, and the optimal solution has the specific feedback form.

The establishment of the well-posedness of (1.1) is addressed in Section 2. The optimal control policy is discussed in Section 3. The paper concludes with a brief discussion.

2. Well-posedness of the hybrid system

The solution to the hybrid system (1.1) is defined by employing the characteristic curve technique [23], as follows.

Definition 2.1. *A three-tuple* $(p(a,t), c_e(t), c_0(t))$ *is called a solution of hybrid system* (1.1) *if it satisfies*

$$p(a,t) = \begin{cases} p(0, t-a)\exp\Big\{-\displaystyle\int_0^a [\mu(s, t-a+s) + m(s, c_0(t-a+s))]\,\mathrm{d}s\Big\}, & a \le t, \\ p_0(a-t)\exp\Big\{-\displaystyle\int_0^t [\mu(a-t+s, s) + m(a-t+s, c_0(s))]\,\mathrm{d}s\Big\}, & a > t, \end{cases} \tag{2.1}$$

$$c_e(t) = \int_0^t u(s)\exp\Big\{-\int_s^t (k_1 P(\sigma) + h_1)\,\mathrm{d}\sigma\Big\}\,\mathrm{d}s \quad \textit{with} \quad P(\sigma) = \int_0^{a_\dagger} p(a,\sigma)\,\mathrm{d}a, \tag{2.2}$$

$$c_0(t) = k_2 \int_0^t c_e(s)\exp\{-h_2(t-s)\}\,\mathrm{d}s. \tag{2.3}$$

Theorem 2.1. *Assume that* (A_1)–(A_4) *hold. Then, for each* $u \in \mathcal{U}$, *hybrid system* (1.1) *has a unique solution* $(p(a,t), c_e(t), c_0(t)) \in \mathcal{X}$. *Here*

$$\mathcal{X} = \left\{ (p, c_e, c_0) \in \mathbf{X} \;\middle|\; \begin{array}{l} 0 \le c_e(t) \le \frac{L}{h_1},\ \ 0 \le c_0(t) \le \frac{k_2 L}{h_1 h_2} \ \ a.e.\ \ t \in [0,T], \\ p(a,t) \ge 0 \ a.e.\ (a,t) \in D \ and \int_0^{a_\dagger} p(a,t)\,\mathrm{d}a \le M. \end{array} \right\},$$

where $\mathbf{X} = L^\infty(0,T; L^1[0,a_\dagger)) \times L^\infty(0,T) \times L^\infty(0,T)$ *and* $M = \|p_0\|_{L^1}\exp\{\bar{\beta}T\}$.

Proof. Define a new norm $\|\cdot\|_*$ on $\mathbf{X}$ by

$$\|(p, c_e, c_0)\|_* = \operatorname{Ess}\sup_{t\in[0,T]} \Big\{ e^{-\lambda t}\Big[\int_0^{a_\dagger} |p(a,t)|\,\mathrm{d}a + |c_e(t)| + |c_0(t)|\Big]\Big\},$$

where $\lambda > 0$ is to be specified. Then $\|\cdot\|$ is equivalent to the product norm on $\mathbf{X}$ and hence $(\mathbf{X}, \|\cdot\|_*)$ is a Banach space.

Define a mapping $\mathcal{A} : \mathcal{X} \to \mathbf{X}$ by $\mathcal{A}(p, c_e, c_0) = (\mathcal{A}_1(p, c_e, c_0), \mathcal{A}_2(p, c_e, c_0), \mathcal{A}_3(p, c_e, c_0))$, where $\mathcal{A}_i(p, c_e, c_0)$, $i = 1, 2, 3$ are defined respectively by the right-hand sides of (2.2)–(2.3). For any $(p, c_e, c_0) \in \mathcal{X}$, we have

$$\begin{aligned}
|\mathcal{A}_2(p, c_e, c_0)|(t) =& \int_0^t u(s) \exp\Big\{ - \int_s^t (k_1 P(\sigma) + h_1)\, \mathrm{d}\sigma \Big\}\, \mathrm{d}s \\
\leq& L \int_0^t \exp\{ - h_1(t-s)\}\, \mathrm{d}s \\
\leq& \frac{L}{h_1} \doteq r_e, \\
|\mathcal{A}_3(p, c_e, c_0)|(t) =& k_2 \int_0^t c_e(s) \exp\{ - h_2(t-s)\}\, \mathrm{d}s \\
\leq& \frac{k_2 L}{h_1} \int_0^t \exp\{ - h_2(t-s)\}\, \mathrm{d}s \\
\leq& \frac{k_2 L}{h_1 h_2} \doteq r_0.
\end{aligned}$$

Moreover, from (2.1), when $0 < t < a_\dagger$, we have

$$\begin{aligned}
\int_0^{a_\dagger} |\mathcal{A}_1(p, c_e, c_0)|(a, t)\, \mathrm{d}a =& \int_0^t p(a, t)\, \mathrm{d}a + \int_t^{a_\dagger} p(a, t)\, \mathrm{d}a \\
\leq& \int_0^t p(0, t-a)\, \mathrm{d}a + \int_t^{a_\dagger} p_0(a-t)\, \mathrm{d}a \\
\doteq& I_1 + I_2.
\end{aligned}$$

For I_1, let $s = t - a$. Then, $s = t$ when $a = 0$, while $s = 0$ when $a = t$. Moreover, we have $\mathrm{d}s = -\mathrm{d}a$. Then

$$\begin{aligned}
I_1 =& \int_0^t p(0, t-a)\, \mathrm{d}a = \int_0^t p(0, s)\, \mathrm{d}s \\
=& \int_0^t \Big[\int_0^{a_\dagger} \beta(a, s)[1 - b(a, c_0(s))]\omega(a) p(a, s)\, \mathrm{d}a \Big]\, \mathrm{d}s \\
\leq& \int_0^t \bar{\beta} \Big[\int_0^{a_\dagger} p(a, s)\, \mathrm{d}a \Big]\, \mathrm{d}s.
\end{aligned}$$

For I_2, let $s = a - t$. Then, $s = 0$ when $a = t$ while $s = a_\dagger - t$ when $a = a_\dagger$. Moreover, we have $\mathrm{d}s = \mathrm{d}a$. Then

$$I_2 = \int_t^{a_\dagger} p_0(a-t)\, \mathrm{d}a = \int_0^{a_\dagger - t} p_0(s)\, \mathrm{d}s \leq \int_0^{a_\dagger} p_0(s)\, \mathrm{d}s = \|p_0\|_{L^1}.$$

Thus, when $0 < t < a_\dagger$, we have

$$\int_0^{a_\dagger} |\mathcal{A}_1(p, c_e, c_0)|(a, t)\, \mathrm{d}a \leq \|p_0\|_{L^1} + \int_0^t \bar{\beta} \Big[\int_0^{a_\dagger} p(a, s)\, \mathrm{d}a \Big]\, \mathrm{d}s.$$

In applying Gronwall's inequality to the aforementioned inequality, the subsequent conclusion is reached

$$\int_0^{a_\dagger} \mathcal{A}_1(p, c_e, c_0)\,\mathrm{d}a \le \|p_0\|_{L^1} \exp\{\bar{\beta}T\} = M.$$

For $a_\dagger < t < T$, the aforementioned inequality remains valid, and the proof is rendered more straightforward. Therefore, it has been demonstrated that $\mathcal{A}$ constitutes a mapping from set $\mathcal{X}$ to itself.

Further, for any $x^i \doteq (p^i, c_e^i, c_0^i) \in \mathcal{X}$, $i = 1, 2$, from (2.2) and (2.3), we have

$$\begin{aligned}
|\mathcal{A}_2(X^1) - \mathcal{A}_2(X^2)|(t) \le& Lk_1 \int_0^t \int_s^t |P^1(\sigma) - P^2(\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}s\\
=& Lk_1 \int_0^t \int_s^t \left| \int_0^{a_\dagger} p^1(a,\sigma)\,\mathrm{d}a - \int_0^{a_\dagger} p^2(a,\sigma)\,\mathrm{d}a \right| \mathrm{d}\sigma\,\mathrm{d}s\\
\le& M_1 \int_0^t \int_0^{a_\dagger} |p^1(a,s) - p^2(a,s)|\,\mathrm{d}a\,\mathrm{d}s,\\
|\mathcal{A}_3(X^1) - \mathcal{A}_3(X^2)|(t) \le& k_2 \int_0^t |c_e^1(s) - c_e^2(s)| \exp\{-h_2(t-s)\}\,\mathrm{d}s\\
\le& M_2 \int_0^t |c_e^1(s) - c_e^2(s)|\,\mathrm{d}s,
\end{aligned}$$

where $M_1 = k_1 LT$ and $M_2 = k_2$. Moreover, from (2.1), when $0 < t < a_\dagger$, we have

$$\begin{aligned}
&\int_0^{a_\dagger} |\mathcal{A}_1(x^1) - \mathcal{A}_1(x^2)|(a,t)\,\mathrm{d}a\\
\le& \int_0^t |p^1(0,t-a) - p^2(0,t-a)|\,\mathrm{d}a\\
&+ \int_0^t p^2(0,t-a) \int_0^a |m(s, c_0^1(t-a+s)) - m(s, c_0^2(t-a+s))|\,\mathrm{d}s\,\mathrm{d}a\\
&+ \int_t^{a_\dagger} p_0(a-t) \int_0^t |m(a-t+s, c_0^1(s)) - m(a-t+s, c_0^2(s))|\,\mathrm{d}s\,\mathrm{d}a\\
\le& \int_0^t |p^1(0,t-a) - p^2(0,t-a)|\,\mathrm{d}a\\
&+ C_m(r_0) \int_0^t p^2(0,t-a) \int_0^a |c_0^1(t-a+s) - c_0^2(t-a+s)|\,\mathrm{d}s\,\mathrm{d}a\\
&+ C_m(r_0) \int_t^{a_\dagger} p_0(a-t) \int_0^t |c_0^1(s) - c_0^2(s)|\,\mathrm{d}s\,\mathrm{d}a\\
\doteq& I_3 + I_4 + I_5.
\end{aligned}$$

For I_3, a similar discussion as that in I_1, we have

$$I_3 = \int_0^t |p^1(0,t-a) - p^2(0,t-a)|\,\mathrm{d}a = \int_0^t |p^1(0,s) - p^2(0,s)|\,\mathrm{d}s$$

$$
\begin{aligned}
&= \int_0^t \left| \int_0^{a_\dagger} \beta(a,s)[1-b(a,c_0^1(s))]\omega(a)p^1(a,s)\,\mathrm{d}a - \int_0^{a_\dagger} \beta(a,s)[1-b(a,c_0^2(s))]\omega(a)p^2(a,s)\,\mathrm{d}a \right| \mathrm{d}s \\
&\le \int_0^t \bar{\beta}\Big[\int_0^{a_\dagger} |p^1(a,s)-p^2(a,s)|\,\mathrm{d}a + \int_0^{a_\dagger} |b(a,c_0^1(s)) - b(a,c_0^2(s))| p^2(a,s)\,\mathrm{d}a \Big] \mathrm{d}s \\
&\le \int_0^t \bar{\beta}\Big[\int_0^{a_\dagger} |p^1(a,s)-p^2(a,s)|\,\mathrm{d}a + C_b(r_0) \int_0^{a_\dagger} |c_0^1(s) - c_0^2(s)| p^2(a,s)\,\mathrm{d}a \Big] \mathrm{d}s \\
&\le \ \bar{\beta} \int_0^t \int_0^{a_\dagger} |p^1(a,s)-p^2(a,s)|\,\mathrm{d}a\,\mathrm{d}s + C_b(r_0)\bar{\beta}M \int_0^t |c_0^1(s) - c_0^2(s)|\,\mathrm{d}s.
\end{aligned}
$$

For I_4, let $r = t - a + s$. Consequently, when $s = 0$, $r = t - a$; and when $s = a$, $r = t$. Moreover, we have $\mathrm{d}r = \mathrm{d}s$. Then

$$
\begin{aligned}
I_4 =& C_m(r_0) \int_0^t p^2(0, t-a) \int_0^a |c_0^1(t-a+s) - c_0^2(t-a+s)|\,\mathrm{d}s\,\mathrm{d}a \\
=& C_m(r_0) \int_0^t p^2(0, t-a) \int_{t-a}^t |c_0^1(r) - c_0^2(r)|\,\mathrm{d}r\,\mathrm{d}a.
\end{aligned}
$$

Further, let $\tau = t - a$. Then, when $a = 0$, $\tau = t$; and when $a = t$, $\tau = 0$. Moreover, we have $\mathrm{d}a = -\mathrm{d}\tau$. Then

$$
\begin{aligned}
I_4 =& C_m(r_0) \int_0^t p^2(0,\tau) \int_\tau^t |c_0^1(r) - c_0^2(r)|\,\mathrm{d}r\,\mathrm{d}a \\
=& C_m(r_0) \int_0^t \Big[\int_0^{a_\dagger} \beta(a,\tau)[1-b(a,c_0^2(\tau))]\omega(a)p^2(a,\tau)\,\mathrm{d}a \Big] \int_\tau^t |c_0^1(r) - c_0^2(r)|\,\mathrm{d}r\,\mathrm{d}a \\
\le& C_m(r_0)\bar{\beta}MT \int_0^t |c_0^1(r) - c_0^2(r)|\,\mathrm{d}r.
\end{aligned}
$$

For I_5, a similar discussion as that in I_2, we have

$$
I_5 = C_m(r_0) \int_t^{a_\dagger} p_0(a-t) \int_0^t |c_0^1(s) - c_0^2(s)|\,\mathrm{d}s\,\mathrm{d}a \le C_m(r_0)\|p_0\|_{L^1} \int_0^t |c_0^1(s) - c_0^2(s)|\,\mathrm{d}s.
$$

Thus, when $0 < t < a_\dagger$, we have

$$
\int_0^{a_\dagger} |\mathcal{A}_1(x^1) - \mathcal{A}_1(x^2)|(a,t)\,\mathrm{d}a \le M_3\Big[\int_0^t \int_0^{a_\dagger} |p^1(a,s) - p^2(a,s)|\,\mathrm{d}a\,\mathrm{d}s + \int_0^t |c_0^1(s) - c_0^2(s)|\,\mathrm{d}s \Big],
$$

where $M_3 = \max\left\{\bar{\beta},\ C_b(r_0)\bar{\beta}M + C_m(r_0)\bar{\beta}MT + C_m(r_0)\|p_0\|_{L^1}\right\}$ is a positive constant. When $a_\dagger < t < T$, the aforementioned inequality remains valid, and the proof is rendered more straight-forward. Thus,

$$
\begin{aligned}
&\|\mathcal{A}(x^1) - \mathcal{A}(x^2)\|_* \\
=& \operatorname{Ess}\sup_{t\in[0,T]} \left\{ e^{-\lambda t}\Big[\int_0^{a_\dagger} |\mathcal{A}_1(x^1) - \mathcal{A}_1(x^2)|(a,t)\,\mathrm{d}a + |\mathcal{A}_2(x^1) - \mathcal{A}_2(x^2)|(t) + |\mathcal{A}_3(x^1) - \mathcal{A}_3(x^2)|(t) \Big] \right\} \\
\le& M_4 \operatorname{Ess}\sup_{t\in[0,T]} \left\{ e^{-\lambda t} \int_0^t e^{\lambda s} \left\{ e^{-\lambda s}\Big[|c_e^1 - c_e^2|(s) + |c_0^1 - c_0^2|(s) + \int_0^{a_\dagger} |p^1(a,s) - p^2(a,s)|\,\mathrm{d}a \Big] \right\} \mathrm{d}s \right\}
\end{aligned}
$$

$$\leq \frac{M_4}{\lambda}\|x^1 - x^2\|_*,$$

where $M_4 = M_1 + M_2 + M_3$. Choosing $\lambda > M_4$, it is noted that $\mathcal{A}$ is a contraction on $(\mathcal{X}, \|\cdot\|_*)$.

Pursuant to the fixed-point theorem, there exists a unique fixed point (p, c_e, c_0) for $\mathcal{A}$ within $\mathcal{X}$, and this point is necessarily the solution of system (1.1). □

Allow the following theorem to clarify how solutions are continuously dependent on the control variable.

Theorem 2.2. *Assume that* (A_1)–(A_4) *hold. Then there exist positive constants* K_1 *and* K_2 *such that*

$$\|p^1 - p^2\|_{L^\infty(0,T;L^1(0,a_\dagger))} + \|c_e^1 - c_e^2\|_{L^\infty([0,T])} + \|c_0^1 - c_0\|_{L^\infty([0,T])} \leq K_1 T\|u^1 - u^2\|_{L^\infty([0,T])},$$

$$\|p^1 - p^2\|_{L^1(D)} + \|c_e^1 - c_e^2\|_{L^1([0,T])} + \|c_0^1 - c_0\|_{L^1([0,T])} \leq K_1 T\|u^1 - u^2\|_{L^1([0,T])}.$$

Here $x^i = (p^i, c_e^i, c_0^i)$ *is the solution of the hybrid system* (1.1) *corresponding to* $u^i \in \mathcal{U}$ $(i = 1, 2)$.

Proof. From (2.2)–(2.3), it follows that

$$\begin{aligned}
|c_0^1(t) - c_0^2(t)| =& \left|k_2\int_0^t c_e^1(s)\exp\{-h_2(t-s)\}\,\mathrm{d}s - k_2\int_0^t c_e^2(s)\exp\{-h_2(t-s)\}\,\mathrm{d}s\right| \\
\leq & k_2\int_0^t |c_e^1(s) - c_e^2(s)|\,\mathrm{d}s \\
=& M_2\int_0^t |c_e^1(s) - c_e^2(s)|\,\mathrm{d}s, \\
|c_e^1(t) - c_e^2(t)| \leq & \int_0^t |u^1(s) - u^2(s)|\,\mathrm{d}s + Lk_1\int_0^t\int_s^t |P^1(\sigma) - P^2(\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}s \\
\leq & \int_0^t |u^1(s) - u^2(s)|\,\mathrm{d}s + Lk_1 T\int_0^t\int_0^{a_\dagger} |p^1(a,s) - p^2(a,s)|\,\mathrm{d}a\,\mathrm{d}s \\
=& \int_0^t |u^1(s) - u^2(s)|\,\mathrm{d}s + M_1\int_0^t\int_0^{a_\dagger} |p^1(a,s) - p^2(a,s)|\,\mathrm{d}a\,\mathrm{d}s.
\end{aligned}$$

Moreover, in a manner akin to the demonstration of Theorem 2.1, we can infer from (2.1) that

$$\int_0^{a_\dagger} |\mathcal{A}_1(x^1) - \mathcal{A}_1(x^2)|(a,t)\,\mathrm{d}a \leq M_3\Big[\int_0^t\int_0^{a_\dagger} |p^1(a,s) - p^2(a,s)|\,\mathrm{d}a\,\mathrm{d}s + \int_0^t |c_0^1(s) - c_0^2(s)|\,\mathrm{d}s\Big].$$

Let

$$A(t) \doteq \int_0^{a_\dagger} |p^1(a,t) - p^2(a,t)|\,\mathrm{d}a + |c_0^1(t) - c_0^2(t)| + |c_e^1(t) - c_e^2(t)|.$$

Thus,

$$A(t) \leq M_5\int_0^t A(s)\,\mathrm{d}s + \int_0^t |u^1(s) - u^2(s)|\,\mathrm{d}s,$$

where $M_5 = \max\{M_1 + M_3, M_2\}$. The result follows immediately from the above analysis and Gronwall's inequality. □

3. Optimal contraception control

Consider (p^u, c_e^u, c_0^u) as the solution of the hybrid system (1.1) for $u \in \mathcal{U}$. In this part, we delve into the analysis of the optimization issue presented as follows:

$$\min_{u\in\mathcal{U}} J(u). \tag{3.1}$$

Here

$$J(u) = \frac{\sigma_1}{2}\int_0^{a_\dagger} [p^u(a,T) - \bar{p}(a)]^2 \,\mathrm{d}a + \sigma_2 \int_0^T c_e^u(t)\,\mathrm{d}t + \frac{\sigma_3}{2}\int_0^T u^2(t)\,\mathrm{d}t,$$

where constants $\sigma_i > 0,\ i = 1,2,3.$ $\bar{p} \in L^\infty(0,a_\dagger)$ is a given ideal distribution of vermin. Thus, an optimal policy for (3.1) is one that, for vermin, the final size falls as close to the ideal distribution as possible while the cost of control and the total amount of sterilant in the environment are as low as possible. In the sequel, for any $u \in \mathcal{U}$, let $\mathcal{T}_{\mathcal{U}}(u)$ and $\mathcal{N}_{\mathcal{U}}(u)$ be the tangent cone and normal cone of $\mathcal{U}$ at element u, respectively.

Theorem 3.1 (Conditions on optimality)**.** *Assume that* (A_1)–(A_2) *hold. Let* u^* *be an optimal policy for the optimization problem* (3.1)*. Then*

$$u^*(t) = \mathcal{F}\left\{\frac{\sigma_1 q_2(t)}{\sigma_3}\right\}, \tag{3.2}$$

where the mapping $\mathcal{F}$ *is given by*

$$(\mathcal{F}\eta)(t) = \begin{cases} 0, & \eta(t) < 0,\\ \eta(t), & 0 \le \eta(t) \le L,\\ L, & \eta(t) > L, \end{cases} \tag{3.3}$$

and (q_1, q_2, q_3) *is the solution of*

$$\begin{cases} \dfrac{\partial q_1}{\partial t} + \dfrac{\partial q_1}{\partial a} = k_1 c_e^*(t) q_2(t) + \Big[\mu(a,t) + m(a, c_0^*(t))\Big] q_1(a,t) \\ \qquad\qquad -\beta(a,t)\Big[1 - b(a, c_0^*(t))\Big]\omega(a) q_1(0,t), \\ \dfrac{\mathrm{d}q_2(t)}{\mathrm{d}t} = (k_1 P^*(t) + h_1) q_2(t) - k_2 q_3(t) - v(t) q_2(t) + \dfrac{\sigma_2}{\sigma_1}, \\ \dfrac{\mathrm{d}q_3(t)}{\mathrm{d}t} = h_2 q_3(t) + \displaystyle\int_0^{a_\dagger} m_{c_0}(a, c_0^*(t)) p^*(a,t) q_1(a,t)\,\mathrm{d}a \\ \qquad\qquad + q_1(0,t)\displaystyle\int_0^{a_\dagger} \beta(a,t)\omega(a) b_{c_0}(a, c_0^*(t)) p^*(a,t)\,\mathrm{d}a, \\ q_1(a_\dagger, t) = 0,\ \ q_1(a,T) = \bar{p}(a) - p^*(a,T),\ \ q_2(T) = q_3(T) = 0,\ \ (a,t) \in D, \end{cases} \tag{3.4}$$

in which m_{c_0} *and* b_{c_0} *are the derivatives of* m *and* b *with respect to* c_0*, respectively. Here* (p^*, c_e^*, c_0^*) *is the solution of system* (1.1) *with* $u = u^*$ *and* $P^*(t) = \int_0^{a_\dagger} p^*(a,t)\,\mathrm{d}a$.

Proof. The existence of the unique bounded solution for system (3.4) can be investigated using the same approach as for system (1.1). For each $v \in \mathcal{T}_{\mathcal{U}}(u^*)$, we have $u^\varepsilon \triangleq u^* + \varepsilon v \in \mathcal{U}$ for sufficiently small $\varepsilon > 0$. Let $(p^\varepsilon, c_e^\varepsilon, c_0^\varepsilon)$ and (p^*, c_e^*, c_0^*) be solutions of system (1.1) corresponding to u^ε and u^*, respectively. Therefore, the optimality of u^* indicates that $J(u^*) \le J(u^\varepsilon)$, in other words,

$$\frac{\sigma_1}{2}\int_0^{a_\dagger}\Big[(p^\varepsilon(a,T)-\bar{p}(a))^2-(p^*(a,T)-\bar{p}(a))^2\Big]\,\mathrm{d}a+\sigma_2\int_0^T\Big[c_e^\varepsilon(t)-c_e^*(t)\Big]\,\mathrm{d}t$$
$$+\frac{\sigma_3}{2}\int_0^T\Big[(u^*(t)+\varepsilon v(t))^2-(u^*(t))^2\Big]\,\mathrm{d}t\ge 0.$$

Then, we can obtain

$$\sigma_1\int_0^{a_\dagger}[p^*(a,T)-\bar{p}(a)]z_1(a,T)\,\mathrm{d}a+\sigma_2\int_0^T z_2(t)\,\mathrm{d}t+\sigma_3\int_0^T u^*(t)v(t)\,\mathrm{d}t\ge 0, \tag{3.5}$$

where

$$z_1(a,t)=\lim_{\varepsilon\to 0^+}\frac{p^\varepsilon(a,t)-p^*(a,t)}{\varepsilon},\quad z_2(t)=\lim_{\varepsilon\to 0^+}\frac{c_e^\varepsilon(t)-c_e^*(t)}{\varepsilon},\quad z_3(t)=\lim_{\varepsilon\to 0^+}\frac{c_0^\varepsilon(t)-c_0^*(t)}{\varepsilon}.$$

Theorem 2.2 implies that $z_1(a,t)$, $z_2(t)$, $z_3(t)$ do make sense [27]. Moreover, from (1.1), it follows that (z_1, z_2, z_3) satisfies

$$\begin{cases}\dfrac{\partial z_1(a,t)}{\partial t}+\dfrac{\partial z_2(a,t)}{\partial a}=-[\mu(a,t)+m(a,c_0^*(t))]z_1(a,t)-m_{c_0}(a,c_0^*(t))p^*(a,t)z_3(t), & (a,t)\in D,\\ \dfrac{\mathrm{d}z_2(t)}{\mathrm{d}t}=v(t)-(k_1P^*(t)+h_1)z_2(t)-k_1c_e^*(t)\displaystyle\int_0^{a_\dagger}z_1(a,t)\,\mathrm{d}a, & t\in(0,T),\\ \dfrac{\mathrm{d}z_3(t)}{\mathrm{d}t}=k_2z_2(t)-h_2z_3(t), & t\in(0,T),\\ z_1(0,t)=\displaystyle\int_0^{a_\dagger}\beta(a,t)\omega(a)\Big\{[1-b(a,c_0^*(t))]z_1(a,t)-b_{c_0}(a,c_0^*(t))p^*(a,t)z_3(t)\Big]\,\mathrm{d}a, & t\in(0,T),\\ z_1(a,0)=0,\quad z_2(0)=z_3(0)=0, & a\in[0,a_\dagger).\end{cases} \tag{3.6}$$

Multiplying the first three equations in the above system by q_1, q_2 and q_3, respectively and integrating the resultants on $D, [0,T], [0,T]$, one can

$$\int_0^T\int_0^{a_\dagger}D_\varphi q_1(a,t)z_1(a,t)\,\mathrm{d}a\,\mathrm{d}t+\sum_{i=2}^{3}\int_0^T\frac{\mathrm{d}q_i(t)}{\mathrm{d}t}z_i(t)\,\mathrm{d}t$$
$$=\int_0^T\int_0^{a_\dagger}\Big\{[\mu(a,t)+m(a,c_0^*(t))]q_1(a,t)-\beta(a,t)\omega(a)[1-b(a,c_0^*(t))]q_1(0,t)+k_1c_e^*(t)q_2(t)\Big\}z_1(a,t)\,\mathrm{d}a\,\mathrm{d}t$$
$$+\int_0^T\Big\{(k_1P^*(t)+h_1)q_2(t)-k_2q_3(t)\Big\}z_2(t)\,\mathrm{d}t+\int_0^T\Big\{q_1(0,t)\int_0^{a_\dagger}\beta(a,t)\omega(a)b_{c_0}(a,c_0^*(t))p^*(a,t)\,\mathrm{d}a$$
$$+\int_0^{a_\dagger}m_{c_0}(a,c_0^*(t))p^*(a,t)q_1(a,t)\,\mathrm{d}a+h_2q_3(t)\Big\}z_3(t)\,\mathrm{d}t$$
$$-\int_0^{a_\dagger}(p^*(a,T)-\bar{p}(a))z_1(a,T)\,\mathrm{d}a-\int_0^T v(t)q_2(t)\,\mathrm{d}t.$$

Similarly, multiplying the first three equations of system (3.4) by z_1, z_2, and z_3, respectively, and integrating the resultants on $D, [0,T], [0,T]$, one obtains

$$\begin{aligned}
&\int_0^T \int_0^{a_\dagger} D_\varphi q_1(a,t) z_1(a,t)\,\mathrm{d}a\,\mathrm{d}t + \sum_{i=2}^{3} \int_0^T \frac{\mathrm{d}q_i(t)}{\mathrm{d}t} z_i(t)\,\mathrm{d}t \\
=& \int_0^T \int_0^{a_\dagger} \Big\{ [\mu(x,t) + m(a, c_0^*(t))] q_1(a,t) - \beta(a,t)\omega(a)[1 - b(a, c_0^*(t))] q_1(0,t) + k_1 c_e^*(t) q_2(t) \Big\} z_1(a,t)\,\mathrm{d}a\,\mathrm{d}t \\
&+ \int_0^T \Big\{ (k_1 P^*(t) + h_1) q_2(t) - k_2 q_3(t) \Big\} z_2(t)\,\mathrm{d}t + \int_0^T \Big\{ q_1(0,t) \int_0^{a_\dagger} \beta(a,t)\omega(a) b_{c_0}(a, c_0^*(t)) p^*(a,t)\,\mathrm{d}a \\
&+ \int_0^{a_\dagger} m_{c_0}(a, c_0^*(t)) p^*(a,t) q_1(a,t)\,\mathrm{d}a + h_2 q_3(t) \Big\} z_3(t)\,\mathrm{d}t + \frac{\sigma_2}{\sigma_1} \int_0^T z_2(t)\,\mathrm{d}t.
\end{aligned}$$

By comparing the above two formulas, it can be concluded that

$$\sigma_1 \int_0^{a_\dagger} [p^*(a,T) - \bar{p}(a)] z_1(a,T)\,\mathrm{d}a + \sigma_2 \int_0^T z_2(t)\,\mathrm{d}t = -\sigma_1 \int_0^T v(t) q_2(t)\,\mathrm{d}t. \tag{3.7}$$

Thus, for each $v \in T_{\mathcal{U}}(u^*)$, it follows from (3.5) and (3.7) that

$$\int_0^T [\sigma_1 q_2(t) - \sigma_3 u^*(t)] v(t)\,\mathrm{d}t \le 0.$$

Hence, $[\sigma_1 q_2(t) - \sigma_3 u^*(t)] \in \mathcal{N}_{\mathcal{U}}(u^*)$. Then the structure of normal cone (see [28]) gives the desired result. □

Drawing on the same approach used in the proof of Theorem 2.2, we have the following lemma.

Lemma 3.1. *There is a positive constant K_3 such that*

$$\|q_1 - q_1'\|_{L^\infty(D)} + \|q_2 - q_2'\|_{L^\infty([0,T])} + \|q_3 - q_3'\|_{L^\infty([0,T])} \le K_3 T \|u_1 - u_2\|_{L^\infty([0,T])},$$

where (q_1, q_2, q_3) and (q_1', q_2', q_3') are solutions to (3.4) *with (p^*, c_e^*, c_0^*) replaced by (p, c_e, c_0) and (p', c_e', c_0'), respectively. Here (p, c_e, c_0) and (p', c_e', c_0') are solutions of* (1.1) *with u and $u' \in \mathcal{U}$, respectively.*

Theorem 3.2 (Existence of optimal control)**.** *Assume that* (A_1)*–*(A_4) *hold. If σ_3 is large enough or σ_1 is small enough, then the optimization problem* (3.1) *has a unique solution $u^* \in \mathcal{U}$.*

Proof. Define the mapping $\mathcal{B} : \mathcal{U} \longrightarrow L^\infty([0,T])$ by

$$(\mathcal{B}u)(t) = \mathcal{F}\Big\{\frac{\sigma_1 q_2(t)}{\sigma_3}\Big\}.$$

It is easy to show that $\mathcal{B}$ maps $\mathcal{U}$ into itself. Moreover, for any u, $u' \in \mathcal{U}$, from Lemma 3.1, we have

$$\begin{aligned}
\|(\mathcal{B}u) - (\mathcal{B}u')\|_{L^\infty[0,T]} =& \Big\| \mathcal{F}\Big\{\frac{\sigma_1 q_2}{\sigma_3}\Big\} - \mathcal{F}\Big\{\frac{\sigma_1 q_2'}{\sigma_3}\Big\} \Big\|_{L^\infty[0,T]} \\
\le& \frac{\sigma_1}{\sigma_3} \|q_2 - q_2'\|_{L^\infty[0,T]}
\end{aligned}$$

$$\leq \frac{\sigma_1 K_3 T}{\sigma_3} \|u - u'\|_{L^\infty[0,T]}.$$

Thus, if $\sigma_3^{-1}\sigma_1 T K_3 < 1$, then $\mathcal{B}$ owns a unique fixed point $\bar{u} \in \mathcal{U}$.

Next, we show $\bar{u} \in \mathcal{U}$ is the optimal contraception policy. Define the mapping

$$\tilde{J}(u) = \begin{cases} J(u), & u \in \mathcal{U}, \\ +\infty, & u \notin \mathcal{U}. \end{cases}$$

It is not difficult to understand that the smallest elements of mappings $\tilde{J}$ and J are the same. Thus, we only need to show $\tilde{J}(\bar{u}) = \inf\{\tilde{J}(u) : u \in \mathcal{U}\}$. Similar to the discussion of [22, Lemma 4.2], we know that $\tilde{J}(u)$ is lower semi-continuous.

From Ekeland's variational principle and Lemma 3.1, it follows that for each $\varepsilon > 0$, there exists $u_\varepsilon \in \mathcal{U}$ such that

$$\tilde{J}(u_\varepsilon) \leq \inf_{u \in \mathcal{U}} \tilde{J}(u) + \varepsilon, \tag{3.8}$$

$$\tilde{J}(u_\varepsilon) \leq \inf_{u \in \mathcal{U}} \Big\{\tilde{J}(u) + \sqrt{\varepsilon}\|u_\varepsilon - u\|_{L^1([0,T])} : u \in \mathcal{U}\Big\}. \tag{3.9}$$

Thus the perturbed functional $\tilde{J}_\varepsilon(u) = \tilde{J}(u) + \sqrt{\varepsilon}\|u_\varepsilon - u\|_{L^1([0,T])}$ attains its infimum at u_ε. Then, with a similar argument as that in Theorem 3.1 and using [16, Lemma 2.4], we know that there is $\theta_\varepsilon \in L^\infty([0,T])$ satisfying $|\theta_\varepsilon(t)| \leq 1$ such that

$$u_\varepsilon(t) = \mathcal{F}\Big\{\frac{\sigma_1 q_2^\varepsilon(t)}{\sigma_3} + \frac{\sqrt{\varepsilon}\theta_\varepsilon(t)}{\sigma_3}\Big\},$$

where $(q_1^\varepsilon, q_2^\varepsilon, q_3^\varepsilon)$ is the solution of (3.4) with $(p^*, c_e^*, c_0^*) = (p^\varepsilon, c_e^\varepsilon, c_0^\varepsilon)$ and $(p^\varepsilon, c_e^\varepsilon, c_0^\varepsilon)$ is the solution of (1.1) with $u = u_\varepsilon$. Clearly,

$$\begin{aligned}\|u_\varepsilon - \mathcal{B}u_\varepsilon\|_{L^\infty[0,T]} =& \left\|\mathcal{F}\Big\{\frac{\sigma_1 q_2^\varepsilon}{\sigma_3} + \frac{\sqrt{\varepsilon}\theta_\varepsilon}{\sigma_3}\Big\} - \mathcal{F}\Big\{\frac{\sigma_1 q_2^\varepsilon}{\sigma_3}\Big\}\right\|_{L^\infty[0,T]} \\ \leq& \sigma_3^{-1}\sqrt{\varepsilon}\,\|\theta_\varepsilon\|_{L^\infty[0,T]} \\ \leq& \sigma_3^{-1}\sqrt{\varepsilon}.\end{aligned}$$

This, together with $\mathcal{B}\bar{u} = \bar{u}$ ($\bar{u}$ is the fixed point of $\mathcal{B}$), implies

$$\|\bar{u} - u_\varepsilon\|_{L^\infty[0,T]} \leq \sigma_3^{-1}\sigma_1 K_3 T\|\bar{u} - u_\varepsilon\|_{L^\infty[0,T]} + \sigma_3^{-1}\sqrt{\varepsilon}.$$

If σ_3 is large enough or σ_1 is small enough (i.e. $\sigma_3^{-1}\sigma_1 K_3 T < 1$), then

$$\|\bar{u} - u_\varepsilon\|_{L^\infty[0,T]} \leq \frac{\sigma_3^{-1}\sqrt{\varepsilon}}{1 - \sigma_3^{-1}\sigma_1 K_3 T}.$$

Hence, $u_\varepsilon \to \bar{u}$ as $\varepsilon \to 0^+$. Further, by Lemma 3.1, $\tilde{J}(\bar{u}) = \inf_{u \in U} \tilde{J}(u)$. Thus $\bar{u} \in \mathcal{U}$ is the optimal policy.

4. Conclusions

For vermin populations, reducing their breeding rate is considered the most effective means of managing excess rodent populations, as compared to traditional chemical poisoning methods. As mentioned earlier, Liu and Liu [21, 22] have investigated optimal contraception control problems for vermin population models with size structure. However, in both works, it is assumed that the female sterilant applied at any time is completely eaten by vermin, and the control variable is the average amount of sterilant consumed by a single individual. This is obviously unreasonable. This paper discusses the optimal contraception control problem for an age-dependent vermin population hybrid system, in which the vermin becomes sterile through ingestion of female sterilant released by humans into the environment. The hybrid system is shown to have a unique nonnegative bounded solution by means of the fixed-point theory. Moreover, the contraception control problem (3.1) has a unique optimal solution, and the optimal solution has a feedback form as shown in (3.2)–(3.3).

In this paper, the control variable is the amount of female sterilant put into the environment. The objective functional describes that the final total size of vermin should be as small as possible while the control cost and the total amount of female sterilant in the environment are as low as possible. The derived optimal policy provides a logical approach for the strategic application of sterilants in effective vermin control efforts. This approach ensures that while sterilants are being administered, the set goals are achieved with a concurrent reduction in costs and environmental impact. By employing these optimality criteria, we can enhance the creation of efficient and cost-effective pest management procedures.

In real ecosystems, vermin populations are often disturbed by environmental noise. In view of this, we will incorporate random factors into our subsequent model construction to investigate their impact on the dynamics of rodent populations. Conducting an in-depth analysis of the controllability of this system is an important task, and we plan to address it in a separate paper in our future research.

Author contributions

Xin Yi: Responsible for the review and editing of the manuscript; Rong Liu: Responsible for preparation of the original draft of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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