Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 2589–2601.
DOI:10.3934/math.2025120
Received: 12 December 2024
Revised: 16 January 2025
Accepted: 23 January 2025
Published: 13 February 2025

Research article

Homological conjectures and stable equivalences of Morita type

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Abstract: Let A and B be two finite-dimensional algebras over an algebraically closed field. Suppose that A and B are stably equivalent of Morita type; we prove that A satisfies the Auslander–Reiten conjecture (resp. Gorenstein projective conjecture, strong Nakayama conjecture, Auslander–Gorenstein conjecture, Nakayama conjecture, Gorenstein symmetric conjecture) if and only if B does so. This can provide new classes of algebras satisfying homological conjectures, and we give an example to illustrate it.

Keywords: Auslander–Reiten conjecture; Gorenstein projective conjecture; strong Nakayama conjecture; generalized Nakayama conjecture; stable equivalence of Morita type; Gorenstein projective module
Mathematics Subject Classification: 16E10, 16E30

1. Introduction

Throughout this paper, all algebras are finite-dimensional algebras over an algebraically closed k, and all modules are finitely generated left modules unless stated otherwise. Let A be a finite-dimensional k-algebra. That is, A is a ring with an identity element such that A has a finite-dimensional k-vector space structure compatible with the multiplication of the ring. For example, the set $M_n(k)$ of all $n \times n$ square matrices with coefficients in k is a finite-dimensional k-algebra with respect to the usual matrix addition and multiplication. We denote by $\operatorname{mod} A$ the category of all finitely generated A-modules. For $X \in \operatorname{mod} A$, we denote by $\operatorname{pd}_A X$ (resp. $\operatorname{id}_A X$) the projective (resp. injective) dimension of X. We write $\operatorname{add}_A X$ to be the full subcategory of $\operatorname{mod} A$ consisting of the direct summands of the finite copies of X.

The following conjectures are important in homological algebra and the representation theory of finite-dimensional k-algebras.

Auslander–Reiten conjecture: *An A-module X with $\operatorname{Ext}_A^{\geq 1}(X, X) = 0 = \operatorname{Ext}_A^{\geq 1}(X, A)$ is projective.*

Gorenstein projective conjecture: *A Gorenstein projective A-module X is projective if* $\mathrm{Ext}_A^{\geq 1}(X, X) = 0$.
Strong Nakayama conjecture: *If X is an A-module such that* $\mathrm{Ext}_A^{\geq 0}(X, A) = 0$*, then we have* $X = 0$.
Generalized Nakayama conjecture: *Let S be a simple A-module. Then, there exists* $i \geq 0$ *such that* $\mathrm{Ext}_A^i(S, A) \neq 0$.

Let A be a finite-dimensional k-algebra, and let

$$0 \to A \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} \cdots$$

be a minimal injective resolution of ${}_AA$.
Auslander–Gorenstein conjecture: *If* $\mathrm{pd}_A I_i \leq i$ *for any* $i \geq 0$*, then A is a Gorenstein algebra (that is,* $\mathrm{id}_A A < \infty$ *and* $\mathrm{id} A_A < \infty$*).*
Nakayama conjecture: *If* I_i *is projective for any* $i \geq 0$*, then A is self-injective.*
Gorenstein symmetric conjecture: $\mathrm{id}_A A = \mathrm{id} A_A$.

There are close relationships among homological conjectures mentioned above; we refer the reader to [2,5,14,20,21]. At present, there are only a handful of algebras that have been proved to satisfy these homological conjectures. Many authors have investigated whether these homological conjectures are valid for two finite-dimensional k-algebras that are closely related. For instance, it was proved in [5] that the Auslander–Reiten conjecture and the Gorenstein projective conjecture hold under singular equivalences of finite-dimensionalalgebras induced by adjoint pairs. Pan in [15] showed that the Auslander–Reiten conjecture holds under derived equivalences.

In studying the representation theory of finite groups, Broué in [4] introduced the notion of stable equivalences of Morita Type, which is not only a special case of stable equivalences and that of separable equivalences but also tightly related to derived equivalences. Rickard showed in [16] that for self-injective algebras, derived equivalences imply stable equivalences of Morita Type. It is well known that two algebras, which are stably equivalent of Morita Type, share many interesting invariants, such as representation dimensions, extension dimensions, finitistic dimensions, ϕ-dimensions, ψ-dimensions, and so on (see [10, 11, 13, 18, 19] for detail).

In this paper, we study the above-mentioned homological conjectures under stable equivalences of Morita Type, and obtain more invariants as follows.
Main Theorem: (Theorems 3.4, 3.5, 3.10, and 3.12) *Let A and B be finite-dimensional k-algebras such that A and B are stably equivalent of Morita Type. Then, A satisfies the Auslander–Reiten conjecture (resp. Gorenstein projective conjecture, strong Nakayama conjecture, generalized Nakayama conjecture, Auslander–Gorenstein conjecture, Nakayama conjecture, Gorenstein symmetric conjecture) if and only if B does so.*

The paper is organized as below: In Section 2, we give some notations and some preliminary results that are often used in this paper. The proof of the main theorem will be given in Section 3, and we give an example to explicate the results.

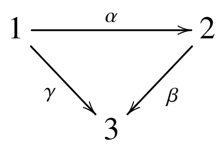
2. Preliminaries

Let A be a finite-dimensional k-algebra, and $X \in \mathrm{mod}\, A$. We use $\mathrm{D}(X)$ to denote the standard duality of X.

Definition 2.1. ([4]) *Let A and B be two finite-dimensional k-algebras. A and B are stably equivalent of Morita Type if there exist bimodules* $_AM_B$ *and* $_BN_A$ *such that*
1) M and N are projective as one-sided modules;
2) $M \otimes_B N \cong A \oplus P$ as (A, A)-bimodules for some projective (A, A)-bimodule P;
3) $N \otimes_A M \cong B \oplus Q$ as (B, B)-bimodules for some projective (B, B)-bimodule Q.

For the convenience of the readers, we give an easy example to understand stable equivalences of Morita Type. We refer to [12] for more information.

Example 2.2. *Let A be a finite-dimensional k-algebra given by the quiver*



with relation $\alpha\beta\gamma\alpha = \beta\gamma\alpha\beta = \gamma\alpha\beta\gamma = 0$.

It is not hard to check that A is a symmetric quasi-hereditary algebra.

And the finite-dimensional k-algebra B is given by the quiver

$$\underset{1}{\cdot} \underset{\delta}{\overset{\rho}{\rightleftarrows}} \underset{2}{\cdot} \underset{\nu}{\overset{\mu}{\rightleftarrows}} \underset{3}{\cdot}$$

with relation $\rho\mu = \nu\delta = \delta\rho - \mu\nu = 0$

B is a symmetric algebra. It follows from [12] that A and B are stably equivalent of Morita Type.

Recall from [3] that two finite-dimensional k-algebras A and B are said to be *stably equivalent* if their stable module categories $\underline{\mathrm{mod}}A$ and $\underline{\mathrm{mod}}B$ are equivalent as additive categories. It follows that the stable equivalence of Morita Type is a significant class of stable equivalences (see [18, Section 4] for detail).

Remark 2.3. *Let A and B be two finite-dimensional k-algebras, and let* $_AM_B$ *and* $_BN_A$ *be finitely generated projective as one-sided modules. Suppose that there exist bimodules* $_AX_A$ *and* $_BY_B$ *and bimodule isomorphisms.*

$$_AM \otimes_B N_A \cong_A A_A \oplus_A X_A \text{ and } _B(N \otimes_A M)_B \cong_B B_B \oplus_B Y_B.$$

1) If $_AX_A$ and $_BY_B$ are the zero modules, then A is Morita equivalent to B.
2) If both $_AX_A$ and $_BY_B$ have finite projective dimension as bimodules, then A and B are singularly equivalent of Morita type [5].
3) If $_AX_A$ and $_BY_B$ are usual bimodules, then A and B are separably equivalent [9].
4) The following chain of implications holds.

Morita equivalence
$\Downarrow$
Stable equivalence of Morita Type
$\Downarrow$
Singular equivalence of Morita Type
$\Downarrow$
Separable equivalence

Let A and B be stably equivalent of Morita Type defined as in Definition 2.1. We write $\mathrm{T}_N = N \otimes_A - : \mathrm{mod}\, A \to \mathrm{mod}\, B$ and $\mathrm{T}_M = M \otimes_B - : \mathrm{mod}\, B \to \mathrm{mod}\, A$, respectively. The functors T_P and T_Q are defined similarly. The next result is due to [18].

Lemma 2.4. *Let A and B be stably equivalent of Morita Type defined as in Definition 2.1; then the following holds:*

1) The bimodules M and N are projective generators as one-side modules;
2) $\mathrm{T}_N, \mathrm{T}_M, \mathrm{T}_P$, and T_Q are exact functors;
3) The images of the functors T_P and T_Q consist of projective modules;
4) The functors T_M and T_N take projective modules to projective modules;
5) $\mathrm{T}_M \circ \mathrm{T}_N \to \mathrm{Id}_{\mathrm{mod}\, A} \oplus \mathrm{T}_P$ and $\mathrm{T}_N \circ \mathrm{T}_M \to \mathrm{Id}_{\mathrm{mod}\, B} \oplus \mathrm{T}_Q$ are natural isomorphisms.

Take a minimal projective presentation $P_1 \xrightarrow{f} P_0 \to X \to 0$ of X in $\mathrm{mod}\, A$. Recall from [1, 3] that $\mathrm{CokerHom}_A(f, A)$ is said to be *the transpose* of X, denoted by $\mathrm{Tr}(X)$. Recall from [6] that X is called *Gorenstein projective* if $\mathrm{Ext}_A^{\geq 1}(X, A) = 0 = \mathrm{Ext}_{A^{op}}^{\geq 1}(\mathrm{Tr}(X), A)$. *The Gorenstein projective dimension* of X, denoted by $\mathrm{Gpd}_A X$, is defined as $\inf \{n|$ there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to X \to 0$ with each G_i Gorenstein projective $\}$.

We also list some homological facts needed in the later proofs.

Lemma 2.5. ([17]) *Let A and B be two finite-dimensional k-algebras, and* $X \in \mathrm{mod}\, A$ *and* $Y \in \mathrm{mod}\, B$. *Suppose that M is an* (A, B)*-bimodule with* ${}_AM$ *and* M_B *projective. Then, for any positive integer n, we have*

1) $\mathrm{Ext}_A^n(M \otimes_B Y, X) \cong \mathrm{Ext}_B^n(Y, \mathrm{Hom}_A(M, X))$;
2) there exists a right B-module isomorphism $\mathrm{Hom}_A(X, M) \cong \mathrm{Hom}_A(X, A) \otimes_A M$;
3) there exists a right B-module isomorphism $\mathrm{Ext}_A^n(X, M) \cong \mathrm{Ext}_A^n(X, A) \otimes_A M$.

According to [3, Section 2, P7], we have the following observation.

Lemma 2.6. *Let* $0 \to C \xrightarrow{f_0} I_0 \to \cdots \xrightarrow{f_n} I_n \to \cdots$ *be a minimal injective resolution of* $C \in \mathrm{mod}\, A$. *If* $L \in \mathrm{add}C$ *with a minimal injective resolution*

$$0 \to L \xrightarrow{g_0} I'_0 \to \cdots \xrightarrow{g_n} I'_n \to \cdots$$

then $I'_i \in \mathrm{add}I_i$ for all $i \geq 0$.

3. Results

In this section, we always assume that A and B are stably equivalent of Morita type linked by bimodules ${}_AM_B$ and ${}_BN_A$. That is, there exist projective bimodules ${}_AP_A$ and ${}_BQ_B$ and bimodule isomorphisms $M \otimes_B N \cong A \oplus P$ and $N \otimes_A M \cong B \oplus Q$. According to Lemma 2.4, 5), there are natural isomorphisms $\mathrm{T}_N \circ \mathrm{T}_M \cong \mathrm{Id}_{\mathrm{mod}\, A} \oplus \mathrm{T}_P$ and $\mathrm{T}_M \circ \mathrm{T}_N \cong \mathrm{Id}_{\mathrm{mod}\, B} \oplus \mathrm{T}_Q$. For convenience, we consider $\mathrm{T}_M \circ \mathrm{T}_N$ and $\mathrm{Id}_{\mathrm{mod}\, A} \oplus \mathrm{T}_P$, $\mathrm{T}_N \circ \mathrm{T}_M$ and $\mathrm{Id}_{\mathrm{mod}\, B} \oplus \mathrm{T}_Q$ to be equal, respectively.

Lemma 3.1. 1) *An A-module I is injective if and only if so is a B-module* $N \otimes_A I$.
2) *A right A-module J is injective if and only if so is* $J \otimes_A M$ *as a right B-module.*

Proof. We only prove 1), and the proof of 2) is similar.

1) The only if part of the assertion is due to the proof of [18, Theorem 4.1]. Now, we give a brief proof. Suppose that I is an injective A-module. Take a short exact sequence in $\operatorname{mod} B$.

$$0 \to X \xrightarrow{f} Y \to Z \to 0. \tag{3.1}$$

For any B-module homomorphism $g : X \to \mathrm{T}_N(I)$, we claim that there exists a B-module homomorphism $\alpha : Y \to \mathrm{T}_N(I)$ such that $g = \alpha f$.

As T_M is an exact functor by Lemma 2.4 2), the exact sequence (3.1) gives an exact sequence in $\operatorname{mod} A$

$$0 \to \mathrm{T}_M(X) \xrightarrow{\mathrm{T}_M(f)} \mathrm{T}_M(Y) \to \mathrm{T}_M(Z) \to 0.$$

Since $\mathrm{T}_M(g) : \mathrm{T}_M(X) \to \mathrm{T}_M\mathrm{T}_N(I) = I \oplus \mathrm{T}_P(I)$, and I is an injective A-module by assumption, there exists an A-module homomorphism $h : \mathrm{T}_M(Y) \to I$ such that $(\mathrm{Id}_I, 0)\mathrm{T}_M(g) = h\mathrm{T}_M(f)$. Hence, $(\mathrm{Id}_{\mathrm{T}_N(I)}, 0)\mathrm{T}_N\mathrm{T}_M(g) = \mathrm{T}_N(h)\mathrm{T}_N\mathrm{T}_M(f)$, where $\mathrm{T}_N(h) : \mathrm{T}_N\mathrm{T}_M(Y) \to \mathrm{T}_N(I)$. Note that $\mathrm{T}_N\mathrm{T}_M(Y) = Y \oplus \mathrm{T}_Q(Y)$. Let $\mathrm{T}_N(h) = (\alpha, \beta) : Y \oplus \mathrm{T}_Q(Y) \to \mathrm{T}_N(Y)$, then we have

$$(\mathrm{Id}_{\mathrm{T}_N(I)}, 0)\begin{pmatrix} g & 0 \\ 0 & \mathrm{T}_Q(g) \end{pmatrix} = (\alpha, \beta)\begin{pmatrix} f & 0 \\ 0 & \mathrm{T}_Q(f) \end{pmatrix},$$

which yields $g = \alpha f$ as claimed. Thus, $N \otimes_A I$ is an injective B-module.

Conversely, assume that $N \otimes_A I$ is an injective B-module. By a similar argument of the only if part, one has that $M \otimes_B N \otimes_A I$ is an injective A-module. Because I is isomorphic to a direct summand of $M \otimes_B N \otimes_A I$, denoted by ${}_AI|_A(M \otimes_B N \otimes_A I)$, by Lemma 2.4 5), I is an injective A-module. □

Proposition 3.2. 1) $\mathrm{id}_A X = \mathrm{id}_B(N \otimes_A X)$ *for an A-module X. In particular,* $\mathrm{id}_A A = \mathrm{id}_B B$.
2) $\mathrm{id} T_A = \mathrm{id}(T \otimes_A M)_B$ *for a right A-module T. In particular,* $\mathrm{id} A_A = \mathrm{id} B_B$.

Proof. We prove only part 1), and part 2) is proved analogously.

Without loss of generality, we assume that $\mathrm{id}_A X = m < \infty$. Then, there exists an injective resolution of X

$$0 \to X \to I_0 \to I_1 \to \cdots \to I_m \to 0,$$

which gives rise to an exact sequence in $\operatorname{mod} B$:

$$0 \to N \otimes_A X \to N \otimes_A I_0 \to N \otimes_A I_1 \to \cdots \to N \otimes_A I_m \to 0,$$

where $N \otimes_A I_i$ is an injective B-module for any $0 \le i \le m$ by Lemma 3.1 1). This means $\mathrm{id}_B(N \otimes_A X) \le m = \mathrm{id}_A X$. Similarly, we get $\mathrm{id}_A(M \otimes_B N \otimes_A X) \le \mathrm{id}_B(N \otimes_A X)$. Since ${}_AX|_A(M \otimes_B N \otimes_A X)$ by Lemma 2.4 5), we have $\mathrm{id}_A X \le \mathrm{id}_A(M \otimes_B N \otimes_A X) \le \mathrm{id}_B(N \otimes_A X)$ as desired.

By the above discussion, we have $\mathrm{id}_A A = \mathrm{id}_B(N \otimes_A A) = \mathrm{id}_B N$. On the other hand, because N is a projective generator for B-modules by Lemma 2.4 1), one gets $\mathrm{id}_B N = \mathrm{id}_B B$. It follows that $\mathrm{id}_A A = \mathrm{id}_B B$. □

Recall that a finite-dimensional k-algebra A is said to be *a Gorenstein algebra*, if A has finite left and right self-injective dimensions. According to Proposition 3.2, the following is obtained directly.

Theorem 3.3. *Let A and B be stably equivalent of Morita Type. Then,*

1) A is a Gorenstein algebra if and only if so is B.
2) A satisfies the Gorenstein symmetric conjecture if and only if B does so.

Proof. 1) Follows from Proposition 3.2 directly.
2) By Proposition 3.2, we have $\mathrm{id}_A A = \mathrm{id}_B B$ and $\mathrm{id} A_A = \mathrm{id} B_B$. So the assertion follows. □

Theorem 3.4. *Let A and B be stably equivalent of Morita Type. Then,*

1) A satisfies the Auslander–Gorenstein conjecture if and only if B does so;
2) A satisfies the Nakayama conjecture if and only if B does so.

Proof. Let

$$0 \to A \xrightarrow{f_0} I_0 \to \cdots \xrightarrow{f_n} I_n \to \cdots \tag{3.2}$$

be a minimal injective resolution of ${}_A A$, and let

$$0 \to B \xrightarrow{g_0} E_0 \to \cdots \xrightarrow{g_m} E_m \to \cdots \tag{3.3}$$

be a minimal injective resolution of ${}_B B$. We claim that $\mathrm{pd}_A I_i = \mathrm{pd}_B E_i$ for any $i \geq 0$. Applying the functor $N \otimes_A -$ to the sequence (3.2) gives an exact sequence in $\mathrm{mod}\, B$:

$$0 \to_B N \xrightarrow{N\otimes f_0} N_A \otimes_A I_0 \to \cdots \xrightarrow{N_A \otimes f_n} N \otimes_A I_n \to \cdots$$

where $N \otimes_A I_i$ is an injective B-module for any $i \geq 0$ by Lemma 3.1 1). It follows from Lemma 2.6 that $E_i \in \mathrm{add}(N \otimes_A I_i)$ for all i. Note that the functor $N \otimes_A -$ takes projective modules to projective modules by Lemma 2.4 4), one gets $\mathrm{pd}_B E_i \leq \mathrm{pd}_B(N \otimes_A I_i) \leq \mathrm{pd}_A(I_i)$, for any $i \geq 0$. Similarly, we have $\mathrm{pd}_A I_i \leq \mathrm{pd}_B E_i$ for any $i \geq 0$. And our claim is obtained.
1) Assume that A satisfies the Auslander–Gorenstein conjecture. If $\mathrm{pd}_B E_i \leq i$ for any $i \geq 0$, then $\mathrm{pd}_A I_i \leq i$ for all $i \geq 0$ by the above discussion, and hence A is a Gorenstein algebra by assumption. It follows from Theorem 3.3 1) that B is a Gorenstein algebra, which means that B satisfies the Auslander–Gorenstein conjecture.

The converse is proved dually.
2) Suppose that A satisfies the Nakayama conjecture. If E_i is projective for any $i \geq 0$, then each I_i is a projective A-module for any $i \geq 0$ by the discussion above. So, A is self-injective by assumption. It follows from Proposition 3.2 1) that B is self-injective.

Dually, it can be verified that A satisfies the Nakayama conjecture when B does so. □

As applications of Lemma 3.1, we have the following conclusions.

Lemma 3.5. 1) *An A-module X is projective if and only if so is a B-module* $\mathrm{Hom}_A(M, X)$. *In particular,* $\mathrm{Hom}_A({}_A M_B, {}_A A)$ *is a projective B-module.*
2) *A B-module Y is projective if and only if so is an A-module* $\mathrm{Hom}_B(N, Y)$. *In particular,* $\mathrm{Hom}_B({}_B N_A, B)$ *is a projective A-module.*

Proof. We only prove 1); the proof of 2) is similar.

1) Suppose that X is a projective A-module. It is clear that $\mathrm{D}(X)$ is a right injective A-module, and hence $\mathrm{D}(X)\otimes_A M$ is a right injective B-module by Lemma 3.1 2). From the right B-module isomorphism $\mathrm{D}(\mathrm{Hom}_A(M, X)) \cong \mathrm{D}(X) \otimes_A M$, we know that $\mathrm{D}(\mathrm{Hom}_A(M, X))$ is an injective right B-module, which implies that $\mathrm{Hom}_A(M, X)$ is a projective left B-module.

Similarly, we prove that X is a projective A-module when $\mathrm{Hom}_A(M, X)$ does so.

In particular, since A is a projective A-module, $\mathrm{Hom}_A({}_AM_B, {}_AA)$ is a projective B-module. □

Lemma 3.6. *Let X be an A-module and n a positive integer. If* $\mathrm{Ext}^n_A(X, A) = 0$*, then* $\mathrm{Ext}^n_B(N\otimes_A X, B) = 0$.

Proof. Assume $\mathrm{Ext}^n_A(X, A) = 0$. One checks easily that $\mathrm{Ext}^n_A(X, L) = 0$ for any projective A-module L. Since N_A and ${}_A\mathrm{Hom}_B(N, B)$ are projective by Lemma 3.5 2), we have

$$\begin{aligned}&\mathrm{Ext}^n_B(N \otimes_A X, B)\\ &\cong \mathrm{Ext}^n_A(X, \mathrm{Hom}_B(N, B))(\text{by Lemma 2.5 (1)})\\ &= 0.\end{aligned}$$

□

Proposition 3.7. *Let X and Y be A-modules and n a positive integer. If* $\mathrm{Ext}^n_A(X, Y) = 0 = \mathrm{Ext}^n_A(X, A)$*, then* $\mathrm{Ext}^n_B(N \otimes_A X, N \otimes_A Y) = 0$.

Proof. Assume that $\mathrm{Ext}^1_A(X, Y) = 0 = \mathrm{Ext}^1_A(X, A)$. It is not hard to check that $\mathrm{Ext}^1_A(X, L) = 0$ for any projective A-module L. Since $\mathrm{T}_M\mathrm{T}_N(X) = X \oplus \mathrm{T}_P(X)$ and $\mathrm{T}_M\mathrm{T}_N(Y) = Y \oplus \mathrm{T}_P(Y)$ and $\mathrm{T}_P(X)$ and $\mathrm{T}_P(Y)$ are projective A-modules by Lemma 2.4 3), there exist isomorphisms $\mathrm{Ext}^1_A(\mathrm{T}_M\mathrm{T}_N(X), \mathrm{T}_M\mathrm{T}_N(Y)) = \mathrm{Ext}^1_A(X \oplus \mathrm{T}_P(X), Y \oplus \mathrm{T}_P(Y)) \cong \mathrm{Ext}^1_A(X, Y) \oplus \mathrm{Ext}^1_A(X, \mathrm{T}_P(Y)) = 0$ by assumption.

Let ε be any element of $\mathrm{Ext}^1_B(\mathrm{T}_N(X), \mathrm{T}_N(Y))$. Represent ε by a short exact sequence in $\mathrm{mod}\, B$.

$$0 \to \mathrm{T}_N(Y) \to K \xrightarrow{f} \mathrm{T}_N(X) \to 0. \tag{3.4}$$

We claim that the sequence (3.4) is split. Indeed, applying the exact functor T_M to the sequence (3.4) gives an exact sequence in $\mathrm{mod}\, A$:

$$0 \to \mathrm{T}_M\mathrm{T}_N(Y) \to \mathrm{T}_M(K) \xrightarrow{\mathrm{T}_M(f)} \mathrm{T}_M\mathrm{T}_N(X) \to 0,$$

which is split because of $\mathrm{Ext}^1_A(\mathrm{T}_M\mathrm{T}_N(X), \mathrm{T}_M\mathrm{T}_N(Y)) = 0$. Then, there exists $g \in \mathrm{Hom}_A(\mathrm{T}_M\mathrm{T}_N(X), \mathrm{T}_M(K))$ such that $\mathrm{T}_M(\mathrm{Id}_{\mathrm{T}_N(X)}) = \mathrm{T}_M(f)g$. Thus, $\mathrm{T}_N\mathrm{T}_M(\mathrm{Id}_{\mathrm{T}_N(X)}) = \mathrm{T}_N\mathrm{T}_M(f)\mathrm{T}_N(g)$, where $\mathrm{T}_N(g) : \mathrm{T}_N(X)\oplus \mathrm{T}_Q\mathrm{T}_N(X) \to K \oplus \mathrm{T}_Q(K)$. Set $\mathrm{T}_N(g) = \begin{pmatrix} \alpha & \beta \\ \sigma & \tau \end{pmatrix}$, then we have

$$\begin{pmatrix} \mathrm{Id}_{\mathrm{T}_N(X)} & 0 \\ 0 & \mathrm{T}_Q(\mathrm{Id}_{\mathrm{T}_N(X)}) \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & \mathrm{T}_Q(f) \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \sigma & \tau \end{pmatrix}.$$

This means $\mathrm{Id}_{\mathrm{T}_N(X)} = f\alpha$, and our claim is obtained. It follows that $\mathrm{Ext}^1_B(N \otimes_A X, N \otimes_A Y) = 0$.

Suppose $\mathrm{Ext}^n_A(X, Y) = 0 = \mathrm{Ext}^n_A(X, A)$ for $n > 1$. By dimension shifting and by assumption, one gets isomorphisms $0 = \mathrm{Ext}^n_A(X, Y) \cong \mathrm{Ext}^1_A(\Omega^{n-1}(X), Y)$ and $0 = \mathrm{Ext}^n_A(X, A) \cong \mathrm{Ext}^1_A(\Omega^{n-1}(X), A)$. From the above step, we have $\mathrm{Ext}^1_B(\mathrm{T}_N(\Omega^{n-1}(X)), \mathrm{T}_N(Y)) = 0$. On the other hand, since T_N is exact and takes projective A-modules to projective B-modules by Lemma 2.4 2) and 3), there are isomorphisms $\mathrm{Ext}^n_B(\mathrm{T}_N(X), \mathrm{T}_N(Y)) \cong \mathrm{Ext}^1_B(\Omega^{n-1}(\mathrm{T}_N(X)), \mathrm{T}_N(Y)) \cong \mathrm{Ext}^1_B(\mathrm{T}_N(\Omega^{n-1}(X)), \mathrm{T}_N(Y)) = 0$ as desired. □

Lemma 3.8. *Let X be an A-module. Then there is a projective right B-module Q' such that*

$$(\mathrm{Tr}X) \otimes_A \mathrm{Hom}_B(N, B) \cong \mathrm{Tr}(N \otimes_A X) \oplus Q'$$

as right B-modules.

Proof. Take a minimal projective presentation of X

$$P_1 \to P_0 \to X \to 0. \tag{3.5}$$

Applying the functor $()^* = \mathrm{Hom}_A(-, A)$ induces an exact sequence

$$P_0^* \to P_1^* \to \mathrm{Tr}X \to 0.$$

On the other hand, one obtains the exact sequence

$$N \otimes_A P_1 \to N \otimes_A P_0 \to N \otimes_A X \to 0, \tag{3.6}$$

and the exact sequence

$$(N \otimes_A P_0)^\dagger \to (N \otimes_A P_1)^\dagger \to \mathrm{Tr}(N \otimes_A X) \oplus Q' \to 0$$

for some projective right B-module Q', where $()^\dagger = \mathrm{Hom}_B(-, B)$.

Since N_A and ${}_A\mathrm{Hom}_B(N, B)$ are projective by Lemmas 2.4 1) and 3.5 2), for any A-module Y we have isomorphisms

$$\begin{aligned}
&\mathrm{Hom}_B(N \otimes_A Y, B) \\
&\cong \mathrm{Hom}_A(Y, \mathrm{Hom}_B(N, B))(\text{by the adjoint isomorphism}) \\
&\cong \mathrm{Hom}_A(Y, A) \otimes_A \mathrm{Hom}_B(N, B)(\text{Lemma 2.5 (2)}).
\end{aligned}$$

Hence, applying $\mathrm{Hom}_B(-, B)$ to the sequence (3.6) and $- \otimes N^\dagger$ to the sequence (3.5), respectively, we get a commutative diagram with exact arrows

$$\begin{array}{ccccccc}
(N \otimes_A P_0)^\dagger & \longrightarrow & (N \otimes_A P_1)^\dagger & \longrightarrow & \mathrm{Tr}(N \otimes_A X) \oplus Q' & \longrightarrow & 0 \\
\cong\downarrow & & \cong\downarrow & & \downarrow & & \\
P_0^* \otimes_A N^\dagger & \longrightarrow & P_1^* \otimes_A N^\dagger & \longrightarrow & \mathrm{Tr}(X) \otimes_A N^\dagger & \longrightarrow & 0
\end{array}$$

This induces a right B-module isomorphism $\mathrm{Tr}(N \otimes_A X) \oplus Q' \cong \mathrm{Tr}(X) \otimes_A N^\dagger$ by five-lemma. □

Proposition 3.9. *Let A and B be stably equivalent of Morita Type. Then an A-module G is Gorenstein projective if and only if so is a B-module $N \otimes_A G$.*

Proof. Assume that G is a Gorenstein projective A-module, we have $\mathrm{Ext}_A^{\geq 1}(G, A) = 0 = \mathrm{Ext}_A^{\geq 1}(\mathrm{Tr}(G)_A, A_A)$. Lemma 3.6 yields $\mathrm{Ext}_B^{\geq 1}(N \otimes_A G, B) = 0$. On the other hand, since ${}_BN$ is projective, there exists a (B, A)-bimodule isomorphism $\mathrm{Hom}_B(\mathrm{Hom}_B(N, B), B) \cong N$. And hence, for all $n \geq 1$, we have

$$\mathrm{Ext}_B^n(\mathrm{Tr}(N \otimes_A G), B)$$

$$
\begin{aligned}
&\cong \mathrm{Ext}^n_B(\mathrm{Tr}G \otimes_A \mathrm{Hom}_B(N, B), B) \quad \text{(by Lemma 3.8)}\\
&\cong \mathrm{Ext}^n_A(\mathrm{Tr}G, \mathrm{Hom}_B(\mathrm{Hom}_B(N, B), B)) \quad \text{(by Lemma 2.5, 1))}\\
&\cong \mathrm{Ext}^n_A(\mathrm{Tr}G, N)\\
&= 0 \quad \text{(because } N \text{ is a projective right } A\text{-module).}
\end{aligned}
$$

This implies that $N \otimes_A G$ is a Gorenstein projective B-module.

Conversely, assume that $N \otimes_A G$ is a Gorenstein projective B-module. Then, $M \otimes_B N \otimes_A G$ is a Gorenstein projective A-module. It follows from [7, Theorem 2.5] that G is a Gorenstein projective A-module because ${}_AG|_A(M \otimes_B N \otimes_A G)$ by Lemma 2.4 5). □

Theorem 3.10. *Let A and B be stably equivalent of Morita Type. Then,*

1) A satisfies the Auslander–Reiten conjecture if and only if B does so;
2) A satisfies the Gorenstein projective conjecture if and only if B does so;
3) A satisfies the strong Nakayama conjecture if and only if B does so;

Proof. We prove only the "if " part, the proof of the "only if " part is analogous.

1) Assume that B satisfies the Auslander–Reiten conjecture. Let X be an A-module satisfying $\mathrm{Ext}^{\geq 1}_A(X, X) = 0 = \mathrm{Ext}^{\geq 1}_A(X, A)$. By Lemma 3.6 and Proposition 3.7, we have $\mathrm{Ext}^{\geq 1}_B(N \otimes_A X, N \otimes_A X) = 0 = \mathrm{Ext}^{\geq 1}_B(N \otimes_A X, B)$. So, $N \otimes_A X$ is a projective B-module by assumption. It follows from Lemma 2.4 4) that $M \otimes_B N \otimes_A X$ is a projective A-module, which shows that X is a projective A-module, for ${}_AX|_A(M \otimes_B N \otimes_A X)$ by Lemma 2.4 5).

2) Suppose that B satisfies the Gorenstein projective conjecture. Let G be a Gorenstein projective A-module with $\mathrm{Ext}^{i\geq 1}_A(G, G) = 0$. According to Proposition 3.9 and Lemma 3.6, it follows that $N \otimes_A G$ is a Gorenstein projective B-module satisfying $\mathrm{Ext}^{i\geq 1}_B(N \otimes_A G, N \otimes_A G) = 0$. So, $N \otimes_A G$ is a projective B-module by assumption, and hence $M \otimes_B N \otimes_A G$ is a projective A-module by Lemma 2.4 4). Thus, we obtain that G is a projective A-module, for ${}_AG|_A(M \otimes_B N \otimes_A G)$ by Lemma 2.4 5).

3) Assume that B satisfies the strong Nakayama conjecture. Let X be an A-module satisfying $\mathrm{Ext}^{\geq 0}_A(X, A) = 0$. By Lemma 3.6, one has $\mathrm{Ext}^{\geq 1}_B(N \otimes_A X, B) = 0$. On the other hand, because $\mathrm{Hom}_B(N, B)$ is a projective A-module by Lemma 3.5 2), there are isomorphisms $\mathrm{Hom}_B(N \otimes_A X, B) \cong \mathrm{Hom}_A(X, \mathrm{Hom}_B(N, B)) \cong \mathrm{Hom}_A(X, A) \otimes_A \mathrm{Hom}_B(N, B) = 0$ by the adjoint isomorphism and Lemma 2.5 2). It follows that $N \otimes_A X = 0$ by assumption, and hence $X = 0$ since N_A is a projective generator from Lemma 2.4 1). □

Lemma 3.11. *Let X be an A-module.*

(1) If $\mathrm{Hom}_A(X, A) = 0$, then we have $P \otimes_A X = 0$;
(2) If X is a simple A-module with $\mathrm{Hom}_A(X, A) = 0$, then $N \otimes_A X$ is a simple B-module.

Proof. 1) Assume that the assertion would not hold. We have $\mathrm{Hom}_A(P \otimes_A X, A) \neq 0$, because $P \otimes_A X$ is a projective A-module by Lemma 2.4 3). By the definition of stable equivalences of Morita type, there exist isomorphisms $\mathrm{Hom}_A(M \otimes_B N \otimes_A X, A) \cong \mathrm{Hom}_A(X \oplus (P \otimes_A X), A) \cong \mathrm{Hom}_A(P \otimes_A X, A) \neq 0$.

On the other hand, since $\mathrm{Hom}_A(M, A)$ is a projective B-module and $\mathrm{Hom}_B(N, B)$ is a projective A-module, respectively, by Lemma 3.5, we have

$$\mathrm{Hom}_A(M \otimes_B N \otimes_A X, A)$$

$$\cong \mathrm{Hom}_B(N \otimes_A X, \mathrm{Hom}_A(M, A)) \quad \text{(by the adjoint isomorphism)}$$
$$\cong \mathrm{Hom}_B(N \otimes_A X, B) \otimes_B \mathrm{Hom}_A(M, A) \quad \text{(by Lemma 2.5 2))}$$
$$\cong \mathrm{Hom}_A(X, \mathrm{Hom}_B(N, B)) \otimes_B \mathrm{Hom}_A(M, A) \quad \text{(by the adjoint isomorphism)}$$
$$\cong \mathrm{Hom}_A(X, A) \otimes_A \mathrm{Hom}_B(N, B) \otimes_B \mathrm{Hom}_A(M, A) \quad \text{(by Lemma 2.5 2)).}$$

So, $\mathrm{Hom}_A(M \otimes_B N \otimes_A X, A) = 0$ by assumption. This leads to a contradiction. Therefore, we obtain $P \otimes_A X = 0$.

2) According to the assumption and (1), it follows that $P \otimes_A X = 0$. Then, we have $M \otimes_B N \otimes_A X \cong X$, which implies that $M \otimes_B N \otimes_A X$ is a simple A-module.

Take any nonzero submodule K of $N \otimes_A X$. The inclusion map $f : K \hookrightarrow N \otimes_A X$ induces an exact sequence in $\mathrm{mod}\, B$

$$0 \to K \to N \otimes_A X \to L \to 0.$$

Because M_B is a projective generator for B-modules, the functor $M \otimes_B -$ is exact and faithful, and hence one gets an exact sequence in $\mathrm{mod}\, A$

$$0 \to M \otimes_B K \to M \otimes_B N \otimes_A X \to M \otimes_B L \to 0$$

with $M \otimes_B K \neq 0$. Since $M \otimes_B N \otimes_A X$ is a simple A-module, then $M \otimes_A L = 0$, which yields $L = 0$. Thus, $N \otimes_A X$ is a simple B-module. □

Theorem 3.12. *Let A and B be stably equivalent of Morita type. Then, A satisfies the generalized Nakayama conjecture if and only if B does so.*

Proof. Assume that B satisfies the generalized Nakayama conjecture. Let S be any simple A-module. If $\mathrm{Hom}_A(S, A) \neq 0$, we are done.

If $\mathrm{Hom}_A(S, A) = 0$, then $N \otimes_A S$ is a simple B-module by Lemma 3.11 2). Note that $\mathrm{Hom}_B(N, B)$ is a projective A-module. Then, there exist isomorphisms $\mathrm{Hom}_B(N \otimes_A S, B) \cong \mathrm{Hom}_A(S, \mathrm{Hom}_B(N, B)) \cong \mathrm{Hom}_A(S, A) \otimes_A \mathrm{Hom}_B(N, B) = 0$ by the adjoint isomorphism and by Lemma 2.5 2). Hence, there exists an integer $n \geq 1$ such that $\mathrm{Ext}^n_B(N \otimes_A S, B) \neq 0$ by assumption. It follows from Lemma 3.6 that $\mathrm{Ext}^n_A(S, A) \neq 0$. □

We conclude with an example to illustrate our results.

Example 3.13. Let k be an algebraically closed field, and let Λ and Γ be finite-dimensional k-algebras by the following quivers with relations:

$$\Lambda \qquad \underset{1}{\cdot} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \underset{2}{\cdot} \quad \text{with relation} \quad \alpha\beta\alpha\beta = 0,$$

and

$$\Gamma \qquad \underset{1}{\cdot} \underset{y}{\overset{x}{\rightleftarrows}} \underset{2}{\cdot} \circlearrowleft_z \quad \text{with relation} \quad xy = xz = zy = z^2 - yx = 0.$$

It follows from [13, Section 5, Example] that Λ and Γ are stably equivalent of Morita type. Note that Λ is a Nakayama algebra, and indecomposable projective and injective Λ-modules are

$$P(1) = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, P(2) = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} = I(2) \quad \text{and} \quad I(1) = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}.$$

Thus, we obtain a minimal injective resolution of $P(1)$:

$$0 \to P(1) \to I(2) \to I(2) \to I(1) \to 0,$$

which yields $\mathrm{id}_\Lambda \Lambda = 2$ and $\mathrm{pd}_\Lambda I(1) = 2$. Similarly, we have $\mathrm{id}\Lambda_\Lambda = 2$. This implies that Λ satisfies the Gorenstein symmetric conjecture, Auslander–Gorenstein conjecture, and strong Nakayama conjecture by [8, Theorem 2]. So that the generalized Nakayama conjecture holds on Λ. On the other hand, since Λ is of finite representation type, one has that Λ satisfies the Auslander–Reiten conjecture by [2, Proposition 1.3]. And hence the Gorenstein projective conjecture holds on Λ by [14]. Therefore, we obtain that Γ satisfies the Gorenstein symmetric conjecture, Auslander–Gorenstein conjecture, Auslander–Reiten conjecture, Gorenstein projective conjecture, strong Nakayama conjecture, and the generalized Nakayama conjecture by Theorem 3.3 2), Theorem 3.4 1), Theorem 3.10, and Theorem 3.12.

4. Conclusions

In this paper, we mainly show that many famous homological conjectures are preserved by algebras that are stably equivalent of Morita type. Our findings contribute to providing new algebras satisfying homological conjectures. This gives support for the validity of these homological conjectures.

It has been known that the Auslander–Reiten conjecture and the Gorenstein projective conjecture hold under singular equivalences induced by adjoint pairs. In the future work, we will study whether the homological conjectures hold under any singular equivalences, even under separable equivalences.

Author contributions

Juxiang Sun: contributed the creative ideals and proof techniques for this paper; Guoqiang Zhao: consulted the relevant background of the paper and composed the article, encompassing the structure of the article and the modification of grammar. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was financially supported by NSFC (Grant No. 12061026), Foundation for University Key Teacher by Henan Province (Grant No. 2019GGJS204), and the Key Project of Henan Educational Committee (Grant No. 25B110022).

Conflict of interest

The authors declare no competing interests.

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