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Research article

The travelling wave phenomena of the space-time fractional Whitham-Broer-Kaup equation

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Abstract: In the present work, we consider the space-time fractional Whitham-Broer-Kaup (FWBK) equation and then find its analytical solutions under the framework of the Riccati-Bernoulli sub-ordinary differential equation method along with the Bäcklund transformation. The derived solutions are described in terms of the hyperbolic rational and trigonometric functions and resolve unique wave features. A unique feature of this work is the investigation of the impact of the fractional order on wave steepness in $2D$, $3D$, and contour plots. Also, it should be pointed that the found relationship shows that the change of the fractional order parameter really describes the traveling periodic waves, for example, Stokes waves, with potential importance for the analysis of wave propagation and stability in the nonlinear fractional structures. This work carries theoretical developments of fractional calculus and ideas for applications in fluid dynamics and wave mechanics.

Keywords: fractional partial differential equations; fractional Whitham-Broer-Kaup equation; Bäcklund transformation; Riccatti equation; travelling wave solutions
Mathematics Subject Classification: 34G20, 35R11

1. Introduction

Nonlinear partial differential equations (NPDEs) have been central in the development of many branches of applied science for the last few decades [1–3]. These equations are basic for analyzing complicated nonlinear phenomena in different systems and, therefore, the appearance of a plethora of nonlinear models [4–6]. As for these equations, a number of approaches have been suggested to derive numerical, analytical, and semi-analytical solutions. The techniques used include: spline scheme [7], finite difference method [8], Hirota bilinear method [9], variational iteration method [10], expansion methods [11], sine cosine expansions [12], Riccati differential equation method [13], residual power series method [14], and the modified simple equation method (SEM) [15]. Nonlinear dynamic systems, and travelling waves in particular, have attracted much attention in the past few years due to their important applications in optics, fluid mechanics, and material science [16, 17]. In addition, there are models and simulations of physical field phenomena, such as wave electro rotating magnetic field fluid dynamics vibrations, and many others have often relied on generalized systems algebraic equations constrains and non linear fractional order models [18].

Recently, fractional partial differential equations (FPDEs) have been generalized or created since these equations have large application in the reorganization of the pattern, control theory, image processing, signal processing, and system identification. These equations offer better description and modeling of these complicated dynamic systems than conventional ones that exhaust free energies of particles where memory effects and long-range interactions exist. Many engineering problems are nonlinear in nature, and this makes it difficult to get solutions, whether numerical or analytical [19–21]. A wide range of PDEs are used in the description of the dynamics of various systems, and a great deal of emphasis has been placed on the efforts aimed at the identification of the effective and fast converging approximations for these equations. To obtain analytical soliton solutions for FPDEs, more complicated techniques, including Lie symmetry analysis [22], the Jacobi elliptic function method [23], the Khater method [24], and the (ψ–φ)-expansion method [35], have been used. These techniques provide exact soliton solutions and provide significant information regarding the dynamics of system involving fractional integers [25–29].

For a better understanding of a much broader variety of phenomena in areas of physics, such as hydrodynamics, acoustics, and various branches of engineering, it becomes crucial to consider different forms of solutions, specifically in relation to traveling waves [30–34]. The traveling wave solutions involve fractional derivatives giving rise to the theory of nonlinear fractional partial differential equations (NFPDEs). These solutions are stationary in phase, and their form and speed remain intact while traveling, thereby offering organization where unexpected motion prevails. Therefore, they are of crucial importance in the theoretical analysis and in practical complex application contexts [36, 37]. Travelling waves are basic signal delivery system bearers of essential details in other network functions or environments. For the discovery of new traveling wave solutions, various advanced techniques and strategies are established to study and control such competitive systems [38–40].

The Riccati-Bernoulli sub-ordinary differential equation (sub-ODE) method used in conjunction with Bäcklund transformations has been identified as an efficient and competent approach for obtaining new solutions of NLFDEs [41–43]. This makes it possible to easily model and explore analytical solutions of a number of NLFDEs involving fractional derivatives. However, to the authors knowledge, an application of the Riccati-Bernoulli sub-ODE method has not been used for solving the space-time FWBK equation. Space-time FWBK equation is a much considerable mathematical model for studying wave phenomenon in nonlinear and dispersive media with memory. The fractional derivative implies a new approach to describe the considered system and its response to certain anomalous and nonlocality, which is impossible in the framework of an integer-order model. All these characteristics are important in defining various physical processes in fluid dynamics, nonlinear optics, and many other processes. This study seeks to fill that gap by developing a wider range of closed form traveling wave solutions for this model. Using the Bäcklund transformation procedure we obtain a set of solutions of the kind which are expressed by means of trigonometrical, hyperbolic, and rational functions. Furthermore, by means of $3D$, $2D$, and contours representation, the physical aspects of the solutions are also provided in terms of amplitude, propagation characteristics, and structural responses. It is apparent that these results may have important applications in wave mechanics, fluid dynamics, and other fields of science and engineering. In addition, the conformable fractional derivative (CFD) is much simpler and more practically useful than the other fractional derivatives for describing initial and boundary conditions. Different from the other fractional derivatives like Riemann-Liouville or Caputo where many of them have some complicated expressions and nonlocality, the CFD offers a coherent and simple format that is reminiscent of calculus. This compatibility is important to the analytical handling of the FDEs while at the same time retaining properties such as the chain rule and Leibniz rule. While the CFD may lack flexibility in some of the more sophisticated modeling applications, the flexibility that it does offer, combined with the ease with which calculations can be performed and the physical interpretation of the results, makes it far more suitable for the goals of this paper. These features allow for a clear and efficient investigation of the fractional space-time dynamics of FWBK, thus providing the foundation for the analysis of wave behavior in nonlinear and dispersive systems. The CFD has been proposed by Khalil et al. [44] in 2014 with the following definitions in order to give a more accurate and efficient approach to fractional order system modeling due to its closer resemblance with the Newtonian derivative.

$$T^{\alpha}(f)(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon}, \quad 0 < \alpha \leq 1. \tag{1.1}$$

$$\begin{cases} T^{\alpha}(af + bg) = aT^{\alpha}(f) + bT^{\alpha}(g). \\ T^{\alpha}(fg)(x) = f(x)T^{\alpha}(g)(x) + g(x)T^{\alpha}(f)(x). \\ T^{\alpha}\left(\dfrac{f}{g}\right)(x) = \dfrac{g(x)T^{\alpha}(f)(x) - f(x)T^{\alpha}(g)(x)}{g(x)^2}. \\ T^{\alpha}(f(g(x))) = f'(g(x))T^{\alpha}(g)(x). \end{cases} \tag{1.2}$$

$$\begin{cases} T^{\alpha}(x^n) = nx^{n-\alpha}, \quad n \in \mathbb{R},\ x > 0. \\ T^{\alpha}(e^{kx}) = kx^{1-\alpha}e^{kx}, \quad k \in \mathbb{R}. \\ T^{\alpha}(\sin(kx)) = kx^{1-\alpha}\cos(kx). \\ T^{\alpha}(\cos(kx)) = -kx^{1-\alpha}\sin(kx). \\ T^{\alpha}(C) = 0, \quad C \in \mathbb{R}. \end{cases} \tag{1.3}$$

$$\begin{cases} I^{\alpha}[f(x)] = \displaystyle\int_a^x \tau^{\alpha-1} f(\tau)\, d\tau, \quad 0 < \alpha \leq 1. \\ I^{\alpha}[af(x) + bg(x)] = aI^{\alpha}[f(x)] + bI^{\alpha}[g(x)]. \\ T^{\alpha}(I^{\alpha}[f(x)]) = f(x). \\ \displaystyle\int_a^b \tau^{\alpha-1} f(\tau)T^{\alpha}[g(\tau)]\, d\tau = [\tau^{\alpha-1} f(\tau)g(\tau)]_a^b - \int_a^b g(\tau)T^{\alpha}[f(\tau)]\, d\tau. \\ I^{\alpha}[x^n] = \dfrac{x^{n+\alpha}}{n+\alpha}, \quad n + \alpha \neq 0. \end{cases} \tag{1.4}$$

This method removes a number of shortcomings observed in previous definitions of fractional derivative making the former suitable for solving NFPDEs. In my study, the CFD is used to derive the traveling wave solutions of space-time FWBK equation in terms of trigonometric, hyperbolic, and rational functions. With the use of Riccati-Bernoulli sub-ODE method plus Bäcklund transformations, I derived solutions that provide more understanding on the wave phenomenon especially on the effect of fractional order variables. Due to the CFD solution adopted in this paper, the waves amplitude, and propagation, as well as the responses of the structures, are described with higher precision and plotted in $3D$, $2D$, and contour views. However, these findings go far beyond theoretical results with applying the theoretical advancement to engineering and technology sectors, such as fluid dynamics, waves physics, and engineering.

2. Methodology

In this section, we provide a brief overview of the pertinent theoretical foundation of the Riccati-Bernoulli sub-ODE method: the general approach. Let us consider the general form of a fractional partial differential equation (FPDE), given as:

$$Z\left(z, D_t^{\alpha} z, D_x^{\alpha} z, D_t^{2\alpha} z, D_x^{2\alpha} z, \ldots\right) = 0. \tag{2.1}$$

For convenience in the integration process Eq (2.1) is transformed into the form of a nonlinear ordinary differential equation, with help of the transformation, $G(\psi) = g(t, x_1, x_2, x_3, \ldots, x_n)$.

$$F\left(f, \frac{df}{d\psi}, \frac{d^2 f}{d\psi^2}, \frac{d^3 f}{d\psi^3}, \ldots\right) = 0. \tag{2.2}$$

The variable $\psi = \psi(t, x_1, x_2, x_3, \ldots, x_n)$, can be parameterized into different forms depending with the specifications of the problem. We assume the formal solution form for Eq (2.2):

$$F(x,t) = f(\psi) = \sum_{j=-n}^{n} a_j H(\zeta)^j, \tag{2.3}$$

where constants (a_i) are decided with $a_n \neq 0$ or $a_{-n} \neq 0$ and $H(\zeta)$ is derived from a Bäcklund transformation:

$$H(\zeta) = \frac{-Wk_2 + k_1 S(\zeta)}{k_1 + k_2 S(\zeta)}, \tag{2.4}$$

where W, k_1, and k_2 are constants with $(k_2 \neq 0)$ and $H(\zeta)$ satisfies the Riccati equation:

$$\frac{d\phi}{d\xi} = S + \phi(\zeta)^2. \tag{2.5}$$

The balance parameter (n) in Eq (2.3) is determined from the highest-order derivative and the leading nonlinear term, from Eq (2.2). This minimizes any prejudice developed in the growth of one part of the equation while creating a perfect balance of the linear and nonlinear aspects. If Eq (2.3) is put into Eq (2.2) or into its integral equivalent, a string of terms containing $\phi(\zeta)$ to the various powers arises. Comparing the coefficients of these terms we get the system of algebraic equations with respect to (a_i) and other parameters. The obtained equation can be compared to that of an ideal behaviour for the constant coefficients of a stable system to achieve an ideal constant coefficients. Analyzing the situation, we employ a computational assistant, such as Maple, to determine the actual values of the unknown coefficients and parameters in the system. By replacing (a_i) and the other parameters in

Eq (2.3) and putting the general solution $\phi(\zeta)$ of Eq (2.3) into the original FPDE Eq (2.1), travelling wave solutions are derived. The emerging families of travelling wave solutions are listed below [45]:

$$\phi(\zeta)=\begin{cases}-\sqrt{-W}\tanh(\sqrt{-W}\zeta), & \text{as } W<0,\\ -\sqrt{-W}\coth(\sqrt{-W}\zeta), & \text{as } W<0,\end{cases}$$
$$\phi(\zeta)=-\frac{1}{\zeta}, \quad \text{as } W=0, \tag{2.6}$$
$$\phi(\zeta)=\begin{cases}\sqrt{\zeta}\tan(\sqrt{W}\zeta), & \text{as } W>0,\\ -\sqrt{W}\cot(\sqrt{W}\zeta), & \text{as } W>0.\end{cases}$$

The generalize trigonometric and hyperbolic functions are expressed as:

$$\tanh_{uv}(\zeta)=\frac{ue^{\zeta}-ve^{-\zeta}}{ue^{\zeta}+ve^{-\zeta}},$$
$$\coth_{uv}(\zeta)=\frac{ue^{\zeta}+ve^{-\zeta}}{ue^{\zeta}-ve^{-\zeta}},$$
$$\tan_{uv}(\zeta)=-i\frac{ue^{i\zeta}-ve^{-i\zeta}}{ue^{i\zeta}+ve^{-i\zeta}},$$
$$\cot_{uv}(\zeta)=i\frac{ue^{i\zeta}+ve^{-i\zeta}}{ue^{i\zeta}-ve^{-i\zeta}},$$

where u and v are positive constants.

3. Execution of the problem

Fractional calculus and notably the FPDEs are fundamental for describing real-life processes, which provide the capability of observing a small change in flow within different systems. Sun et al. [46] discuss the importance of FPDEs and the areas to which they can be applied, including biology, signal processing, electrical engineering, and control systems. In fact, the analysis of these applications often yields analytical solutions that provide quantitative descriptions of qualitative aspects of the processes involved. The Riccati-Bernoulli sub-ODE along with Bäcklund transformation method is an effective procedure to search for analytical solutions for FPDEs and a lot of equations has been studied in this context. In our current study, we get closed form solutions for one of the basic fractional system connected with water wave motion. More precisely, we consider the space-time FWBK equations first defined by Whitham [47] and later slightly modified by Broer [48] and Kaup [49], presented in the following form:

$$\begin{aligned}&D_t^{\alpha}F+D_x^{\alpha}G+FD_x^{\alpha}F+\mu D_x^{2\alpha}F=0, \qquad 0\leq\alpha<1,\\ &D_t^{\alpha}G+D_x^{\alpha}(FG)+\lambda D_x^{3\alpha}F-\mu D_x^{2\alpha}G=0.\end{aligned} \tag{3.1}$$

The values that α, μ and λ define in the space-time FWBK equation also define the nature of the waves. The fractional order parameter α controls the extent of non-locality and memory effects in both time and space, smaller values of α correspond to strong memory effects and can consequently affect steepness, stability, and wave propagation of the wave. The parameter μ defines the degree of the second order spatial dispersion, which defines the behavior of the wave, its spread, and smoothening;

thus, the higher value of μ, the stronger the dispersion is the bigger the influence on the wave structure is. Finally, λ is related to third-order spatial dispersion, which characterizes the impact of other high-order wave interactions and plays a role in wave breaking, instabilities, and nonlinearity of the system. All these parameters allow the model to capture more detailed wave dynamics in the presence of fractional effects. Indeed, by adopting the traveling wave transformation $F(x,t) = f(\psi)$, where $\psi = \frac{x^\alpha}{\alpha} - \frac{\omega t^\alpha}{\alpha}$, the governing Eq (3.1) is reformulated as it follows, and it is possible to go to a less reformulated form of the whole system.

$$\begin{aligned} -\omega\frac{df}{d\psi} + \frac{dg}{d\psi} + f\frac{df}{d\psi} + \mu\frac{d^2 f}{d\psi^2} = 0. \\ -\omega\frac{dg}{d\psi} + \frac{d(f.g)}{d\psi} + \lambda\frac{d^3 f}{d\psi^3} - \mu\frac{d^2 g}{d\psi^2} = 0. \end{aligned} \tag{3.2}$$

The operation carried out to Eq (3.2) leads to a nonlinear ordinary differential equation as shown above. When this equation has been integrated with respect to (ψ) taking the constant of integration as zero the following form is found with reference to (ψ).

$$\begin{aligned} g = \omega f - \frac{1}{2}(f)^2 - \mu\frac{df}{d\psi} = 0. \\ -\omega g + (f.g) + \lambda\frac{d^2 f}{d\psi^2} - \mu\frac{dg}{d\psi} = 0. \end{aligned} \tag{3.3}$$

By substituting the first part of Eq (3.3), we get the following result:

$$-\omega^2 f + \frac{3}{2}\omega(f)^2 - \frac{1}{2}(f)^3 + \left(\mu + \lambda^2\right)\frac{d^2 f}{d\psi^2} = 0. \tag{3.4}$$

To find the value of the balance number (n), we use the homogeneous balance method

$$D\left[\frac{d^q f}{d\zeta^q}\right] = n + q$$

and

$$D\left[f^p\left(\frac{d^q f}{d\zeta^q}\right)^s\right] = np + s(q+n)$$

for Eq (3.4) and obtain the value ($n = 1$). By applying the solution procedure discussed above, we derive the corresponding algebraic system.

$$\begin{aligned}
&\phi^0 : 4\mu^2 a_{-1} {k_2}^6 W^2 + 4\lambda a_{-1} {k_2}^6 W^2 - {a_{-1}}^3 {k_2}^6 = 0, \\
&\phi^1 : -3\omega {a_{-1}}^2 {k_2}^6 W + 3 {a_{-1}}^2 {k_2}^6 a_0 W = 0, \\
&\phi^2 : -3 {a_{-1}}^2 {k_2}^6 a_1 W^2 + 4\mu^2 a_{-1} {k_2}^6 W^3 + 4\lambda a_{-1} {k_2}^6 W^3 \\
&\qquad - 2\omega^2 a_{-1} {k_2}^6 W^2 - 3 a_{-1} {k_2}^6 {a_0}^2 W^2 + 6\omega a_{-1} {k_2}^6 a_0 W^2 = 0, \\
&\phi^3 : 6 a_{-1} {k_2}^6 a_0 a_1 W^3 - 6\omega a_{-1} {k_2}^6 a_1 W^3 + {a_0}^3 W^3 {k_2}^6 - 3\omega {a_0}^2 W^3 {k_2}^6 + 2\omega^2 a_0 W^3 {k_2}^6 = 0, \\
&\phi^4 : 4\mu^2 a_1 {k_2}^6 W^5 + 4\lambda a_1 {k_2}^6 W^5 - 3 a_{-1} {k_2}^6 {a_1}^2 W^4 \\
&\qquad + 6\omega a_0 a_1 W^4 {k_2}^6 - 2\omega^2 a_1 W^4 {k_2}^6 - 3 {a_0}^2 a_1 W^4 {k_2}^6 = 0, \\
&\phi^5 : -3\omega {a_1}^2 W^5 {k_2}^6 + 3 a_0 {a_1}^2 W^5 {k_2}^6 = 0, \\
&\phi^6 : 4\lambda a_1 {k_2}^6 W^6 + 4\mu^2 a_1 {k_2}^6 W^6 - {a_1}^3 W^6 {k_2}^6 = 0.
\end{aligned} \tag{3.5}$$

The equations resulted throughout this process form a system of algebraic equations, and they can be solved using computational assistance, such as Maple, Mathematica, MATLAB or any other comparable software assistance. This allows for the identification of several distinct solution cases, which are presented as follows:

Case 1.

$$a_1 = 2\sqrt{\mu^2+\lambda}, \omega = a_0, a_0 = a_0, a_{-1} = 1/4\,\frac{{a_0}^2}{\sqrt{\mu^2+\lambda}}, W = 1/8\,\frac{{a_0}^2}{\mu^2+\lambda}. \tag{3.6}$$

Case 2.

$$\begin{aligned} &a_1 = 0, a_0 = -i\sqrt[4]{4\,{a_{-1}}^2\mu^2 + 4\,{a_{-1}}^2\lambda}, a_{-1} = a_{-1},\\ &\omega = -i\sqrt[4]{4\,{a_{-1}}^2\mu^2 + 4\,{a_{-1}}^2\lambda}, W = 1/2\,\frac{\sqrt{{a_{-1}}^2\mu^2 + {a_{-1}}^2\lambda}}{\mu^2+\lambda}. \end{aligned} \tag{3.7}$$

The two cases developed from the algebraic system offer conceptualization for recognizing and categorizing diverse sets of solutions relative to (W). The observed behaviors reveal the temporal evolution of the system and the corresponding physical dynamics depending on fractional order parameter. In addition, these families include all interactions in the system and show how (W) influences travelling behavior and stability. This analysis provides important information regarding the dynamics of the flow and the resulting waves, with important consequences for the behavior of nonlinear systems.

Family 1. The following travelling wave solutions are obtained for Case 1 and $M < 0$.

$$\begin{aligned} F_1(x,t) = 1/4\,&\frac{{a_0}^2\left(k_1 - 1/4\,k_2\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\tanh\left(1/4\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)}{\frac{1}{\sqrt{\mu^2+\lambda}}\left(-1/8\,\frac{{a_0}^2 k_2}{\mu^2+\lambda} - 1/4\,k_1\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\tanh\left(1/4\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)} + a_0\\ &+\frac{2\sqrt{\mu^2+\lambda}\left(-1/8\,\frac{{a_0}^2 k_2}{\mu^2+\lambda} - 1/4\,k_1\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\tanh\left(1/4\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)}{\left(k_1 - 1/4\,k_2\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\tanh\left(1/4\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)} \end{aligned} \tag{3.8}$$

or

$$\begin{aligned} F_2(x,t) = 1/4\,&\frac{{a_0}^2\left(k_1 - 1/4\,k_2\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\coth\left(1/4\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)}{\frac{1}{\sqrt{\mu^2+\lambda}}\left(-1/8\,\frac{{a_0}^2 k_2}{\mu^2+\lambda} - 1/4\,k_1\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\coth\left(1/4\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)} + a_0\\ &+\frac{2\sqrt{\mu^2+\lambda}\left(-1/8\,\frac{{a_0}^2 k_2}{\mu^2+\lambda} - 1/4\,k_1\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\coth\left(1/4\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)}{\left(k_1 - 1/4\,k_2\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\coth\left(1/4\sqrt{-2\,\frac{{a_0}^2}{\mu^2+\lambda}}\left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)}. \end{aligned} \tag{3.9}$$

Family 2. The following travelling wave solutions are obtained for Case 1 and $M > 0$.

$$F_3(x,t) = 1/4\, \frac{{a_0}^2 \left(k_1 + 1/4\, k_2 \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \tan\left(1/4\, \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)}{\frac{1}{\sqrt{\mu^2+\lambda}} \left(-1/8\, \frac{{a_0}^2 k_2}{\mu^2+\lambda} + 1/4\, k_1 \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \tan\left(1/4\, \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)} + a_0$$
$$+ \frac{2\, \sqrt{\mu^2+\lambda} \left(-1/8\, \frac{{a_0}^2 k_2}{\mu^2+\lambda} + 1/4\, k_1 \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \tan\left(1/4\, \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)}{\left(k_1 + 1/4\, k_2 \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \tan\left(1/4\, \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)} \tag{3.10}$$

or

$$F_4(x,t) = 1/4\, \frac{{a_0}^2 \left(k_1 - 1/4\, k_2 \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \cot\left(1/4\, \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)}{\frac{1}{\sqrt{\mu^2+\lambda}} \left(-1/8\, \frac{{a_0}^2 k_2}{\mu^2+\lambda} - 1/4\, k_1 \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \cot\left(1/4\, \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)} + a_0$$
$$+ \frac{2\, \sqrt{\mu^2+\lambda} \left(-1/8\, \frac{{a_0}^2 k_2}{\mu^2+\lambda} - 1/4\, k_1 \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \cot\left(1/4\, \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)}{\left(k_1 - 1/4\, k_2 \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \cot\left(1/4\, \sqrt{2} \sqrt{\frac{{a_0}^2}{\mu^2+\lambda}} \left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)\right)\right)}. \tag{3.11}$$

Family 3. The following travelling wave solutions are obtained for Case 1 and $M = 0$.

$$F_5(x,y,t) = 1/4\, \frac{{a_0}^2 \left(k_1 - k_2 \left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)^{-1}\right)}{\frac{1}{\sqrt{\mu^2+\lambda}} \left(-1/8\, \frac{{a_0}^2 k_2}{\mu^2+\lambda} - k_1 \left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)^{-1}\right)}$$
$$+ a_0 + \frac{2\, \sqrt{\mu^2+\lambda} \left(-1/8\, \frac{{a_0}^2 k_2}{\mu^2+\lambda} - k_1 \left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)^{-1}\right)}{\left(k_1 - k_2 \left(\frac{x^\alpha}{\alpha} - \frac{a_0 t^\alpha}{\alpha}\right)^{-1}\right)}. \tag{3.12}$$

Family 4. The following travelling wave solutions are obtained for Case 2, $M < 0$ and

$$\psi = \frac{x^\alpha}{\alpha} + \frac{i \sqrt[4]{4\, {a_{-1}}^2 \mu^2 + 4\, {a_{-1}}^2 \lambda}\, t^\alpha}{\alpha}.$$

$$F_6(x,t) = \frac{a_{-1} \left(k_1 - 1/2\, k_2 \sqrt{-2\, \frac{\sqrt{{a_{-1}}^2 \mu^2 + {a_{-1}}^2 \lambda}}{\mu^2+\lambda}} \tanh\left(1/2\, \sqrt{-2\, \frac{\sqrt{{a_{-1}}^2 \mu^2 + {a_{-1}}^2 \lambda}}{\mu^2+\lambda}} \psi\right)\right)}{\left(-1/2\, \frac{\sqrt{{a_{-1}}^2 \mu^2 + {a_{-1}}^2 \lambda}\, k_2}{\mu^2+\lambda} - 1/2\, k_1 \sqrt{-2\, \frac{\sqrt{{a_{-1}}^2 \mu^2 + {a_{-1}}^2 \lambda}}{\mu^2+\lambda}} \tanh\left(1/2\, \sqrt{-2\, \frac{\sqrt{{a_{-1}}^2 \mu^2 + {a_{-1}}^2 \lambda}}{\mu^2+\lambda}} \psi\right)\right)}$$
$$- i \sqrt[4]{4\, {a_{-1}}^2 \mu^2 + 4\, {a_{-1}}^2 \lambda} \tag{3.13}$$

or

$$F_7(x,t) = \frac{a_{-1}\left(k_1 - 1/2\,k_2\sqrt{-2\,\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}}{\mu^2+\lambda}}\coth\left(1/2\,\sqrt{-2\,\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}}{\mu^2+\lambda}}\psi\right)\right)}{\left(-1/2\,\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}k_2}{\mu^2+\lambda} - 1/2\,k_1\sqrt{-2\,\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}}{\mu^2+\lambda}}\coth\left(1/2\,\sqrt{-2\,\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}}{\mu^2+\lambda}}\psi\right)\right)} - i\sqrt[4]{4\,a_{-1}{}^2\mu^2 + 4\,a_{-1}{}^2\lambda}. \tag{3.14}$$

Family 5. The following travelling wave solutions are obtained for Case 2 and $M > 0$.

$$F_8(x,t) = \frac{a_{-1}\left(k_1 + 1/2\,k_2\sqrt{2}\sqrt{\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}}{\mu^2+\lambda}}\tan\left(1/2\,\sqrt{2}\sqrt{\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}}{\mu^2+\lambda}}\psi\right)\right)}{\left(-1/2\,\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}k_2}{\mu^2+\lambda} + 1/2\,k_1\sqrt{2}\sqrt{\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}}{\mu^2+\lambda}}\tan\left(1/2\,\sqrt{2}\sqrt{\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}}{\mu^2+\lambda}}\psi\right)\right)} - i\sqrt[4]{4\,a_{-1}{}^2\mu^2 + 4\,a_{-1}{}^2\lambda} \tag{3.15}$$

or

$$F_9(x,t) = \frac{a_{-1}\left(k_1 - 1/2\,k_2\sqrt{2}\sqrt{\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}}{\mu^2+\lambda}}\cot\left(1/2\,\sqrt{2}\sqrt{\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}}{\mu^2+\lambda}}\psi\right)\right)}{\left(-1/2\,\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}k_2}{\mu^2+\lambda} - 1/2\,k_1\sqrt{2}\sqrt{\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}}{\mu^2+\lambda}}\cot\left(1/2\,\sqrt{2}\sqrt{\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}}{\mu^2+\lambda}}\psi\right)\right)} - i\sqrt[4]{4\,a_{-1}{}^2\mu^2 + 4\,a_{-1}{}^2\lambda}. \tag{3.16}$$

Family 6: The following travelling wave solutions are obtained for Case 2 and ($M = 0$).

$$F_{10}(x,t) = \frac{a_{-1}\left(k_1 - \frac{k_2}{\psi}\right)}{\left(-1/2\,\frac{\sqrt{a_{-1}{}^2\mu^2+a_{-1}{}^2\lambda}k_2}{\mu^2+\lambda} - \frac{k_1}{\psi}\right)} - i\sqrt[4]{4\,a_{-1}{}^2\mu^2 + 4\,a_{-1}{}^2\lambda}. \tag{3.17}$$

4. Results and discussion

This work proposes a new approach to study the space-time FWBK equation, which plays an important role in wave dynamics and stability in fluid dynamics and wave mechanics. By applying the Riccati-Bernoulli sub-ODE approach in combination with the Bäcklund transformation, the study finds the analytical solutions in terms of hyperbolic, rational, and trigonometric functions which characterizes different types of waves. The solutions are given in the form of traveling waves where temporal and spatial behavior can be seen periodically implemented in $3D$, contour, and $2D$ plots to explain the involvement of fractional parameters. The outcome shows that adjusting the fractional order parameter raises the wave steepness cut-off even more, which gives the wave profile a sharper fragmentation and fractality in contour diagrams. Besides, this research improves the general theoretical knowledge of wave mechanics by using a fractional calculus approach and sheds light on the behaviour of periodic travelling waves that are important in analyzing various aspects in ocean engineering, hydrodynamics, and nonlinear optics. The developed insights help to realize the capability of waves in fractional systems and their application in intricate conditions.

As observed from Table 1, the present method based on Riccati-Bernoulli sub-ODE and Bäcklund transformation has better efficiency than the SEM in solving FWBK equation. In particular, the Riccati-Bernoulli method produces a rich set of solutions in hyperbolic, rational, and trigonometric forms to describe the oscillation of the fractional systems for $W < 0$, $W > 0$, and $W = 0$, respectively. However, SEM provides primarily only hyperbolic-type solutions and utilizes the balance principle; therefore, it cannot be applied to more complex parameters. In addition, the envisaged Riccati-Bernoulli method is more sensitive to the values of fractional-order parameters to offered deeper understanding of the wave steepness and stability. This has made the Riccati-Bernoulli approach more reliable and fit to be applied on different nonlinear fractional system, as observed in the comparative study. The analytical solution of the space-time FWBK equation for the case of the traveling wave is given in $3D$, $2D$, and contour plots with a peak in Figure 1. The integer-order wave structure is illustrative of a stable, perfect symmetry as manifested in the $3D$ plot. Also, it is clear from the 2D plots that as the fractional order parameter (α) increases, the steepness of the wave and its localization increases, signifying its bright wave nature. Besides, the contour plot shows the detailed patterns of the solution with the curve fractal patterns. These dynamics make the solution highly relevant for use in ocean engineering, hydrodynamics, nonlinear optics, meteorology, and biomedical engineering especially in terms of steep localized and fractal wave behavior.

Table 1. Comparison of the space-time FWBK equation with the alternative approach, specifically the SEM [15].

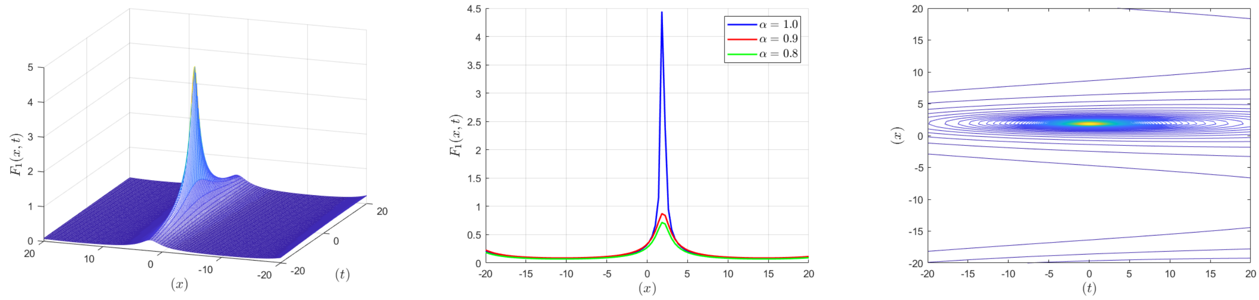
Case	Riccati-Bernoulli sub-ODE along with the Bäcklund transformation	SEM
Case I: $W < 0$	$F_6 = \frac{a_{-1}\left(k_1-1/2\,k_2\sqrt{-2\,\frac{\sqrt{{a_{-1}}^2\mu^2+{a_{-1}}^2\lambda}}{\mu^2+\lambda}}\tanh\left(1/2\sqrt{-2\,\frac{\sqrt{{a_{-1}}^2\mu^2+{a_{-1}}^2\lambda}}{\mu^2+\lambda}}\psi\right)\right)}{\left(-1/2\,\frac{\sqrt{{a_{-1}}^2\mu^2+{a_{-1}}^2\lambda}k_2}{\mu^2+\lambda}-1/2\,k_1\sqrt{-2\,\frac{\sqrt{{a_{-1}}^2\mu^2+{a_{-1}}^2\lambda}}{\mu^2+\lambda}}\tanh\left(1/2\sqrt{-2\,\frac{\sqrt{{a_{-1}}^2\mu^2+{a_{-1}}^2\lambda}}{\mu^2+\lambda}}\psi\right)\right)}$ $-i\sqrt[4]{4\,{a_{-1}}^2\mu^2+4\,{a_{-1}}^2\lambda}$,	$u_{W1} = 2\sqrt{\gamma+\beta^2}\frac{\frac{c}{\frac{c}{\sqrt{\gamma+\beta^2}}}c_1e^{\frac{c}{\sqrt{\gamma+\beta^2}}\xi}}{c_0+c_1e^{\frac{c}{\sqrt{\gamma+\beta^2}}\xi}}$ $= \frac{2cc_1e^{\frac{c}{\sqrt{\gamma+\beta^2}}(x-ct)}}{c_0+c_1e^{\frac{c}{\sqrt{\gamma+\beta^2}}(x-ct)}}$.
Case II: $W > 0$	$F_9 = \frac{a_{-1}\left(k_1-1/2\,k_2\sqrt{2}\sqrt{\frac{\sqrt{{a_{-1}}^2\mu^2+{a_{-1}}^2\lambda}}{\mu^2+\lambda}}\cot\left(1/2\,\sqrt{2}\sqrt{\frac{\sqrt{{a_{-1}}^2\mu^2+{a_{-1}}^2\lambda}}{\mu^2+\lambda}}\psi\right)\right)}{\left(-1/2\,\frac{\sqrt{{a_{-1}}^2\mu^2+{a_{-1}}^2\lambda}k_2}{\mu^2+\lambda}-1/2\,k_1\sqrt{2}\sqrt{\frac{\sqrt{{a_{-1}}^2\mu^2+{a_{-1}}^2\lambda}}{\mu^2+\lambda}}\cot\left(1/2\,\sqrt{2}\sqrt{\frac{\sqrt{{a_{-1}}^2\mu^2+{a_{-1}}^2\lambda}}{\mu^2+\lambda}}\psi\right)\right)}$ $-i\sqrt[4]{4\,{a_{-1}}^2\mu^2+4\,{a_{-1}}^2\lambda}$,	$u_{W2} = \frac{2cc_0}{c_0+c_1e^{\frac{-c}{\sqrt{\gamma+\beta^2}}(x-ct)}}$.
Case III: $W = 0$	$F_{10} = \frac{a_{-1}\left(k_1-\frac{k_2}{\psi}\right)}{\left(-1/2\,\frac{\sqrt{{a_{-1}}^2\mu^2+{a_{-1}}^2\lambda}k_2}{\mu^2+\lambda}-\frac{k_1}{\psi}\right)}-i\sqrt[4]{4\,{a_{-1}}^2\mu^2+4\,{a_{-1}}^2\lambda}$,	$u_{W3} = \frac{2cc_0}{c_0+c_1e^{\frac{c}{\sqrt{\gamma+\beta^2}}(x-ct)}}$.

Figure 2 shows the traveling wave solution of the space-time FWBK equation in the form of $3D$, $2D$, and contour plots, as well which is a periodic solution with two peaks. The $3D$ plot ($\alpha = 1$) depicts central symmetric wave form with clear two peak structures signifying stable periodicity. Upon increasing the fractional order parameter (α) in the $2D$ plot, the extension of the corresponding peaks towards the base plane is obvious in addition to the steepness and localization of the waveform that appears to become sharper and denser. The contour plot represents diverse patterns that are again self-similar, representing the wave pattern they have. Such dual hump kinetics are useful in ocean engineering for simulating multi-crested waves, hydrodynamic systems, nonlinear optics, meteorology, and biomedical engineering, and also prove the efficiency of fractional models in analyzing erratic localized and fractal wave forms.

The periodical travelling wave solution of the FWBK equation in $3D$, $2D$, and contour plots with single peak is presented in Figure 3. The $3D$ plot of the solitary wave with an integer order indicates that the wave has stability with one major hump at the middle of the parabolic curve. As it can be noted in the $2D$ plot, the wave tightens and steepens as α rises with the focus on the sharp change in values. Similar to the contour plot also exhibit fractal like behavior and can be used to identify the complexities of the solution. Single-peak dynamics are useful for problems in solitary wave simulation, fluid dynamics, optics and photonics, meteorology, and biomedical engineering in order to demonstrate applicability of the fractional models for steep, narrow, and fractal wave motion analysis.

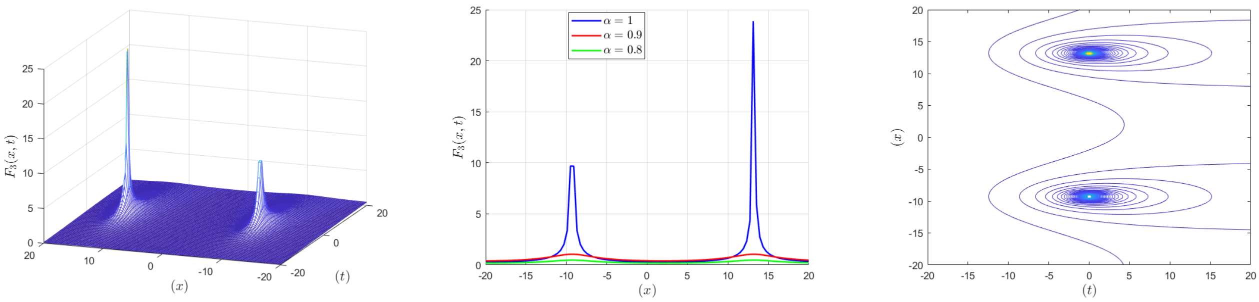
The periodic traveling wave solution of the space-time FWBK equation, in $3D$, $2D$, and contour plots with two peaks is shown in Figure 4. The integer-order $3D$ plot presents a stable profile with two characteristic maxima and it corresponds to a perpetual wave configuration. It is observed from the $2D$ plot that as the value of fractional order parameter (α) increases, both the peaks become steeper and are more localized, showing the transition in the wave profile is faster. The contour plot also shows more fractal shapes where there are many small patterns, giving the appearance of complexity in the solution, and self-similarity. Since these two-peak dynamics can be used the modeling of multi-crest waves, hydrodynamics, steep, and localized waves with fractal characteristics, in the area of ocean engineering, nonlinear optics, meteorology, and biomedical engineering, the presented fractional-order model can be of interest.

These results are not only theoretical but also hold enormous implication in many fields of engineering and technology. For example, the results can be used for fluid waves in turbulent flows, in oceanography for current and wave pattern analysis, and for investigating waves breaking. In wave physics, they help in explaining the non linear wave phenomena associated with optical fibers, water surface waves, and shock wave front. Furthermore, the results can be used in engineering to design efficient energy systems, to anticipate the wave patterns in seacoast constructions, and to improve the analysis of substances under dynamic loads. These applications demonstrate how the proposed theoretical developments will be helpful and applicable in tackling various problems.



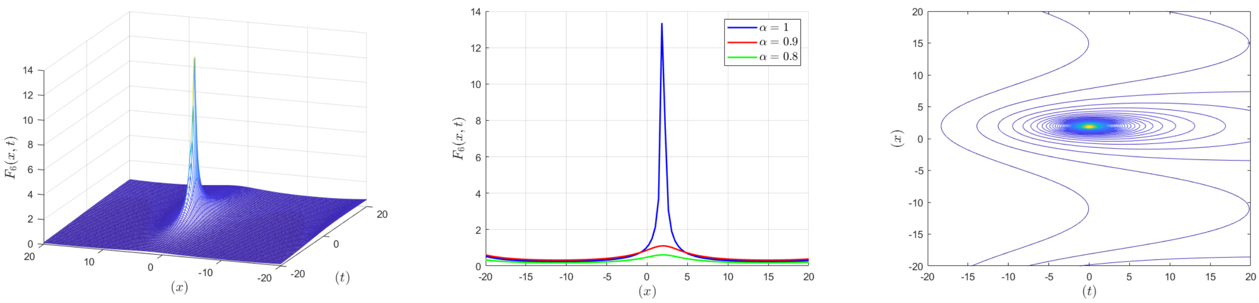
(a) **(b)** **(c)**

Figure 1. Figure showing $3D$, contour, and $2D$ fractional–order variations of the function $F_1(x,t)$.



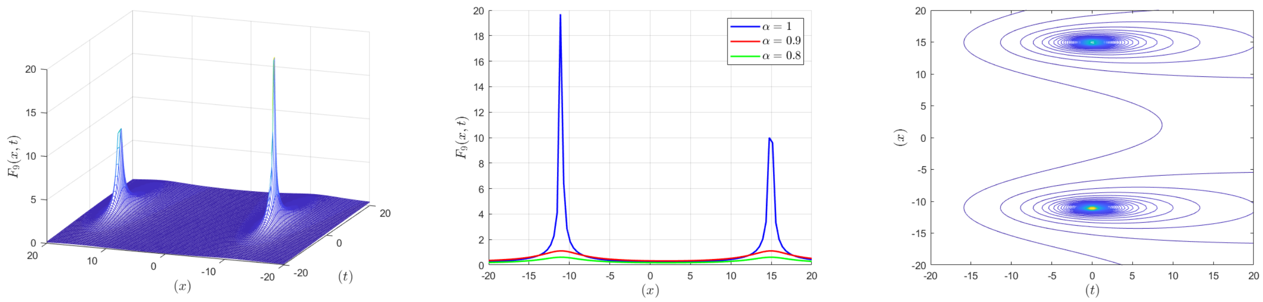
(a) **(b)** **(c)**

Figure 2. Figure showing $3D$, contour, and $2D$ fractional–order variations of the function $F_3(x,t)$.



(a) **(b)** **(c)**

Figure 3. Figure showing $3D$, contour, and $2D$ fractional–order variations of the function $F_6(x,t)$.



(a) **(b)** **(c)**

Figure 4. Figure showing $3D$, contour, and $2D$ fractional–order variations of the function $F_9(x,t)$.

5. Conclusions

Finally, this work also presents a new approach to solve the space-time FWBK equation using the Riccati-Bernoulli sub-ODE method and Bäcklund transformation. The obtained solutions in terms of hyperbolic, rational, and trigonometric functions clearly indicate how the fractional order control parameter affects the steepness of oscillating travelling waves. Eventually, the contour plots highlight the fractal appearance which appears as a result of the high fractional order parameter.

The results of this work contribute to the understanding of wave characteristics in nonlinear fractional systems and can be beneficial for the application in the area of ocean engineering, hydrodynamic, and nonlinear optics. The study not only helps to improve the existing knowledge regarding wave behaviors in the fractional systems but also points to the directions for further application.

Further work should be dedicated to expanding such an approach towards the closed systems with multiple nonlinear equations, studying bifurcation of solutions depending from the change of the fractional indices, as well as future investigations into the higher-order fractional differential equations. Moreover, this framework can be used to investigate multi-phase flow, wave breaking and shock wave propagation in fractional systems, which could be instrumental in material science and biomedical engineering applications.

Author contributions

H.G.: Conceptualization, Visualization, Funding, Data curation, Resources, Writing-review & editing; A.A.H.A.: Formal analysis, Project administration, Data curation, validation; A.H.H.: Investigation, Validation, Resources, Software. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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