Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 2466–2491.
DOI:10.3934/math.2025115
Received: 31 December 2024
Revised: 26 January 2025
Accepted: 08 February 2025
Published: 12 February 2025

Research article

A fractional Halanay inequality for neutral systems and its application to Cohen-Grossberg neural networks

Mohammed D. Kassim*

Department of Basic Engineering Sciences, College of Engineering, Imam Abdulrahman Bin Faisal University, P.O. Box 1982, Dammam 34151, Saudi Arabia

* **Correspondence:** Email: mdkassim@iau.edu.sa.

Abstract: We expand the Halanay inequality to accommodate fractional-order systems incorporating both discrete and distributed neutral delays. By establishing specific conditions, we demonstrate that the solutions of these systems converge to zero at a Mittag-Leffler rate. Our analysis is versatile, accommodating a wide range of delay kernels. This versatility extends the applicability of our findings to fractional Cohen-Grossberg neural networks, offering valuable insights into their stability and dynamical behavior.

Keywords: Halanay inequality; Cohen-Grossberg neural network system; Caputo fractional derivative; neutral delay; Mittag-Leffler stability
Mathematics Subject Classification: 92B20, 26A33, 34D20

1. Introduction

We generalize the classical Halanay inequality to encompass fractional-order systems with both discrete and distributed neutral delays. This inequality, originally formulated for integer-order systems, is now generalized to non-integer orders.

Lemma 1.1. *Consider a nonnegative function $w(t)$ that satisfies the inequality*

$$w'(t) \leq -K_1 w(t) + K_2 \sup_{t-\tau\leq s\leq t} w(s),\ t \geq a,$$

where $0 < K_2 < K_1$. Under these conditions, positive constants K_3 and K_4 exist such that

$$w(t) \leq K_3 e^{-K_4(t-a)},\ t \geq a.$$

Halanay first introduced this inequality while studying the stability of a specific differential equation [10]

$$\upsilon'(t) = -A\upsilon(t) + B\upsilon(t-\tau),\ \tau > 0.$$

Since then, the inequality has been generalized to include variable coefficients and delays of varying magnitude, both bounded and unbounded [1, 25, 26]. These generalizations have found applications in Hopfield neural networks and the analysis of Volterra functional equations, particularly in the context of problems described by the following system [12, 16, 27]:

$$\begin{cases} x_i'(t) = -c_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(t-\tau)) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + I_i, \ t > 0, \\ x_i(t) = \phi_i(t), \ -\tau \le t \le 0, \ i = 1, ..., n. \end{cases}$$

Such problems arise in various fields, including parallel computing, cryptography, image processing, combinatorial optimization, signal theory, and geology [15, 17, 18].

Additionally, a generalization of the Halanay inequality to systems with distributed delays is presented in [21]:

$$w'(x) \le -B(x)w(x) + A(x) \int_0^\infty k(s)w(x-s)\, ds, \ x \ge 0.$$

The solutions exhibit exponential decay if the kernels satisfy the conditions

$$\int_0^\infty e^{\beta s} k(s) ds < \infty,$$

for some $\beta > 0$, and

$$A(x) \int_0^\infty k(s)\, ds \le B(x) - C, \ C > 0, \ x \in R.$$

See also [22] for further details.

This study broadens the scope of Halanay's inequality to encompass fractional-order systems. The justification for using fractional derivatives is provided in [2,3]. We also consider neutral delays, where delays appear in the leading derivative. Specifically, we analyze the stability of the following problem:

$$\begin{cases} D_C^{\varphi,\alpha} \left[w(t) - p w(t-\upsilon) \right] \le -q w(t) + \int_a^t w(r) k(t-r)\, dr, \ p > 0, \ 0 < \alpha < 1, \ \upsilon, \ t > a, \\ w(t) = \varpi(t), \ a - \upsilon \le t \le a. \end{cases} \tag{1.1}$$

We establish sufficient conditions on the kernel k to guarantee Mittag-Leffler stability, ensuring that the solutions satisfy

$$w(t) \le A E_\alpha(-q \left[\varphi(t) - \varphi(a) \right]^\alpha), \ t > a.$$

We provide examples of function families that satisfy our assumptions. As an application, we consider a fractional-order Cohen-Grossberg neural network system with neutral delays [9]. This system represents a more general form of the traditional Hopfield neural network.

There is extensive research on the existence, stability, and long-term behavior of Cohen-Grossberg neural network systems. Our focus is on research that specifically addresses networks with time delays or fractional-order dynamics. For integer-order neutral Cohen-Grossberg systems, refer to [5, 7, 24]. The fractional case with discrete delays was explored in [14]. While the Halanay inequality has been adapted for fractional-order systems with discrete delays in [4, 11, 28], we are unaware of any work addressing our specific problem (1.1).

The techniques used for integer-order systems are not directly applicable to the fractional-order case. For example, the Mittag-Leffler functions lack the semigroup property, and estimating the expression $E_\alpha(-q(\varphi(t-\upsilon)-\varphi(a))^\alpha)/E_\alpha(-q(\varphi(t)-\varphi(a))^\alpha)$ is challenging for convergence analysis. The ideal decay rate would be $E_\alpha(-q(\varphi(t)-\varphi(a))^\alpha)$, but the neutral delay introduces new challenges, particularly near ν. Approximating with $(\varphi(t)-\varphi(a))^{-\alpha}$ (using Mainardi's conjecture) does not fully resolve these issues.

This paper is organized into eight sections, beginning with background information in Section 2. Section 3 presents our inequality for systems with discrete time delays, and Section 4 discusses two potential kernel functions. Section 5 investigates a fractional Halanay inequality in the presence of distributed neutral delays. Solutions of arbitrary signs for the problem in Section 3 are addressed in Section 6, and Section 7 applies our results to a Cohen-Grossberg system with neutral delays. Section 8 provides the conclusion, summarizing the findings and highlighting directions for future research.

2. Preliminaries

This section provides fundamental definitions and lemmas essential for the subsequent analysis. Throughout the paper, we consider $[a,b]$ to be an infinite or finite interval, and φ to be an n-continuously differentiable function on $[a,b]$ such that φ is increasing and $\varphi'(\varkappa)\neq 0$ on $[a,b]$.

Definition 2.1. *The φ-Riemann-Liouville fractional integral of a function ω with respect to a function φ is defined as*

$$I^{\varphi,\alpha}\omega(z)=\frac{1}{\Gamma(\alpha)}\int_a^z[\varphi(z)-\varphi(s)]^{\alpha-1}\omega(s)\varphi'(s)\,ds,\ \alpha>0,\ z>a$$

provided that the right side exists.

Definition 2.2. *The φ-Caputo derivative of order $\alpha>0$ is defined by*

$$D_C^{\varphi,\alpha}\omega(\varkappa)=I^{\varphi,n-\alpha}\left(\frac{1}{\varphi'(\varkappa)}\frac{d}{d\varkappa}\right)^n\omega(\varkappa),$$

which can be expressed equivalently as

$$D_C^{\varphi,\alpha}\omega(\varkappa)=\frac{1}{\Gamma(n-\alpha)}\int_a^\varkappa[\varphi(\varkappa)-\varphi(\tau)]^{n-\alpha-1}\varphi'(\tau)\omega_\varphi^{[n]}(\tau)\,d\tau,\ \varkappa>a,$$

where

$$\omega_\varphi^{[n]}(\varkappa)=\left(\frac{1}{\varphi'(\varkappa)}\frac{d}{d\varkappa}\right)^n\omega(\varkappa),\ n=-[-\alpha].$$

Particularly, when $0<\alpha<1$

$$\begin{aligned}D_C^{\varphi,\alpha}\omega(\varkappa) &= I^{\varphi,1-\alpha}\left(\frac{1}{\varphi'(\varkappa)}\frac{d}{d\varkappa}\right)\omega(\varkappa)\\ &= \frac{1}{\Gamma(1-\alpha)}\int_a^\varkappa[\varphi(\varkappa)-\varphi(\tau)]^{-\alpha}\omega'(\tau)\,d\tau.\end{aligned}$$

The Mittag-Leffler functions used in this context are defined as follows:

$$E_\alpha(y):=\sum_{n=0}^\infty\frac{y^n}{\Gamma(1+\alpha n)},\ \mathrm{Re}(\alpha)>0,$$

and

$$E_{\alpha,\beta}(y) := \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(\beta + \alpha n)}, \ \mathrm{Re}(\beta) > 0, \ \mathrm{Re}(\alpha) > 0.$$

Lemma 2.1. *[13] The Cauchy problem*

$$\begin{cases} D_C^{\varphi,\alpha} y(\zeta) = \lambda y(\zeta), \ 0 < \alpha \le 1, \ \zeta > a, \ \lambda \in \mathbb{R} \\ y(a) = y_a, \end{cases} \tag{2.1}$$

has the solution

$$y(\zeta) = y_a E_\alpha \left(\lambda \left[\varphi(\zeta) - \varphi(a)\right]^\alpha\right), \ \zeta \ge a.$$

Lemma 2.2. *[13] The Cauchy problem*

$$\begin{cases} D_C^{\varphi,\alpha} y(\zeta) = \lambda y(\zeta) + h(\zeta), \ 0 < \alpha \le 1, \ \lambda \in \mathbb{R}, \ \zeta > a, \\ y(a) = y_a \in \mathbb{R}, \end{cases} \tag{2.2}$$

admits the solution for $\zeta \ge a$

$$y(\zeta) = y_a E_\alpha \left(\lambda \left[\varphi(\zeta) - \varphi(a)\right]^\alpha\right) + \int_a^\zeta \left[\varphi(\zeta) - \varphi(s)\right]^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left[\varphi(\zeta) - \varphi(s)\right]^\alpha\right) \varphi'(s) h(s) \, ds.$$

Lemma 2.3. *For $\lambda, \ \nu, \ \omega > 0$, the following inequality is valid for all $z > a$:*

$$\int_a^z \left[\varphi(s) - \varphi(a)\right]^{\lambda-1} \left[\varphi(z) - \varphi(s)\right]^{\nu-1} e^{-\omega\left[\varphi(s)-\varphi(a)\right]} \varphi'(s) \, ds \le C \left[\varphi(z) - \varphi(a)\right]^{\nu-1},$$

where

$$C = \max\left\{1, 2^{1-\nu}\right\} \Gamma(\lambda) \left[1 + \lambda(\lambda+1)/\nu\right] \omega^{-\lambda}.$$

Proof. For $z > a$, let

$$I(z) = \left[\varphi(z) - \varphi(a)\right]^{1-\nu} \int_a^z \left[\varphi(s) - \varphi(a)\right]^{\lambda-1} \left[\varphi(z) - \varphi(s)\right]^{\nu-1} e^{-\omega\left[\varphi(s)-\varphi(a)\right]} \varphi'(s) \, ds.$$

Set $\xi \left[\varphi(z) - \varphi(a)\right] = \varphi(s) - \varphi(a)$. Then, $\left[\varphi(z) - \varphi(a)\right] d\xi = \varphi'(s) \, ds$ and

$$I(z) = \left[\varphi(z) - \varphi(a)\right]^\lambda \int_0^1 (1-\xi)^{\nu-1} \xi^{\lambda-1} e^{-\omega\xi\left[\varphi(z)-\varphi(a)\right]} d\xi, \ z > a.$$

As for $0 \le \xi < 1/2$, we have $(1-\xi)^{\nu-1} \le \max\left\{1, 2^{1-\nu}\right\}$, therefore

$$\begin{aligned} I(z) \le \ & \max\left\{1, 2^{1-\nu}\right\} \left[\varphi(z) - \varphi(a)\right]^\lambda \int_0^{1/2} \xi^{\lambda-1} e^{-\omega\xi\left[\varphi(z)-\varphi(a)\right]} d\xi \\ & + \left[\varphi(z) - \varphi(a)\right]^\lambda \int_{1/2}^1 (1-\xi)^{\nu-1} \xi^{\lambda-1} e^{-\omega\xi\left[\varphi(z)-\varphi(a)\right]} d\xi. \end{aligned} \tag{2.3}$$

Let $u = \omega\xi \left[\varphi(z) - \varphi(a)\right]$. Then, $d\xi = \left[\varphi(z) - \varphi(a)\right]^{-1} \omega^{-1} du$ and

$$\left[\varphi(z) - \varphi(a)\right]^\lambda \int_0^{1/2} \xi^{\lambda-1} e^{-\omega\xi\left[\varphi(z)-\varphi(a)\right]} d\xi \le \omega^{-\lambda} \int_0^\infty u^{\lambda-1} e^{-u} du = \omega^{-\lambda} \Gamma(\lambda). \tag{2.4}$$

If $1 \leq \omega\xi\left[\varphi(z)-\varphi(a)\right]$, then

$$e^{\omega\xi\left[\varphi(z)-\varphi(a)\right]} \geq \frac{\left[\omega\xi\left[\varphi(z)-\varphi(a)\right]\right]^{1+[\lambda]}}{\Gamma([\lambda]+2)} \geq \frac{\left[\omega\xi\left[\varphi(z)-\varphi(a)\right]\right]^{\lambda}}{\Gamma(\lambda+2)}.$$

Therefore, when $1/2 < \xi \leq 1$,

$$\xi^{\lambda-1}e^{-\omega\xi\left[\varphi(z)-\varphi(a)\right]} \leq \xi^{\lambda-1}\frac{\Gamma(2+\lambda)}{\left[\omega\xi\left[\varphi(z)-\varphi(a)\right]\right]^{\lambda}} \leq \frac{2\omega^{-\lambda}\Gamma(\lambda+2)}{\left[\varphi(z)-\varphi(a)\right]^{\lambda}},$$

and consequently

$$\begin{aligned}
& \left[\varphi(z)-\varphi(a)\right]^{\lambda}\int_{1/2}^{1}(1-\xi)^{\nu-1}\xi^{\lambda-1}e^{-\omega\xi\left[\varphi(z)-\varphi(a)\right]}d\xi \\
\leq\; & \left[\varphi(z)-\varphi(a)\right]^{\lambda}\int_{1/2}^{1}(1-\xi)^{\nu-1}\frac{2\omega^{-\lambda}\Gamma(2+\lambda)}{\left[\varphi(z)-\varphi(a)\right]^{\lambda}}d\xi \\
=\; & 2\omega^{-\lambda}\Gamma(2+\lambda)\int_{1/2}^{1}(1-\xi)^{\nu-1}d\xi = \frac{2^{1-\nu}\omega^{-\lambda}\Gamma(\lambda+2)}{\nu}.
\end{aligned}$$

When $\omega\xi\left[\varphi(z)-\varphi(a)\right] < 1$, it implies that $\left[\omega\xi\left[\varphi(z)-\varphi(a)\right]\right]^{\lambda} < 1 \leq e^{\omega\xi\left[\varphi(z)-\varphi(a)\right]}$. Consequently,

$$\begin{aligned}
& \left[\varphi(z)-\varphi(a)\right]^{\lambda}\int_{1/2}^{1}\xi^{\lambda-1}(1-\xi)^{\nu-1}e^{-\omega\xi\left[\varphi(z)-\varphi(a)\right]}d\xi \\
<\; & \left[\varphi(z)-\varphi(a)\right]^{\lambda}\int_{1/2}^{1}\xi^{\lambda-1}(1-\xi)^{\nu-1}\left[\omega\xi\left[\varphi(z)-\varphi(a)\right]\right]^{-\lambda}d\xi \\
<\; & 2\omega^{-\lambda}\int_{1/2}^{1}(1-\xi)^{\nu-1}d\xi = 2^{1-\nu}\frac{\omega^{-\lambda}}{\nu}.
\end{aligned} \tag{2.5}$$

Taking into account (2.3)–(2.5), we infer that

$$\begin{aligned}
I(z) &\leq \max\left\{1,2^{1-\nu}\right\}\omega^{-\lambda}\Gamma(\lambda) + \frac{2^{1-\nu}\omega^{-\lambda}\Gamma(\lambda+2)}{\nu} \\
&\leq \max\left\{1,2^{1-\nu}\right\}\omega^{-\lambda}\Gamma(\lambda)\left(1+\frac{\lambda(\lambda+1)}{\nu}\right),\ z>a.
\end{aligned}$$

The proof is complete.

□

Lemma 2.4. *[8, (4.4.10), (4.9.4)] For $\beta > 0$, $\nu > 0$, and $\lambda, \lambda^* \in \mathbb{C}$, $\lambda \neq \lambda^*$, we have*

$$\begin{aligned}
& \int_0^{\varkappa} z^{\beta-1}E_{\alpha,\beta}(\lambda z^{\alpha})(\varkappa-z)^{\nu-1}E_{\alpha,\nu}(\lambda^*(\varkappa-z)^{\alpha})dz \\
=\; & \frac{\lambda E_{\alpha,\beta+\nu}(\lambda\varkappa^{\alpha}) - \lambda^* E_{\alpha,\beta+\nu}(\lambda^*\varkappa^{\alpha})}{\lambda-\lambda^*}\varkappa^{\beta+\nu-1},
\end{aligned}$$

and for $\sigma > 0$, $\gamma > 0$,

$$I^{\sigma}z^{\gamma-1}E_{\alpha,\gamma}(pz^{\alpha})(\varkappa) = \varkappa^{\sigma+\gamma-1}E_{\alpha,\sigma+\gamma}(p\varkappa^{\alpha}).$$

Lemma 2.5. *For* $\beta > 0$, $\nu > 0$, *and* $\lambda, \lambda^* \in \mathbb{C}$, $\lambda \neq \lambda^*$, *we have*

$$\begin{aligned} & \int_a^{\varkappa} E_{\alpha,\beta}(\lambda\left[\varphi(z)-\varphi(a)\right]^{\alpha})\left[\varphi(\varkappa)-\varphi(z)\right]^{\nu-1}\left[\varphi(z)-\varphi(a)\right]^{\beta-1} \\ & \times E_{\alpha,\nu}(\lambda^*\left[\varphi(\varkappa)-\varphi(z)\right]^{\alpha})\varphi'(z)\,dz \\ = \; & \left[\varphi(\varkappa)-\varphi(a)\right]^{\beta+\nu-1}\frac{\lambda^* E_{\alpha,\beta+\nu}(\lambda^*\left[\varphi(\varkappa)-\varphi(a)\right]^{\alpha})-\lambda E_{\alpha,\beta+\nu}(\lambda\left[\varphi(\varkappa)-\varphi(a)\right]^{\alpha})}{\lambda^*-\lambda}, \end{aligned}$$

and for $\sigma > 0$, $\gamma > 0$,

$$\begin{aligned} I^{\varphi,\sigma}\left[\varphi(z)-\varphi(a)\right]^{\gamma-1}E_{\alpha,\gamma}(p\left[\varphi(z)-\varphi(a)\right]^{\alpha})(\varkappa) \; = \; & \left[\varphi(\varkappa)-\varphi(a)\right]^{\sigma+\gamma-1} \\ & \times E_{\alpha,\sigma+\gamma}(p\left[\varphi(\varkappa)-\varphi(a)\right]^{\alpha}). \end{aligned} \tag{2.6}$$

Proof. Let $u = \varphi(\varkappa) - \varphi(z)$. Then,

$$\begin{aligned} & \int_a^{\varkappa} E_{\alpha,\beta}(\lambda\left[\varphi(z)-\varphi(a)\right]^{\alpha})\left[\varphi(\varkappa)-\varphi(z)\right]^{\nu-1}\left[\varphi(z)-\varphi(a)\right]^{\beta-1} \\ & \times E_{\alpha,\nu}(\lambda^*\left[\varphi(\varkappa)-\varphi(z)\right]^{\alpha})\varphi'(z)\,dz \\ = \; & \int_0^{\varphi(\varkappa)-\varphi(a)} E_{\alpha,\beta}(\lambda\left[\varphi(\varkappa)-\varphi(a)-u\right]^{\alpha})\left[\varphi(\varkappa)-\varphi(a)-u\right]^{\beta-1}u^{\nu-1}E_{\alpha,\nu}(\lambda^* u^{\alpha})du. \end{aligned}$$

At this point, we can utilize Lemma 2.4 to derive the following:

$$\begin{aligned} & \int_a^{\varkappa} E_{\alpha,\beta}(\lambda\left[\varphi(z)-\varphi(a)\right]^{\alpha})\left[\varphi(\varkappa)-\varphi(z)\right]^{\nu-1}\left[\varphi(z)-\varphi(a)\right]^{\beta-1} \\ & \times E_{\alpha,\nu}(\lambda^*\left[\varphi(\varkappa)-\varphi(z)\right]^{\alpha})\varphi'(z)\,dz \\ = \; & \left[\varphi(\varkappa)-\varphi(a)\right]^{\beta+\nu-1}\frac{\lambda^* E_{\alpha,\beta+\nu}(\lambda^*\left[\varphi(\varkappa)-\varphi(a)\right]^{\alpha})-\lambda E_{\alpha,\beta+\nu}(\lambda\left[\varphi(\varkappa)-\varphi(a)\right]^{\alpha})}{\lambda^*-\lambda}. \end{aligned}$$

To prove the second formula in the lemma, we have

$$\begin{aligned} & I^{\varphi,\sigma}\left[\varphi(z)-\varphi(a)\right]^{\gamma-1}E_{\alpha,\gamma}(p\left[\varphi(z)-\varphi(a)\right]^{\alpha})(\varkappa) \\ = \; & \frac{1}{\Gamma(\sigma)}\int_a^{\varkappa} E_{\alpha,\gamma}(p\left[\varphi(z)-\varphi(a)\right]^{\alpha})\left[\varphi(\varkappa)-\varphi(z)\right]^{\sigma-1}\left[\varphi(z)-\varphi(a)\right]^{\gamma-1}\varphi'(z)\,dz. \end{aligned}$$

From the first formula in the lemma, with $\beta = \gamma$, $\nu = \sigma$, $\lambda = p$, $\lambda^* = 0$, we obtain

$$\begin{aligned} & I^{\varphi,\sigma}\left[\varphi(z)-\varphi(a)\right]^{\gamma-1}E_{\alpha,\gamma}(p\left[\varphi(z)-\varphi(a)\right]^{\alpha})(\varkappa) \\ = \; & \frac{1}{\Gamma(\sigma)}\int_a^{\varkappa}\left[\varphi(z)-\varphi(a)\right]^{\gamma-1}E_{\alpha,\gamma}(p\left[\varphi(z)-\varphi(a)\right]^{\alpha})\left[\varphi(\varkappa)-\varphi(z)\right]^{\sigma-1}\varphi'(z)\,dz \\ = \; & \left[\varphi(\varkappa)-\varphi(a)\right]^{\gamma+\sigma-1}E_{\alpha,\gamma+\sigma}(p\left[\varphi(\varkappa)-\varphi(a)\right]^{\alpha}), \end{aligned}$$

where we have used

$$E_{\alpha,\sigma}(\lambda^*\left[\varphi(\varkappa)-\varphi(z)\right]^{\alpha}) = \frac{1}{\Gamma(\sigma)}.$$

□

Mainardi's conjecture. [19] For fixed γ with $0 < \gamma < 1$, the following holds:

$$\frac{1}{1+q\Gamma(1-\gamma)t^{\gamma}} \leq E_{\gamma}(-qt^{\gamma}) \leq \frac{1}{q\Gamma(1+\gamma)^{-1}t^{\gamma}+1}, \; q, \; t \geq 0. \tag{2.7}$$

This result was later established in [6, 23].

3. Halanay inequality for fractional-order systems with both discrete neutral delays and distributed delays

To start, we will introduce the concept of Mittag-Leffler stability.

Definition 3.1. *For $0 < \alpha < 1$, a solution $v(z)$ is defined as α-Mittag-Leffler stable if there exist positive constants A and γ such that*

$$\|v(z)\| \leq AE_{\alpha}(-\gamma\left[\varphi(z) - \varphi(a)\right]^{\alpha}),\ z > a,$$

where $\|.\|$ represents a specific norm.

Theorem 3.1. *Let $u(t)$ be a nonnegative function fulfilling the conditions*

$$D_{C}^{\varphi,\alpha}\left[u(t) - pu(t-\upsilon)\right] \leq -qu(t) + \int_{a}^{t} u(s)k(t-s)\,ds,\ 0 < \alpha < 1,\ t > a, \tag{3.1}$$

with the initial condition

$$u(t) = \varpi(t) \geq 0,\ a - \upsilon \leq t \leq a, \tag{3.2}$$

where k is a nonnegative function integrable over its domain, and $q > 0$. Assume $p > 0$, and that k satisfies the following inequality for some $M > 0$:

$$\begin{aligned} &\int_{a}^{t} E_{\alpha,\alpha}\left(-q\left[\varphi(t) - \varphi(s)\right]^{\alpha}\right)\left[\varphi(t) - \varphi(s)\right]^{\alpha-1} \\ &\times\left(\int_{a}^{s} E_{\alpha}\left(-q\left[\varphi(\sigma) - \varphi(a)\right]^{\alpha}\right)k(s-\sigma)d\sigma\right)\varphi'(s)\ ds \\ \leq\ & ME_{\alpha}\left(-q\left[\varphi(t) - \varphi(a)\right]^{\alpha}\right),\ t > a. \end{aligned} \tag{3.3}$$

Further, assume that the constant M satisfies

$$M < 1 - \frac{1}{(\varphi(a+\upsilon) - \varphi(a))^{\alpha}}\left(\frac{1}{q} + \Gamma(1-\alpha)\left[\varphi(a+3\upsilon) - \varphi(a)\right]^{\alpha}\right)p, \tag{3.4}$$

with the additional condition

$$\frac{1}{(\varphi(a+\upsilon) - \varphi(a))^{\alpha}}\left(\frac{1}{q} + \Gamma(1-\alpha)\left[\varphi(a+3\upsilon) - \varphi(a)\right]^{\alpha}\right)p < 1. \tag{3.5}$$

Then, $u(t)$ exhibits Mittag-Leffler decay, i.e.,

$$u(t) \leq CE_{\alpha}\left(-q\left[\varphi(t) - \varphi(a)\right]^{\alpha}\right),\ t > a$$

for some constant $C > 0$.

Proof. Solutions of (3.1) and (3.2) will be compared to those of

$$\begin{cases} D_{C}^{\varphi,\alpha}\left[w(t) - pw(t-\upsilon)\right] = -qw(t) + \int_{a}^{t} w(s)k(t-s)ds,\ 0 < \alpha < 1,\ t > a, \\ w(t) = \varpi(t) \geq 0,\ a - \upsilon \leq t \leq a. \end{cases} \tag{3.6}$$

The equation presented in (3.6) can be expressed equivalently as

$$D_C^{\varphi,\alpha}\left[w(t)-pw(t-\upsilon)\right]=-q\left[w(t)-pw(t-\upsilon)\right]+\int_a^t k(t-s)w(s)\,ds-qpw(t-\upsilon),\ t>a.$$

This permits to profit from the form

$$\begin{aligned}w(t)-pw(t-\upsilon) &= \left[\varpi(a)-p\varpi(a-\upsilon)\right]E_\alpha(-q\left[\varphi(t)-\varphi(a)\right]^\alpha)\\ &\quad+\int_a^t\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha)\\ &\quad\times\left(-qpw(s-\upsilon)+\int_a^s k(s-\sigma)w(\sigma)d\sigma\right)\varphi'(s)\,ds.\end{aligned}$$

Capitalizing on the nonnegativity of the solution, we find for $t>a$,

$$\begin{aligned}w(t) &\le \varpi(a)E_\alpha(-q(\varphi(t)-\varphi(a))^\alpha)+pw(t-\upsilon)+\int_a^t E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha)\\ &\quad\times\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}\left(\int_a^s k(s-\sigma)w(\sigma)d\sigma\right)\varphi'(s)\ ds. \end{aligned}\tag{3.7}$$

Therefore, for $t>a$,

$$\begin{aligned}&\frac{w(t)}{E_\alpha(-q(\varphi(t)-\varphi(a))^\alpha)}\\ \le\ &\varpi(a)+\frac{p}{E_\alpha(-q(\varphi(t)-\varphi(a))^\alpha)}w(t-\upsilon)\\ &+\frac{1}{E_\alpha(-q(\varphi(t)-\varphi(a))^\alpha)}\int_a^t\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha)\\ &\times\left(\int_a^s k(s-\sigma)E_\alpha\left(-q(\varphi(\sigma)-\varphi(a))^\alpha\right)\frac{w(\sigma)}{E_\alpha\left(-q(\varphi(\sigma)-\varphi(a))^\alpha\right)}d\sigma\right)\varphi'(s)\,ds,\end{aligned}$$

and

$$\begin{aligned}\frac{w(t)}{E_\alpha(-q(\varphi(t)-\varphi(a))^\alpha)} &\le \varpi(a)+\frac{p}{E_\alpha(-q(\varphi(t)-\varphi(a))^\alpha)}w(t-\upsilon)\\ &\quad+\frac{1}{E_\alpha(-q(\varphi(t)-\varphi(a))^\alpha)}\int_a^t\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}\\ &\quad\times E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha)\\ &\quad\times\left(\int_a^s k(s-\sigma)E_\alpha\left(-q(\varphi(\sigma)-\varphi(a))^\alpha\right)d\sigma\right)\varphi'(s)\,ds\\ &\quad\times\sup_{a\le\sigma\le t}\frac{w(\sigma)}{E_\alpha\left(-q(\varphi(\sigma)-\varphi(a))^\alpha\right)}\\ &\le \varpi(a)+\frac{p}{E_\alpha(-q(\varphi(t)-\varphi(a))^\alpha)}w(t-\nu)\\ &\quad+M\sup_{a\le\sigma\le t}\frac{w(\sigma)}{E_\alpha\left(-q(\varphi(\sigma)-\varphi(a))^\alpha\right)}.\end{aligned}$$

We will repeatedly utilize the following estimation:

$$\begin{aligned}
& \frac{1}{E_{\alpha}(-q\left(\varphi(t)-\varphi(a)\right)^{\alpha})}\int_a^t E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^{\alpha})\left[\varphi(t)-\varphi(s)\right]^{\alpha-1} \\
& \times\left(\int_a^s k(s-\sigma)w(\sigma)d\sigma\right)\varphi'(s)\,ds \\
= & \frac{1}{E_{\alpha}(-q\left(\varphi(t)-\varphi(a)\right)^{\alpha})}\int_a^t E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^{\alpha})\left[\varphi(t)-\varphi(s)\right]^{\alpha-1} \\
& \times\left(\int_a^s E_{\alpha}\left(-q\left[\varphi(\sigma)-\varphi(a)\right]^{\alpha}\right)k(s-\sigma)\frac{w(\sigma)}{E_{\alpha}\left(-q\left[\varphi(\sigma)-\varphi(a)\right]^{\alpha}\right)}d\sigma\right)\varphi'(s)\,ds \\
\leq & M\sup_{a\leq\sigma\leq t}\frac{w(\sigma)}{E_{\alpha}\left(-q\left(\varphi(\sigma)-\varphi(a)\right)^{\alpha}\right)},\ t>a. \qquad (3.8)
\end{aligned}$$

Then, for $t > a$, the following inequality holds:

$$\begin{aligned}
\frac{w(t)}{E_{\alpha}(-q\left[\varphi(t)-\varphi(a)\right]^{\alpha})} \leq\ & \varpi(a)+\frac{p}{E_{\alpha}(-q\left[\varphi(t)-\varphi(a)\right]^{\alpha})}w(t-\upsilon) \\
& +M\sup_{a\leq\sigma\leq t}\frac{w(\sigma)}{E_{\alpha}\left(-q\left[\varphi(\sigma)-\varphi(a)\right]^{\alpha}\right)}. \qquad (3.9)
\end{aligned}$$

This inequality will serve as our initial reference.

For $t\in[a,a+\upsilon]$, since $E_{\alpha}(-q\left[\varphi(t)-\varphi(a)\right]^{\alpha})$ is decreasing, it follows that

$$E_{\alpha}(-q\left[\varphi(t)-\varphi(a)\right]^{\alpha})\geq E_{\alpha}(-q\left[\varphi(a+\upsilon)-\varphi(a)\right]^{\alpha}),$$

and hence

$$\begin{aligned}
\frac{w(t)}{E_{\alpha}(-q\left[\varphi(t)-\varphi(a)\right]^{\alpha})} \leq\ & \left(1+\frac{p}{E_{\alpha}(-q\left[\varphi(a+\upsilon)-\varphi(a)\right]^{\alpha}}\right)\sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma) \\
& +M\sup_{a\leq\sigma\leq t}\frac{w(\sigma)}{E_{\alpha}\left(-q\left[\varphi(\sigma)-\varphi(a)\right]^{\alpha}\right)},
\end{aligned}$$

or

$$(1-M)\frac{w(t)}{E_{\alpha}(-q\left(\varphi(t)-\varphi(a)\right)^{\alpha})}\leq\left(1+\frac{p}{E_{\alpha}(-q\left(\varphi(a+\upsilon)-\varphi(a)\right)^{\alpha}}\right)\sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma). \qquad (3.10)$$

If $t\in[a+\upsilon,a+2\upsilon]$, owing to relations (3.9) and (3.10), we find

$$\begin{aligned}
\frac{w(t)}{E_{\alpha}(-q\left(\varphi(t)-\varphi(a)\right)^{\alpha})} \leq\ & \sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma)+\frac{p}{1-M}\left(1+\frac{p}{E_{\alpha}(-q\left(\varphi(a+\upsilon)-\varphi(a)\right)^{\alpha}}\right) \\
& \times\frac{E_{\alpha}(-q\left(\varphi(t-\upsilon)-\varphi(a)\right)^{\alpha})}{E_{\alpha}(-q\left(\varphi(t)-\varphi(a)\right)^{\alpha})}\sup_{a-\nu\leq\sigma\leq a}\varpi(\sigma) \\
& +M\sup_{a\leq\sigma\leq t}\frac{w(\sigma)}{E_{\alpha}\left(-q\left(\varphi(\sigma)-\varphi(a)\right)^{\alpha}\right)}.
\end{aligned}$$

Observe that

$$\frac{E_{\alpha}(-q\left(\varphi(t-\upsilon)-\varphi(a)\right)^{\alpha})}{E_{\alpha}(-q\left(\varphi(t)-\varphi(a)\right)^{\alpha})} \leq \frac{1}{E_{\alpha}(-q\left(\varphi(t)-\varphi(a)\right)^{\alpha})}$$

$$\begin{aligned}&\leq \frac{1}{E_\alpha(-q(\varphi(2\upsilon+a)-\varphi(a))^\alpha)}\\&\leq 1+q\Gamma(1-\alpha)(\varphi(2\upsilon+a)-\varphi(a))^\alpha =: A.\end{aligned}\tag{3.11}$$

Therefore,

$$\begin{aligned}\frac{w(t)}{E_\alpha(-q(\varphi(t)-\varphi(a))^\alpha)} &\leq \left[1+\frac{A\,(E_\alpha(-q(\varphi(\upsilon+a)-\varphi(a))^\alpha+p)\,p}{E_\alpha(-q(\varphi(\upsilon+a)-\varphi(a))^\alpha(1-M)}\right]\sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma)\\&\quad+M\sup_{a\leq\sigma\leq t}\frac{w(\sigma)}{E_\alpha(-q(\varphi(\sigma)-\varphi(a))^\alpha)},\end{aligned}$$

and consequently,

$$\begin{aligned}&\frac{w(t)}{E_\alpha(-q(\varphi(t)-\varphi(a))^\alpha)}(1-M)\\&\leq\left[1+\frac{A}{(1-M)}p+\frac{A}{E_\alpha(-q[\varphi(a+\upsilon)-\varphi(a)]^\alpha)(1-M)}p^2\right]\sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma).\end{aligned}\tag{3.12}$$

Notice that we will write (3.12) as

$$\begin{aligned}\frac{w(t)}{E_\alpha(-q(\varphi(t)-\varphi(a))^\alpha)}(1-M) &\leq \frac{A}{E_\alpha(-q(\varphi(\upsilon+a)-\varphi(a))^\alpha)}\\&\quad\times\left[1+\frac{p}{1-M}+\left(\frac{p}{1-M}\right)^2\right]\sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma).\end{aligned}\tag{3.13}$$

When $t\in[a+2\nu,a+3\nu]$, the estimations

$$\frac{\varphi(t)-\varphi(a)}{\varphi(t-\upsilon)-\varphi(a)}\leq\frac{\varphi(a+3\upsilon)-\varphi(a)}{\varphi(a+\upsilon)-\varphi(a)},$$

together with (2.7), imply for $t\geq a+2\nu$,

$$\begin{aligned}\frac{E_\alpha(-q(\varphi(t-\upsilon)-\varphi(a))^\alpha)}{E_\alpha(-q(\varphi(t)-\varphi(a))^\alpha)} &\leq \frac{1+q(\varphi(t)-\varphi(a))^\alpha\,\Gamma(1-\alpha)}{1+q(\varphi(t-\upsilon)-\varphi(a))^\alpha\,\Gamma(1+\alpha)^{-1}}\\&\leq \frac{1+q(\varphi(t)-\varphi(a))^\alpha\,\Gamma(1-\alpha)}{q\Gamma(1+\alpha)^{-1}(\varphi(t-\upsilon)-\varphi(a))^\alpha}\\&\leq \frac{\Gamma(1+\alpha)}{q(\varphi(t-\upsilon)-\varphi(a))^\alpha}+\frac{\Gamma(1+\alpha)(\varphi(t)-\varphi(a))^\alpha\,\Gamma(1-\alpha)}{(\varphi(t-\upsilon)-\varphi(a))^\alpha}\\&\leq \frac{\Gamma(1+\alpha)}{q(\varphi(a+\upsilon)-\varphi(a))^\alpha}\\&\quad+\frac{(\varphi(a+3\upsilon)-\varphi(a))^\alpha\,\Gamma(1+\alpha)\Gamma(1-\alpha)}{(\varphi(a+\upsilon)-\varphi(a))^\alpha}\\&\leq \frac{\Gamma(1+\alpha)}{(\varphi(a+\upsilon)-\varphi(a))^\alpha}\\&\quad\times\left(\frac{1}{q}+\Gamma(1-\alpha)(\varphi(a+3\upsilon)-\varphi(a))^\alpha\right),\end{aligned}\tag{3.14}$$

Notice that $\Gamma(1+\alpha)$ can be approximated by one.

By virtue of relations (3.13) and (3.14), having in mind (3.9), we infer

$$\begin{aligned}\frac{w(t)}{E_{\alpha}(-q(\varphi(t)-\varphi(a))^{\alpha})} &\leq \varpi(a)+\frac{p}{1-M}\frac{E_{\alpha}(-q(\varphi(t-\upsilon)-\varphi(a))^{\alpha})}{E_{\alpha}(-q(\varphi(t)-\varphi(a))^{\alpha})} \\ &\times\frac{A}{E_{\alpha}(-q(\varphi(a+\upsilon)-\varphi(a))^{\alpha})} \\ &\times\left[1+\frac{p}{1-M}+\left(\frac{p}{1-M}\right)^{2}\right]\sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma) \\ &+M\sup_{a\leq\sigma\leq t}\frac{w(\sigma)}{E_{\alpha}(-q(\varphi(\sigma)-\varphi(a))^{\alpha})},\end{aligned}$$

or

$$\begin{aligned}\frac{w(t)}{E_{\alpha}(-q[\varphi(t)-\varphi(a)]^{\alpha})}(1-M) &\leq \sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma)\left\{1+\frac{p}{1-M}\frac{AV}{E_{\alpha}(-q[\varphi(\upsilon+a)-\varphi(a)]^{\alpha})}\right. \\ &\left.\times\left[1+\frac{p}{1-M}+\left(\frac{p}{1-M}\right)^{2}\right]\right\}, \end{aligned}\tag{3.15}$$

where

$$V:=\frac{1}{(\varphi(a+\upsilon)-\varphi(a))^{\alpha}}\left(\frac{1}{q}+\Gamma(1-\alpha)[\varphi(a+3\upsilon)-\varphi(a)]^{\alpha}\right).$$

As

$$\frac{AV}{E_{\alpha}(-q[\varphi(a+\upsilon)-\varphi(a)]^{\alpha})}>1,$$

we can rewrite Eq (3.15) as follows:

$$\begin{aligned}(1-M)\frac{w(t)}{E_{\alpha}(-q[\varphi(t)-\varphi(a)]^{\alpha})} &\leq \sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma)\frac{A}{E_{\alpha}(-q[\varphi(\upsilon+a)-\varphi(a)]^{\alpha})} \\ &\times\left\{1+\frac{pV}{1-M}+\left(\frac{pV}{1-M}\right)^{2}+\left(\frac{pV}{1-M}\right)^{3}\right\}.\end{aligned}$$

We now make the following claim.
Claim. For $t\in[a+(n-1)\upsilon, a+n\upsilon]$,

$$\begin{aligned}(1-M)\frac{w(t)}{E_{\alpha}(-q(\varphi(t)-\varphi(a))^{\alpha})} &\leq \frac{A}{E_{\alpha}(-q(\varphi(\upsilon+a)-\varphi(a))^{\alpha})} \\ &\times\sum_{k=0}^{n}\left(\frac{pV}{1-M}\right)^{k}\sup_{a-\nu\leq\sigma\leq a}\varpi(\sigma).\end{aligned}$$

It is evident that the assertion is valid for the cases $n=1$, 2, and 3. Assume that it holds for n, i.e., on $[a+(n-1)\upsilon, a+n\upsilon]$. Now, let $t\in[a+n\upsilon, a+\upsilon(n+1)]$. Utilizing (3.9), we derive

$$\begin{aligned}\frac{w(t)}{E_{\alpha}(-q[\varphi(t)-\varphi(a)]^{\alpha})} &\leq \sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma)+\frac{pE_{\alpha}(-q[\varphi(t-\upsilon)-\varphi(a)]^{\alpha})}{(1-M)E_{\alpha}(-q[\varphi(t)-\varphi(a)]^{\alpha})} \\ &\times\frac{A}{E_{\alpha}(-q[\varphi(a+\upsilon)-\varphi(a)]^{\alpha})}\sum_{k=0}^{n}\left(\frac{pV}{1-M}\right)^{k}\sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma)\end{aligned}$$

$$+M \sup_{a\leq\sigma\leq t} \frac{w(\sigma)}{E_\alpha\left(-q\left[\varphi(\sigma)-\varphi(a)\right]^\alpha\right)}.$$

and by (3.14)

$$\begin{aligned}
&\frac{w(t)}{E_\alpha(-q\left(\varphi(t)-\varphi(a)\right)^\alpha)}(1-M)\\
\leq\;& \left[1+\frac{Vp}{1-M}\frac{A}{E_\alpha\left(-q\left(\varphi(a+\upsilon)-\varphi(a)\right)^\alpha\right)}\sum\nolimits_{k=0}^{n}\left(\frac{Vp}{1-M}\right)^k\right]\sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma)\\
\leq\;& \frac{A}{E_\alpha\left(-q\left(\varphi(a+\upsilon)-\varphi(a)\right)^\alpha\right)}\left[1+\sum\nolimits_{k=1}^{n+1}\left(\frac{Vp}{1-M}\right)^k\right]\sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma)\\
=\;& \frac{A}{E_\alpha\left(-q\left[\varphi(a+\upsilon)-\varphi(a)\right]^\alpha\right)}\sum\nolimits_{k=0}^{n+1}\left(\frac{Vp}{1-M}\right)^k\sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma).
\end{aligned}$$

Therefore, the claim holds true. Then, for $t > a$,

$$\begin{aligned}
w(t) \;\leq\;& \left[\frac{A}{E_\alpha\left(-q\left[\varphi(a+\upsilon)-\varphi(a)\right]^\alpha\right)(1-M)}\sum\nolimits_{k=0}^{\infty}\left(\frac{pV}{1-M}\right)^k\sup_{a-\upsilon\leq\sigma\leq a}\varpi(\sigma)\right]\\
&\times E_\alpha(-q\left(\varphi(t)-\varphi(a)\right)^\alpha).
\end{aligned}\tag{3.16}$$

The series in (3.16) converges due to (3.4) and (3.5). The proof is complete. □

4. Examples

In this section, we identify two classes of functions that satisfy the conditions of the theorem.
First class: Consider the set of functions k that fulfill the following inequality for all $s \geq a$:

$$\int_a^s E_\alpha(-q\left[\varphi(\sigma)-\varphi(a)\right]^\alpha)k(s-\sigma)d\sigma \leq C_1\left[\varphi(s)-\varphi(a)\right]^{\lambda-1},\ C_1,\lambda>0. \tag{4.1}$$

The family of functions $k(t-s)$ defined as

$$k(t-s) \leq C_2\left[\varphi(t)-\varphi(s)\right]^{-\alpha} e^{-b\left[\varphi(s)-\varphi(a)\right]}\varphi'(s)$$

satisfies the specified relation when the constants b and C_2 are carefully chosen. Indeed, since

$$E_\alpha(-qt^\alpha) \leq \frac{1}{1+\frac{qt^\alpha}{\Gamma(1+\alpha)}} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)+qt^\alpha} \leq \frac{\Gamma(1+\alpha)}{qt^\alpha},\ t>0, \tag{4.2}$$

it follows that

$$\begin{aligned}
&\int_a^s E_\alpha(-q\left[\varphi(\sigma)-\varphi(a)\right]^\alpha)k(s-\sigma)\,d\sigma\\
\leq\;& \frac{C_2\Gamma(1+\alpha)}{q}\int_a^s\left[\varphi(\sigma)-\varphi(a)\right]^{-\alpha}\left[\varphi(s)-\varphi(\sigma)\right]^{-\alpha}e^{-b\left[\varphi(\sigma)-\varphi(a)\right]}\varphi'(\sigma)\,d\sigma\\
\leq\;& \frac{2^\alpha C_2\Gamma(1+\alpha)\Gamma(1-\alpha)\left[3-\alpha\right]b^{\alpha-1}}{q}\left[\varphi(s)-\varphi(a)\right]^{-\alpha},\ s>a.
\end{aligned}$$

Therefore, (4.1) holds with

$$C_1 := \frac{2^{\alpha} C_2 \Gamma(1+\alpha)\Gamma(1-\alpha)\,[3-\alpha]\, b^{\alpha-1}}{q},\ \lambda := 1-\alpha.$$

By applying formula (2.6), we obtain

$$\begin{aligned}
&\int_a^t E_{\alpha,\alpha}(-q\,[\varphi(t)-\varphi(s)]^{\alpha})\,[\varphi(t)-\varphi(s)]^{\alpha-1} \\
&\times\left(\int_a^s k(s-\sigma)\,E_{\alpha}(-q\,[\varphi(\sigma)-\varphi(a)]^{\alpha})d\sigma\right)\varphi'(s)\,ds \\
\leq\ & C_1 \int_a^t E_{\alpha,\alpha}(-q\,[\varphi(t)-\varphi(s)]^{\alpha})\,[\varphi(t)-\varphi(s)]^{\alpha-1}\,[\varphi(s)-\varphi(a)]^{-\alpha}\,\varphi'(s)\,ds \\
\leq\ & C_1\Gamma(\alpha)E_{\alpha,1}(-q\,[\varphi(t)-\varphi(a)]^{\alpha}).
\end{aligned} \tag{4.3}$$

To ensure that assumption (3.4) is met, we can select C_1 (or C_2 for the specific example) such that

$$C_1\Gamma(\alpha) < 1 - \frac{1}{(\varphi(a+\upsilon)-\varphi(a))^{\alpha}}\left(\frac{1}{q} + \Gamma(1-\alpha)\,[\varphi(a+3\upsilon)-\varphi(a)]^{\alpha}\right)p.$$

Second class: Assume that $k(t-s) \leq C_3\,[\varphi(t)-\varphi(s)]^{\alpha-1}\,E_{\alpha,\alpha}(-b\,[\varphi(t)-\varphi(s)]^{\alpha})\varphi'(s)$ for some $b > 0$ and $C_3 > 0$ to be determined. A double use of (2.6) and (4.2) gives

$$\begin{aligned}
& C_3 \int_a^t [\varphi(t)-\varphi(s)]^{\alpha-1}\,E_{\alpha,\alpha}(-q\,[\varphi(t)-\varphi(s)]^{\alpha}) \\
&\times\left(\int_a^s [\varphi(s)-\varphi(\sigma)]^{\alpha-1}\,E_{\alpha,\alpha}(-b\,[\varphi(s)-\varphi(\sigma)]^{\alpha})E_{\alpha}(-q\,[\varphi(\sigma)-\varphi(a)]^{\alpha})\varphi'(\sigma)\,d\sigma\right)\varphi'(s)\,ds \\
\leq\ & \frac{C_3\Gamma(1+\alpha)}{q}\int_a^t E_{\alpha,\alpha}(-q\,[\varphi(t)-\varphi(s)]^{\alpha})\,[\varphi(t)-\varphi(s)]^{\alpha-1} \\
&\times\left(\int_a^s [\varphi(s)-\varphi(\sigma)]^{\alpha-1}\,E_{\alpha,\alpha}(-b\,[\varphi(s)-\varphi(\sigma)]^{\alpha})\,[\varphi(\sigma)-\varphi(a)]^{-\alpha}\,\varphi'(\sigma)\,d\sigma\right)\varphi'(s)\,ds \\
\leq\ & \frac{C_3\Gamma(\alpha)\Gamma(1+\alpha)}{q}\int_a^t E_{\alpha,\alpha}(-q\,[\varphi(t)-\varphi(s)]^{\alpha}\,E_{\alpha,1}(-b\,[\varphi(s)-\varphi(a)]^{\alpha}\,[\varphi(t)-\varphi(s)]^{\alpha-1}\,\varphi'(s)\,ds \\
\leq\ & \frac{C_3\Gamma^2(1+\alpha)\Gamma(\alpha)}{qb}\int_a^t E_{\alpha,\alpha}(-q\,[\varphi(t)-\varphi(s)]^{\alpha}\,[\varphi(t)-\varphi(s)]^{\alpha-1}\,[\varphi(s)-\varphi(a)]^{-\alpha}\,\varphi'(s)\,ds \\
\leq\ & \frac{C_3\Gamma^2(1+\alpha)\Gamma^2(\alpha)}{qb}E_{\alpha,1}(-q\,[\varphi(t)-\varphi(a)]^{\alpha}.
\end{aligned} \tag{4.4}$$

Clearly, $M = \frac{C_3\Gamma^2(1+\alpha)\Gamma^2(\alpha)}{qb}$. It suffices now to impose the condition on C_3 and/or the constant b in order to fulfill the condition on M.

5. Fractional Halanay inequality with both distributed neutral delays and distributed delays

In this section, we will examine the inequality that arises when the neutral delay is distributed,

$$\begin{cases} D_C^{\varphi,\alpha}\left[u(t)-p\int_a^t u(s)g(t-s)\,ds\right]\leq -qu(t)+\int_a^t u(s)k(t-s)\,ds,\ t,\upsilon>a,\ 0<\alpha<1,\ p>0, \\ u(t)=u_0\geq 0,\ t\in[a-\upsilon,a], \end{cases} \tag{5.1}$$

which we will contrast with

$$\begin{cases} D_C^{\varphi,\alpha}\left[w(t)-p\int_a^t w(s)g(t-s)\,ds\right]=-qw(t)+\int_a^t w(s)k(t-s)\,ds,\ t,\upsilon>a,\ 0<\alpha<1,\ p>0, \\ w(t)=w_0=u_0\geq 0,\ t\in[a-\upsilon,a]. \end{cases} \tag{5.2}$$

We assume g is a continuous function (to be determined later) and that the solutions are nonnegative.

Let us reformulate this as

$$\begin{cases} D_C^{\varphi,\alpha}\left[w(t)-p\int_a^t w(s)g(t-s)\,ds\right]=-q\left[w(t)-p\int_a^t w(s)g(t-s)\,ds\right] \\ -qp\int_a^t w(s)g(t-s)\,ds+\int_a^t w(s)k(t-s)\,ds,\ t,\upsilon>a,\ 0<\alpha<1,\ p>0 \\ w(t)=w_0\geq 0,\ t\in[a-\upsilon,a]. \end{cases}$$

Therefore,

$$\begin{aligned} w(t)-p\int_a^t w(s)g(t-s)\,ds &= E_\alpha(-q\left[\varphi(t)-\varphi(a)\right]^\alpha)w_0 \\ &\quad+\int_a^t\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha) \\ &\quad\times\left(-qp\int_a^s g(s-\sigma)w(\sigma)\,d\sigma+\int_a^s k(s-\sigma)w(\sigma)\,d\sigma\right)\varphi'(s)\,ds, \end{aligned}$$

and, for $t>a$,

$$\begin{aligned} w(t) &\leq E_\alpha(-q\left[\varphi(t)-\varphi(a)\right]^\alpha)w_0+p\int_a^t g(t-s)\,w(s)ds+\int_a^t E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha) \\ &\quad\times\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}\left(\int_a^s k(s-\sigma)w(\sigma)\,d\sigma\right)\varphi'(s)\,ds. \end{aligned} \tag{5.3}$$

Dividing both sides of (5.3) by $E_\alpha(-q\left[\varphi(t)-\varphi(a)\right]^\alpha)$, we find

$$\begin{aligned} \frac{w(t)}{E_\alpha(-q\left[\varphi(t)-\varphi(a)\right]^\alpha)} &= w_0+\frac{p}{E_\alpha(-q\left[\varphi(t)-\varphi(a)\right]^\alpha)}\int_a^t w(s)\,g(t-s)ds \\ &\quad+\frac{1}{E_\alpha(-q\left[\varphi(t)-\varphi(a)\right]^\alpha)}\int_a^t\left[\varphi(t)-\varphi(s)\right]^{\alpha-1} \\ &\quad\times E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha) \\ &\quad\times\left(\int_a^s k(s-\sigma)E_\alpha(-q\left[\varphi(\sigma)-\varphi(a)\right]^\alpha)d\sigma\right)\varphi'(s)\,ds \\ &\quad\times\sup_{a\leq\sigma\leq t}\frac{w(\sigma)}{E_\alpha(-q\left[\varphi(\sigma)-\varphi(a)\right]^\alpha)}, \end{aligned}$$

or, for $t > a$,

$$\begin{aligned}\frac{w(t)}{E_\alpha(-q\left[\varphi(t)-\varphi(a)\right]^\alpha)} &\leq w_0+\frac{p}{E_\alpha(-q\left[\varphi(t)-\varphi(a)\right]^\alpha)}\int_a^t g(t-s)E_\alpha(-q\left[\varphi(s)-\varphi(a)\right]^\alpha)\\ &\quad\times\left(\frac{w(s)}{E_\alpha(-q\left[\varphi(s)-\varphi(a)\right]^\alpha)}\right)ds\\ &\quad+M\sup_{a\leq\sigma\leq t}\frac{w(\sigma)}{E_\alpha(-q\left[\varphi(\sigma)-\varphi(a)\right]^\alpha)}.\end{aligned}$$

The relation

$$\frac{p}{E_\alpha(-q\left[\varphi(t)-\varphi(a)\right]^\alpha)}\int_a^t g(t-s)E_\alpha(-q\left[\varphi(s)-\varphi(a)\right]^\alpha)\,ds\leq M^*,$$

is assumed for some $M^* > 0$. Then,

$$\frac{w(t)}{E_\alpha(-q\left[\varphi(t)-\varphi(a)\right]^\alpha)}\leq w_0+(M^*+M)\sup_{a\leq\sigma\leq t}\frac{w(\sigma)}{E_\alpha(-q\left[\varphi(\sigma)-\varphi(a)\right]^\alpha)},\ t>a,$$

and

$$w(t)\leq\frac{w_0}{1-M^*-M}E_\alpha(-q\left[\varphi(t)-\varphi(a)\right]^\alpha),\ t>a,$$

in the case that

$$M^*+M<1.$$

Example. Take k as above, and select g fulfilling

$$g(t-s)\leq C_4\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}E_{\alpha,\alpha}(-c\left[\varphi(t)-\varphi(s)\right]^\alpha)\varphi'(s),$$

for some $C_4, c > q$. Then,

$$\begin{aligned}\int_a^t E_\alpha(-q\left[\varphi(s)-\varphi(a)\right]^\alpha)g(t-s)ds &\leq \frac{\Gamma(1+\alpha)}{q}\int_a^t\left[\varphi(s)-\varphi(a)\right]^{-\alpha}g(t-s)ds\\ &\leq \frac{C_4\Gamma(1+\alpha)}{q}\int_a^t E_{\alpha,\alpha}(-c\left[\varphi(t)-\varphi(s)\right]^\alpha)\\ &\quad\times\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}\left[\varphi(s)-\varphi(a)\right]^{-\alpha}\varphi'(s)\,ds\\ &\leq C_4\frac{\Gamma(1+\alpha)\Gamma(\alpha)}{q}E_{\alpha,1}(-q\left[\varphi(t)-\varphi(a)\right]^\alpha),\ t>a.\end{aligned}$$

A value for M^* would be

$$M^*=\frac{C_4p\Gamma(1+\alpha)\Gamma(\alpha)}{q}.$$

Therefore, we have proved the following theorem.

Theorem 5.1. *Let $u(t)$ be a nonnegative solution of (5.1), where q and p are positive and k and g are continuous functions with $k(t), g(t)\geq 0$ for all t such that*

$$\int_a^t E_{\alpha,\alpha}\left(-q\left[\varphi(t)-\varphi(s)\right]^\alpha\right)\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}$$

$$\times\left(\int_a^s E_\alpha\left(-q\left[\varphi(\sigma)-\varphi(a)\right]^\alpha\right)k(s-\sigma)\,d\sigma\right)\varphi'(s)\,ds$$
$$\leq\quad ME_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right),\ t>a,$$

$$p\int_a^t g(t-s)E_\alpha(-q\left[\varphi(s)-\varphi(a)\right]^\alpha)\,ds\leq M^*E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right),\ t>a,$$

hold for some M, $M^*>0$ with

$$M^*+M<1.$$

Then, we can find a positive constant C such that

$$w(t)\leq CE_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right),\ t>a.$$

6. Solutions that may be positive or negative

Before delving into applications, it is important to note that previous research on Halanay inequalities, including our earlier work, often assumes that solutions are non-negative. This supposition is sufficient for applications like neural networks without time delays. To determine the stability of the equilibrium solution, we can simplify the problem by shifting the equilibrium point to the origin using a variable transformation and then analyzing the magnitude of the solutions. However, when dealing with systems that have time delays, this approach becomes more complex. Directly proving stability for solutions that can be positive or negative presents new challenges, as time delays now appear within convolution integrals. The necessary estimations are more intricate and require careful analysis.

Now, we return to

$$\begin{cases} D_C^{\varphi,\alpha}\left[u(t)-pu(t-\upsilon)\right]\leq -qu(t)+\int_a^t k(t-s)u(s)\,ds,\ p>0,\ 0<\alpha<1,\ t,\upsilon>a,\\ u(t)=\varpi(t)\geq 0,\ a-\upsilon\leq t\leq a,\end{cases}$$

with $|\varpi(s)|\leq w_0E_\alpha(-q(\varphi(s+\upsilon)-\varphi(a))^\alpha)$ for $s\in[a-\upsilon,a]$, $w_0>0$. To clarify these concepts, let us suppose that $1>p>0$, and examine the following expression:

$$\begin{aligned} w(t)-pw(t-\upsilon) &= \left[\varpi(a)-p\varpi(a-\upsilon)\right]E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)\\ &\quad+\int_a^t\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha)\\ &\quad\times\left(-qpw(s-\upsilon)+\int_a^s k(s-\sigma)w(\sigma)d\sigma\right)\varphi'(s)\,ds.\end{aligned}$$

Then, for $t>a$

$$\begin{aligned} |w(t)| &\leq 2w_0E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)+p\,|w(t-\upsilon)|\\ &\quad+qp\int_a^t E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha)\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}|w(s-\upsilon)|\,\varphi'(s)\,ds\\ &\quad+\int_a^t E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha)\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}\\ &\quad\times\left(\int_a^s k(s-\sigma)\,|w(\sigma)|\,d\sigma\right)\varphi'(s)\,ds. \end{aligned}\tag{6.1}$$

For $t \in [a, a+\upsilon]$,

$$\begin{aligned}\frac{|w(t)|}{E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)} &\le 3w_0 + \frac{qpw_0}{E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)}\int_a^t \left[\varphi(t)-\varphi(s)\right]^{\alpha-1} \\ &\quad \times E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha)E_\alpha(-q\left[\varphi(s)-\varphi(a)\right]^\alpha)\varphi'(s)\,ds \\ &\quad + M\sup_{a\le\sigma\le t}\frac{w(\sigma)}{E_\alpha\left(-q\left[\varphi(\sigma)-\varphi(a)\right]^\alpha\right)},\end{aligned}$$

where M is defined as in Eq (3.3). Again, as

$$\begin{aligned}&\int_a^t E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha)\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}E_\alpha(-q\left[\varphi(s)-\varphi(a)\right]^\alpha)\varphi'(s)\,ds \\ \le\ &\frac{\Gamma(1+\alpha)}{q}\int_a^t E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha)\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}\left[\varphi(s)-\varphi(a)\right]^{-\alpha}\ \varphi'(s)\,ds \\ \le\ &\frac{\Gamma(1+\alpha)\Gamma(\alpha)}{q}E_{\alpha,1}\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right),\end{aligned} \tag{6.2}$$

we can write

$$\begin{aligned}\frac{|w(t)|}{E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)} &\le 3w_0 + w_0\Gamma(1+\alpha)\Gamma(\alpha)p \\ &\quad + M\sup_{a\le\sigma\le t}\frac{w(\sigma)}{E_\alpha\left(-q\left[\varphi(\sigma)-\varphi(a)\right]^\alpha\right)},\end{aligned}$$

or

$$(1-M)\frac{|w(t)|}{E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)} \le 3w_0 + w_0\Gamma(1+\alpha)\Gamma(\alpha)p. \tag{6.3}$$

If $t \in [a+\upsilon, a+2\upsilon]$, we first observe that

$$\begin{aligned}|w(t-\upsilon)| &\le \frac{3w_0 + w_0\Gamma(1+\alpha)\Gamma(\alpha)p}{(1-M)} \\ &\quad \times\frac{E_\alpha\left(-q\left[\varphi(t-\upsilon)-\varphi(a)\right]^\alpha\right)}{E_\alpha(-q\left[\varphi(t)-\varphi(a)\right]^\alpha)}E_\alpha(-q\left[\varphi(t)-\varphi(a)\right]^\alpha) \\ &\le A\frac{3w_0 + w_0\Gamma(1+\alpha)\Gamma(\alpha)p}{(1-M)}E_\alpha(-q\left(\varphi(t)-\varphi(a)\right)^\alpha),\end{aligned}$$

where A is as in (3.11). Using the fact that

$$w_0 \le A\frac{3w_0 + w_0\Gamma(1+\alpha)\Gamma(\alpha)p}{1-M},$$

and relations (6.1) and (6.3), we get

$$\begin{aligned}|w(t)| &\le 2w_0E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right) \\ &\quad + pA\frac{3w_0 + w_0\Gamma(1+\alpha)\Gamma(\alpha)p}{(1-M)}E_\alpha(-q\left(\varphi(t)-\varphi(a)\right)^\alpha) \\ &\quad + qpA\frac{3w_0 + w_0\Gamma(1+\alpha)\Gamma(\alpha)p}{(1-M)}\end{aligned}$$

$$\times \int_a^t E_{\alpha,\alpha}(-q\,[\varphi(t)-\varphi(s)]^{\alpha})\,[\varphi(t)-\varphi(s)]^{\alpha-1}$$
$$\times E_{\alpha}(-q\,(\varphi(s)-\varphi(a))^{\alpha})\varphi'(s)\,ds$$
$$+\int_a^t E_{\alpha,\alpha}(-q\,[\varphi(t)-\varphi(s)]^{\alpha})\,[\varphi(t)-\varphi(s)]^{\alpha-1}$$
$$\times\left(\int_a^s k(s-\sigma)\,|w(\sigma)|\,d\sigma\right)\varphi'(s)\,ds.$$

Next, in view of (6.2), we find

$$\begin{aligned}
|w(t)| \leq\ & 2w_0 E_{\alpha}\left(-q\,[\varphi(t)-\varphi(a)]^{\alpha}\right) \\
& + pA\frac{3w_0 + w_0\Gamma(\alpha)\Gamma(\alpha+1)p}{(1-M)} E_{\alpha}(-q\,(\varphi(t)-\varphi(a))^{\alpha}) \\
& + qpA\frac{3w_0 + w_0\Gamma(1+\alpha)\Gamma(\alpha)p}{(1-M)} \\
& \times \frac{\Gamma(1+\alpha)\Gamma(\alpha)}{q} E_{\alpha}\left(-q\,[\varphi(t)-\varphi(a)]^{\alpha}\right) \\
& + \int_a^t [\varphi(t)-\varphi(s)]^{\alpha-1} E_{\alpha,\alpha}(-q\,[\varphi(t)-\varphi(s)]^{\alpha}) \\
& \times\left(\int_a^s k(s-\sigma)\,|w(\sigma)|\,d\sigma\right)\varphi'(s)\,ds.
\end{aligned}$$

or

$$\begin{aligned}
(1-M)\frac{|w(t)|}{E_{\alpha}\left(-q\,[\varphi(t)-\varphi(a)]^{\alpha}\right)} \leq\ & 2w_0 + \frac{pAw_0\,[1+\Gamma(1+\alpha)\Gamma(\alpha)]}{(1-M)} \\
& \times (3+\Gamma(1+\alpha)\Gamma(\alpha)p) \\
\leq\ & 2w_0 + \frac{3Aw_0\,[1+\Gamma(1+\alpha)\Gamma(\alpha)]}{(1-M)}p \\
& + \frac{Aw_0\,[1+\Gamma(1+\alpha)\Gamma(\alpha)]^2}{(1-M)}p^2. \qquad (6.4)
\end{aligned}$$

For $t \in [a+2\upsilon, a+3\upsilon]$, by virtue of (3.14),

$$\begin{aligned}
\frac{E_{\alpha}(-q\,(\varphi(t-\upsilon)-\varphi(a))^{\alpha})}{E_{\alpha}(-q\,(\varphi(t)-\varphi(a))^{\alpha})} &\leq \frac{1}{(\varphi(a+\upsilon)-\varphi(a))^{\alpha}} \\
\times\left(\frac{1}{q}+\Gamma(1-\alpha)\,(\varphi(a+3\upsilon)-\varphi(a))^{\alpha}\right) &= :V>1,
\end{aligned}$$

and therefore

$$\begin{aligned}
|w(t)| \leq\ & 2w_0 E_{\alpha}\left(-q\,[\varphi(t)-\varphi(a)]^{\alpha}\right) + \frac{pV}{(1-M)} \\
& \times\left[2w_0 + \frac{3Aw_0\,[1+\Gamma(1+\alpha)\Gamma(\alpha)]}{(1-M)}p + \frac{Aw_0\,[1+\Gamma(1+\alpha)\Gamma(\alpha)]^2}{(1-M)}p^2\right] \\
& \times E_{\alpha}(-q\,(\varphi(t)-\varphi(a))
\end{aligned}$$

$$+\frac{pV\Gamma(1+\alpha)\Gamma(\alpha)}{(1-M)}$$
$$\times\left[2w_0+\frac{3Aw_0\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]}{(1-M)}p+\frac{Aw_0\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]^2}{(1-M)}p^2\right]$$
$$\times E_\alpha(-q\left(\varphi(t)-\varphi(a)\right)$$
$$+\int_a^t\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha)\left(\int_a^s k(s-\sigma)\left|w(\sigma)\right|d\sigma\right)\varphi'(s)\,ds.$$

So,

$$\begin{aligned}
&(1-M)\frac{|w(t)|}{E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)}\\
\le\ & 2w_0+\frac{pV}{(1-M)}\left[2w_0+\frac{3Aw_0\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]}{(1-M)}p+\frac{Aw_0\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]^2}{(1-M)}p^2\right]\\
&+\frac{pV\Gamma(1+\alpha)\Gamma(\alpha)}{(1-M)}\\
&\times\left[2w_0+\frac{3Aw_0\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]}{(1-M)}p+\frac{Aw_0\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]^2}{(1-M)}p^2\right],
\end{aligned}$$

or

$$\begin{aligned}
(1-M)\frac{|w(t)|}{E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)} \le\ & 2w_0+2w_0\frac{pV\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]}{1-M}\\
&+3w_0A\frac{p^2V\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]^2}{(1-M)^2}\\
&+\frac{Aw_0V\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]^3}{(1-M)^2}p^3. \qquad (6.5)
\end{aligned}$$

Writing (6.5) in the form

$$\begin{aligned}
\frac{|w(t)|}{E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)}(1-M) \le\ & 2w_0+2w_0\frac{pV\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]}{1-M}\\
&+3w_0A\left(\frac{pV\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]}{(1-M)}\right)^2\\
&+w_0A\left(\frac{pV\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]}{1-M}\right)^3\\
\le\ & 3w_0A\left[1+\frac{pV\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]}{1-M}\right.\\
&+\left(\frac{pV\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]}{1-M}\right)^2\\
&\left.+\left(\frac{pV\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]}{1-M}\right)^3\right] \qquad (6.6)
\end{aligned}$$

provides the basis for our next claim.

Claim. On the interval $[a+(n-1)\upsilon, a+n\upsilon]$, it is clear that

$$\frac{|w(t)|}{E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)}\left(\frac{1-M}{w_0}\right)\leq 3A\sum_{k=0}^{n}\left(\frac{Vp\left[1+\Gamma(1+\alpha)\Gamma(\alpha)\right]}{1-M}\right)^k.$$

The validity of the claim for $n = 1, 2$, and 3 is established by Eqs (6.3), (6.4), and (6.6). Let $t \in [a+n\upsilon, a+(n+1)\upsilon]$. Then from (6.1),

$$\begin{aligned}|w(t)| \leq\ & 2w_0E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)\\ &+3ApV\frac{w_0}{1-M}\sum_{k=0}^{n}\left(\frac{Vp\left[\Gamma(\alpha)\Gamma(1+\alpha)+1\right]}{1-M}\right)^kE_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)\\ &+3Ap\Gamma(1+\alpha)\Gamma(\alpha)\frac{w_0V}{1-M}\sum_{k=0}^{n}\left(\frac{Vp\left[\Gamma(\alpha)\Gamma(1+\alpha)+1\right]}{1-M}\right)^k\\ &\times E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)\\ &+\int_a^t E_{\alpha,\alpha}(-q\left[\varphi(t)-\varphi(s)\right]^\alpha)\left[\varphi(t)-\varphi(s)\right]^{\alpha-1}\left(\int_a^s k(s-\sigma)\left|w(\sigma)\right|d\sigma\right)\varphi'(s)\,ds,\end{aligned}$$

or

$$\begin{aligned}\left(\frac{1-M}{w_0}\right)\frac{|w(t)|}{E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)} \leq\ & 2+\frac{3VpA}{1-M}\sum_{k=0}^{n}\left(\frac{Vp\left[\Gamma(1+\alpha)\Gamma(\alpha)+1\right]}{1-M}\right)^k\\ &+\frac{3VpA}{1-M}\Gamma(1+\alpha)\Gamma(\alpha)\\ &\times\sum_{k=0}^{n}\left(\frac{Vp\left[\Gamma(1+\alpha)\Gamma(\alpha)+1\right]}{1-M}\right)^k.\end{aligned}$$

Then,

$$\begin{aligned}&\left(\frac{1-M}{w_0}\right)\frac{|w(t)|}{E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)}\\ \leq\ & 3A\left\{1+\left[\Gamma(1+\alpha)\Gamma(\alpha)+1\right]\frac{pV}{1-M}\sum_{k=0}^{n}\left(\frac{Vp\left[\Gamma(1+\alpha)\Gamma(\alpha)+1\right]}{1-M}\right)^k\right\},\end{aligned}$$

i.e.,

$$\left(\frac{1-M}{w_0}\right)\frac{|w(t)|}{E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)}\leq 3A\left\{1+\sum_{k=0}^{n}\left(\frac{Vp\left[\Gamma(1+\alpha)\Gamma(\alpha)+1\right]}{1-M}\right)^{1+k}\right\}.$$

Thus,

$$\left(\frac{1-M}{w_0}\right)\frac{|w(t)|}{E_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right)}\leq 3A\sum_{k=0}^{n+1}\left(\frac{Vp\left[\Gamma(1+\alpha)\Gamma(\alpha)+1\right]}{1-M}\right)^k,$$

demonstrating that the assertion holds. Moreover, the series converges if the following condition is satisfied:

$$\frac{1+\Gamma(1+\alpha)\Gamma(\alpha)}{1-M}Vp<1.$$

We have just proved the following result.

Theorem 6.1. *Suppose that $u(t)$ is a solution of*

$$\begin{cases} D_C^{\varphi,\alpha}\left[u(t)-pu(t-\upsilon)\right] \leq -qu(t)+\int_0^t k(t-s)u(s)\,ds,\ t,\upsilon>a,\ p>0,\ 0<\alpha<1, \\ u(t)=\varpi(t),\ a-\upsilon\leq t\leq a, \end{cases}$$

with $|\varpi(t)| \leq E_\alpha(-q(\varphi(t+\upsilon)-\varphi(a))^\alpha)$, $a-\upsilon\leq t\leq a$, $q>0$, $p>0$, *and k is a nonnegative function verifying*

$$\begin{aligned} &\int_a^t [\varphi(t)-\varphi(s)]^{\alpha-1} E_{\alpha,\alpha}\left(-q\left[\varphi(t)-\varphi(s)\right]^\alpha\right) \\ &\times\left(\int_a^s E_\alpha(-q\left[\varphi(\sigma)-\varphi(a)\right]^\alpha)k(s-\sigma)d\sigma\right)\varphi'(s)\,ds \\ \leq\ & ME_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right),\ t>a, \end{aligned}$$

for some M such that

$$M<1-\left[\Gamma(1+\alpha)\Gamma(\alpha)+1\right]Vp,$$

with

$$\left[\Gamma(1+\alpha)\Gamma(\alpha)+1\right]Vp<1.$$

Then,

$$|w(t)| \leq CE_\alpha\left(-q\left[\varphi(t)-\varphi(a)\right]^\alpha\right),\ t>a,$$

where $C>0$ is a positive constant.

7. Application in neural network theory

Neural networks are a fundamental part of artificial intelligence and are widely used to address complex problems in various fields. In this work, we utilize our findings to analyze the behavior of Cohen-Grossberg neural networks. Specifically, we consider the following problems:

$$\begin{cases} D_C^{\varphi,\alpha}\left[x_i(t)-px_i(t-\upsilon)\right] = -h_i\left(x_i(t)\right)\Big[g_i\left(x_i(t)\right)-\sum\limits_{j=1}^n a_{ij}f_j\left(x_j(t)\right)-\sum\limits_{j=1}^n b_{ij}l_j\left(x_j(t-\tau)\right) \\ -\sum\limits_{j=1}^n d_{ij}\int_a^\infty k_j(s)\,\Phi_j\left(x_j(t-s)\right)ds-I_i\Big],\ \ t,\upsilon>a,\ p>0 \\ x_i(t)=x_{i0}(t),\ t\in[a-\upsilon,a],\ i=1,2,...,n, \end{cases}$$

and

$$\begin{cases} D_C^{\varphi,\alpha}\left[x_i(t)-p\int_a^t x_i(s)\psi_i(t-s)ds\right] = -h_i\left(x_i(t)\right)\Big[g_i\left(x_i(t)\right)-\sum\limits_{j=1}^n b_{ij}l_j\left(x_j(t-\tau)\right) \\ -\sum\limits_{j=1}^n a_{ij}f_j\left(x_j(t)\right)-\sum\limits_{j=1}^n d_{ij}\int_a^\infty \Phi_j\left(x_j(t-s)\right)k_j(s)\,ds-I_i\Big],\ \ t>a,\ p>0, \\ x_i(0)=x_{i0}(t),\ t\leq a,\ i=1,2,...,n, \end{cases}$$

where $x_{i.}(t)$ stands the state of the ith neuron, n is the number of neurons, g_i is a suitable function, h_i represents an amplification function, b_{ij}, a_{ij}, d_{ij} represent the weights or strengths of the connections from the jth neuron to the ith neuron, I_i is the external input to the ith neuron, ψ_i are the neutral delay kernels, f_j, l_j, Φ_j denote the signal transmission functions, υ is the neutral delay, τ corresponds to the transmission delay, ϕ_i is the history of the ith state, and k_j denotes the delay kernel function. These systems represent a general class of Cohen-Grossberg neural networks with both continuously distributed and discrete delays. To streamline our analysis and highlight our key findings, we have opted to examine simpler systems with fixed time delays. More complex scenarios involving variable delays or multiple delays can be explored in future research. To simplify our analysis, let us examine the simpler case

$$\begin{cases} D_C^{\varphi,\alpha}\left[x_i(t) - px_i(t-\upsilon)\right] = -h_i\left(x_i(t)\right)\left[g_i\left(x_i(t)\right) - \sum\limits_{j=1}^{n} d_{ij}\int_a^\infty k_j(s) f_j\left(x_j(t-s)\right)ds - I_i\right], \\ x_i(t) = x_{i0}(t),\ t \in [a-\upsilon, a],\ i = 1,2,...,n, \end{cases} \tag{7.1}$$

for $t, \upsilon > a$, $p > 0$.

We adopt the following standard assumptions.

(A1) The functions f_i are assumed to satisfy the Lipschitz condition

$$|f_i(x) - f_i(y)| \leq L_i |x-y| \text{ for every } x, y \in \mathbb{R} \text{ and for each } i = 1,2,...,n,$$

where L_i denotes the Lipschitz constant corresponding to the function f_i.

(A2) The delay kernel functions k_j are nonnegative and exhibit piecewise continuity. Additionally, each k_j has a finite integral over its domain, expressed as $\kappa_j = \int_a^\infty k_j(s)\,ds < \infty$, for $j = 1,...,n$.

(A3) The functions g_i have derivatives that are uniformly bounded by a constant G. Specifically,

$$\left|g_i'(z)\right| \leq G, \text{ for all } z \in \mathbb{R} \text{ and for each } i = 1,2,...,n,$$

where $G > 0$ is a fixed constant.

(A4) The functions h_i are strictly positive and continuous, and they satisfy the following bounds:

$$0 < \underline{\beta}_i \leq h_i(z) \leq \overline{\beta}_i, \text{ for all } z \in \mathbb{R} \text{ and } i = 1,2,...,n.$$

For simplicity, we suppose that the initial values $x_{i0}(t)$ are all zero for times before a.

Definition 7.1. *The point* $x^* = \left(x_1^*, x_2^*, ..., x_n^*\right)^T$ *is said to be an equilibrium if, for each* $i = 1,2,...,n$, *it satisfies the equation*

$$\begin{aligned} g_i\left(x_i^*\right) &= \sum_{j=1}^{n} a_{ij} f_j\left(x_j^*\right) + \sum_{j=1}^{n} d_{ij}\int_a^\infty k_j(s) f_j\left(x_j^*\right)ds + I_i \\ &= \sum_{j=1}^{n}\left(a_{ij} + d_{ij}\kappa_j\right) f_j\left(x_j^*\right) + I_i,\ t > a. \end{aligned}$$

Previous studies have shown that an equilibrium exists and is unique. To facilitate our analysis, we translate the equilibrium point to the origin of the coordinate system by using the substitution

$x(t) - x^* = y(t)$. This leads to the following:

$$\begin{cases} D_C^{\varphi,\alpha}\left[y_i(t) - py_i(t-\upsilon)\right] = -h_i\left(x_i^* + y_i(t)\right)\Big[\, g_i\left(y_i(t) + x_i^*\right) \\ \qquad\qquad - \sum\limits_{j=1}^{n} d_{ij}\int_a^t f_j\left(x_j^* + y_j(t-s)\right)k_j(s)\,ds - I_i\Big], \;\; t > a,\; i = 1, ..., n, \\ y_i(t) = \psi_i(t) := \phi_i(t) - x_i^*,\; t\in[a-\upsilon, a],\; i = 1, ..., n, \end{cases}$$

or

$$\begin{cases} D_C^{\varphi,\alpha}\left[y_i(t) - py_i(t-\upsilon)\right] = -H_i(y_i(t))\left[G_i(y_i(t))\right. \\ \left. - \sum\limits_{j=1}^{n} d_{ij}\int_0^t F_j\left(y_j(t-s)\right)k_j(s)\,ds\right], \;\; t > a,\; i = 1, ..., n, \\ y_i(t) = \psi_i(t) := \phi_i(t) - x_i^*,\; t\in[a-\upsilon, a],\; i = 1, ..., n, \end{cases}$$

where

$$F_i(y_i(t)) = f_i\left(y_i(t) + x_i^*\right) - f_i\left(x_i^*\right),\; G_i(y_i(t)) = g_i\left(y_i(t) + x_i^*\right) - g_i(x^*)$$
$$H_i(y_i(t)) = h_i\left(y_i(t) + x_i^*\right), \;\; t > a,\; i = 1, ..., n.$$

Using the mean value theorem, the following inequality can be established:

$$D_C^{\varphi,\alpha}\left|y_i(t) - py_i(t-\upsilon)\right| \le sgn\left[y_i(t) - py_i(t-\upsilon)\right] D_C^{\varphi,\alpha}\left[y_i(t) - py_i(t-\upsilon)\right]$$
$$= -sgn\left[y_i(t) - py_i(t-\upsilon)\right] H_i(y_i(t))\left[g_i'(\bar{x}_i(t))\,y_i(t) - \sum_{j=1}^{n} d_{ij}\int_a^\infty F_j\left(y_j(t-s)\right)k_j(s)\,ds\right].$$

By subtracting and adding the term $pg_i'(\bar{x}_i(t))\,y_i(t-\upsilon)$, we obtain

$$D_C^{\varphi,\alpha}\left|y_i(t) - py_i(t-\upsilon)\right| \le -sgn\left[y_i(t) - py_i(t-\upsilon)\right] H_i(y_i(t))\Big[g_i'(\bar{x}_i(t))\left[y_i(t) - py_i(t-\upsilon)\right]$$
$$+pg_i'(\bar{x}_i(t))\,y_i(t-\upsilon) - \sum_{j=1}^{n} d_{ij}\int_a^\infty F_j\left(y_j(t-s)\right)k_j(s)\,ds\Big], \;\; t > a,\; i = 1, ..., n,$$

or

$$D_C^{\varphi,\alpha}\left|y_i(t) - py_i(t-\upsilon)\right| \le -H_i(y_i(t))\,G\left|y_i(t) - py_i(t-\upsilon)\right| + pGH_i(y_i(t))\left|y_i(t-\upsilon)\right|$$
$$+H_i(y_i(t))\sum_{j=1}^{n} d_{ij}\int_a^\infty k_j(s)\,L_j\left|y_j(t-s)\right|ds, \;\; t > a,\; i = 1, 2, ..., n.$$

Therefore,

$$D_C^{\varphi,\alpha}\left|y_i(t) - py_i(t-\upsilon)\right| \le -G\underline{\beta}_i\left|y_i(t) - py_i(t-\upsilon)\right| + pG\overline{\beta}_i\left|y_i(t-\upsilon)\right|$$
$$+\overline{\beta}_i\sum_{j=1}^{n} L_j d_{ij}\int_a^\infty k_j(s)\left|y_j(t-s)\right|ds, \;\; t > a,\; i = 1, ..., n.$$

Finally, we consider the equation for w_i and rewrite it in the following form:

$$\left|w_i(t) - pw_i(t-\upsilon)\right| = E_\alpha(-G\underline{\beta}_i\left[\varphi(t) - \varphi(a)\right]^\alpha)\left|\Phi_i(a) - p\Phi_i(a-\upsilon)\right|$$
$$+\int_a^t\left[\varphi(t) - \varphi(s)\right]^{\alpha-1}E_{\alpha,\alpha}(-q\left[\varphi(t) - \varphi(s)\right]^\alpha)$$
$$\times\left(pG\overline{\beta}_i\left|w_i(t-\upsilon)\right| + \overline{\beta}_i\sum_{j=1}^{n} L_j d_{ij}\int_0^\infty k_j(s)\left|w_j(t-s)\right|ds\right)\varphi'(s)\,ds, \;\; t > a,\; i = 1, 2, ..., n.$$

The Mittag-Leffler stability of this problem follows directly from our earlier result.

8. Conclusions

We have investigated a general Halanay inequality of fractional order with distributed delays, incorporating delays of neutral type. General sufficient conditions were established to guarantee the Mittag-Leffler stability of the solutions, supported by illustrative examples. The rate of stability obtained appears to be the best achievable, consistent with previous findings in fractional-order problems.

Furthermore, we applied our theoretical results to a practical problem, demonstrating their applicability. Our analysis suggests that these results can be extended to more general cases, such as variable delays or systems involving additional terms. It is worth noting that the conditions on the various parameters within the system could potentially be improved, as we did not focus on optimizing the estimations and bounds. In this regard, exploring optimal bounds for the delay coefficient p and the kernel k would be an interesting direction for future research.

Use of Generative-AI tools declaration

The author declares that have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author sincerely appreciates the financial support and facilities provided by Imam Abdulrahman Bin Faisal University.

Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

References

1. C. T. H. Baker, A. Tang, *Generalized Halanay inequalities for Volterra functional differential equations and discretized versions, Invited plenary talk*, Volterra Centennial Meeting, UTA Arlington, June, 1996.
2. A. Boroomand, M. B. Menhaj, Fractional-order Hopfield neural networks, *Int. Conf. Neural Inf. Process.*, **5506** (2008), 883–890.
3. R. Caponetto, L. Fortuna, D. Porto, Bifurcation and chaos in noninteger order cellular neural networks, *Int. J. Bifurc. Chaos*, **8** (1998), 1527–1539.
4. B. Chen, J. Chen, Razumikhin-type stability theorems for functional fractional-order differential systems and applications, *Appl. Math. Comput.*, **254** (2015), 63–69. https://doi.org/10.1016/j.amc.2014.12.010
5. C. J. Cheng, T. L. Liao, J. J. Yan, C. C. Hwang, Globally asymptotic stability of a class of neutral-type neural networks with delays, *IEEE T. Syst. Man Cy. B*, **36** (2006), 1191–1195. https://doi.org/10.1109/TSMCB.2006.874677

6. M. Concezzi, R. Spigler, Some analytical and numerical properties of the Mittag-Leffler functions, *Fract. Calc. Anal. Appl.*, **18** (2015), 64–94. https://doi.org/10.1515/fca-2015-0006
7. O. Faydasicok, S. Arik, An analysis of global stability of a class of neutral-type neural systems with time delays, *Internat. J. Math. Comput. Sci.*, **6** (2012), 391–395.
8. R. Gorenflo, A. A. Kilbas, F. Mainardi, S. Y. Rogosin, *Mittag-Leffler functions, related topics and applications*, Berlin: Springer, 2014. https://doi.org/10.1007/978-3-662-43930-2
9. S. Grossberg, Nonlinear neural networks: Principles, mechanisms, and architectures, *Neural Networks*, **1** (1988), 17–61.
10. A. Halanay, *Differential equations*, New York: Academic Press, 1996.
11. B. B. He, H. C. Zhou, Y. Q. Chen, C. H. Kou, Asymptotical stability of fractional order systems with time delay via an integral inequality, *IET Control Theory A.*, **12** (2018), 1748–1754. https://doi.org/10.1049/iet-cta.2017.1144
12. L. V. Hien, V. Phat, H. Trinh, New generalized Halanay inequalities with applications to stability of nonlinear non-autonomous time-delay systems, *Nonlinear Dynam.*, **82** (2015), 563–575. https://doi.org/10.1007/s11071-015-2176-0
13. F. Jarad, T. Abdeljawad, Generalized fractional derivatives and Laplace transform, *Discrete Cont. Dyn.-s*, **13** (2020), 709–722.
14. Y. Ke, C. Miao, Stability analysis of fractional-order Cohen-Grossberg neural networks with time delay, *Int. J. Comput. Math.*, **92** (2015), 1102–1113. http://dx.doi.org/10.1080/00207160.2014.935734
15. R. P. Lippmann, D. M. Shahian, Coronary artery bypass risk prediction using neural networks, *Ann. Thorac. Surg.*, **63** (1997), 1635–1643.
16. B. Liu, W. Lu, T. Chen, Generalized Halanay inequalities and their applications to neural networks with unbounded time-varying delays, *IEEE T. Neural Networ.*, **22** (2011), 1508–1513. https://doi.org/10.1109/TNN.2011.2160987
17. X. Lou, Q. Ye, B. Cui, Exponential stability of genetic regulatory networks with random delays, *Neurocomputing*, **73** (2010), 759–769.
18. X. Luan, P. Shi, F. Liu, Robust adaptive control for greenhouse climate using neural networks, *Int. J. Robust Nonlin.*, **21** (2011), 815–826. https://doi.org/10.1002/hipo.20797
19. F. Mainardi, On some properties of the Mittag-Leffler function $E_\alpha(-t^\alpha)$, completely monotone for $t > 0$ with $0 < \alpha < 1$, *Discrete Cont. Dyn.-B*, **10** (2014), 2267–2278. https://doi.org/10.48550/arXiv.1305.0161
20. M. W. Michalski, *Derivatives of noninteger order and their applications*, Ph. D. Thesis, Polska Akademia Nauk, 1993.
21. S. Mohamed, K. Gopalsamy, Continuous and discrete Halanay-type inequalities, *B. Aust. Math. Soc.*, **61** (2000), 371–385.
22. P. Niamsup, Stability of time-varying switched systems with time-varying delay, *Nonlinear Anal.-Hybri.*, **3** (2009), 631–639.

23. T. Simon, Comparing Frechet and positive stable laws, *Electron. J. Probab.*, **19** (2014), 1–25. https://doi.org/10.1214/EJP.v19-3058

24. M. S. Ali, S. Saravanan, M. E. Rani, S. Elakkia, J. Cao, A. Alsaedi, et al., Asymptotic stability of Cohen-Grossberg BAM neutral type neural networks with distributed time varying delays, *Neural Process. Lett.*, **46** (2017), 991–1007. https://doi.org/10.1007/s11063-017-9622-6

25. H. Tian, The exponential asymptotic stability of singularly perturbed delay differential equations with a bounded lag, *J. Math. Anal. Appl.*, **270** (2002), 143–149. https://doi.org/10.1016/S0022-247X(02)00056-2

26. L. Wen, Y. Yu, W. Wang, Generalized Halanay inequalities for dissipativity of Volterra functional differential equations, *J. Math. Anal. Appl.*, **347** (2008), 169–178. https://doi.org/10.1016/j.jmaa.2008.05.007

27. Z. Zeng, J. Wang, X. Liao, Global asymptotic stability and global exponential stability of neural networks with unbounded time-varying delays, *IEEE T. Circuits-II*, **52** (2005), 168–173. https://doi.org/10.1109/TCSII.2004.842047

28. B. Zheng, Z. Wang, Mittag-Leffler synchronization of fractional-order coupled neural networks with mixed delays, *Appl. Math. Comput.*, **430** (2022), 127303. https://doi.org/10.1016/j.amc.2022.127303

AIMS Press

© 2025 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)