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Research article

Approximation by the heat kernel of the solution to the transport-diffusion equation with the time-dependent diffusion coefficient

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Abstract: In this paper, we examined the transport-diffusion equation in $\mathbb{R}^d$, where the diffusion is represented by the Laplace operator multiplied by a function $\kappa(t)$ dependent on time. We transformed the equation using the inverse function of $s(t) = \int_0^t \kappa(t')dt'$. This transformation allowed us to construct a family of approximate solutions by using the heat kernel and translation corresponding to the transport in each step of time discretization. We proved the uniform convergence of these approximate solutions and their first and second derivatives with respect to the spatial variables. We also showed that the limit function satisfies the transport-diffusion equation in the space $\mathbb{R}^d$.

Keywords: transport-diffusion equation; diffusion coefficient depending on time; approximation by heat kernel
Mathematics Subject Classification: 35K58, 35K15

1. Introduction

As a parabolic equation, the transport-diffusion equation was treated in many studies and well-consolidated methods are known [7]. Concerning the behavior of the solution of the transport-diffusion equation when the diffusion coefficient tends to zero, results have been obtained using the stochastic representation of the solution of a parabolic equation [4]. However, these results are expressed in the language of probability theory, and it might not be straightforward to translate them into the language of mathematical analysis. Furthermore, in this framework, the treatment of non-linear terms is not easy, see [9–11].

In recent years, a method inspired by the idea of the stochastic representation of the solution but formulated in the language of mathematical analysis without using probability concepts has been proposed. First, the convergence of approximate solutions constructed by the heat kernel at each discrete time step toward the solution of the transport-diffusion equation in $\mathbb{R}^d$ with a constant diffusion coefficient was demonstrated [12, 13]. Then, this result was generalized to the case of the equation considered in the half-space with homogeneous Dirichlet boundary condition [2, 5] and with homogeneous Neumann boundary condition [6]. Furthermore, using the approximate solutions of this type, the convergence of the solution of the transport-diffusion equation in $\mathbb{R}^d$ toward that of the transport equation were shown [1, 3]. On the other hand, in [8], approximate solutions for the transport-diffusion equation in $\mathbb{R}^d$ and their limit function are considered, and it was proved that the limit function belongs to the Hölder space corresponding to the regularity of the given functions and satisfies the equation.

In this paper, we consider the transport-diffusion equation in $\mathbb{R}^d$ where the diffusion is represented by the Laplace operator multiplied by a function $\kappa(t)$. More precisely, we consider the equation

$$\partial_t u(t,x) + v(t,x)\cdot\nabla u(t,x) = \kappa(t)\Delta u(t,x) + f(t,x,u(t,x)).$$

Here and throughout, $v\cdot\nabla = \sum_{i=1}^d v_i\partial_{x_i}$, and $\Delta = \sum_{i=1}^d \partial_{x_i}^2$.

Let us recall that in [12] and [13], the family of approximate solutions was defined on the discretized time family $\{t_k^{[n]}\}_{k=0}^\infty$, $n = 1, 2, \cdots$,

$$0 = t_0^{[n]} < t_1^{[n]} < \cdots < t_k^{[n]} < t_{k+1}^{[n]} < \cdots, \qquad t_k^{[n]} = k2^{-n},$$

using the heat kernel, i.e., the fundamental solution of the heat equation

$$\Theta_n(x) = \frac{1}{(4\pi\kappa\delta_n)^{d/2}}\exp(-\frac{|x|^2}{4\kappa\delta_n}), \qquad \delta_n = \frac{1}{2^n} = t_k^{[n]} - t_{k-1}^{[n]},$$

over each interval $[t_{k-1}^{[n]}, t_k^{[n]}]$ of the time discretization. The specific property $\Theta_n(x) = (\Theta_{n+1}*\Theta_{n+1})(x)$ of Gaussian functions was the technical basis in the demonstration of the convergence of the approximate solutions. But, if κ varies with the time t, this technique cannot be applied directly.

To overcome this difficulty arising from the non-constancy of the coefficient $\kappa(t)$, we transform the equation using the inverse function of the function $s(t) = \int_0^t \kappa(t')dt'$. This allows us to construct a family of approximate solutions similarly to the research in [12, 13]. However, to demonstrate their convergence, it is essential to examine carefully the inequalities used, in which the consequences of the transformation of the equation also step in. In what follows, we aim to identify a reasonably large class of functions $\kappa(t)$ for which we can obtain the convergence of the approximate solutions to the solution of the transport-diffusion equation.

2. General result with $\kappa = \kappa(t)$

Let us consider the equation for the unknown function $u(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$

$$\partial_t u(t,x) + v(t,x)\cdot\nabla u(t,x) = \kappa(t)\Delta u(t,x) + f(t,x,u(t,x)), \quad t>0,\ x\in\mathbb{R}^d, \tag{2.1}$$

where $\kappa(t) : \mathbb{R}_+ \to \mathbb{R}_+$, $v(t,x) : \mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}^d$, and $f(t,x,u) : \mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R} \to \mathbb{R}$ are given functions. Eq (2.1) will be envisaged with the initial condition

$$u(0,x) = u_0(x) \qquad x\in\mathbb{R}^d. \tag{2.2}$$

For the function $\kappa(t)$, we assume that

$$\kappa(t) > 0 \quad \text{a.e. in } \mathbb{R}_+, \tag{2.3}$$

and for any sequence $\{[a_n, b_n[\,\}_{n=1}^{\infty}$ of disjoint intervals contained in $\mathbb{R}_+$, we have

$$\forall \varepsilon > 0 \ \ \exists \delta > 0 \ \text{ such that, if } \ \sum_{n=1}^{\infty} \int_{a_n}^{b_n} \kappa(t)dt \leq \delta \text{ then } \ \sum_{n=1}^{\infty} (b_n - a_n) \leq \varepsilon, \tag{2.4}$$

$$\kappa(\cdot) \in L^1_{\rm loc}(\mathbb{R}_+). \tag{2.5}$$

The conditions (2.3) and (2.5) imply that the function

$$s(t) = \int_0^t \kappa(t')dt' \tag{2.6}$$

is invertible. We also note that the condition (2.4) implies that the inverse function of $s(t)$, denoted by $t(s)$, is absolutely continuous. As for the conditions on the functions $v(t, x)$ and $f(t, x, u)$, we will specify them in the statement of the theorem.

Next, we will consider a family of approximate solutions for Eq (2.1). For their definition, we will use the notation

$$\delta_n = 2^{-n}, \qquad n = 1, 2, \cdots, \tag{2.7}$$

and for each n, the function

$$\Theta_n(x) = \frac{1}{(4\pi\delta_n)^{d/2}} \exp(-\frac{|x|^2}{4\delta_n}), \qquad x \in \mathbb{R}^d. \tag{2.8}$$

We also introduce the class of functions

$$\Lambda = \{\, \varphi : D \to \mathbb{R}, \text{ continuous}, \sum_{n=1}^{\infty} \lambda_{\tau,n}(\varphi) < \infty, \ \ \forall \tau > 0 \,\}, \tag{2.9}$$

where $D = \mathbb{R}_+ \times \mathbb{R}^d$ or $D = \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$ and

$$\lambda_{\tau,n}(\varphi) = \sup\{\, |\varphi(r_1, x) - \varphi(r_2, x)| \ : \ r_1, r_2 \in [0, \tau], \ x \in \mathbb{R}^d, \ |r_1 - r_2| \leq \delta_n \,\} \tag{2.10}$$

if $D = \mathbb{R}_+ \times \mathbb{R}^d$, and

$$\lambda_{\tau,n}(\varphi) = \sup\{\, |\varphi(r_1, x, u) - \varphi(r_2, x, u)| \ : \ r_1, r_2 \in [0, \tau], \ x \in \mathbb{R}^d, \ u \in \mathbb{R}, \ |r_1 - r_2| \leq \delta_n \,\} \tag{2.11}$$

if $D = \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$.

In this paper, we use the notations

$$D^{\nu}_x = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \cdots \partial x_d^{\nu_d}}, \qquad D^{\nu}_{x,u} = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \cdots \partial x_d^{\nu_d} \partial u^{\nu_{d+1}}},$$

where

$$|\nu| = \sum_{j=1}^{d} \nu_j \quad \text{ if } \ \nu = (\nu_1, \cdots, \nu_d) \in \mathbb{N}^d,$$

$$|\nu| = \sum_{j=1}^{d+1} \nu_j \quad \text{if} \;\; \nu = (\nu_1, \cdots, \nu_d, \nu_{d+1}) \in \mathbb{N}^{d+1}.$$

Furthermore, we denote by $C_b(D)$ the class of continuous and bounded functions defined on the domain D.

The general result of the present work is the following theorem.

Theorem 1. *Suppose that the function $\kappa(t)$ satisfies conditions (2.3)–(2.5), and the functions $v(t, x)$ and $f(t, x, u)$ (with the function $\kappa(t)$) satisfy the conditions:*

$$\frac{1}{\kappa(t)} D_x^\nu v(t, x) \in C_b([0, \tau] \times \mathbb{R}^d)) \quad \forall \nu \in \mathbb{N}^d, \; |\nu| \le 3, \;\; \forall \tau > 0, \tag{2.12}$$

$$\frac{1}{\kappa(t(s))} D_x^\nu v(t(s), x) \in \Lambda \quad \forall \nu \in \mathbb{N}^d, \; |\nu| \le 2, \tag{2.13}$$

$$\frac{1}{\kappa(t)} \frac{D_{x,u}^\nu f(t, x, u)}{1 + |u|} \in C_b([0, \tau] \times \mathbb{R}^d \times \mathbb{R}) \quad \forall \nu \in \mathbb{N}^{d+1}, \; |\nu| \le 3, \; \forall \tau > 0, \tag{2.14}$$

$$\frac{1}{\kappa(t(s))} D_{x,u}^\nu f(t(s), x, u) \in \Lambda \quad \forall \nu \in \mathbb{N}^{d+1}, \; |\nu| \le 2, \tag{2.15}$$

$$D_x^\nu u_0(x) \in C_b(\mathbb{R}^d) \quad \forall \nu \in \mathbb{N}^d, \; |\nu| \le 3. \tag{2.16}$$

(In (2.13) and (2.15), the functions are considered as functions of (s, x) and (s, x, u), respectively.)

If we define

$$s_k^{[n]} = k\delta_n, \quad t_k^{[n]} = t(s_k^{[n]}), \quad n = 1, 2, \ldots, \quad k = 0, 1, 2, \ldots, \tag{2.17}$$

then, for any $\tau > 0$, the functions $u^{[n]}(t, x)$ defined by

$$u^{[n]}(t_0^{[n]}, x) = u_0(x), \tag{2.18}$$

$$u^{[n]}(t_k^{[n]}, x) = \int_{\mathbb{R}^d} \Theta_n(y) u^{[n]}(t_{k-1}^{[n]}, x - \delta_n \frac{1}{\kappa(t_k^{[n]})} v(t_k^{[n]}, x) - y)\, dy \tag{2.19}$$

$$+ \delta_n \frac{1}{\kappa(t_{k-1}^{[n]})} f(t_{k-1}^{[n]}, x, u^{[n]}(t_{k-1}^{[n]}, x)), \qquad k = 1, 2, \cdots,$$

$$u^{[n]}(t, x) = \frac{s_k^{[n]} - s(t)}{\delta_n} u^{[n]}(t_{k-1}^{[n]}, x) + \frac{s(t) - s_{k-1}^{[n]}}{\delta_n} u^{[n]}(t_k^{[n]}, x) \quad \text{for } t_{k-1}^{[n]} \le t \le t_k^{[n]} \tag{2.20}$$

(with $s(t)$ and $s_k^{[n]}$ defined by (2.6) and (2.17)), and their first and second derivatives with respect to $x \in \mathbb{R}^d$, converge uniformly on $[0, \tau] \times \mathbb{R}^d$ toward a function $u(t, x)$ and its first and second derivatives with respect to $x \in \mathbb{R}^d$, and the limit function $u(t, x)$ satisfy Eq (2.1) and the initial condition (2.2) in the sense of integral equality:

$$- \int_0^\infty u(t, x) \frac{d}{dt} \varphi(t) dt - u_0(x)\varphi(0) + \int_0^\infty v(t, x) \cdot \nabla u(t, x) \varphi(t) dt \tag{2.21}$$

$$= \int_0^\infty (\kappa(t) \Delta u(t, x) + f(t, x, u)) \varphi(t) dt$$

for any $\varphi(\cdot) \in C^1(\mathbb{R}_+)$ such that $\varphi(t) = 0$ for $t \ge \tau_1$ with $\tau_1 > 0$.

3. Case of functions given with Hölder continuity

Since the conditions (2.13) and (2.15) are formulated through the inverse function $t(s)$, to have a more concrete idea, we mention a class of functions $(\kappa(t), v(t,x), f(t,x,u))$ that satisfies the conditions (2.13) and (2.15).

Lemma 2. *Suppose that the function $\kappa(t)$ satisfies the conditions* (2.3) *and* (2.5)*, and for each $\tau > 0$, there exists a number $\alpha = \alpha(\tau)$, $0 < \alpha < 1$, and a constant $C_1 = C_1(\tau)$ such that*

$$t_2 - t_1 \le C_1 \left(\int_{t_1}^{t_2} \kappa(t) dt \right)^{\alpha} \quad \forall t_1, t_2 \in [0, \tau],\ t_1 < t_2. \tag{3.1}$$

Furthermore, suppose that for each $\tau > 0$, there exist numbers $\beta = \beta(\tau)$ and $\gamma = \gamma(\tau)$ such that $0 < \beta < 1$ and $0 < \gamma < 1$, and the functions $v(t,x)$ and $f(t,x,u)$ satisfy the relations:

$$\sup_{0 \le t_1 < t_2 \le \tau, x \in \mathbb{R}^d} \frac{|D_x^\nu v(t_2,x) - D_x^\nu v(t_2,x)|}{(t_2 - t_1)^\beta} < \infty, \quad \forall \nu \in \mathbb{N}^d,\ |\nu| \le 2, \tag{3.2}$$

$$\sup_{0 \le t_1 < t_2 \le \tau, x \in \mathbb{R}^d, u \in \mathbb{R}} \frac{|D_{x,u}^\nu f(t_2,x,u) - D_x^\nu f(t_2,x,u)|}{(t_2 - t_1)^\gamma} < \infty, \quad \forall \nu \in \mathbb{N}^{d+1},\ |\nu| \le 2. \tag{3.3}$$

Then the functions $v(t(s),x)$ and $f(t(s),x,u)$ (with the function $\kappa(t(s))$) satisfy the conditions (2.13) *and* (2.15).

Proof. Fix $\tau > 0$. From (3.1) and (3.2), we deduce that there exists a constant $C_2 = C_2(\tau)$ such that

$$\lambda_{\tau,n}\left(\frac{1}{\kappa(t(\cdot))} D_x^\nu v(t(\cdot),\cdot)\right) \le C_2 \sup_{0 \le t_1 < t_2 \le \tau,\ s(t_2) - s(t_1) \le \delta_n} |t_2 - t_1|^\beta \le C_2 C_1^\beta (\delta_n)^{\beta\alpha}.$$

Similarly, from (3.1) and (3.3), we deduce that there exists a constant $C_3 = C_3(\tau)$ such that

$$\lambda_{\tau,n}\left(\frac{1}{\kappa(t(\cdots))} D_x^\nu f(t(\cdots),\cdot,\cdot)\right) \le C_3 \sup_{0 \le t_1 < t_2 \le \tau,\ s(t_2) - s(t_1) \le \delta_n} |t_2 - t_1|^\gamma \le C_3 C_1^\gamma (\delta_n)^{\gamma\alpha}.$$

Since $0 < \beta\alpha < 1$ and $0 < \gamma\alpha < 1$, it follows that

$$\sum_{n=1}^{\infty} \lambda_{\tau,n}\left(\frac{1}{\kappa(t(\cdot))} D_x^\nu v(t(\cdot),\cdot)\right) \le C_2 C_1^\beta \sum_{n=1}^{\infty} (\delta_n)^{\beta\alpha} < \infty,$$

$$\sum_{n=1}^{\infty} \lambda_{\tau,n}\left(\frac{1}{\kappa(t(\cdot))} D_{x,u}^\nu f(t(\cdot),\cdot,\cdot)\right) \le C_3 C_1^\gamma \sum_{n=1}^{\infty} (\delta_n)^{\gamma\alpha} < \infty,$$

which means that the functions $v(t(s),x)$ and $f(t(s),x,u)$ (with the function $\kappa(t(s))$) satisfy conditions (2.13) and (2.15). The lemma is proved. □

With Lemma 2 proved, under the assumption of the validity of Theorem 1, we can state the following result, which immediately follows from Theorem 1 and Lemma 2.

Corollary 3. *Suppose that the functions $\kappa(t)$, $v(t,x)$, $f(t,x,u)$, and $u_0(x)$ satisfy the conditions* (2.3)–(2.5), (2.12), (2.14), (2.16)*, and* (3.1)–(3.3). *Then, for any $\tau > 0$, the functions $u^{[n]}(t,x)$ defined by* (2.18)–(2.20) *(with $s(t)$ and $s_k^{[n]}$ defined by* (2.6) *and* (2.17)*) and their first and second derivatives with respect to $x \in \mathbb{R}^d$ converge uniformly in $[0,\tau] \times \mathbb{R}^d$ to a function $u(t,x)$ and its first and second derivatives with respect to $x \in \mathbb{R}^d$, and the limit function $u(t,x)$ satisfies equation* (2.1) *and the initial condition* (2.2) *in the sense of integral equality* (2.21).

4. Theorem for the equation with the κ constant

We will prove Theorem 1 by transforming Eq (2.1) into an equation with the diffusion coefficient $\kappa = 1$. Theorem 1 will result from the theorem for the equation with the constant diffusion coefficient κ, which we will prove subsequently.

Consider the equation

$$\partial_t u(t,x) + v(t,x)\cdot\nabla u(t,x) = \Delta u(t,x) + f(t,x,u(t,x)), \qquad t>0,\ x\in\mathbb{R}^d, \tag{4.1}$$

and the initial condition

$$u(0,x) = u_0(x), \qquad x\in\mathbb{R}^d. \tag{4.2}$$

The first term on the right-hand side of (4.1) can be $\kappa\Delta u(t,x)$ instead of $\Delta u(t,x)$, but this generalization is almost immediate. Therefore, here we consider the equation in the form of (4.1). For this problem, we have the following theorem.

Theorem 4. *Suppose that*

$$D_x^\nu v(t,x)\in C_b([0,\tau]\times\mathbb{R}^d)) \quad \forall\nu\in\mathbb{N}^d,\ |\nu|\le 3,\ \ \forall\tau>0, \tag{4.3}$$

$$D_x^\nu v(t,x)\in\Lambda \quad \forall\nu\in\mathbb{N}^d,\ |\nu|\le 2, \tag{4.4}$$

$$\frac{D_{x,u}^\nu f(t,x,u)}{1+|u|}\in C_b([0,\tau]\times\mathbb{R}^d) \quad \forall\nu\in\mathbb{N}^{d+1},\ |\nu|\le 3,\ \ \forall\tau>0, \tag{4.5}$$

$$D_{x,u}^\nu f(t,x,u)\in\Lambda \quad \forall\nu\in\mathbb{N}^{d+1},\ |\nu|\le 2, \tag{4.6}$$

$$D_x^\nu u_0(x)\in C_b(\mathbb{R}^d) \quad \forall\nu\in\mathbb{N}^d,\ |\nu|\le 3, \tag{4.7}$$

where Λ is the function class defined by (2.9)*. Let us also define*

$$t_k^{[n]} = k\delta_n, \qquad \delta_n = 2^{-n}. \tag{4.8}$$

Then, for any $\tau>0$, the functions $u^{[n]}(t,x)$ defined by

$$u^{[n]}(t_0^{[n]},x) = u_0(x), \tag{4.9}$$

$$\begin{aligned} u^{[n]}(t_k^{[n]},x) = &\int_{\mathbb{R}^d}\Theta_n(y)u^{[n]}(t_{k-1}^{[n]}, x-\delta_n v(t_k^{[n]},x)-y)dy \\ &+\delta_n f(t_{k-1}^{[n]},x,u^{[n]}(t_{k-1}^{[n]},x)), \quad k=1,2,\cdots, \end{aligned} \tag{4.10}$$

$$u^{[n]}(t,x) = \frac{t_k^{[n]}-t}{\delta_n}u^{[n]}(t_{k-1}^{[n]},x) + \frac{t-t_{k-1}^{[n]}}{\delta_n}u^{[n]}(t_k^{[n]},x) \quad \text{for } t_{k-1}^{[n]}\le t\le t_k^{[n]}, \tag{4.11}$$

and their first and second derivatives with respect to $x\in\mathbb{R}^d$, converge uniformly in $[0,\tau]\times\mathbb{R}^d$ to a function $u(t,x)$ and its first and second derivatives with respect to $x\in\mathbb{R}^d$, and the limit function $u(t,x)$ satisfies Eq (4.1) *and the initial condition in the sense of the integral equality*

$$-\int_0^\infty u(t,x)\frac{d}{dt}\varphi(t)dt - u_0(x)\varphi(0) + \int_0^\infty v(t,x)\cdot\nabla u(t,x)\varphi(t)dt = \int_0^\infty(\Delta u(t,x)+f(t,x,u))\varphi(t)dt \tag{4.12}$$

for every $\varphi(\cdot)\in C^1(\mathbb{R}_+)$ such that $\varphi(t)=0$ for $t\ge\tau_1$ with a $\tau_1>0$.

The proof of Theorem 4 follows the scheme developed in [12, 13]. The demonstration of the estimates of the approximate solutions and their derivatives with respect to $x \in \mathbb{R}^d$ does not differ from that presented in [12, 13] only for simple modifications. However, to prove the convergence of the approximate solutions, we need to specify the consequence of the conditions (4.4) and (4.6), which was not explicitly exposed in [12, 13]. Therefore, before we begin the essential part of the proof of Theorem 4, let us revisit the estimates of the approximate solutions.

Lemma 5. *Assume that the hypotheses of Theorem 4 hold. Let $u^{[n]}(t,x)$ be the functions defined by* (2.18)–(2.20). *Then, there exist functions $\Phi_0(t)$, $\Phi_1(t)$, $\Phi_2(t)$, and $\Phi_3(t)$ that are continuous on $\mathbb{R}_+$, increasing, independent of n, and such that we have*

$$\sup_{x\in\mathbb{R}^d} |u^{[n]}(t,x)| \leq \Phi_0(t), \tag{4.13}$$

$$\sum_{|\nu|=1} \sup_{x\in\mathbb{R}^d} \left| D_x^\nu u^{[n]}(t,x) \right| \leq \Phi_1(t), \tag{4.14}$$

$$\sum_{|\nu|=2} \sup_{x\in\mathbb{R}^d} \left| D_x^\nu u^{[n]}(t,x) \right| \leq \Phi_2(t), \tag{4.15}$$

$$\sum_{|\nu|=3} \sup_{x\in\mathbb{R}^d} \left| D_x^\nu u^{[n]}(t,x) \right| \leq \Phi_3(t) \tag{4.16}$$

for all $t \geq 0$ and for all $n \in \mathbb{N}\backslash\{0\}$.

Proof. The existence of $\Phi_0(t)$ satisfying (4.13) can be established similarly to Lemma 5 in [5] and Lemma 1 in [2]. We note that if we fix $\tau > 0$, according to condition (4.5), the function $D_{x,u}^\nu f(t,x,u)$ with $|\nu| \leq 3$ is continuous and bounded in $[0,\tau]\times\mathbb{R}^d$. Hence, the lemma can be demonstrated in a manner entirely analogous to [12] and [13]. □

5. Proof of Theorem 4

Having recalled the necessary estimates of the approximate solutions and their derivatives, we now proceed with the proof of Theorem 4.

Proof. We will demonstrate it in stages: Step 1 – Convergence of the approximate solutions.
Step 2 – Convergence of their first derivative. Step 3 – Convergence of their second derivatives.
Step 4 – Passage to the limit.
To simplify the presentation, we introduce the fellowing notations:

$$\lambda_{\tau,n}(v) = \max_{|\nu|\leq 2} \lambda_{\tau,n}\left(\frac{1}{\kappa(t(\cdot))} D_x^\nu v(t(\cdot),\cdot)\right),$$

$$\lambda_{\tau,n}(f) = \max_{|\nu|\leq 2} \lambda_{\tau,n}\left(\frac{1}{\kappa(t(\cdots))} D_x^\nu f(t(\cdots),\cdot,\cdot)\right),$$

$$\overline{\lambda}_{\tau,n} = \max(\lambda_{\tau,n}(v), \lambda_{\tau,n}(f))$$

(for the symbol $\lambda_{\tau,n}(\cdot)$, see (2.10) and (2.11)). Moreover, in different inequalities, we simply denote C (or C') as constants that may depend on τ but do not depend on n, when it is not necessary to specify them.

Step 1 –Convergence of the approximate solutions: We will prove that, for any $\tau > 0$, the functions $u^{[n]}(t,x)$ converge uniformly to a function $u(t,x)$ on $[0,\tau]\times\mathbb{R}^d$ as $n\to\infty$.

Let us first examine the difference between $u^{[n+1]}(t^{[n+1]}_{2k},x)$ and $u^{[n]}(t^{[n]}_{k},x)$ for $n = 1,2,\cdots$ and $k = 0,1,\cdots$ (recalling that $t^{[n+1]}_{2k} = t^{[n]}_{k}$ (see (4.8))). Applying the definition (4.10) twice and using the notation

$$\xi^{[n']}_{k'}(x,y) = x - \delta_{n'} v(t^{[n']}_{k'},x) - y \tag{5.1}$$

for $k' = k+1$ (or $2k+1$ or $2k+2$) and $n' = n$ (or $n+1$), we have

$$u^{[n+1]}(t^{[n+1]}_{2k+2},x) = I^{[n+1]}_{2k} + J^{[n+1]}_{a,2k} + J^{[n+1]}_{b,2k}, \tag{5.2}$$

where

$$I^{[n+1]}_{2k} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Theta_{n+1}(y_1)\Theta_{n+1}(y_2)u^{[n+1]}(t^{[n+1]}_{2k},\xi^*(y_1,y_2))dy_1dy_2, \tag{5.3}$$

$$\xi^*(y_1,y_2) = \xi^{[n+1]}_{2k+2}(x,y_1) - \delta_{n+1}v(t^{[n+1]}_{2k+1},\xi^{[n+1]}_{2k+2}(x,y_1)) - y_2,$$

$$J^{[n+1]}_{a,2k} = \delta_{n+1}\int_{\mathbb{R}^d}\Theta_{n+1}(y)$$

$$\times f(t^{[n+1]}_{2k},\xi^{[n+1]}_{2k+2}(x,y),u^{[n+1]}(t^{[n+1]}_{2k},\xi^{[n+1]}_{2k+2}(x,y)))dy, \tag{5.4}$$

$$J^{[n+1]}_{b,2k} = \delta_{n+1}f(t^{[n+1]}_{2k+1},x,U_1+U_2), \tag{5.5}$$

$$U_1 = \int_{\mathbb{R}^d}\Theta_{n+1}(y)u^{[n+1]}(t^{[n+1]}_{2k},\xi^{[n+1]}_{2k+1}(x,y))dy,$$

$$U_2 = \delta_{n+1}f(t^{[n+1]}_{2k},x,u^{[n+1]}(t^{[n+1]}_{2k},x)).$$

Therefore, recalling once again the definition (4.10) and the relation $\delta_n = 2\delta_{n+1}$, we have

$$u^{[n+1]}(t^{[n+1]}_{2k+2},x) - u^{[n]}(t^{[n]}_{k+1},x) = D_{(0)} + D_{(a)} + D_{(b)}, \tag{5.6}$$

where

$$D_{(0)} = I^{[n+1]}_{2k} - \int_{\mathbb{R}^d}\Theta_n(y)u^{[n]}(t^{[n]}_k,\xi^{[n]}_{k+1}(x,y))dy,$$

$$D_{(a)} = J^{[n+1]}_{a,2k} - \delta_{n+1}f(t^{[n]}_k,x,u^{[n]}(t^{[n]}_k,x)),$$

$$D_{(b)} = J^{[n+1]}_{b,2k} - \delta_{n+1}f(t^{[n]}_k,x,u^{[n]}(t^{[n]}_k,x)).$$

To estimate $D_{(0)}$, we note that, thanks to the well-known property of Gaussian functions, we have

$$\int_{\mathbb{R}^d}\Theta_n(z)u^{[n]}(t^{[n]}_k,\xi^{[n]}_{k+1}(x,z))dz = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Theta_{n+1}(y_1)\Theta_{n+1}(y_2)u^{[n]}(t^{[n]}_k,\xi^{[n]}_{k+1}(x,y_1+y_2))dy_1dy_2$$

and thus

$$D_{(0)} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Theta_{n+1}(y_1)\Theta_{n+1}(y_2)\times(u^{[n+1]}(t^{[n+1]}_{2k},\xi^*(y_1,y_2)) - u^{[n]}(t^{[n]}_k,\xi^{[n]}_{k+1}(x,y_1+y_2)))dy_1dy_2.$$

Now, based on the definition of $\lambda_{\tau,n+1}(\cdot)$, we have

$$|v(t_{2k+1}^{[n+1]},\xi)-v(t_{k+1}^{[n]},\xi)|=|v(t_{2k+1}^{[n+1]},\xi)-v(t_{2k+1}^{[n+1]}+\delta_{n+1},\xi)|\leq\lambda_{\tau,n+1}(v),$$

and recalling the expression for $\xi^*(y_1,y_2)$ and $\xi_{k+1}^{[n]}(x,y_1+y_2)$, we obtain

$$\begin{aligned}|\xi^*(y_1,y_2)-\xi_{k+1}^{[n]}(x,y_1+y_2)|&=\delta_{n+1}|v(t_{2k+1}^{[n+1]},\xi_{2k+2}^{[n+1]}(x,y_1))-v(t_{k+1}^{[n]},x)|\\&\leq\delta_{n+1}(\lambda_{\tau,n+1}(v)+\sup|\nabla v|(\delta_{n+1}\sup|v|+|y_1|)).\end{aligned}$$

Note that we simply write $\sup|\nabla v|$ instead of $\sup\limits_{(t,y)\in[0,\tau]\times\mathbb{R}^d}|\nabla v(t,x)|$, wherever this abbreviated notation does not cause ambiguity. It follows that

$$\begin{aligned}&\left|u^{[n+1]}(t_{2k}^{[n+1]},\xi^*(y_1,y_2))-u^{[n]}(t_k^{[n]},\xi_{k+1}^{[n]}(x,y_1+y_2))\right|\\&\leq\sup_{y\in\mathbb{R}^d}|u^{[n+1]}(t_{2k}^{[n+1]},y)-u^{[n]}(t_k^{[n]},y)|+\sup|\nabla u^{[n+1]}|(\delta_{n+1}\lambda_{\tau,n+1}(v)\\&+\delta_{n+1}\sup|\nabla v|(\delta_{n+1}\sup|v|+|y_1|)).\end{aligned}\tag{5.7}$$

Therefore, considering Lemma 5 and the relation $\int_{\mathbb{R}^d}\Theta_{n+1}(y_1)|y_1|dy_1=\dfrac{2\sqrt{\delta_{n+1}}}{\sqrt{\pi}}$, we deduce that there exists a constant K_1 such that

$$|D_{(0)}|\leq\sup_{y\in\mathbb{R}^d}|u^{[n+1]}(t_{2k}^{[n+1]},y)-u^{[n]}(t_k^{[n]},y)|+K_1\delta_{n+1}(\lambda_{\tau,n+1}(v)+\delta_{n+1}^{1/2})\sup|\nabla u^{[n+1]}|.\tag{5.8}$$

Regarding $D_{(a)}$, using the inequality

$$\begin{aligned}&\left|f(t_{2k}^{[n+1]},\xi_{2k+2}^{[n+1]}(x,y),u^{[n+1]}(t_{2k}^{[n+1]},\xi_{2k+2}^{[n+1]}(x,y)))-f(t_k^{[n]},x,u^{[n]}(t_k^{[n]},x))\right|\\&\leq\Big(\sup|\nabla_xf|+\sup|\partial_uf|\sup|\nabla u^{[n+1]}|\Big)\Big(\delta_{n+1}\sup|v|+|y|\Big)\\&+\sup|\partial_uf|\,|u^{[n+1]}(t_{2k}^{[n+1]},x)-u^{[n]}(t_k^{[n]},x)|,\end{aligned}$$

we easily obtain

$$|D_{(a)}|\leq K_2(\delta_{n+1}^2+\delta_{n+1}^{3/2})+K_2\delta_{n+1}|u^{[n+1]}(t_{2k}^{[n+1]},x)-u^{[n]}(t_k^{[n]},x)|\tag{5.9}$$

with a constant K_2 independent of n.

As for $D_{(b)}$, recalling the expression of U_1 and U_2 and considering the relation

$$|f(t_{2k+1}^{[n+1]},x,u^{[n+1]}(t_{2k}^{[n+1]},x))-f(t_{2k}^{[n+1]},x,u^{[n+1]}(t_{2k}^{[n+1]},x))|\leq\lambda_{\tau,n+1}(f)\ \ (see(2.11)),$$

we have

$$\begin{aligned}&\left|f(t_{2k+1}^{[n+1]},x,U_1+U_2)-f(t_k^{[n]},x,u^{[n]}(t_k^{[n]},x))\right|\\&\leq\sup|\partial_uf|\sup|\nabla u^{[n+1]}|(\delta_{n+1}\sup|v|+|y|)+\delta_{n+1}\sup|f|+\lambda_{\tau,n+1}(f)\\&+\sup|\partial_uf|\,|u^{[n+1]}(t_{2k}^{[n+1]},x)-u^{[n]}(t_k^{[n]},x)|.\end{aligned}$$

We deduce that there exists a constant K_3 independent of n such that

$$|D_{(b)}| \le K_3(\delta_{n+1}^2 + \delta_{n+1}(\lambda_{\tau,n+1}(f) + \delta_{n+1}^{1/2})) + K_3\delta_{n+1}|u^{[n+1]}(t_{2k}^{[n+1]}, x) - u^{[n]}(t_k^{[n]}, x)|. \tag{5.10}$$

Finally, from the relations (5.2), (5.6), and (5.8)–(5.10), we deduce that there exists a constant K_4 independent of n such that

$$|u^{[n+1]}(t_{2k+2}^{[n+1]}, x) - u^{[n]}(t_{k+1}^{[n]}, x)| \le (1 + K_4\delta_{n+1}) \sup_{y\in\mathbb{R}^d} |u^{[n+1]}(t_{2k}^{[n+1]}, y) - u^{[n]}(t_k^{[n]}, y)| + K_4\delta_{n+1}(\overline{\lambda}_{\tau,n+1} + \delta_{n+1}^{1/2}).$$

Therefore, if we set

$$Y_k = \sup_{x\in\mathbb{R}^d} |u^{[n+1]}(t_{2k}^{[n+1]}, x) - u^{[n]}(t_k^{[n]}, x)|, \tag{5.11}$$

we have

$$Y_{k+1} \le (1 + K_4\delta_{n+1})Y_k + K_4\delta_{n+1}(\overline{\lambda}_{\tau,n+1} + \delta_{n+1}^{1/2}), \tag{5.12}$$

where, considering the relation $Y_0 = 0$, we obtain

$$Y_k \le \delta_{n+1}(\overline{\lambda}_{\tau,n+1} + \delta_{n+1}^{1/2})K_4 \sum_{j=0}^{k} (1 + K_4\delta_{n+1})^{k-j} \le (\overline{\lambda}_{\tau,n+1} + \delta_{n+1}^{1/2})e^{tK_4}, \tag{5.13}$$

or

$$\sup_{x\in\mathbb{R}^d} |u^{[n+1]}(t, x) - u^{[n]}(t, x)| \le (\overline{\lambda}_{\tau,n+1} + \delta_{n+1}^{1/2})e^{tK_4} \qquad \text{for } 0 \le t \le \tau. \tag{5.14}$$

As under the assumptions (4.4) and (4.6) (also refer to (2.9)), we have

$$\sum_{n=1}^{\infty} (\overline{\lambda}_{\tau,n+1} + \delta_{n+1}^{1/2}) < \infty,$$

and considering also the independence of K_4 from n, it is evident that the inequality (5.14) and the definition (2.20) imply that the sequence $u^{[n]}(t, x)$ converges uniformly on $[0, \tau] \times \mathbb{R}^d$ as $n \to \infty$.

Step 2 –Convergence of the first derivatives of the approximate solutions –We will demonstrate that, for any $\tau > 0$, the functions $\partial_{x_i} u^{[n]}(t, x)$, $i = 1, \cdots, d$, converge to $\partial_{x_i} u(t, x)$ (where $u(t, x)$ is the limit function obtained in Step 1) uniformly on $[0, \tau] \times \mathbb{R}^d$ as $n \to \infty$.

We put

$$w_{i,k}^{[1,n]}(x) = \frac{\partial}{\partial x_i} u^{[n]}(t_k^{[n]}, x), \quad i = 1, \cdots, d, \tag{5.15}$$

and we will examine $w_{i,2k+2}^{[1,n+1]}(x) - w_{i,k+1}^{[1,n]}(x)$. To simplify the notation, let us introduce abbreviated notations:

$$u_k^{[n]}(x) = u^{[n]}(t_k^{[n]}, x), \quad f'_{i,k}(x, u^{[n]}(t_k^{[n]}, x)) = \frac{\partial f(t_k^{[n]}, x, u)}{\partial x_i}\Big|_{u=u^{[n]}(t_k^{[n]},x)},$$

and similarly for $f'_{u,k}(x, u^{[n]}(t_k^{[n]}, x))$. Additionally, denote by $\xi_{k',j}^{[n']}(x, y)$ the j-th component of the vector $\xi_{k'}^{[n']}(x, y)$ (see (5.1)).

By taking the derivative of both sides of equation (4.10) with respect to x_i, we obtain

$$w_{i,k}^{[1,n]}(x) = \int_{\mathbb{R}^d} \Theta_n(y)\Big[w_{i,k-1}^{[1,n]}(\xi(x, y)) - \delta_n \sum_{j=1}^{d} \frac{\partial v_j(t_k^{[n]}, x)}{\partial x_i} w_{j,k-1}^{[1,n]}(\xi(x, y))\Big]dy \tag{5.16}$$

$$+\,\delta_n f'_{u,k-1}(x, u^{[n]}(t^{[n]}_{k-1}, x))w^{[1,n]}_{i,k-1}(x) + \delta_n f'_{i,k-1}(x, u^{[n]}(t^{[n]}_{k-1}, x)).$$

On the other hand, by deriving with respect to x_i the right-hand side of (5.2), which corresponds to the terms given in (5.3)–(5.5), we get

$$w^{[1,n+1]}_{i,2k+2}(x) = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \Theta_{n+1}(y_1)\Theta_{n+1}(y_2)\Big[w^{[1,n+1]}_{i,2k}(\xi^*) - \delta_{n+1}\sum_{j=1}^{d} w^{[1,n+1]}_{j,2k}(\xi^*)(\partial_{x_i} v_j(t^{[n+1]}_{2k+2}, x) \tag{5.17}$$
$$+ \sum_{l=1}^{d} \partial_{x_l} v_j(t^{[n+1]}_{2k+1}, \xi^{[n+1]}_{2k+2}(x, y_1))\partial_{x_i}\xi^{[n+1]}_{2k+2,l}(x, y_1))\Big]dy_1 dy_2$$
$$+ \delta_{n+1}\int\limits_{\mathbb{R}^d} \Theta_{n+1}(y_1)\sum_{j=1}^{d}\Big[f'_{j,2k}(\xi^{[n+1]}_{2k+2}(x, y_1), u^{[n+1]}_{2k}(\xi^{[n+1]}_{2k+2}(x, y_1)))$$
$$+ f'_{u,2k}(\xi^{[n+1]}_{2k+2}(x, y_1), u^{[n+1]}_{2k}(\xi^{[n+1]}_{2k+2}(x, y_1)))w^{[1,n+1]}_{j,2k}(\xi^{[n+1]}_{2k+2}(x, y_1))\Big]\partial_{x_i}\xi^{[n+1]}_{2k+2,j}(x, y_1)dy_1$$
$$+ \delta_{n+1} f'_{i,2k+1}(x, u^{[n+1]}_{2k+1}(x)) + \delta_{n+1} f'_{u,2k+1}(x, u^{[n+1]}_{2k+1}(x))w^{[n+1]}_{i,2k+1}(x),$$

where $\xi^* = \xi^*(y_1, y_2)$ (see (5.3)).

From (5.16) and (5.17), it follows that

$$w^{[1,n+1]}_{i,2k+2}(x) - w^{[1,n]}_{i,k+1}(x) = \sum_{p=1}^{7} Z_p, \tag{5.18}$$

where

$$Z_1 = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \Theta_{n+1}(y_1)\Theta_{n+1}(y_2) \times (w^{[1,n+1]}_{i,2k}(\xi^*) - w^{[1,n]}_{i,k}(\xi^{[n]}_{k+1}(x, y_1 + y_2)))dy_1 dy_2, \tag{5.19}$$

$$Z_2 = \delta_{n+1}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \Theta_{n+1}(y_1)\Theta_{n+1}(y_2)\sum_{j=1}^{d} \zeta_{2,j}(x, y_1, y_2)dy_1 dy_2, \tag{5.20}$$

$$\zeta_{2,j}(x, y_1, y_2) = -\partial_{x_i} v_j(t^{[n+1]}_{2k+2}, x)w^{[1,n+1]}_{j,2k}(\xi^*) + \partial_{x_i} v_j(t^{[n]}_{k+1}, x)w^{[1,n]}_{j,k}(\xi^{[n]}_{k+1}(x, y_1 + y_2)),$$

$$Z_3 = \delta_{n+1}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \Theta_{n+1}(y_1)\Theta_{n+1}(y_2)\sum_{j=1}^{d} \zeta_{3,j}(x, y_1, y_2)dy_1 dy_2, \tag{5.21}$$

$$\zeta_{3,j}(x, y_1, y_2) = -\sum_{l=1}^{d} \partial_{x_l} v_j(t^{[n+1]}_{2k+1}, \xi^{[n+1]}_{2k+2}(x, y_1))\partial_{x_i}\xi^{[n+1]}_{2k+2,l}(x, y_1)w^{[1,n+1]}_{j,2k}(\xi^*)$$
$$+\partial_{x_i} v_j(t^{[n]}_{k+1}, x)w^{[1,n]}_{j,k}(\xi^{[n]}_{k+1}(x, y_1 + y_2)),$$

$$Z_4 = \delta_{n+1}\int\limits_{\mathbb{R}^d} \Theta_{n+1}(y_1)\sum_{j=1}^{d} \zeta_{4,j}(x, y_1)dy_1 - \delta_{n+1} f'_{i,k}(x, u^{[n]}_k(x)), \tag{5.22}$$

$$\zeta_{4,j}(x, y_1) = f'_{j,2k}(\xi^{[n+1]}_{2k+2}(x, y_1), u^{[n+1]}_{2k}(\xi^{[n+1]}_{2k+2}(x, y_1)))\partial_{x_i}\xi^{[n+1]}_{2k+2,j}(x, y_1),$$

$$Z_5 = \delta_{n+1} \int\limits_{\mathbb{R}^d} \Theta_{n+1}(y_1) \sum_{j=1}^{d} \zeta_{5,j}(x, y_1) dy_1 - \delta_{n+1} f'_{u,k}(x, u_k^{[n]}(x)) w_{i,k}^{[1,n]}(x), \tag{5.23}$$

$$\zeta_{5,j}(x, y_1) = f'_{u,2k}(\xi_{2k+2}^{[n+1]}(x, y_1), u_{2k}^{[n+1]}(\xi_{2k+2}^{[n+1]}(x, y_1))) \times w_{j,2k}^{[1,n+1]}(\xi_{2k+2}^{[n+1]}(x, y_1)) \partial_{x_i} \xi_{2k+2,j}^{[n+1]}(x, y_1),$$

$$Z_6 = \delta_{n+1}(f'_{i,2k+1}(x, u_{2k+1}^{[n+1]}(x)) - f'_{i,k}(x, u_k^{[n]}(x))), \tag{5.24}$$

$$Z_7 = \delta_{n+1}(f'_{u,2k+1}(x, u_{2k+1}^{[n+1]}(x)) w_{i,2k+1}^{[1,n+1]}(x) - f'_{u,k}(x, u_k^{[n]}(x)) w_{i,k}^{[1,n]}(x)). \tag{5.25}$$

As

$$\xi^* - \xi_{k+1}^{[n]}(x, y_1 + y_2) = \delta_{n+1}(v(t_{k+1}^{[n]}, x) - v(t_{2k+1}^{[n+1]}, \xi_{2k+2}^{[n+1]}(x, y_1))),$$

similarly to (5.7), we have

$$\begin{aligned} |w_{i,2k}^{[1,n+1]}(\xi^*) - w_{i,k}^{[1,n]}(\xi_{k+1}^{[n]}(x, y_1 + y_2))| \le \sup_{y \in \mathbb{R}^d} |w_{i,2k}^{[1,n+1]}(y) - w_{i,k}^{[1,n]}(y)| + \sup |\nabla w_{i,2k}^{[1,n+1]}| \\ \times (C\delta_{n+1}\lambda_{\tau,n+1}(v) + \delta_{n+1} \sup |\nabla v|(\delta_{n+1} \sup |v| + |y_1|)). \end{aligned} \tag{5.26}$$

Recall that according to Lemma 5, $|\nabla w_{i,2k}^{[1,n+1]}|$ is bounded by a constant independent of n. Thus, similarly to obtaining (5.8), we get

$$|Z_1| \le \sup_{y \in \mathbb{R}^d} |w_{i,2k}^{[1,n+1]}(y) - w_{i,k}^{[1,n]}(y)| + C'\delta_{n+1}(\lambda_{\tau,n+1}(v) + \delta_{n+1}^{1/2}). \tag{5.27}$$

Let us introduce the notation

$$Y_k^{[1]} = \sum_{i=1}^{d} \sup_{x \in \mathbb{R}^d} |w_{i,2k}^{[1,n+1]}(x) - w_{i,k}^{[1,n]}(x)|. \tag{5.28}$$

Recalling that $t_{2k+2}^{[n+1]} = t_{k+1}^{[n]}$ and using the inequality (5.26), which, written with j instead of i, is valid for all $j \in \{1, \cdots, d\}$, we obtain

$$|Z_2| \le \delta_{n+1} C(Y_k^{[1]} + C\delta_{n+1}(\lambda_{\tau,n+1}(v) + \delta_{n+1}^{1/2})). \tag{5.29}$$

Regarding $\zeta_{3,j}(x, y_1, y_2)$ that appears under the integration sign in (5.21), considering the relation

$$\partial_{x_i} \xi_{2k+2,l}^{[n+1]}(x, y_1) = \delta_{il} - \delta_{n+1} \partial_{x_i} v_l(t_{2k+2}^{[n+1]}, x), \tag{5.30}$$

where δ_{il} is the Kronecker symbol, we have

$$\begin{aligned} \zeta_{3,j}(x, y_1, y_2) = \delta_{n+1} \sum_{l=1}^{d} \partial_{x_l} v_j(t_{2k+1}^{[n+1]}, \xi_{2k+2}^{[n+1]}(x, y_1)) \partial_{x_i} v_l(t_{2k+2}^{[n+1]}, x) w_{j,2k}^{[1,n+1]}(\xi^*) \\ + (\partial_{x_i} v_j(t_{k+1}^{[n]}, \xi_{2k+2}^{[n+1]}(x, y_1)) - \partial_{x_i} v_j(t_{2k+1}^{[n+1]}, \xi_{2k+2}^{[n+1]}(x, y_1))) w_{j,2k}^{[1,n+1]}(\xi^*) \\ + (\partial_{x_i} v_j(t_{k+1}^{[n]}, x) - \partial_{x_i} v_j(t_{k+1}^{[n]}, \xi_{2k+2}^{[n+1]}(x, y_1))) w_{j,2k}^{[1,n+1]}(\xi^*) \end{aligned} \tag{5.31}$$

$$+ \partial_{x_i} v_j(t_{k+1}^{[n]}, x)(w_{j,2k}^{[1,n+1]}(\xi_{k+1}^{[n]}(x, y_1 + y_2)) - w_{j,2k}^{[1,n+1]}(\xi^*))$$
$$+ \partial_{x_i} v_j(t_{k+1}^{[n]}, x)(w_{j,k}^{[1,n]}(\xi_{k+1}^{[n]}(x, y_1 + y_2)) - w_{j,2k}^{[1,n+1]}(\xi_{k+1}^{[n]}(x, y_1 + y_2))).$$

As

$$|\partial_{x_i} v_j(t_{k+1}^{[n]}, \xi) - \partial_{x_i} v_j(t_{2k+1}^{[n+1]}, \xi)| \le \lambda_{\tau,n+1}(v) \qquad (\xi \in \mathbb{R}^d),$$
$$|\partial_{x_i} v_j(t_{k+1}^{[n]}, x) - \partial_{x_i} v_j(t_{k+1}^{[n]}, \xi_{2k+2}^{[n+1]}(x, y_1))| \le \sup |\nabla \partial_{x_i} v_j|(\delta_{n+1} \sup |v| + |y_1|),$$

given (5.26) and the values uniformly bounded by the hypotheses and Lemma 5, we have

$$|Z_3| \le C\delta_{n+1}(\lambda_{\tau,n+1}(v) + \delta_{n+1}^{1/2}) + \delta_{n+1} C Y_k^{[1]}. \tag{5.32}$$

Regarding Z_4, Z_5, Z_6, and Z_7, recalling the notation convention $f'_{\cdot,2k}(x,u) = f'_{\cdot,k}(x,u)$ and taking into account the relation

$$|f'_{\cdot,k}(x^{(1)}, u^{(1)}) - f'_{\cdot,k}(x^{(2)}, u^{(2)})| \le C(|x^{(1)} - x^{(2)}| + |u^{(1)} - u^{(2)}|)$$

and relations (4.6) and (5.30), we have

$$|Z_4 + Z_5 + Z_6 + Z_7| \le C\delta_{n+1}(\lambda_{\tau,n+1}(f) + \delta_{n+1}^{1/2}) + \delta_{n+1} C(Y_k + Y_k^{[1]}), \tag{5.33}$$

where Y_k is defined in (5.11).

By summing up the inequalities (5.27), (5.29), (5.32), and (5.33), we have

$$|w_{i,2k+2}^{[1,n+1]}(x) - w_{i,k+1}^{[1,n]}(x)| \le \sum_{p=1}^{7} |Z_p|$$
$$\le \sup_{y\in\mathbb{R}^d} |w_{i,2k}^{[1,n+1]}(y) - w_{i,k}^{[1,n]}(y)| + C\delta_{n+1}(\overline{\lambda}_{\tau,n+1} + \delta_{n+1}^{1/2}) + \delta_{n+1} C(Y_k + Y_k^{[1]}).$$

As this inequality holds for any $x \in \mathbb{R}^d$, summing up this inequality for $i = 1, \cdots, d$, we get

$$Y_{k+1}^{[1]} \le Y_k^{[1]} + C\delta_{n+1}(\overline{\lambda}_{\tau,n+1} + \delta_{n+1}^{1/2}) + \delta_{n+1} C(Y_k + Y_k^{[1]}). \tag{5.34}$$

If we set

$$\widetilde{Y}_k^{[1]} = Y_k + Y_k^{[1]}, \tag{5.35}$$

then, by adding (5.34) and (5.12), we obtain

$$\widetilde{Y}_{k+1}^{[1]} \le \widetilde{Y}_k^{[1]} + C'\delta_{n+1}(\overline{\lambda}_{\tau,n+1} + \delta_{n+1}^{1/2}) + \delta_{n+1} C' \widetilde{Y}_k^{[1]}. \tag{5.36}$$

Therefore, similarly to the derivation of (5.13) (and thus (5.14)), we obtain

$$\widetilde{Y}_k^{[1]} \le (\overline{\lambda}_{\tau,n+1} + \delta_{n+1}^{1/2}) e^{tC'}. \tag{5.37}$$

As, due to (2.9), (4.4), and (4.6), we have $\sum_{n=1}^{\infty} (\overline{\lambda}_{\tau,n+1} + \delta_{n+1}^{1/2}) < \infty$, from inequality (5.37) and definition (2.20), we conclude that the sequence $w_i^{[1,n]}(t,x) = \frac{\partial}{\partial x_i} u^{[n]}(t,x)$ converges uniformly on $[0,\tau] \times \mathbb{R}^d$.

Step 3 –Convergence of second derivatives of approximate solutions –We will demonstrate that, for any $\tau > 0$, the functions $\frac{\partial^2}{\partial x_i \partial x_j} u^{[n]}(t,x)$, $i, j = 1, \cdots, d$, converge to $\frac{\partial^2}{\partial x_i \partial x_j} u(t,x)$ ($u(t,x)$ being the limiting function obtained in Step 1) uniformly on $[0,\tau] \times \mathbb{R}^d$ as $n \to \infty$.

Let us define

$$w_{ij,k}^{[2,n]}(x) = \frac{\partial}{\partial x_j} w_{i,k}^{[1,n]}(x) = \frac{\partial^2}{\partial x_j \partial x_i} u^{[n]}(t_k^{[n]}, x) \tag{5.38}$$

and initially estimate

$$w_{ij,2k+2}^{[2,n+1]}(x) - w_{ij,k+1}^{[2,n]}(x) = \sum_{p=1}^{7} \partial_{x_j} Z_p, \tag{5.39}$$

where Z_p, $p = 1, \cdots, 7$, are the terms defined in (5.19)–(5.25).

Recalling that for the l-th component ξ_l^* of $\xi^* = \xi^*(y_1, y_2)$ (see (5.3)), we have

$$\begin{aligned}\partial_{x_j} \xi_l^* = \delta_{jl} - \delta_{n+1} \partial_{x_j} v_l(t_{2k+2}^{[n+1]}, x) - \delta_{n+1} \partial_{x_j} v_l(t_{2k+1}^{[n+1]}, \xi_{2k+2}^{[n+1]}(x, y_1)) \\ + \delta_{n+1}^2 \sum_{q=1}^{d} \partial_{x_q} v_l(t_{2k+1}^{[n+1]}, \xi_{2k+2}^{[n+1]}(x, y_1)) \partial_{x_j} v_q(t_{2k+2}^{[n+1]}, x)\end{aligned} \tag{5.40}$$

(see (5.30)), whereas for the l-th component $\xi_{k+1,l}^{[n]}(x, y_1 + y_2)$ of $\xi_{k+1}^{[n]}(x, y_1 + y_2)$, we have

$$\partial_{x_j} \xi_{k+1,l}^{[n]}(x, y_1 + y_2) = \delta_{jl} - 2\delta_{n+1} \partial_{x_j} v_l(t_{k+1}^{[n]}, x). \tag{5.41}$$

Using relations (5.40) and (5.41) and the hypotheses on the regularity of $v(t,x)$, reasoning similarly to obtain (5.27), we get

$$|\partial_{x_j} Z_1| \leq (1 + C\delta_{n+1}) \sup_{y \in \mathbb{R}^d} |w_{ij,2k}^{[2,n+1]}(y) - w_{ij,k}^{[2,n]}(y)| + C\delta_{n+1}(\lambda_{\tau,n+1}(v) + \delta_{n+1}^{1/2}). \tag{5.42}$$

Notice that for $p = 2$ and $p = 3$, we have

$$\partial_{x_j} Z_p = \delta_{n+1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Theta_{n+1}(y_1) \Theta_{n+1}(y_2) \sum_{l=1}^{d} \partial_{x_j} \zeta_{p,l}(x, y_1, y_2) dy_1 dy_2$$

and for $p = 2$, we decompose $\zeta_{2,l}$ as

$$\begin{aligned}\zeta_{2,l} = \partial_{x_i} v_l(t_{k+1}^{[n]}, x) \big(w_{l,k}^{[1,n]}(\xi_{k+1}^{[n]}(x, y_1 + y_2)) - w_{l,2k}^{[1,n+1]}(\xi_{k+1}^{[n]}(x, y_1 + y_2)) \big) \\ + \partial_{x_i} v_l(t_{k+1}^{[n]}, x) \big(w_{l,2k}^{[1,n+1]}(\xi_{k+1}^{[n]}(x, y_1 + y_2)) - w_{l,2k}^{[1,n+1]}(\xi^*) \big).\end{aligned}$$

Recall that a similar decomposition of $\zeta_{3,l}$ was made in (5.31). By examining the terms in which $\zeta_{2,l}$ and $\zeta_{3,l}$ decompose and deriving each term with respect to x_j, we can notice, particularly with the help of (5.30) and (5.40), that the absolute value of the derivative of each term is bounded by

$$\delta_{n+1} C$$

or

$$\lambda_{\tau,n+1}(v) C$$

or

$$C(\delta_{n+1} + |y_1|)$$

or

$$CY_k^{[1]}$$

or

$$CY_k^{[2]},$$

where $Y_k^{[1]}$ is the function defined by (5.28), while $Y_k^{[2]}$ is defined by

$$Y_k^{[2]} = \sum_{l,j=1}^{d} \sup_{x\in\mathbb{R}^d} |w_{lj,2k}^{[2,n+1]}(x) - w_{lj,k}^{[2,n]}(x)|. \tag{5.43}$$

Using the properties of the integral of Θ_{n+1} over $\mathbb{R}^d$, we deduce that

$$|\partial_{x_j} Z_2 + \partial_{x_j} Z_3| \le C\delta_{n+1}(\lambda_{\tau,n+1}(v) + \delta_{n+1}^{1/2}) + \delta_{n+1} C(Y_k^{[1]} + Y_k^{[2]}). \tag{5.44}$$

Regarding $\partial_{x_j} Z_p$, $p = 4, \cdots, 7$, applying the considerations used earlier to each term found in the expression of $\partial_{x_j} Z_p$, we readily obtain the inequality

$$|\partial_{x_j}(Z_4 + Z_5 + Z_6 + Z_7)| \le C\delta_{n+1}(\lambda_{\tau,n+1}(f) + \delta_{n+1}^{1/2}) + \delta_{n+1} C(Y_k + Y_k^{[1]} + Y_k^{[2]}), \tag{5.45}$$

where Y_k is the function defined by (5.11).

However, we can proceed similarly to the final part of the demonstration in Step 2. That is, by combining the inequalities (5.42), (5.44), and (5.45), considering that the inequality obtained is valid for $|w_{ij,2k+2}^{[2,n+1]}(x) - w_{ij,k+1}^{[2,n]}(x)|$ for all $x \in \mathbb{R}^d$, and summing up the inequality obtained for $i = 1, \cdots, d$, we obtain

$$Y_{k+1}^{[2]} \le Y_k^{[2]} + C(\delta_{n+1}^2 + \delta_{n+1}(\overline{\lambda}_{\tau,n+1} + \delta_{n+1}^{1/2})) + \delta_{n+1} C(Y_k + Y_k^{[1]} + Y_k^{[2]}). \tag{5.46}$$

If we set

$$\widetilde{Y}_k^{[2]} = Y_k + Y_k^{[1]} + Y_k^{[2]}, \tag{5.47}$$

then, by combining (5.12), (5.34), and (5.46), we obtain

$$\widetilde{Y}_{k+1}^{[2]} \le \widetilde{Y}_k^{[2]} + C'(\delta_{n+1}^2 + \delta_{n+1}(\overline{\lambda}_{\tau,n+1} + \delta_{n+1}^{1/2})) + \delta_{n+1} C' \widetilde{Y}_k^{[2]}. \tag{5.48}$$

Therefore, we obtain

$$\widetilde{Y}_k^{[2]} \le (\overline{\lambda}_{\tau,n+1} + \delta_{n+1}^{1/2}))e^{tC'}, \tag{5.49}$$

from which we deduce that the sequence $w_{ij}^{[1,n]}(t,x) = \frac{\partial^2}{\partial x_i \partial x_j} u^{[n]}(t,x)$ converges uniformly on $[0,\tau] \times \mathbb{R}^d$.

Step 4 —Convergence to the limit —

First, let us demonstrate that, given $\tau > 0$, for $t_1^{[n]} \le t_k^{[n]} \le \tau$, we have

$$\frac{u^{[n]}(t_k^{[n]}, x) - u^{[n]}(t_{k-1}^{[n]}, x)}{\delta_n} = -v(t,x) \cdot \nabla u^{[n]}(t_{k-1}^{[n]}, x) + \Delta u^{[n]}(t_{k-1}^{[n]}, x) + f(t_{k-1}^{[n]}, x, u^{[n]}(t_{k-1}^{[n]}, x)) + R \tag{5.50}$$

with

$$|R| \le (\delta_n^2 + \delta_n^{1/2})C. \tag{5.51}$$

Indeed, according to Taylor's formula, we have

$$\begin{aligned} u^{[n]}(t_{k-1}^{[n]}, x - \delta_n v(t,x) + y) &= u^{[n]}(t_{k-1}^{[n]}, x) - \delta_n v(t,x) \cdot \nabla u^{[n]}(t_{k-1}^{[n]}, x) + y \cdot \nabla u^{[n]}(t_{k-1}^{[n]}, x) \\ &+ \frac{1}{2}\sum_{i,j=1}^{d} [\delta_n^2 v_i(x) v_j(t,x) - 2\delta_n v_i(t,x) y_j + y_i y_j] \frac{\partial^2 u^{[n]}(t_{k-1}^{[n]}, x)}{\partial x_i \partial x_j} \\ &+ \frac{1}{6}\sum_{i,j,h=1}^{d} \mu_i \mu_j \mu_h \frac{\partial^3 u^{[n]}(t_{k-1}^{[n]}, \widetilde{x})}{\partial x_i \partial x_j \partial x_h}, \end{aligned} \tag{5.52}$$

where $\mu_i = -\delta_n v_i - y_i$ (and similarly for μ_j and μ_h), while $\widetilde{x}$ is a point between x and $x - \delta_n v(t_k^{[n]}, x) - y$.

However, since

$$\int_{\mathbb{R}^d} \Theta_n(y) y_j \, dy = 0, \qquad \int_{\mathbb{R}^d} \Theta_n(y) y_i y_j \, dy = 0 \ \text{ if } i \neq j, \qquad \int_{\mathbb{R}^d} \Theta_n(y) y_i^2 \, dy = 2\delta_n,$$

we have

$$\int_{\mathbb{R}^d} \Theta_n(y) y \cdot \nabla u^{[n]}(t_{k-1}^{[n]}, x) dy = 0,$$

$$\begin{aligned} &\int_{\mathbb{R}^d} \Theta_n(y) \Big[\frac{1}{2}\sum_{i,j=1}^{d} (\delta_n^2 v_i(t,x) v_j(t,x) - 2\delta_n v_i(t,x) y_j + y_i y_j) \frac{\partial^2 u^{[n]}(t_{k-1}^{[n]}, x)}{\partial x_i \partial x_j}\Big] dy \\ &= \delta_n \Delta u^{[n]}(t_{k-1}^{[n]}, x) + \delta_n^2 \frac{1}{2}\sum_{i,j=1}^{d} v_i(t,x) v_j(t,x) \frac{\partial^2 u^{[n]}(t_{k-1}^{[n]}, x)}{\partial x_i \partial x_j}. \end{aligned}$$

On the other hand, since we have $|\mu_i| \leq \delta_n |v| + |y|$ and similarly for μ_j and μ_h, there exists a constant C such that

$$\left| \frac{1}{6}\sum_{i,j,h=1}^{3} \mu_i \mu_j \mu_h \frac{\partial^3 u^{[n]}(t_{k-1}^{[n]}, x)}{\partial x_i \partial x_j \partial x_h} \right| \leq C(\delta_n |v| + |y|)^3 \left| \frac{\partial^3 u^{[n]}(t_{k-1}^{[n]}, x)}{\partial x_i \partial x_j \partial x_h} \right|.$$

Since, according to Lemma 5, the third derivatives of $u^{[n]}$ are uniformly bounded, we have

$$\left| \int_{\mathbb{R}^d} \Theta_n(y) \frac{1}{6}\sum_{i,j,h=1}^{3} \mu_i \mu_j \mu_h \frac{\partial^3 u^{[n]}(t_{k-1}^{[n]}, x)}{\partial x_i \partial x_j \partial x_h} \, dy \right| \leq (\delta_n^3 + \delta_n^{3/2}) C'.$$

We deduce that

$$u^{[n]}(t_k^{[n]}, x) - u^{[n]}(t_{k-1}^{[n]}, x) = -\delta_n v(t,x) \cdot \nabla u^{[n]}(t_{k-1}^{[n]}, x) + \delta_n \kappa \Delta u^{[n]}(t_{k-1}^{[n]}, x) + \delta_n F(t_{k-1}^{[n]}, x, u^{[n]}(t_{k-1}^{[n]}, x)) + R'$$

where

$$|R'| \leq (\delta_n^3 + \delta_n^{3/2}) C;$$

therefore, dividing both sides of this equality by δ_n, we obtain (5.50) with (5.51).

From (5.50), it follows that there exists a constant L independent of n such that

$$|u^{[n]}(t_1, x) - u^{[n]}(t_2, x)| \leq L|t_1 - t_2| \quad \forall t_1, t_2 \in [0, \tau - 1], \ \forall x \in \mathbb{R}^d, \tag{5.53}$$

$$|u(t_1,x)-u(t_2,x)|\le L|t_1-t_2| \quad \forall t_1,t_2\in[0,\tau-1],\ \forall x\in\mathbb{R}^d, \tag{5.54}$$

where $u(t,x)$ is the limit function of the sequence $u^{[n]}(t,x)$.

Indeed, according to Lemma 5 and the obvious relation $\delta_n \le \delta_1$, the absolute value of the right-hand side of (5.50) is bounded by a constant L that does not depend on n. Therefore, the inequality (5.53) follows from the definition (2.20). Furthermore, inequality (5.54) results from (5.53) and the uniform convergence of $u^{[n]}(t,x)$ to $u(t,x)$.

Now we are ready to conclude the proof of Theorem 4. Consider the function

$$\psi^{[n]}(t,x) = \frac{t_{k+1}^{[n]}-t}{\delta_n}\Big(\frac{u^{[n]}(t_{k+1}^{[n]},x)-u^{[n]}(t_k^{[n]},x)}{\delta_n}\Big) \tag{5.55}$$
$$+\frac{t-t_k^{[n]}}{\delta_n}\Big(\frac{u^{[n]}(t_{k+2}^{[n]},x)-u^{[n]}(t_{k+1}^{[n]},x)}{\delta_n}\Big), \qquad \text{for } t_k^{[n]}<t<t_{k+1}^{[n]},\ \ k=0,1,\cdots.$$

It is immediately evident that $\psi^{[n]}(t,x)$ is continuous concerning t, and according to the definition (2.20), we have

$$\psi^{[n]}(t,x) = \frac{(t_{k+1}^{[n]}-t)}{\delta_n}\frac{\partial u^{[n]}(t,x)}{\partial t} + \frac{(t-t_k^{[n]})}{\delta_n}\frac{\partial u^{[n]}(t+\delta_n,x)}{\partial t}, \qquad \text{for } t_k^{[n]}<t<t_{k+1}^{[n]}.$$

Furthermore, according to relations (5.50) along with (5.51) and definition in (2.20), we have

$$\psi^{[n]}(t,x) = -v(t,x)\cdot\nabla u^{[n]}(t,x)+\Delta u^{[n]}(t,x)+f(t,x,u^{[n]}(t,x))+R \tag{5.56}$$

where

$$|R|\le(\delta_n^2+\delta_n^{1/2})C.$$

Consider now a function $\varphi(\cdot)\in C^\infty([0,\infty[)$ such that $\varphi(t)=0$ for $t\ge\tau_1$ with $\tau_1>0$. By multiplying both sides of (5.56) by the function $\varphi(t)$ and integrating with respect to t, we obtain

$$\int_0^\infty \psi^{[n]}(t,x)\varphi(t)dt = \int_0^\infty(-v(t,x)\cdot\nabla u^{[n]}(t,x)+\Delta u^{[n]}(t,x)+f(t,x,u^{[n]}(t,x))+R)\varphi(t)dt. \tag{5.57}$$

By virtue of (5.51) (also refer to (5.56)) and what we have proved in Steps 1, 2, and 3, the right-hand side of equality (5.57) tends to

$$\int_0^\infty(-v(t,x)\cdot\nabla u(t,x)+\Delta u(t,x)+f(t,x,u(t,x)))\varphi(t)dt.$$

On the other hand, if we set

$$\Psi^{[n]}(t,x)=\frac{1}{2}(u^{[n]}(t_0^{[n]},x)+u^{[n]}(t_1^{[n]},x))+\int_0^t\psi^{[n]}(t',x)dt',$$

by performing integration by parts, the first term of (5.57) transforms into

$$-\int_0^\infty\Psi^{[n]}(t,x)\varphi'(t)dt-\frac{1}{2}(u^{[n]}(t_0^{[n]},x)+u^{[n]}(t_1^{[n]},x))\varphi(0)\equiv I.$$

Now, by explicit calculation, we observe that

$$\Psi^{[n]}(t,x) = \frac{1}{2}(u^{[n]}(t_k^{[n]},x) + u^{[n]}(t_{k+1}^{[n]},x)) - \Big(\frac{1}{2} - \frac{(t_{k+1}^{[n]}-t)^2}{2\delta_n^2}\Big)u^{[n]}(t_k^{[n]},x)$$
$$+ \Big(\frac{1}{2} - \frac{(t_{k+1}^{[n]}-t)^2}{2\delta_n^2} - \frac{(t-t_k^{[n]})^2}{2\delta_n^2}\Big)u^{[n]}(t_{k+1}^{[n]},x) + \frac{(t-t_k^{[n]})^2}{2\delta_n^2}u^{[n]}(t_{k+2}^{[n]},x)$$

for $t_k^{[n]} < t \le t_{k+1}^{[n]}$ and $k = 0, 1, 2, \cdots$. From this expression of $\Psi^{[n]}(t,x)$, the uniform convergence of $u^{[n]}(t,x)$ to $u(t,x)$ (Step 1), and the relation (5.53), we can conclude that

$$\Psi^{[n]}(t,x) \to u(t,x) \quad \text{uniformly on } [0,\tau]\times\mathbb{R}^d \quad \text{as } n\to\infty$$

for any $\tau > 0$. Furthermore, as $u^{[n]}(t_0^{[n]},x) = u_0(x)$ for all n and for all $x \in \mathbb{R}^d$, considering (5.53), we have

$$\frac{1}{2}(u^{[n]}(t_0^{[n]},x) + u^{[n]}(t_1^{[n]},x)) \to u_0(x) \quad \text{uniformly on } \mathbb{R}^d \quad \text{as } n\to\infty.$$

Therefore, I tends toward

$$-\int_0^\infty u(t,x)\varphi'(t)dt - u_0(x)\varphi(0),$$

which gives us (4.12). The proof of Theorem 4 is complete. □

6. Proof of Theorem 1

Proof. As $t(s)$ is the inverse function of $s(t)$, by virtue of (2.6), we have

$$\frac{dt(s)}{ds} = \frac{1}{\frac{ds(t)}{dt}} = \frac{1}{\kappa(t)}.$$

We thus obtain

$$\partial_s u(t(s),x) = \frac{1}{\kappa(t)}\partial_t u(t,x),$$

allowing us to transform Eq (2.1) into

$$\partial_s u(t(s),x) + \frac{1}{\kappa(t(s))}v(t(s),x)\cdot\nabla u(t(s),x) = \Delta u(t(s),x) + \frac{1}{\kappa(t(s))}f(t(s),x,u(t(s),x)). \tag{6.1}$$

If we set

$$\widetilde{v}(s,x) = \frac{1}{\kappa(t(s))}v(t(s),x), \qquad \widetilde{f}(s,x,u) = \frac{1}{\kappa(t(s))}f(t(s),x,u),$$

according to hypotheses (2.12)–(2.15), the functions $\widetilde{v}(s,x)$ and $\widetilde{f}(s,x,u)$ satisfy the conditions (4.3)–(4.6) by replacing t with s. Therefore, following Theorem 4, the functions $u^{[n]}(s,x)$ defined by (4.9)–(4.11), where s substitutes t, converge uniformly on $[0,\tau]\times\mathbb{R}^d$ for any $\tau > 0$, with their first and second derivatives, to a function $u(s,x)$. The limit function $u(s,x)$ satisfies equation (4.1) (with s replacing t) and the initial condition (2.2) in terms of integral equality

$$-\int_0^\infty u(s,x)\frac{d}{ds}\widetilde{\varphi}(s)ds - u_0(x)\widetilde{\varphi}(0) + \int_0^\infty \widetilde{v}(s,x)\cdot\nabla u(s,x)\widetilde{\varphi}(s)ds \tag{6.2}$$

$$= \int_0^\infty (\Delta u(s,x) + \widetilde{f}(s,x,u))\widetilde{\varphi}(s)ds$$

for any $\widetilde{\varphi}(\cdot) \in C^1([0, \infty[)$ such that $\widetilde{\varphi}(s) = 0$ for $s \geq \tau_{1,s}$ with $\tau_{1,s} > 0$.

Now, let us consider a function $\varphi(\cdot) \in C^1([0,\infty[)$ with $\operatorname{supp}(\varphi(\cdot)) \subset [0,\tau_{1,t}]$ and its composition $\varphi \circ t = \varphi(t(\cdot))$ with the function $t(s)$. Then we have

$$\frac{d}{ds}\varphi(t(s)) = \frac{d}{dt}\varphi(t)\Big|_{t=t(s)} \cdot \frac{dt(s)}{ds} = \frac{d}{dt}\varphi(t)\Big|_{t=t(s)} \cdot \frac{1}{\kappa(t(s))}.$$

Since $t(s)$ is continuous and $\frac{d}{dt}\varphi(t)$ is also continuous by assumption, the function $\frac{d}{dt}\varphi(t)\big|_{t=t(s)}$ is continuous in s. On the other hand, according to condition (2.4), the function $t(s)$ is absolutely continuous, so that $\frac{dt(s)}{ds}$ belongs to $L^1_{\text{loc}}([0,\infty[)$. Consequently,

$$\frac{d}{ds}\varphi(t(s)) \in L^1(]0, s(\tau_{1,t}) + 1[).$$

Hence, there exists a sequence of functions $\{\widetilde{\varphi}_m\}_{m=1}^\infty$ such that

$$\widetilde{\varphi}_m \in C^1(\mathbb{R}_+), \qquad \operatorname{supp}(\widetilde{\varphi}_m) \subset [0, s(\tau_{1,t}) + 1] \qquad \forall m \in \mathbb{N}\backslash\{0\}, \tag{6.3}$$

and that

$$\|\widetilde{\varphi}_m(\cdot) - \varphi(t(\cdot))\|_{W^1_1(]0,s(\tau_{1,t})+1[)} \to 0, \quad |\widetilde{\varphi}_m(s) - \varphi(0)| \to 0, \quad \text{for } m \to \infty. \tag{6.4}$$

According to (6.3), we can substitute $\widetilde{\varphi}(s) = \widetilde{\varphi}_m(s)$ into (6.2). Since $u \in L^\infty([0,\tau] \times \mathbb{R}^d)$ for any $\tau > 0$, considering (6.4), we have

$$\int_0^\infty u(s,x)\frac{d}{ds}\widetilde{\varphi}_m(s)ds \to \int_0^\infty u(s,x)\frac{d}{ds}\varphi(t(s))ds = \int_0^\infty u(t,x)\frac{d}{dt}\varphi(t)dt$$

as $m \to \infty$. Moreover, it can be easily seen that

$$\int_0^\infty \widetilde{v}(s,x) \cdot \nabla u(s,x)\widetilde{\varphi}_m(s)ds \to \int_0^\infty \frac{1}{\kappa(t(s))}v(t(s),x) \cdot \nabla u(t(s),x)\varphi(s)ds = \int_0^\infty v(t,x) \cdot \nabla u(t,x)\varphi(t)dt,$$

$$\int_0^\infty \Delta u(s,x)\widetilde{\varphi}_m(s)ds \to \int_0^\infty \Delta u(t(s),x)\varphi(s)ds = \int_0^\infty \kappa(t)\Delta u(t,x)\varphi(t)dt,$$

$$\int_0^\infty \widetilde{f}(s,x,u)\widetilde{\varphi}_m(s)ds \to \int_0^\infty \frac{1}{\kappa(t(s))}f(t(s),x,u)\varphi(s)ds = \int_0^\infty f(t,x,u)\varphi(t)dt$$

as $m \to \infty$. Hence, it follows that for any function $\varphi(t) \in C^1(\mathbb{R}_+)$ such that there exists a positive number $\tau_{1,t}$ satisfying $\varphi(t) = 0$ for all $t \geq \tau_{1,t}$, it satisfies the relation (2.21). The theorem is proved. □

Author contributions

Lynda Taleb, Rabah Gherdaoui: Writing-original draft, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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