Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 2375–2391.
DOI: 10.3934/math.2025110
Received: 07 October 2024
Revised: 11 January 2025
Accepted: 24 January 2025
Published: 10 February 2025

Research article

Total positivity, Gramian matrices, and Schur polynomials

Pablo Díaz, Esmeralda Mainar and Beatriz Rubio*

Departamento de Matemática Aplicada / IUMA, Universidad de Zaragoza, España

* **Correspondence:** Email: brubio@unizar.es.

Abstract: This paper investigated the total positivity of Gramian matrices associated with general bases of functions. It demonstrated that the total positivity of collocation matrices for totally positive bases extends to their Gramian matrices. Additionally, a bidiagonal decomposition of these Gramian matrices, derived from integrals of symmetric functions, was presented. This decomposition enables the design of algorithms with high relative accuracy for solving linear algebra problems involving totally positive Gramian matrices. For polynomial bases, compact and explicit formulas for the bidiagonal decomposition were provided, involving integrals of Schur polynomials. These integrals, known as Selberg-like integrals, arise naturally in various contexts within Physics and Mathematics.

Keywords: Gramian matrices; bidiagonal decompositions; Schur functions; Selberg-like integrals; totally positive matrices
Mathematics Subject Classification: 05E05, 15A23, 65F05

1. Introduction

Traditionally, a matrix is defined as totally positive (TP) if all its minors are nonnegative, and strictly totally positive (STP) if all its minors are strictly positive (see [1, 11, 25]). It is worth noting that in some literature, TP and STP matrices are referred to as totally nonnegative and totally positive matrices, respectively [9]. A fundamental property of TP matrices is that their product also results in a TP matrix.

A basis $(u_0,\dots,u_n)$ of a given space U of functions defined on $I \subseteq \mathbb{R}$ is said to be totally positive (TP) if, for any sequence of parameters $T := (t_1,\dots,t_{N+1})$ in I with $t_1 < \cdots < t_{N+1}$ and $N \geq n$, the collocation matrix

$$M_T := \big(u_{j-1}(t_i)\big)_{1\leq i\leq N+1;1\leq j\leq n+1} \tag{1.1}$$

is TP.

Collocation matrices form a significant class of structured matrices, which has become a prominent research topic in numerical linear algebra, attracting increasing attention in recent years. Extensive studies have focused on algebraic computations for various types of collocation matrices associated with specific bases of polynomials [2, 3, 7, 19–23], rational bases [4, 26], or bases of functions of the form $t^k e^{\lambda t}$ [16], among others. Once recognized as TP matrices, collocation matrices have found diverse and interesting applications in numerical mathematics, computer-aided geometric design, and statistics.

If U is a Hilbert space of functions equipped with an inner product $\langle\cdot,\cdot\rangle$, and $(u_0,\dots,u_n)$ is a basis of U, the corresponding Gram (or Gramian) matrix is the symmetric matrix $G=\left(g_{i,j}\right)_{1\le i,j\le n+1}$ defined as

$$g_{i,j} := \left\langle u_{i-1}, u_{j-1}\right\rangle, \quad 1\le i,j\le n+1. \tag{1.2}$$

Gramian matrices have a wide range of applications. For example, they can be used to transform non-orthogonal bases of U into orthogonal ones, with the Gramian matrix defined in (1.2) serving as the transformation matrix. In least-squares approximation, where a function is represented as a linear combination of a basis in U, the solution involves solving a system of normal equations. The coefficient matrix for this system is precisely the Gramian matrix (1.2) corresponding to the chosen basis. Furthermore, Gramian matrices are also linked to inverse scattering problems [6], highlighting their significance in applied mathematics and physics.

For certain polynomial bases [18] and bases of the form $t^k e^{\lambda t}$ [16], these matrices have been efficiently represented through bidiagonal factorizations. These factorizations facilitate the design of highly accurate algorithms for addressing key problems in linear algebra.

As shown in [10, 11], any nonsingular TP matrix $A\in\mathbb{R}^{(n+1)\times(n+1)}$ can be expressed as

$$A = F_nF_{n-1}\cdots F_1 D G_1 G_2\cdots G_n, \tag{1.3}$$

where $F_i\in\mathbb{R}^{(n+1)\times(n+1)}$ (respectively, $G_i\in\mathbb{R}^{(n+1)\times(n+1)}$) for $i=1,\dots,n$ are TP lower (respectively, upper) triangular bidiagonal matrices of the form:

$$F_i = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & m_{n+1,n+1-i} & 1 \end{bmatrix}, \quad G_i^T = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \widetilde{m}_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \widetilde{m}_{n+1,n+1-i} & 1 \end{bmatrix},$$

and $D=\operatorname{diag}(p_{i,i})_{1\le i\le n+1}$ is a nonsingular diagonal matrix whose nonzero diagonal entries are called pivots. The off-diagonal entries $m_{i,j}$, known as multipliers, satisfy

$$m_{i,j} = \frac{p_{i,j}}{p_{i-1,j}},$$

with $p_{i,1} := a_{i,1}$, $1\le i\le n+1$, and

$$p_{i,j} := \frac{\det A[i-j+1,\dots,i \mid 1,\dots,j]}{\det A[i-j+1,\dots,i-1 \mid 1,\dots,j-1]}, \tag{1.4}$$

for $1<j\le i\le n+1$, where the submatrix of A formed by rows $i_1,\dots,i_r$ and columns $j_1,\dots,j_s$ is denoted by $A[i_1,\dots,i_r \mid j_1,\dots,j_s]$ and $A[i_1,\dots,i_r]$ denotes the matrix $A[i_1,\dots,i_r \mid i_1,\dots,i_r]$.

Similarly, the entries $\widetilde{m}_{i,j}$ are given by

$$\widetilde{m}_{i,j} = \frac{\widetilde{p}_{i,j}}{\widetilde{p}_{i-1,j}}, \tag{1.5}$$

where $\widetilde{p}_{i,1} := a_{1,i}$, and the terms $\widetilde{p}_{i,j}$ can be computed as in (1.4) for the transpose A^T of the matrix A. For symmetric matrices, it holds that $m_{i,j} = \widetilde{m}_{i,j}$, which implies $G_i = F_i^T$ for $i = 1, \ldots, n$.

The factorization (1.3) offers an explicit expression for the determinant of TP matrices. Furthermore, when its computation avoids inaccurate cancellations, it provides a matrix representation suitable for developing algorithms with high relative accuracy (HRA) to address relevant algebraic problems (cf. [15, 16]). Achieving HRA is crucial, as such algorithms ensure that relative errors are on the order of machine precision and remain unaffected by the matrix dimension or condition number. Outstanding results have been obtained for collocation matrices (see [2–5, 19–22]) as well as for Gramian matrices of bases such as the Poisson and Bernstein bases on the interval $[0,1]$. Similarly, significant progress has been made with non-polynomial bases like $\{x^k e^{\lambda x}\}$ (see [17], and [16]).

A symmetric function is a function in several variables which remains unchanged for any permutation of its variables. In contrast, a totally antisymmetric function changes sign with any transposition of its variables. If M_T is a collocation matrix defined as in (1.1), any minor $\det M_T[i_1,\ldots,i_r|j_1,\ldots,j_s]$ is a totally antisymmetric function of the parameters $t_{i_1},\ldots,t_{i_r}$. The same applies to the minors of the transpose of M_T. These statements may be concisely expressed through the following relations:

$$\begin{aligned} \det M_T[i_1,\ldots,i_r|j_1,\ldots,j_r] &= g(t_{i_1},\ldots,t_{i_r}) \det V_{t_{i_1},\ldots,t_{i_r}}, \\ \det M_T^T[i_1,\ldots,i_r|j_1,\ldots,j_r] &= \widetilde{g}(t_{j_1},\ldots,t_{j_r}) \det V_{t_{j_1},\ldots,t_{j_r}}, \end{aligned}$$

for suitable symmetric functions $g(x_1,\ldots,x_r)$, $\widetilde{g}(x_1,\ldots,x_r)$, and $V_{t_{j_1},\ldots,t_{j_r}}$, the Vandermonde matrix at nodes $t_{j_1},\ldots,t_{j_r}$.

The above observation, together with formula (1.4) for computing the pivots and multipliers of the factorization (1.3), reveals an intriguing connection between symmetric functions and TP bases, previously explored in [7] and [8]. For collocation matrices of polynomial bases, the diagonal pivots and multipliers involved in (1.3) can be expressed in terms of Schur functions, leading to novel insights into the total positivity properties (cf. [7]). Subsequently, [8] extended these results to the class of Wronskian matrices, also deriving their bidiagonal decomposition in terms of symmetric functions.

In this paper, we extend this line of research by considering Gramian matrices (1.2). Due to their inherent symmetry (see (1.4) and (1.5)), computing the pivots and multipliers in the factorization (1.3) reduces to determining minors with consecutive rows and initial consecutive columns, specifically:

$$\det G[i-j+1,\ldots,i|1,\ldots,j], \quad 1 \le j \le i \le n+1. \tag{1.6}$$

These minors will be central objects of study in this work. Moreover, it will be shown that Gramian matrices can be represented as a specific limit of products of matrices involving collocation matrices of the given basis (see formula (2.2) in Section 2). This framework enables us to derive results analogous to those obtained in [7] for collocation matrices, or in [8] for Wronskian matrices now in the context of Gramian matrices. Ultimately, we establish a connection between the total positivity of Gramian matrices and integrals of symmetric functions.

In the following sections, we first demonstrate that any Gramian matrix for a given basis can be written as a limit of products involving the collocation matrices of the system. Consequently, we establish that Gramian matrices of TP bases are themselves TP. These findings are applied in Section 3, where we represent any minor (1.6) as an integral of products of minors of collocation matrices. In Section 4, we further refine this representation for polynomial bases, expressing the minors in terms of integrals of Schur polynomials. These integrals, known as Selberg-like integrals, have been explicitly computed in the literature and arise naturally in various contexts of Physics and Mathematics, such as the quantum Hall effect and random matrix theory. Relevant results on these integrals, essential for our purposes, are summarized in Section 5. Finally, we include an appendix providing the pseudocode of an algorithm designed for the computation of the determinants in (1.6), specifically tailored for polynomial bases.

2. Gramian matrices of totally positive bases

Consider U to be a Hilbert space of functions defined on $J=[a,b]$, equipped with the inner product

$$\langle u,v\rangle := \int_J \kappa(t)\, u(t)\, v(t)\, dt, \tag{2.1}$$

for a weight function κ satisfying $\kappa(t)\geq 0$, for all $t\in J$. In this section, we focus on the Gramian matrix G, as defined in (1.2), corresponding to a basis $(u_0,\dots,u_n)$ of U with respect to the inner product (2.1).

The following result shows that G can be represented as the limit of products involving collocation matrices for $(u_0,\dots,u_n)$ evaluated at equally spaced sequences of parameters on $[a,b]$ and diagonal matrices containing the values of the weight function κ at those parameters.

Lemma 2.1. *Let G be the Gramian matrix* (1.2) *of a basis $(u_0,\dots,u_n)$ with respect to the inner product* (2.1). *Then,*

$$G=\lim_{N\to\infty}\frac{b-a}{N}\, U_N^T K_N U_N, \tag{2.2}$$

where

$$U_N := (u_{j-1}(t_i))_{i=1,\dots,N+1,\, j=1,\dots,n+1}, \quad K_N := diag\big(\kappa(t_i)\big)_{i=1,\dots,N+1}, \tag{2.3}$$

and $t_i := a+(i-1)(b-a)/N$, $i=1,\dots,N+1$, for $N\in\mathbb{N}$.

Proof. Using the definition of the Riemann integral, it is straightforward to verify that the matrix

$$G_N := \frac{b-a}{N} U_N^T K_N U_N \tag{2.4}$$

converges component-wise to the Gramian matrix G as $N\to\infty$. □

Using Lemma 2.1, we establish the total positivity property of Gramian matrices corresponding to TP bases under the inner product (2.1).

Theorem 2.1. *Let $(u_0,\dots,u_n)$ be a TP basis of a space U of functions defined on the interval I. The Gramian matrix G, as defined in* (1.2)*, is TP if $J\subseteq I$.*

Proof. Let us consider the compact interval $J = [a,b]$. If $(u_0,\dots,u_n)$ is a TP basis on I, it remains TP on $J \subseteq I$. Consequently, the matrices U_N and K_N defined in (2.3) are TP for all $N \in \mathbb{N}$. Therefore, the matrix G_N in (2.4) is TP for all $N \in \mathbb{N}$, as it is the product of TP matrices.

Let us analyze the sign of the $r \times r$ minor $\det G[i_1,\dots,i_r | j_1,\dots,j_r]$ corresponding to rows $1 \le i_1 < \cdots < i_r \le n+1$ and columns $1 \le j_1 < \cdots < j_r \le n+1$. Since $G_N = (G^N_{i,j})_{1\le i\le n+1; 1\le j\le n+1}$ is TP, we have

$$0 \le \det G_N[i_1,\dots,i_r|j_1,\dots,j_r] = \sum_{\sigma\in S_r} \operatorname{sgn}(\sigma)\, G^N_{i_1 j_{\sigma(1)}} \cdots G^N_{i_r j_{\sigma(r)}}, \quad N \in \mathbb{N}, \tag{2.5}$$

where S_r denotes the group of permutations of $\{1,\dots,r\}$ and $\operatorname{sgn}(\sigma)$ is the signature of the permutation σ, taking the value 1 if σ is even and -1 if σ is odd. Recall that a permutation is even (or odd) if it can be expressed as the product of an even (or odd) number of transpositions.

From (2.2), we have $\lim_{N\to\infty} G^N_{i,j} = G_{i,j}$ and so,

$$G^N_{i,j} = G_{i,j} + \varepsilon^N_{i,j}, \quad \lim_{N\to\infty} \varepsilon^N_{i,j} = 0, \tag{2.6}$$

for $1 \le i,j \le n+1$. Using (2.5) and (2.6), we derive

$$\begin{aligned} 0 \;&\le\; \det G_N[i_1,\dots,i_r|j_1,\dots,j_r] = \sum_{\sigma\in S_r} \operatorname{sgn}(\sigma) \prod_{\ell=1}^{r} (G_{i_\ell, j_{\sigma(\ell)}} + \varepsilon^N_{i_\ell, j_{\sigma(\ell)}}) \\ &=\; \det G[i_1,\dots,i_r|j_1,\dots,j_r] + \sum_{k=1}^{r}\sum_{\sigma\in S_r} \operatorname{sgn}(\sigma) \varepsilon^N_{i_k, j_{\sigma(k)}} \prod_{\ell\ne k} G^N_{i_\ell, j_{\sigma(\ell)}} \\ &\le\; \det G[i_1,\dots,i_r|j_1,\dots,j_r] + \sum_{k=1}^{r}\sum_{\sigma\in S_r} |\varepsilon^N_{i_k, j_{\sigma(k)}}| \prod_{\ell\ne k} |G^N_{i_\ell, j_{\sigma(\ell)}}|, \quad N \in \mathbb{N}. \end{aligned}$$

By defining

$$\varepsilon_N := \max\{|\varepsilon^N_{i,j}| \mid i = i_1,\dots,i_r,\ j = j_1,\dots,j_r\}, \quad \psi_N := \max\{|G^N_{i,j}| \mid i = i_1,\dots,i_r,\ j = j_1,\dots,j_r\},$$

we have

$$0 \le \det G[i_1,\dots,i_r|j_1,\dots,j_r] + \sum_{k=1}^{r}\sum_{\sigma\in S_r} \varepsilon_N \psi_N^{r-1} = \det G[i_1,\dots,i_r|j_1,\dots,j_r] + r\cdot r!\varepsilon_N \psi_N^{r-1}, \quad N \in \mathbb{N}.$$

The value ψ_N is clearly bounded and $\varepsilon_N \to 0$ as $N \to \infty$. So,

$$0 = \lim_{N\to\infty} -r\cdot r!\varepsilon_N \psi_N^{r-1} \le \det G[i_1,\dots,i_r|j_1,\dots,j_r].$$

Finally, since any $r \times r$ minor of G is nonnegative, we conclude that G is a TP matrix. □

3. Bidiagonal factorizations of Gramian matrices

In this section, we use formula (2.2) to write the determinants in (1.6) as integrals involving the product of minors of specific collocation matrices associated with the considered basis.

Before presenting the main result of the section, we first prove the following auxiliary lemma on the integrals of general symmetric functions.

Proposition 3.1. *Let $g(x_1,\dots,x_j)$ be a symmetric function. Then*

$$\int_a^b dx_1 \int_{x_1}^b dx_2 \cdots \int_{x_{j-1}}^b dx_j\, g(x_1,\dots,x_j) = \frac{1}{j!}\int_{[a,b]^j} \prod_{l=1}^{j} dx_l\, g(x_1,\dots,x_j). \tag{3.1}$$

Proof. The integration region of the LHS of (3.1) covers all the points $(x_1,\dots,x_j)$ of the hypercube $[a,b]^j$ such that $a \le x_1 \le \cdots \le x_j \le b$. On the other hand, the hypercube is fully covered when considering all the points obtained by permuting the variables, that is,

$$[a,b]^j = \cup_{\sigma\in S_j}\{x_\sigma = (x_{\sigma(1)},\dots,x_{\sigma(j)})\,|\, a \le x_1 \le \cdots \le x_j \le b\}.$$

Two points x_σ and $x_{\sigma'}$, with $\sigma \neq \sigma'$, can be equal only if two or more variables take the same value, say $x_i = x_j$. So, only points lying on a face of a simplex are in the intersection set. But the collection of these points forms a set of null measure. Thus the integral over $[a,b]^j$ can split into $j!$ integrals, each corresponding to a region labeled by a permutation $\sigma \in S_j$. Moreover, since $g(x_1,\dots,x_j)$ is a symmetric function, the permutation of variables does not alter the value of the integral. So, all the $j!$ integrals are identical and (3.1) follows. □

Theorem 3.1. *Let G be the Gramian matrix* (1.2) *of a basis $(u_0,\dots,u_n)$ with respect to the inner product* (2.1). *Then*

$$\det G[i-j+1,\dots,i|1,\dots,j] = \frac{1}{j!}\int_{[a,b]^j}\prod_{l=1}^{j} dx_l\,\kappa(x_l)\det M_X^T[i-j+1,\dots,i|1,\dots,j]\det M_X[1,\dots,j], \tag{3.2}$$

where

$$M_X := \big(u_{j-1}(x_i)\big)_{1\le i,j\le n+1}$$

is the square collocation matrix of $(u_0,\dots,u_n)$ at the sequence of parameters $X=(x_1,\dots,x_{n+1})$.

Proof. Given $N\in\mathbb{N}$, we define an equally spaced partition of $[a,b]$ with $t_i := a+(i-1)(b-a)/N$, $i=1,\dots,N+1$. Using basic properties of determinants, we can derive the following identities for the matrix G_N in (2.4):

$$\begin{aligned}
&\det G_N[i-j+1,\dots,i|1,\dots,j]\\
&= \left(\frac{b-a}{N}\right)^j \begin{vmatrix} \sum_{l=1}^{N+1} u_{i-j+1}(t_l)u_0(t_l)\kappa(t_l) & \dots & \sum_{l=1}^{N+1} u_{i-j+1}(t_l)u_{j-1}(t_l)\kappa(t_l) \\ \vdots & \ddots & \vdots \\ \sum_{l=1}^{N+1} u_i(t_l)u_0(t_l)\kappa(t_l) & \dots & \sum_{l=1}^{N+1} u_i(t_l)u_{j-1}(t_l)\kappa(t_l) \end{vmatrix}\\
&= \left(\frac{b-a}{N}\right)^j \sum_{k_1,\dots,k_j=1}^{N+1}\prod_{l=1}^{j} u_{l-1}(t_{k_l})\kappa(t_{k_l}) \begin{vmatrix} u_{i-j+1}(t_{k_1}) & \dots & u_{i-j+1}(t_{k_j}) \\ \vdots & \ddots & \vdots \\ u_i(t_{k_1}) & \dots & u_i(t_{k_j}) \end{vmatrix}\\
&= \left(\frac{b-a}{N}\right)^j \sum_{k_1<\cdots<k_j}\sum_{\sigma\in S_j}\prod_{l=1}^{j} u_{l-1}(t_{k_{\sigma(l)}})\kappa(t_{k_{\sigma(l)}}) \begin{vmatrix} u_{i-j+1}(t_{k_{\sigma(1)}}) & \dots & u_{i-j+1}(t_{k_{\sigma(j)}}) \\ \vdots & \ddots & \vdots \\ u_i(t_{k_{\sigma(1)}}) & \dots & u_i(t_{k_{\sigma(j)}}) \end{vmatrix}.
\end{aligned} \tag{3.3}$$

In the last line in (3.3), we have taken into account that the determinant cancels whenever two of the dummy variables $(k_1,\dots,k_j)$ take the same value and so, the total sum can be reorganized into ascending sequences and permutations of the variables $(k_1,\dots,k_j)$. Next, we can sum over the permutation group S_j, and obtain

$$\begin{aligned}
&\det G_N[i-j+1,\dots,i|1,\dots,j]\\
&=\left(\frac{b-a}{N}\right)^j\sum_{k_1<\cdots<k_j}\sum_{\sigma\in S_j}\operatorname{sgn}(\sigma)\prod_{l=1}^{j}u_{l-1}(t_{k_{\sigma(l)}})\kappa(t_{k_l})\begin{vmatrix}u_{i-j+1}(t_{k_1}) & \dots & u_{i-j+1}(t_{k_j})\\ \vdots & \ddots & \vdots\\ u_i(t_{k_1}) & \dots & u_i(t_{k_j})\end{vmatrix}\\
&=\left(\frac{b-a}{N}\right)^j\sum_{k_1<\cdots<k_j}\prod_{l=1}^{j}\kappa(t_{k_l})\begin{vmatrix}u_{i-j+1}(t_{k_1}) & \dots & u_{i-j+1}(t_{k_j})\\ \vdots & \ddots & \vdots\\ u_i(t_{k_1}) & \dots & u_i(t_{k_j})\end{vmatrix}\begin{vmatrix}u_1(t_{k_1}) & \dots & u_j(t_{k_1})\\ \vdots & \ddots & \vdots\\ u_1(t_{k_j}) & \dots & u_j(t_{k_j})\end{vmatrix}.
\end{aligned}$$

In general, for any integrable function $g(x_1,\dots,x_j)$ defined on $[a,b]^j$, we have

$$\lim_{N\to\infty}\sum_{k_1<\cdots<k_j}\left(\frac{b-a}{N}\right)^j g(t_{k_1},\dots,t_{k_j})=\int_a^b dx_1\int_{x_1}^b dx_2\cdots\int_{x_{j-1}}^b dx_j\, g(x_1,\dots,x_j).$$

Thus,

$$\begin{aligned}
&\det G[i-j+1,\dots,i|1,\dots,j]=\lim_{N\to\infty}\det G_N[i-j+1,\dots,i|1,\dots,j]\\
&=\int_a^b dx_1\int_{x_1}^b dx_2\cdots\int_{x_{j-1}}^b dx_j\prod_{l=1}^{j}\kappa(x_l)\det M_X^T[i-j+1,\dots,i|1,\dots,j]\det M_X[1,\dots,j]\\
&=\frac{1}{j!}\int_{[a,b]^j}\prod_{l=1}^{j}dx_l\,\kappa(x_l)\det M_X^T[i-j+1,\dots,i|1,\dots,j]\det M_X[1,\dots,j],
\end{aligned}\tag{3.4}$$

where, in the last step of (3.4), we were able to use Proposition 3.1 since the integrand is always a symmetric function in its variables $(x_1,\dots,x_j)$. □

Theorem 3.1 exhibits an explicit connection between Gramian matrices and collocation matrices. Namely, it shows the relation between the determinants (1.6) and the analogous minors of the collocation matrices that can be constructed within the range of integration. Two comments about Theorem 3.1 are in order. First, it serves as a consistency check of Theorem 2.1, since the integral of the product of positive minors and a positive definite function κ is always positive. Thus, the total positivity of a basis $(u_0,\dots,u_n)$ translates into the total positivity of G. Second, since the integrand of (3.4) is a symmetric function, the pivots and multipliers of the bidiagonal decomposition of the Gramian matrix associated to a TP basis can be expressed in terms of integrals of symmetric functions. In the following section, we will flesh out the last statement in the case of polynomial bases, for which the integrand is an explicit linear combination of Schur polynomials.

4. Bidiagonal factorization of Gramian matrices of polynomial bases

Given a partition $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of size $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_p$ and length $l(\lambda) := p$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$, Jacobi's definition of the corresponding Schur polynomial in $n+1$ variables is expressed via Weyl's formula as:

$$s_\lambda(t_1, t_2, \ldots, t_{n+1}) := \det \begin{bmatrix} t_1^{\lambda_1+n} & t_2^{\lambda_1+n} & \ldots & t_{n+1}^{\lambda_1+n} \\ t_1^{\lambda_2+n-1} & t_2^{\lambda_2+n-1} & \ldots & t_{n+1}^{\lambda_2+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{\lambda_{n+1}} & t_2^{\lambda_{n+1}} & \ldots & t_{n+1}^{\lambda_{n+1}} \end{bmatrix} \Big/ \det \begin{bmatrix} 1 & 1 & \ldots & 1 \\ t_1 & t_2 & \ldots & t_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^n & t_2^n & \ldots & t_{n+1}^n \end{bmatrix}. \tag{4.1}$$

By convention, the Schur polynomial associated with the empty partition is defined as $s_\emptyset(t_1, \ldots, t_{n+1}) := 1$. This serves as the multiplicative identity in the algebra of symmetric functions. When considering all possible partitions, Schur polynomials form a basis for the space of symmetric functions, allowing any symmetric function to be uniquely expressed as a linear combination of Schur polynomials.

Let $\mathbf{P}^n(I)$ denote the space of polynomials of degree at most n defined on $I \subseteq \mathbb{R}$, and let $(p_0, \ldots, p_n)$ be a basis of $\mathbf{P}^n(I)$ such that

$$p_{i-1}(t) = \sum_{j=1}^{n+1} a_{i,j} t^{j-1}, \quad t \in I, \quad i = 1, \ldots, n+1. \tag{4.2}$$

We denote by $A = (a_{i,j})_{1 \leq i,j \leq n+1}$ the matrix representing the linear transformation from the basis $(p_0, \ldots, p_n)$ to the monomial polynomial basis of $\mathbf{P}^n(I)$. Specifically,

$$(p_0, \ldots, p_n)^T = A(m_0, \ldots, m_n)^T, \tag{4.3}$$

where $(m_0, \ldots, m_n)$ denotes the monomial basis.

Let M_T be the collocation matrix of $(p_0, \ldots, p_n)$ at $T := (t_1, \ldots, t_{n+1})$ on I with $t_1 < \cdots < t_{n+1}$, defined as

$$M_T := \big(p_{j-1}(t_i)\big)_{1 \leq i,j \leq n+1}. \tag{4.4}$$

The collocation matrix of the monomial polynomial basis $(m_0, \ldots, m_n)$ at T corresponds to the Vandermonde matrix V_T at the chosen nodes:

$$V_T := \big(t_i^{j-1}\big)_{1 \leq i,j \leq n+1}.$$

In [7], it was shown how to express the bidiagonal factorization (1.3) of $M := M_T$ in terms of Schur polynomials and some minors of the change of basis matrix A satisfying (4.3). For this purpose, it was shown that

$$\det M[i-j+1, \ldots, i \,|\, 1, \ldots, j] = \det V_{t_{i-j+1}, \ldots, t_i} \sum_{l_1 > \cdots > l_j} \det A_{[1, \ldots, j \,|\, l_j, \ldots, l_1]} s_{(l_1 - j, \ldots, l_j - 1)}(t_{i-j+1}, \ldots, t_i). \tag{4.5}$$

To effectively apply the product rules of Schur polynomials, we express the linear combination in (4.5) using partitions. Consider partitions $\lambda = (\lambda_1, \ldots, \lambda_j)$, where $\lambda_r = l_r + r - j - 1$ for $r = 1, \ldots, j$. Given that $l_1 > \cdots > l_j$, λ is a well-defined partition. For the minors of matrix A, we use the following notation:

$$A_{[i,\lambda]} := \det A[i-j+1, \ldots, i \,|\, l_j, \ldots, l_1] = \det A[i-j+1, \ldots, i \,|\, \lambda_j + 1, \ldots, \lambda_1 + j]. \tag{4.6}$$

Since the indices satisfy $l_1 > \cdots > l_j$ and $l_k \leq n+1$, for $1 \leq k \leq n+1$, the corresponding partitions will have j parts, each with a maximum length of $n+1-j$. In other words, the sum in (4.5) spans all Young diagrams that fit within a $j \times (n+1-j)$ rectangular box, which can be expressed as:

$$l(\lambda) \leq j, \qquad \lambda_1 \leq n+1-j.$$

With this notation, we have

$$\sum_{l_j < \cdots < l_1} \det A_{[1,\ldots,j \,|\, l_j,\ldots,l_1]} \, s_{(l_1-j,\ldots,l_j-1)}(t_{i-j+1}, \ldots, t_i) = \sum_{\substack{l(\lambda) \leq j \\ \lambda_1 \leq n+1-j}} A_{[j,\lambda]} \, s_\lambda(t_{i-j+1}, \ldots, t_i). \tag{4.7}$$

The derived formula (4.5) for the minors of the collocation matrices M_T of polynomial bases in terms of Schur polynomials, combined with known properties of these symmetric functions, facilitates a comprehensive characterization of total positivity on unbounded intervals for significant polynomial bases $(p_0, \ldots, p_n)$ (cf. [7]). Furthermore, considering Eqs (4.5) and (1.4), the bidiagonal factorization (1.3) of $M_{t_1,\ldots,t_{n+1}}$ was obtained, enhancing high relative accuracy (HRA) computations in algebraic problems involving these matrices.

Equation (3.2) applies to a general basis of functions $(u_0, \ldots, u_n)$. For polynomial bases, fully characterized by the transformation matrix A via (4.3), explicit formulas for the minors (1.6) can be derived in terms of Schur polynomials. In this context, it is important to recall the role of Littlewood-Richardson numbers, denoted $c^{\nu}_{\lambda,\mu}$, which describe the coefficients in the expansion of the product of two Schur polynomials. Specifically, given Schur polynomials s_λ and s_μ associated with partitions λ and μ, their product can be expressed as:

$$s_\lambda(x_1, \ldots, x_j) \cdot s_\mu(x_1, \ldots, x_j) = \sum_\rho c^\rho_{\lambda\mu} \, s_\rho(x_1, \ldots, x_j). \tag{4.8}$$

The following result provides a compact formula for the minors $\det G[i-j+1, \ldots, i \,|\, 1, \ldots, j]$, representing a significant contribution of this paper.

Theorem 4.1. *Let G be the Gramian matrix of a basis $(p_0, \ldots, p_n)$ of $P^n(J)$ with respect to an inner product* (2.1)*. Let A be the matrix of the linear transformation satisfying* (4.3)*. Then*

$$\det G[i-j+1, \ldots, i | 1, \ldots, j] = \sum_{\substack{l(\lambda) \leq j \\ \lambda_1 \leq n+1-j}} \sum_{\substack{l(\mu) \leq j \\ \mu_1 \leq n+1-j}} \sum_{|\rho| = |\lambda| + |\mu|} A_{[j,\lambda]} A_{[i,\mu]} \, c^\rho_{\lambda\mu} f_{\rho,j,\langle\cdot,\cdot\rangle}, \tag{4.9}$$

where the determinants $A_{[j,\lambda]}$ and $A_{[i,\mu]}$ are defined in (4.6)*, $c^\rho_{\lambda\mu}$ are the Littlewood-Richardson numbers,*

$$f_{\rho,j,\langle\cdot,\cdot\rangle} := \frac{1}{j!} \int_{[a,b]^j} \prod_{l=1}^{j} dx_l \, \kappa(x_l) \, (\det V_{x_1,\ldots,x_j})^2 s_\rho(x_1, \ldots, x_j), \tag{4.10}$$

and $V_{x_1,\ldots,x_j}$ denotes the Vandermonde matrix corresponding to the variables $x_1, \ldots, x_j$.

Proof. Consider the general formula for the initial minors of Gramian matrices provided in (3.2). By substituting the minors of the collocation matrices in terms of Schur polynomials, as shown in (4.7), we derive a compact expression for these minors. The derivation relies on the relation (4.8) for the product of Schur polynomials, and the fact that the numbers $c^{\rho}_{\lambda\mu}$ are zero unless $|\rho| = |\lambda| + |\mu|$. □

The evaluation of minors of Gramian matrices using the expression (4.9) involves summing over all partitions λ and μ whose Young diagrams fit within a box of size $j \times (n+1-j)$, as well as over partitions ρ for which the Littlewood-Richardson numbers $c^{\rho}_{\lambda\mu}$ are nonzero. For a general matrix A, the complexity of computing minors through (4.9) grows rapidly with n, primarily due to the need to calculate the Littlewood-Richardson numbers. Although combinatorial methods exist for their computation, these coefficients become increasingly costly to determine as n increases. Indeed, it has been conjectured that no algorithm can compute Littlewood-Richardson numbers in polynomial time [24].

However, this limitation is mitigated when considering lower triangular change of basis matrices A, where the computation of initial minors becomes significantly simpler, as we will now show. Notably, polynomial bases corresponding to lower triangular change of basis matrices constitute a broad and commonly used family.

Corollary 4.1. *Let G be the Gramian matrix* (1.2) *of a basis* $(p_0, \ldots, p_n)$ *of* $P^n(J)$ *with respect to an inner product* (2.1)*. If the matrix A satisfying* (4.3) *is lower triangular, then*

$$\det G[i-j+1, \ldots, i | 1, \ldots, j] = \sum_{\substack{l(\mu) \le j \\ \mu_1 \le n+1-j}} A_{[j,\emptyset]} A_{[i,\mu]} f_{\mu,j,\langle\cdot,\cdot\rangle}, \tag{4.11}$$

where the determinants $A_{[j,\emptyset]}$ and $A_{[i,\mu]}$ are defined in (4.6) *and $f_{\mu,j,\langle\cdot,\cdot\rangle}$ is defined in* (4.10)*.*

Proof. In (4.9), be aware that, for lower triangular matrices A, $A_{[j,\lambda]} \neq 0$ only in the case where $\lambda = \emptyset$. Then use the following property of Littlewood-Richardson numbers $c^{\rho}_{\emptyset\mu} = \delta^{\rho}_{\mu}$.

□

5. Selberg-like integrals and examples

In this section, we address the computation of the values $f_{\mu,j,\langle\cdot,\cdot\rangle}$. For a specific inner product, the integrals in (4.10) have been explicitly calculated and are commonly referred to as Selberg-like integrals. The case of integrals with the inner product

$$\kappa(t) := t^{\alpha}(1-t)^{\beta}, \quad J = [0,1],$$

has been studied by Kadell, among others. In [13, 14], it was found that for $\alpha, \beta > -1$ and a partition $\rho = (\rho_1, \ldots, \rho_j)$, we have

$$\begin{aligned} I_j(\alpha,\beta;\rho) &= \int_{[0,1]^j} \prod_{l=1}^{j} [dx_l\, x_l^{\alpha}(1-x_l)^{\beta}]\, (\det V_{x_1,\ldots,x_j})^2 s_\rho(x_1,\ldots,x_j) \\ &= j! \prod_{1 \le i < k \le j} (\rho_i - \rho_k + k - i) \prod_{i=1}^{j} \frac{\Gamma(\alpha+\rho_i+j-i+1)\Gamma(\beta+j-i+1)}{\Gamma(\alpha+\beta+2j-i+1+\rho_i)}. \end{aligned} \tag{5.1}$$

Also interesting for us is the result for the integral involving the product of two Schur polynomials with the same arguments in the case that $\beta = 0$. In [12], it was found that for $\alpha > -1$ and partitions $\lambda = (\lambda_1, \ldots, \lambda_j)$ and $\mu = (\mu_1, \ldots, \mu_j)$, we have

$$
\begin{aligned}
I_j(\alpha; \lambda, \mu) &= \int_{[0,1]^j} \prod_{l=1}^{j} [dx_l \, x_l^{\alpha}] \, (\det V_{x_1,\ldots,x_j})^2 s_\lambda(x_1, \ldots, x_j) s_\mu(x_1, \ldots, x_j) \\
&= j! \prod_{1 \le i < k \le j} (k - i + \mu_i - \mu_k)(k - i + \lambda_i - \lambda_k) \prod_{i,k=1}^{j} \frac{1}{\alpha + 2j - i - k + 1 + \lambda_i + \mu_k}. \qquad (5.2)
\end{aligned}
$$

Now, the integral (5.2) can be used by substituting

$$
\sum_{|\rho| = |\lambda| + |\mu|} c_{\lambda\mu}^{\rho} f_{\rho, j, \langle\cdot,\cdot\rangle} = \frac{1}{j!} I_j(\alpha; \lambda, \mu)
$$

in (4.9). Thus, for the case $\kappa(t) = t^\alpha$ and $J = [0,1]$, we have

$$
\det G[i - j + 1, \ldots, i | 1, \ldots, j] = \frac{1}{j!} \sum_{\substack{l(\lambda) \le j \\ \lambda_1 \le n+1-j}} \sum_{\substack{l(\mu) \le j \\ \mu_1 \le n+1-j}} A_{[j,\lambda]} A_{[i,\mu]} I_j(\alpha; \lambda, \mu). \qquad (5.3)
$$

So, for this particular case of inner product, the computation of (5.3) does not involve the Littlewood-Richardson numbers, which significantly reduces the computational complexity. This simplification arises from the remarkable properties of (5.2), which implicitly incorporates these numbers.

Algorithm 1 (see Appendix) provides an implementation of (5.3). To illustrate the application of this formula for computing the minors $\det G[i - j + 1, \ldots, i \,|\, 1, \ldots, j]$, we present two notable examples that highlight its efficiency and utility.

5.1. Bernstein mass matrices

Bernstein polynomials, defined as

$$
B_i^n(t) := \binom{n}{i} t^i (1 - t)^{n-i}, \quad i = 0, \ldots, n,
$$

are square-integrable functions with respect to the inner product

$$
\langle f, g \rangle_{\alpha,\beta} := \int_0^1 t^\alpha (1 - t)^\beta f(t) g(t) \, dt, \quad \alpha, \beta > -1. \qquad (5.4)
$$

The Gramian matrix of the Bernstein basis $(B_0^n, \ldots, B_n^n)$ under the inner product (5.4) is denoted as $G^{\alpha,\beta} = (g_{i,j}^{\alpha,\beta})_{1 \le i,j \le n+1}$, where

$$
g_{i,j}^{\alpha,\beta} = \binom{n}{i-1} \binom{n}{j-1} \frac{\Gamma(i + j + \alpha - 1) \Gamma(2n - i - j + \beta + 3)}{\Gamma(2n + \alpha + \beta + 2)},
$$

for $1 \le i, j \le n + 1$, and $\Gamma(x)$ is the Gamma function (see [17]). In the special case where $\alpha = \beta = 0$, the Gramian matrix $M := G^{(0,0)}$ is referred to as the Bernstein mass matrix.

For $n=2$, $G:=G^{0,0}$ is

$$G=\begin{pmatrix}1/5 & 1/10 & 1/30\\ 1/10 & 2/15 & 1/10\\ 1/30 & 1/10 & 1/5\end{pmatrix}.$$

It can be easily checked that $(B_0^2,B_1^2,B_2^2)^T=A(1,t,t^2)^T$, with

$$A=\begin{pmatrix}1 & -2 & 1\\ 0 & 2 & -2\\ 0 & 0 & 1\end{pmatrix}.$$

Be aware that the matrix of change of basis A is not lower triangular, so we cannot use (4.11) to compute the minors. Instead, we will use formula (5.2).

For $\det M[2,3|1,2]$, the partitions used for λ are $\{(1,1),(1,0),(0,0)\}$ and for μ are $\{(1,1),(1,0),(0,0)\}$. From (4.6), we can obtain $A_{[2,\lambda]}$ and $A_{[3,\mu]}$, respectively, as follows:

$$A_{[2,(1,1)]}=\det A[1,2|2,3]=2,\quad A_{[2,(1,0)]}=\det A[1,2|1,3]=-1,\quad A_{[2,(0,0)]}=\det A[1,2|1,2]=2,$$
$$A_{[3,(1,1)]}=\det A[2,3|2,3]=2,\quad A_{[3,(1,0)]}=\det A[2,3|1,3]=0,\quad A_{[3,(0,0)]}=\det A[2,3|1,2]=0.$$

Then, taking into account (5.2) and $I(\lambda,\mu):=I_2(\alpha;\lambda,\mu)$ for $j=2$ and $\alpha=0$, we can obtain

$$\begin{aligned}
&I((1,1),(1,1))=2/240,\\
&I((1,1),(1,0))=I((1,0),(1,1))=2/60,\quad I((1,1),(0,0))=I((0,0),(1,1))=2/72,\\
&I((1,0),(1,0))=8/45, I((1,0),(0,0))=2/12,\\
&I((0,0),(0,0))=2/12.
\end{aligned}$$

Now, by (5.3), we obtain

$$\begin{aligned}
&\det M[2,3|1,2]=\\
&(1/2)\Big(A_{[2,(1,1)]}\big(A_{[3,(1,1)]}I((1,1),(1,1))+A_{[3,(1,0)]}I((1,1),(1,0))+A_{[3,(0,0)]}I((1,1),(0,0))\big)\Big)\\
&+\big(A_{[2,(1,0)]}\big(A_{[3,(1,1)]}I((1,0),(1,1))+A_{[3,(1,0)]}I((1,0),(1,0))+A_{[3,(0,0)]}I((1,0),(0,0))\big)\big)\\
&+\big(A_{[2,(0,0)]}\big(A_{[3,(1,1)]}I((0,0),(1,1))+A_{[3,(1,0)]}I((0,0),(1,0))+A_{[3,(0,0)]}I((0,0),(0,0))\big)\big)=\\
&1/180.
\end{aligned}$$

Following the same reasoning, we can obtain the other determinants.

While the matrix A is not lower triangular, the computation of Littlewood-Richardson numbers can be avoided by using (5.2). Specifically, (5.3) can be efficiently applied to any polynomial basis, provided that the inner product is defined as in (5.4) with $\beta=0$.

As previously noted, the computation of Littlewood-Richardson numbers is also unnecessary for polynomial bases associated with lower triangular matrices A. In such cases, the formula (4.11) is applicable for any inner product, provided the corresponding Selberg-like integrals can be efficiently evaluated. This approach achieves a comparable level of computational efficiency and is demonstrated in the following example, which features a generic recursive basis.

5.2. Recursive bases

The example of recursive bases illustrates that Eq (4.11) can be highly effective for computing the minors $\det G[i-j+1,\dots,i|1,\dots,j]$, especially when the structure of the basis change matrix A facilitates systematic computation of its minors involving consecutive rows.

For given values $b_1,\dots,b_{n+1}$ with $b_i>0$ for $i=1,\dots,n+1$, the recursive basis $(p_0,\dots,p_n)$ is defined by the polynomials:

$$p_i=\sum_{j=1}^{i+1} b_j t^{j-1}, \quad i=0,\dots,n.$$

The corresponding change of basis matrix B, which satisfies $(p_0,\dots,p_n)^T=B(m_0,\dots,m_n)^T$, where $m_i:=t^i$ for $i=0,\dots,n$, is a nonsingular, lower triangular, and the TP matrix is structured as follows:

$$B=\begin{pmatrix} b_1 & 0 & 0 & \cdots & 0 \\ b_1 & b_2 & 0 & \cdots & 0 \\ b_1 & b_2 & b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & b_3 & \cdots & b_{n+1} \end{pmatrix}.$$

Thus, the basis $(p_0,\dots,p_n)$ is TP for $t\in[0,\infty)$. As before, we consider the inner product defined as:

$$\langle f,g\rangle=\int_0^1 f(t)g(t)\,dt,$$

which corresponds to the special case of the inner product (5.4) with $\alpha=\beta=0$.

Let us note that the only nonzero minors of B are

$$B[i-j+1,\dots,i|m,i-j+2,i-j+3,\dots,i]=b_m\prod_{k=2}^{j} b_{i-j+k}, \quad m=1,\dots,i-j+1.$$

This way, the only nonzero contributions to (4.11) come from the partitions $\mu=(\mu_1,\dots,\mu_j)$ with

$$\mu_r=i-j,\ r=1,\dots,j-1, \qquad \mu_j=0,\dots,i-j.$$

Applying (4.11) with (5.1), we obtain

$$\det G[i-j+1,\dots,i|1,\dots,j]=b_1\prod_{k=2}^{j} b_{i-j+k}b_k\prod_{l=1}^{j-1}\frac{(j-l)!^2}{(i-l+1)_j}\sum_{m=0}^{i-j}\frac{1}{(m+1)_j}\prod_{r=1}^{j-1}(i-m-r)\,b_{m+1},$$

where $(x)_n:=x(x+1)\cdots(x+n-1)$ denotes the Pochhammer symbol for ascending factorials.

6. Conclusions

We have shown that Gramian matrices can be expressed as limits of products of collocation matrices associated with the corresponding bases. This formulation allows the total positivity property of the bases to be extended to their Gramian matrices, whose minors can be written in terms of Selberg-like integrals, in the polynomial case.

Several open lines of research will be dealt with in future works. A logical continuation of this work is to consider other internal products. We would like to point out that aside from the conceptual and theoretical interest that Eqs (4.9) and (4.11) may have, their computational operativeness crucially depends on the Selberg-like integrals to be solved. Different bases may be TP for different ranges, and therefore the use of suitable inner products will be necessary. The chosen inner product will enter explicitly the computation of the minors of G in Theorem 4.1 through the multivariable integrals $f_{\rho,j,\langle\cdot,\cdot\rangle}$. For this reason, in the future, it will be desirable to solve other Selberg-like integrals.

Besides the above proposal, Gramian matrices of non-polynomial basis could be considered, and their bidiagonal decomposition studied using Theorem 3.1.

Author contributions

Pablo Díaz: Conceptualization, methodology, investigation, writing—original draft, writing—review and editing; Esmeralda Mainar: Conceptualization, methodology, investigation, writing—original draft, writing—review and editing; Beatriz Rubio: Conceptualization, methodology, investigation, writing—original draft, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was partially supported by Spanish research grants PID2022-138569NB-I00 (MCI/AEI) and RED2022-134176-T (MCI/AEI) and by Gobierno de Aragón (E41_23R).

Conflict of interest

Esmeralda Mainar is the Guest Editor of special issue "Advances in Numerical Linear Algebra: Theory and Methods" for AIMS Mathematics. Esmeralda Mainar was not involved in the editorial review and the decision to publish this article.

The authors declare no conflicts of interest.

References

1. T. Ando, Totally positive matrices, *Linear Algebra Appl.*, **90** (1987), 165–219. https://doi.org/10.1016/0024-3795(87)90313-2
2. J. Delgado, H. Orera, J. M. Peña, Accurate computations with Laguerre matrices, *Numer. Linear Algebra Appl.*, **26** (2019), e2217. https://doi.org/10.1002/nla.2217
3. J. Delgado, H. Orera, J. M. Peña, Accurate algorithms for Bessel matrices, *J. Sci. Comput.*, **80** (2019), 1264–1278. https://doi.org/10.1007/s10915-019-00975-6

4. J. Delgado, J. M. Peña, Accurate computations with collocation matrices of rational bases, *Appl. Math. Comput.*, **219** (2013), 4354–4364. https://doi.org/10.1016/j.amc.2012.10.019

5. J. Demmel, P. Koev, The accurate and efficient solution of a totally positive generalized Vandermonde linear system, *SIAM J. Matrix Anal. Appl.*, **27** (2005), 42–52. https://doi.org/10.1137/S0895479804440335

6. H. Diao, H. Liu, L. Tao, Stable determination of an impedance obstacle by a single far-field measurement, *Inverse Probl.*, **40** (2024), 055005. https://doi.org/10.1088/1361-6420/ad3087

7. P. Díaz, E. Mainar, B. Rubio, Polynomial total positivity and high relative accuracy through Schur polynomials, *J. Sci. Comput.*, **97** (2023), 10. https://doi.org/10.1007/s10915-023-02323-1

8. P. Díaz, E. Mainar, B. Rubio, Totally positive wronskian matrices and symmetric functions, *Axioms*, **13** (2024), 589. https://doi.org/10.3390/axioms13090589

9. S. M. Fallat, C. R. Johnson, *Totally nonnegative matrices*, Princeton: Princeton University Press, 2011. https://doi.org/10.1515/9781400839018

10. M. Gasca, J. M. Peña, A matricial description of Neville elimination with applications to total positivity, *Linear Algebra Appl.*, **202** (1994), 33–53. https://doi.org/10.1016/0024-3795(94)90183-X

11. M. Gasca, J. M. Peña, On factorizations of totally positive matrices, In: *Total positivity and its applications*, Dordrecht: Springer, **359** (1996), 109–130. https://doi.org/10.1007/978-94-015-8674-0_7

12. L. K. Hua, Harmonic analysis of functions of several complex variables in the complex domains, In: *Translations of mathematical monographs*, Providence: American Mathematical Society, **6** (1963).

13. K. W. J. Kadell, An integral for the product of two Selberg-Jack symmetric polynomials, *Compositio Math.*, **87** (1993), 5–43.

14. K. W. J. Kadell, The Selberg-Jack symmetric functions, *Adv. Math.*, **130** (1997), 33–102. https://doi.org/10.1006/aima.1997.1642

15. P. Koev, Accurate computations with totally nonnegative matrices, *SIAM J. Matrix Anal. Appl.*, **29** (2007) , 731–751. https://doi.org/10.1137/04061903X

16. E. Mainar, J. M. Peña, B. Rubio, Accurate computations with matrices related to bases $\{t^i e^{\lambda t}\}$, *Adv. Comput. Math.*, **48** (2022), 38. https://doi.org/10.1007/s10444-022-09954-2

17. E. Mainar, J. M. Peña, B. Rubio, Total positivity and accurate computations with Gram matrices of Bernstein bases, *Numer. Algor.*, **91** (2022), 841–859. https://doi.org/10.1007/s11075-022-01284-0

18. E. Mainar, J. M. Peña, B. Rubio, Total positivity and accurate computations with Gram matrices of Said-Ball bases, *Numer. Linear Algebra Appl.*, **30** (2023), e2521. https://doi.org/10.1002/nla.2521

19. A. Marco, J. J. Martínez, A fast and accurate algorithm for solving Bernstein-Vandermonde linear systems, *Linear Algebra Appl.*, **422** (2007) , 616–628. https://doi.org/10.1016/j.laa.2006.11.020

20. A. Marco, J. J. Martínez, Accurate computations with Said-Ball-Vandermonde matrices, *Linear Algebra Appl.*, **432** (2010), 2894–2908. https://doi.org/10.1016/j.laa.2009.12.034

21. A. Marco, J. J. Martínez, Accurate computations with totally positive Bernstein-Vandermonde matrices, *Electron. J. Linear Algebra*, **26** (2013), 357–380. https://doi.org/10.13001/1081-3810.1658

22. A. Marco, J. J. Martínez, J. M. Peña, Accurate bidiagonal decomposition of totally positive Cauchy Vandermonde matrices and applications, *Linear Algebra Appl.*, **517** (2017), 63–84. https://doi.org/10.1016/j.laa.2016.12.003

23. A. Marco, J. J. Martínez, R. Viaña, Total positivity and least squares problems in the Lagrange basis, *Numer. Linear Algebra Appl.*, **31** (2024), e2554. https://doi.org/10.1002/nla.2554

24. H. Narayanan, On the complexity of computing Kostka numbers and Littlewood-Richardson coefficients, *J. Algebr. Comb.*, **24** (2006), 347–354. https://doi.org/10.1007/s10801-006-0008-5

25. A. Pinkus, Totally positive matrices, In: *Cambridge tracts in mathematics*, Cambridge: Cambridge University Press, **181** (2010). https://doi.org/10.1017/CBO9780511691713

26. Z. Yang, X. X. Ma, Computing eigenvalues of quasi-rational Bernstein-vandermonde matrices to high relative accuracy, *Numer. Linear Algebra Appl.*, **29** (2022), e2421. https://doi.org/10.1002/nla.2421

Supplementary

Implementation of the code of formula (5.3)

Algorithm 1: MATLAB code formula (5.3)

```
Require: alpha, i, j, A
Ensure: G [i-j+1,. . . ,i—1,. . . ,j] (see (5.3))
n = size(A,1)
parts = partitions(j,n)
partsize = size(parts,1)
total = 0
for k1 = 1:partsize
    rho = parts(k1,:)
    sum = 0
    for k2 = 1:partsize
        mu = parts(k2,:)
        sum = sum + Alambda(A,j,i,mu)∗ f(alpha,j,rho,mu)
    end
    G = G + Alambda(A,j,j,rho) ∗ sum
end
function s = s(j,vector)
comb = transpose(nchoosek(1:j,2))
s = 1
for c = comb
    i = c(1)
    k = c(2)
s = s ∗ (vector(i)-vector(k)+k-i)
end
function part = partitionsR(from, level)
part = []
for value = from:-1:0
    if level > 1
        res = partitionsR(min(from,value),level-1)
        part = [part; [value .∗ ones(size(res,1),1) res]]
    else
        part = [part; value]
    end
end
function partitions = partitions(j, n)
partitions = partitionsR(n-j,j)
function I = f(alpha,j,rho,mu)
I = s(j,rho) ∗ s(j,mu)
for i = 1:j
    for k = 1:j
        I = I ∗ 1 / (alpha + 2∗j - i - k + 1 + rho(i) + mu(k))
    end
end
function Alambda = Alambda(A, j, i, lambda)
rows = (i-j+1):i
cols = flip(lambda) + (1:j)
Alambda = det(A(rows,cols))
```