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Research article

Existence of solutions for [p, q]-difference initial value problems: application to the [p, q]-based model of vibrating eardrums

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Abstract: The eardrum is one of the most important organs in the body, and disorders such as infection or injury may affect the proper functioning of the eardrum and lead to hearing problems. In this paper, based on a real-world phenomena, we study some mathematical aspects of an abstract fractional [p, q]-difference equation with initial conditions. Our initial value problem tries to model a vibrating eardrum by using the newly defined fractional Caputo-type [p, q]-derivatives in two nonlinear single-valued and set-valued structures. We obtain a general form of the solutions in the framework of a [p, q]-integral equation, and then we investigate the existence and uniqueness properties with the help of fixed points and the end-points of some special β-α-contractions and compact mappings. Finally, we simulate this version of the vibrating eardrum model by giving two numerical examples to validate the established theorems.

Keywords: set-valued function; fixed-point; end-point; [p, q]-derivative; initial problem; vibrating eardrum model
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1. Introduction

As a widely appreciated and exciting field of mathematics, fractional calculus including fractional and nonfractional differential equations has developed in recent years. Different types of boundary value problems (BVPs) can be studied with the help of these differential equations and simulated dynamical behaviors of real phenomena in the world. All these actions and developments in the field of differential equations have been possible due to some important mathematical features of the derivation and integration operators. In other words, in recent years, various mathematicians have tried to define various nonlocal operators with singular or nonsingular kernels, which can be used in modeling engineering phenomena, chemical interactions, physical events, economic issues related to banking and insurance, the spread of deadly viruses in medical science, and other health and hygiene problems, etc. Among the most widely used operators (and some studies conducted with them), we can mention the use of the Reimann-Liouville derivative in [1], the Caputo derivative in [1], the Hadamard derivative in [1], the Katugampola derivative in [2], the Hilfer derivative in [3], the Caputo-Fabrizio derivative in [4], and the Atangana-Baleanu derivative in [5] and other forms of the integer order derivatives [6].

Quantum calculus, which is referred to as $\mathtt{q}$-calculus in the world of mathematics, created a revolution in the definition of the new concepts of derivative and integral operators without relying on the concept of limits, which today, can be used on discrete spaces or finite sets. This calculus provides the corresponding quantum concepts for most of the previously defined mathematical concepts in which there is no trace of the concept of limits. In the 18th century, Euler defined some $\mathtt{q}$-formulas in this field, but the main spark of these concepts, which were systematically and precisely struck in the field of mathematics, is related to the works presented by Jackson in 1910 [7]. Since then, others have tried to expand this new $\mathtt{q}$-calculus and define new $\mathtt{q}$-analog operators of fractional orders, among whom mathematicians such as Al-Salam [8], Agarwal [9], and Annaby [10] can be mentioned.

In 2013, Tariboon and Ntouyas [11] introduced an $\mathtt{q}$-shifting operator and used it to define a generalized version of the $\mathtt{q}$-derivative (defined by Jackson) on finite intervals. They applied their $\mathtt{q}$-derivative operator for modeling an impulsive difference equation to show its efficiency. Later, we can see that various papers have been published in this field, in which $\mathtt{q}$-derivative operators have been used to check the existence and uniqueness of the solutions of $\mathtt{q}$-difference BVPs many times, like [12–17].

Along with the completion and development of $\mathtt{q}$-calculus, Chakrabarti and Jagannathan [18] used the idea of defining derivative and integral operators without the limit notion, and this time, they got help from two parameters to define new operators. In fact, the new calculus that they founded is known as post-quantum calculus or $[\mathtt{p}, \mathtt{q}]$-calculus. With the assumption of $0 < \mathtt{q} < \mathtt{p} \leq 1$, they were able to introduce $[\mathtt{p}, \mathtt{q}]$-analogs of the above operators. If we pay attention to the basic definitions of $[\mathtt{p}, \mathtt{q}]$-calculus, we find that $\mathtt{q}$-calculus is a special case of $[\mathtt{p}, \mathtt{q}]$-calculus when the parameter $\mathtt{p}$ is equal to one. Bézier surfaces and curves [19], $[\mathtt{p}, \mathtt{q}]$-approximation methods [20,21], and $[\mathtt{p}, \mathtt{q}]$-hypergeometric series [22] are a small part of the applications of $[\mathtt{p}, \mathtt{q}]$-calculus in various mathematical theories. Moreover, in 2020, Soontharanon and Sitthiwirattham [23] conducted research on fractional $[\mathtt{p}, \mathtt{q}]$-calculus and defined the fractional $[\mathtt{p}, \mathtt{q}]$-operators of the two Reimann-Liouville and Caputo types. From this paper on, other researchers wanted to continue their studies on mathematical modeling and existence theory based on the existing operators in fractional $[\mathtt{p}, \mathtt{q}]$-calculus (see [24–27]).

In 2020, Soontharanon et al. [28] considered the $[\mathtt{p}, \mathtt{q}]$-Robin boundary conditions for a Riemann-Liouville type l-th order $[\mathtt{p}, \mathtt{q}]$-integro-difference equation, given as

$$\begin{cases} {}^{\mathrm{R}}\mathcal{D}^{l_0}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(\mathrm{t}) = E(\mathrm{t}, \mathrm{z}(\mathrm{t}), {}^{\mathrm{R}}\mathcal{I}^{l_1}_{[\mathrm{p},\mathrm{q}]}(f\mathrm{z})(\mathrm{t}), {}^{\mathrm{R}}\mathcal{D}^{l_2}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(\mathrm{t})), \quad l_1, l_2 \in (0,1],\ \mathrm{t} \in \Im^L_{[\mathrm{p},\mathrm{q}]},\ l_0 \in (1,2], \\ \zeta_1 \mathrm{z}(a) + \zeta_2 {}^{\mathrm{R}}\mathcal{D}^{l_3}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(a) = h_1(\mathrm{z}(\mathrm{t})), \quad l_3 \in (0,1],\ \zeta_1, \zeta_2 \in \mathbb{R}^+, \\ \delta_1 \mathrm{z}(\frac{L}{\mathrm{p}}) + \delta_2 {}^{\mathrm{R}}\mathcal{D}^{l_3}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(\frac{L}{\mathrm{p}}) = h_2(\mathrm{z}(\mathrm{t})), \quad \delta_1, \delta_2 \in \mathbb{R}^+,\ 0 < \mathrm{q} < \mathrm{p} \leq 1, \end{cases}$$

and they proved some existence theorems for this (p,q)-problem in which $a \in \Im^L_{[\mathrm{p},\mathrm{q}]} - \left\{0, \frac{L}{\mathrm{p}}\right\}$ with $\Im^L_{[\mathrm{p},\mathrm{q}]} := \left\{\frac{\mathrm{q}^\ell}{\mathrm{p}^{\ell+1}} L : \ell \in \mathbb{N}_0\right\} \cup \{0\}$. If $\mathcal{A} = \mathbb{R}^3$, then the nonlinear function $E : \Im^L_{[\mathrm{p},\mathrm{q}]} \times \mathcal{A} \to \mathbb{R}$ is continuous, and $h_1, h_2 : C(\Im^L_{[\mathrm{p},\mathrm{q}]}, \mathbb{R}) \to \mathbb{R}$ and $f \in C(\Im^L_{[\mathrm{p},\mathrm{q}]} \times \Im^L_{[\mathrm{p},\mathrm{q}]}, [0,\infty))$. Moreover, ${}^{\mathrm{R}}\mathcal{D}^{l^*}_{[\mathrm{p},\mathrm{q}]}$ denotes the l^*-th order $[\mathrm{p},\mathrm{q}]$-derivative with $l^* = l_i (i = 0,2,3)$ (of the Riemann-Liouville type).

In 2022, Neang et al. [29] introduced a function $E \in C([0,L] \times \mathbb{R}, \mathbb{R})$ with a simplified domain, and for proving the existence results, they defined an l-th order $[\mathrm{p},\mathrm{q}]$-difference boundary problem of the Caputo type as

$$\begin{cases} {}^{\mathrm{c}}\mathcal{D}^{l}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(\mathrm{t}) = E(\mathrm{p}^l\mathrm{t}, \mathrm{z}(\mathrm{p}^l\mathrm{t})), \ \ l \in (1,2], \ \ 0 \leq \mathrm{t} \leq L, \\ \zeta_1 \mathrm{z}(0) + \zeta_2 \mathcal{D}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(0) = \zeta_3, \quad \zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}, \\ \zeta_4 \mathrm{z}(L) + \zeta_5 \mathcal{D}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(\mathrm{p}L) = \zeta_6, \quad \zeta_4, \zeta_5, \zeta_6 \in \mathbb{R},\ 0 < \mathrm{q} < \mathrm{p} \leq 1. \end{cases}$$

In 2024, Etemad et al. [30] extended the standard Navier problem to two sequential $[\mathrm{p},\mathrm{q}]$-Navier difference and inclusion problems, given by

$$\begin{cases} {}^{\mathrm{c}}\mathcal{D}^{l_1}_{[\mathrm{p},\mathrm{q}]}({}^{\mathrm{c}}\mathcal{D}^{l_2}_{[\mathrm{p},\mathrm{q}]}\mathrm{z})(\mathrm{t}) = E(\mathrm{t}, \mathrm{z}(\mathrm{t}), {}^{\mathrm{c}}\mathcal{D}^{l_2}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(\mathrm{t})), \qquad \mathrm{t} \in \Im^L_{[\mathrm{p},\mathrm{q}]} := [0, \frac{L}{\mathrm{p}}],\ \mathrm{p},\mathrm{q} \in (0,1), \\ \lambda \mathrm{z}(0) = \beta \mathrm{z}(1) = \gamma {}^{\mathrm{c}}\mathcal{D}^{l_2}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(0) = \delta {}^{\mathrm{c}}\mathcal{D}^{l_2}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(1) = 0, \end{cases}$$

and

$$\begin{cases} {}^{\mathrm{c}}\mathcal{D}^{l_1}_{[\mathrm{p},\mathrm{q}]}({}^{\mathrm{c}}\mathcal{D}^{l_2}_{[\mathrm{p},\mathrm{q}]}\mathrm{z})(\mathrm{t}) \in \mathcal{E}(\mathrm{t}, \mathrm{z}(\mathrm{t}), {}^{\mathrm{c}}\mathcal{D}^{l_2}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(\mathrm{t})), \qquad \mathrm{t} \in \Im^L_{[\mathrm{p},\mathrm{q}]} := [0, \frac{L}{\mathrm{p}}],\ \mathrm{p},\mathrm{q} \in (0,1), \\ \lambda \mathrm{z}(0) = \beta \mathrm{z}(1) = \gamma {}^{\mathrm{c}}\mathcal{D}^{l_2}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(0) = \delta {}^{\mathrm{c}}\mathcal{D}^{l_2}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(1) = 0, \end{cases}$$

where $l_1 \in (1,2]$, $l_2 \in (1,2]$ and $\lambda, \beta, \gamma, \delta \in \mathbb{R}^+$. ${}^{\mathrm{c}}\mathcal{D}^{(\cdot)}_{[\mathrm{p},\mathrm{q}]}$ shows the fractional $[\mathrm{p},\mathrm{q}]$-derivative of the Caputo type.

The main advantage of $[\mathrm{p},\mathrm{q}]$-calculus over q-calculus lies in its flexibility in different applications, allowing for a more generalized framework that captures a wider range of mathematical behaviors. More precisely, while q-calculus focuses solely on the quantum parameter q to define sequences and functions, $[\mathrm{p},\mathrm{q}]$-calculus introduces two parameters, p and q [31]. This duality enables more complex interactions between variables, leading to richer results in certain mathematical contexts. The presence of both parameters allows for symmetric treatments of different types of problems. For example, one can explore the relationships between series or special functions that may not be evident when only using a single parameter. The versatility provided by $[\mathtt{p},\mathtt{q}]$-calculations makes $[\mathtt{p},\mathtt{q}]$-operators suitable for diverse fields such as combinatorics, number theory, quantum calculus, and even areas like physics, where two-variable generalizations might model phenomena better than single-variable approaches. By considering these points, we decided start this study by introducing these capabilities for the next models in the context of $[\mathtt{p},\mathtt{q}]$-derivatives.

According to the analysis method used in the $[\mathtt{p},\mathtt{q}]$-difference problems above, and motivated by some real-world phenomena and the aforementioned advantages, in this paper, we aimed to study a Caputo-type $[\mathtt{p},\mathtt{q}]$-based difference initial value problem (taken from the standard model of a vibrating eardrum, which will be completely explained in Section 5), given by

$$\begin{cases} ({}^{c}\mathcal{D}^{\nu}_{[\mathtt{p},\mathtt{q}]}\mathtt{z})(\mathtt{t}) = \mathfrak{f}_{\mathtt{z}}(\mathtt{t},\mathtt{z}(\mathtt{t}),{}^{c}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})), \qquad (\mathtt{t}\in\mathfrak{I}^{L}_{[\mathtt{p},\mathtt{q}]}=[0,\frac{L}{\mathtt{p}}],\ 0<\mathtt{q}<\mathtt{p}\leq 1), \\ \mathtt{z}(\mathtt{t})\big|_{\mathtt{t}=0}=\lambda, \qquad \mathcal{D}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})\big|_{\mathtt{t}=0}=\eta, \end{cases} \tag{1.1}$$

and a Caputo-type $[\mathtt{p},\mathtt{q}]$-based inclusion problem, given by

$$\begin{cases} ({}^{c}\mathcal{D}^{\nu}_{[\mathtt{p},\mathtt{q}]}\mathtt{z})(\mathtt{t}) \in \mathfrak{F}_{\mathtt{z}}(\mathtt{t},\mathtt{z}(\mathtt{t}),{}^{c}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})), \\ \mathtt{z}(\mathtt{t})\big|_{\mathtt{t}=0}=\lambda, \qquad \mathcal{D}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})\big|_{\mathtt{t}=0}=\eta, \end{cases} \tag{1.2}$$

where $L > 0$, $\nu \in (1,2]$, $\mathtt{w} \in (0,1]$, $\lambda,\eta \in \mathbb{R}$, and the given $[\mathtt{p},\mathtt{q}]$-derivatives of the Caputo type are denoted by the symbol ${}^{c}\mathcal{D}^{(\cdot)}_{[\mathtt{p},\mathtt{q}]}$. Moreover $\mathfrak{f}_{\mathtt{z}} : [0,\frac{L}{\mathtt{p}}]\times\mathbb{R}^2 \to \mathbb{R}$ and $\mathfrak{F}_{\mathtt{z}} : [0,\frac{L}{\mathtt{p}}]\times\mathbb{R}^2 \to \mathrm{P}(\mathbb{R})$ are two arbitrary nonlinear continuous single-valued and set-valued functions, respectively, along with some related properties. As we know, in traditional calculus, derivatives measure rates of change concerning time or space using standard definitions based on limits. In this model, the $[\mathtt{p},\mathtt{q}]$-based derivative generalizes this idea by introducing two parameters which influence how we interpret these rates of change under varying conditions or scales.

The main contribution of this study is that we try to establish the existence theorems under a special group of contractions like β-admissible β-α-contractions in two single-valued and set-valued structures for the given nonlinear functions. This method not only shows the applicability of such contractions in fixed point theory but also helps us to establish some related results on the existence of solutions of the given Caputo-type $[\mathtt{p},\mathtt{q}]$-based initial value problems in (1.1) and (1.2). Another aspect of the novelty of this paper is related to the model of a vibrating eardrum using the newly defined $[\mathtt{p},\mathtt{q}]$-based operators for the first time. In fact, we can mention that this model is new and our results give new directions for future studies in relation to the effectiveness of the $[\mathtt{p},\mathtt{q}]$-based operators in the modeling theory.

On the other hand, the use of $[\mathtt{p},\mathtt{q}]$-based operators, particularly in the context of modeling phenomena such as the behavior of eardrums, allows for a more flexible approach to capturing complex behaviors that traditional calculus may not adequately represent. In fact, the parameters $\mathtt{p}$ and $\mathtt{q}$ are control parameters and they can be adjusted to model different sensitivities or responses at varying frequencies or amplitudes of incoming sound waves. This adaptability can help create more accurate simulations that reflect the physiological variations in human hearing. By utilizing both parameters $\mathtt{p}$ and $\mathtt{q}$, researchers can modify the boundary conditions dynamically on the basis of the characteristics. Evidently, while direct literature specifically addressing improvements in traditional eardrum models via $[\mathtt{p},\mathtt{q}]$-based derivatives is limited at this stage (and remains largely conceptual), these points illustrate potential directions where this advanced mathematical framework may enhance existing modeling techniques used in biophysics or auditory research fields when applied thoroughly.

We outline the structure of the paper as follows. Section 2 gives some preliminaries about $[p, q]$-calculus and q-calculus. Section 3 studies the theoretical properties of the solutions of the Caputo-type $[p, q]$-based initial value problem in (1.1). To do this, we use the β-admissible β-α-contractions along with Krasnoselskii's fixed point theorem. Section 4 continues this approach for set-valued functions this time, in relation to the Caputo-type $[p, q]$-based inclusion problem in (1.2). Section 5 gives an application to an $[p, q]$-based model of a vibrating eardum along with two numerical examples. Finally, Section 6 completes this research by giving the conclusions.

2. $[p, q]$-Calculus and q-calculus: preliminaries

In this section, we start by recalling several definitions and theorems in the context of $[p, q]$-calculus and its special version, i.e., q-calculus. In fact, it is notable that all definitions and existing properties given in this section are transformable into the similar ones in q-calculus if we set $p = 1$. From this point forward, we let $q \in (0, 1)$ and $0 < q < p \leq 1$ until the end.

The $[p, q]$-analog for a power function like $(r_1 - r_2)^m$, which is named as a $[p, q]$-power function [23], is formulated by $(r_1 - r_2)^{(0)}_{[p,q]} = 1$ and

$$(r_1 - r_2)^{(m)}_{[p,q]} = \prod_{b=0}^{m-1} (r_1 p^b - r_2 q^b), \quad (r_1, r_2 \in \mathbb{R}, m \in \mathbb{N}_0). \tag{2.1}$$

It is natural that the q-version for a power function is obtained easily from the relations above by assuming that $p = 1$ [32], $(r_1 - r_2)^{(0)}_{[q]} = 1$, and

$$(r_1 - r_2)^{(m)}_{[q]} = \prod_{b=0}^{m-1} (r_1 - r_2 q^b). \tag{2.2}$$

In (2.1) and (2.2), by assuming $m = l \in \mathbb{R}$, the general fractional forms are obtained as

$$(r_1 - r_2)^{(l)}_{[p,q]} = r_1^l \prod_{b=0}^{\infty} \frac{1}{p^l} \left(\frac{1 - (\frac{r_2}{r_1})(\frac{q}{p})^b}{1 - (\frac{r_2}{r_1})(\frac{q}{p})^{l+b}} \right),$$

and

$$(r_1 - r_2)^{(l)}_{[q]} = r_1^l \prod_{b=0}^{\infty} \frac{1 - (\frac{r_2}{r_1}) q^b}{1 - (\frac{r_2}{r_1}) q^{l+b}},$$

for $r_1 \neq 0$, respectively. Note that if $r_2 = 0$, then $(r_1)^{(l)}_{[p,q]} = \frac{1}{p^l} r_1^l$ [23] and $(r_1)^{(l)}_{[q]} = r_1^l$ [32].

Now, the $[p, q]$-number $[r]_{[p,q]}$ and its q-analog $[r]_{[q]}$, along with the $[p, q]$-Gamma function $\Gamma_{[p,q]}(\cdot)$ and its q-analog $\Gamma_{[q]}(\cdot)$ are defined by

$$[0]_{[p,q]} = 0, \quad [r]_{[p,q]} = p^{r-1} [r]_{\frac{q}{p}} = \frac{p^r - q^r}{p - q}, \quad \Gamma_{[p,q]}(l) = \frac{(p - q)^{(l-1)}_{[p,q]}}{(p - q)^{l-1}},$$

and

$$[0]_{[q]} = 0, \quad [r]_{[q]} = q^{r-1} + \cdots + q + 1 = \frac{1-q^r}{1-q} \ (r \neq 0), \quad \Gamma_{[q]}(l) = \frac{(1-q)_{[q]}^{(l-1)}}{(1-q)^{l-1}},$$

for $r \in \mathbb{R}$ and $l \in \mathbb{R}\backslash\mathbb{Z}^{\leq 0}$.

For two types of Gamma functions [23, 32], we have

$$\Gamma_{[p,q]}(l+1) = [l]_{[p,q]}\Gamma_{[p,q]}(l) \quad \text{and} \quad \Gamma_{[q]}(l+1) = [l]_{[q]}\Gamma_{[q]}(l).$$

Definition 2.1. *[23, 33] The* $[p, q]$*-derivative and its* q*-analog, i.e., the* q*-derivative, for a function* z *are given as*

$$\mathcal{D}_{[p,q]}z(t) = \frac{z(pt) - z(qt)}{(p-q)t},$$

and

$$\mathcal{D}_{[q]}z(t) = \Big[\frac{d}{dt}\Big]_{[q]} z(t) = \frac{z(t) - z(qt)}{(1-q)t},$$

respectively.

Definition 2.2. *[23, 33] The* $[p, q]$*-integral and its* q*-analog, i.e., the* q*-integral, for* $z \in C([0, L], \mathbb{R})$ *are given as*

$$\mathcal{I}_{[p,q]}z(t) = \int_0^t z(u) d_{[p,q]}u = (p-q)t \sum_{b=0}^{\infty} \frac{q^b}{p^{b+1}} z\Big[\frac{q^b}{p^{b+1}}t\Big],$$

and

$$\mathcal{I}_{[q]}z(t) = \int_0^t z(u) d_{[q]}u = (1-q)t \sum_{b=0}^{\infty} q^b z[q^b t],$$

respectively.

Definition 2.3. *[23, 34] The lth order* $[p, q]$*-integral and its* q*-analog, of the Riemann-Liouville type, for the function* $z \in C([0, L], \mathbb{R})$ *are defined as*

$$^{R}\mathcal{I}_{[p,q]}^{l}z(t) = \begin{cases} \dfrac{1}{\Gamma_{[p,q]}(l)p^{\binom{l}{2}}} \displaystyle\int_0^t (t-qu)_{[p,q]}^{(l-1)} z\Big[\frac{u}{p^{l-1}}\Big] d_{[p,q]}u, & l > 0, \\ z(t), & l = 0, \end{cases}$$

and

$$^{R}\mathcal{I}_{[q]}^{l}z(t) = \begin{cases} \dfrac{1}{\Gamma_{[q]}(l)} \displaystyle\int_0^t (t-qu)_{[q]}^{(l-1)} z(u)\, d_{[q]}u, & l > 0, \\ z(t), & l = 0, \end{cases}$$

respectively, if both integrals converge.

Definition 2.4. *[23, 34] Let* $\gamma = [l] + 1$*. The lth order* $[p, q]$*-derivative and its* q*-analog, of the Caputo type, for* $z \in C^{(\gamma)}([0, L], \mathbb{R})$ *are defined as*

$$^{c}\mathcal{D}_{[p,q]}^{l}z(t) = {}^{R}\mathcal{I}_{[p,q]}^{\gamma-l}\mathcal{D}_{[p,q]}^{\gamma}z(t) = \frac{1}{\Gamma_{[p,q]}(\gamma-l)p^{\binom{\gamma-l}{2}}} \int_0^t (t-qu)_{[p,q]}^{(\gamma-l-1)} \mathcal{D}_{[p,q]}^{\gamma} z\Big[\frac{u}{p^{\gamma-l-1}}\Big] d_{[p,q]}u,$$

and

$$ {}^{\mathtt{c}}\mathcal{D}^{l}_{[\mathtt{q}]}\mathtt{z}(\mathtt{t}) = {}^{\mathtt{R}}\mathcal{I}^{\gamma-l}_{[\mathtt{q}]}\mathcal{D}^{\gamma}_{[\mathtt{q}]}\mathtt{z}(\mathtt{t}) = \frac{1}{\Gamma_{[\mathtt{q}]}(\gamma-l)}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{qu})^{(\gamma-l-1)}_{[\mathtt{q}]}\mathcal{D}^{\gamma}_{[\mathtt{q}]}\mathtt{z}(\mathtt{u})\,\mathtt{d}_{[\mathtt{q}]}\mathtt{u}, $$

respectively, if both integrals converge.

Some properties are given in the sequel, which are taken from [23].

Lemma 2.5. *[23] Let* $l, \tilde{l} > 0$. *Then*

$(A_{[\mathtt{p},\mathtt{q}]})$ ${}^{\mathtt{R}}\mathcal{I}^{l}_{[\mathtt{p},\mathtt{q}]}[{}^{\mathtt{R}}\mathcal{I}^{\tilde{l}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})] = {}^{\mathtt{R}}\mathcal{I}^{\tilde{l}}_{[\mathtt{p},\mathtt{q}]}[{}^{\mathtt{R}}\mathcal{I}^{l}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})] = {}^{\mathtt{R}}\mathcal{I}^{l+\tilde{l}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t}).$
$(B_{[\mathtt{p},\mathtt{q}]})$ ${}^{\mathtt{c}}\mathcal{D}^{l}_{[\mathtt{p},\mathtt{q}]}[{}^{\mathtt{R}}\mathcal{I}^{l}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})] = \mathtt{z}(\mathtt{t}).$

Lemma 2.6. *[23] Let* $l, \tilde{l} > 0$ *and* $\mathtt{z}(\mathtt{t}) = \mathtt{t}^{\tilde{l}}$. *Then*

$(C_{[\mathtt{p},\mathtt{q}]})$ ${}^{\mathtt{R}}\mathcal{I}^{l}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t}) = \dfrac{\Gamma_{[\mathtt{p},\mathtt{q}]}(\tilde{l}+1)}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\tilde{l}+l+1)}\mathtt{t}^{\tilde{l}+l}.$

$(D_{[\mathtt{p},\mathtt{q}]})$ ${}^{\mathtt{c}}\mathcal{D}^{l}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t}) = \mathtt{p}^{l}\dfrac{\Gamma_{[\mathtt{p},\mathtt{q}]}(\tilde{l}+1)}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\tilde{l}-l+1)}\mathtt{t}^{\tilde{l}-l},\ \ \tilde{l} > l.$

$(E_{[\mathtt{p},\mathtt{q}]})$ $\displaystyle\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{qu})^{(l-1)}_{[\mathtt{p},\mathtt{q}]}\mathtt{u}^{\tilde{l}}\mathtt{d}_{[\mathtt{p},\mathtt{q}]}\mathtt{u} = \mathfrak{B}_{[\mathtt{p},\mathtt{q}]}(\tilde{l}+1, l)\mathtt{t}^{l+\tilde{l}}.$

Theorem 2.7. *[23] Let* $\gamma = [l] + 1$. *Then*

$$ {}^{\mathtt{R}}\mathcal{I}^{l}_{[\mathtt{p},\mathtt{q}]}[{}^{\mathtt{c}}\mathcal{D}^{l}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})] = \mathtt{z}(\mathtt{t}) - \sum_{\mathtt{b}=0}^{\gamma-1}\frac{\mathcal{D}^{\mathtt{b}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(0)}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\mathtt{b}+1)\mathtt{p}^{\binom{l}{2}}}\mathtt{t}^{\mathtt{b}}. $$

In another form, we have

$$ {}^{\mathtt{R}}\mathcal{I}^{l}_{[\mathtt{p},\mathtt{q}]}[{}^{\mathtt{c}}\mathcal{D}^{l}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})] = \mathtt{z}(\mathtt{t}) + \mathtt{c}^*_0 + \mathtt{c}^*_1\mathtt{t} + \cdots + \mathtt{c}^*_{\gamma-1}\mathtt{t}^{\gamma-1}, $$

where $\mathtt{c}^*_{\mathtt{b}} \in \mathbb{R}$; $\mathtt{b} = 0, 1, \ldots, \gamma-1$.

3. On solutions of the $[\mathtt{p},\mathtt{q}]$-based problem in (1.1)

Here, first, we start by introducing a space $\mathbb{K}_* = \{\mathtt{z}(\mathtt{t}) : \mathtt{z}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t}) \in C_{\mathbb{R}}(\Im^{L}_{[\mathtt{p},\mathtt{q}]})\}$ including all the real-valued functions on $\Im^{L}_{[\mathtt{p},\mathtt{q}]}$. It is a Banach space with the norm

$$ \|\mathtt{z}\|_{\mathbb{K}_*} = \sup_{\mathtt{t}\in\Im^{L}_{[\mathtt{p},\mathtt{q}]}}|\mathtt{z}(\mathtt{t})| + \sup_{\mathtt{t}\in\Im^{L}_{[\mathtt{p},\mathtt{q}]}}\left|{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})\right|, $$

for all $\mathtt{z} \in \mathbb{K}_*$. According to the existing properties in this Banach space, we can begin our analysis on the solutions of the Caputo-type $[\mathtt{p},\mathtt{q}]$-based initial value problem in (1.1).

In the following, a lemma is established to give us an equivalent form of the $[\mathtt{p},\mathtt{q}]$-based integral solutions corresponding to the solutions of the Caputo-type $[\mathtt{p},\mathtt{q}]$-based initial value problem (1.1).

Lemma 3.1. *Let* $\mathrm{F} \in C_{\mathbb{R}}(\Im^{L}_{[\mathtt{p},\mathtt{q}]})$, $\nu \in (1,2]$, *and* $\lambda, \eta \in \mathbb{R}^+$. *Then* $\mathtt{z}^*$ *is a solution to the Caputo-type linear* $[\mathtt{p},\mathtt{q}]$*-based initial value problem*

$$ \begin{cases} ({}^{\mathtt{c}}\mathcal{D}^{\nu}_{[\mathtt{p},\mathtt{q}]}\mathtt{z})(\mathtt{t}) = \mathrm{F}(\mathtt{t}), & (\mathtt{t} \in \Im^{L}_{[\mathtt{p},\mathtt{q}]} = [0, \frac{L}{\mathtt{p}}],\ 0 < \mathtt{q} < \mathtt{p} \le 1, L > 0), \\ \mathtt{z}(\mathtt{t})\big|_{\mathtt{t}=0} = \lambda, \quad \mathcal{D}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})\big|_{\mathtt{t}=0} = \eta, \end{cases} \tag{3.1} $$

if and only if z^* *satisfies the* $[p,q]$*-based integral equation*

$$z(t) = \lambda + \eta t + \frac{1}{\Gamma_{[p,q]}(\nu)p^{\binom{\nu}{2}}} \int_0^t (t - qu)_{[p,q]}^{(\nu-1)} F\Big[\frac{u}{p^{\nu-1}}\Big] d_{[p,q]}u. \tag{3.2}$$

Proof. Suppose that z^* is a solution of the Caputo-type linear $[p,q]$-based initial value problem in (3.1), i.e.,

$$({}^c\mathcal{D}_{[p,q]}^{\nu} z^*)(t) = F(t). \tag{3.3}$$

We know that $\nu \in (1,2]$. In this case, if the νth order $[p,q]$-based integral of the Riemann-Liouville type acts on (3.3), then, by Theorem 2.7, we have

$$z^*(t) = \frac{1}{\Gamma_{[p,q]}(\nu)p^{\binom{\nu}{2}}} \int_0^t (t - qu)_{[p,q]}^{(\nu-1)} F\Big[\frac{u}{p^{\nu-1}}\Big] d_{[p,q]}u + c_0^* + c_1^* t,$$

for each $c_0^*, c_1^* \in \mathbb{R}$. The first condition, i.e., $z(t)\big|_{t=0} = \lambda$, gives a straightforward output as $c_0^* = \lambda$. Therefore,

$$z^*(t) = \frac{1}{\Gamma_{[p,q]}(\nu)p^{\binom{\nu}{2}}} \int_0^t (t - qu)_{[p,q]}^{(\nu-1)} F\Big[\frac{u}{p^{\nu-1}}\Big] d_{[p,q]}u + \lambda + c_1^* t. \tag{3.4}$$

On the other hand,

$$\mathcal{D}_{[p,q]} z^*(t) = \frac{1}{\Gamma_{[p,q]}(\nu - 1)p^{\binom{\nu-1}{2}}} \int_0^t (t - qu)_{[p,q]}^{(\nu-2)} F\Big[\frac{u}{p^{\nu-2}}\Big] d_{[p,q]}u + c_1^*. \tag{3.5}$$

According to (3.5) and the second condition, i.e., $\mathcal{D}_{[p,q]} z(t)\big|_{t=0} = \eta$, we get $c_1^* = \eta$. As a result, (3.4) becomes

$$z^*(t) = \lambda + \eta t + \frac{1}{\Gamma_{[p,q]}(\nu)p^{\binom{\nu}{2}}} \int_0^t (t - qu)_{[p,q]}^{(\nu-1)} F\Big[\frac{u}{p^{\nu-1}}\Big] d_{[p,q]}u. \tag{3.6}$$

We see that z^* satisfies the $[p,q]$-based integral equation (3.2). The converse is established by direct calculations and it completes the proof. □

According to the above lemma and in view of the main Caputo-type $[p,q]$-based initial value problem in (1.1), an operator $G : \mathbb{K}_* \to \mathbb{K}_*$ can be defined as

$$(Gz)(t) = \lambda + \eta t + \frac{1}{\Gamma_{[p,q]}(\nu)p^{\binom{\nu}{2}}} \int_0^t (t - qu)_{[p,q]}^{(\nu-1)} \mathfrak{f}_z\Big[\frac{u}{p^{\nu-1}}, z(\frac{u}{p^{\nu-1}}), {}^c\mathcal{D}_{[p,q]}^{w} z(\frac{u}{p^{\nu-1}})\Big] d_{[p,q]}u.$$

With this definition, it is known that z^*, which is the solution of the Caputo-type $[p,q]$-based initial value problem in (1.1), will be a fixed point of the operator G.

For simplicity, we can keep the following notation in mind:

$$A_1 = \frac{(\frac{L}{p})^{\nu}}{\Gamma_{[p,q]}(\nu + 1)}, \qquad A_2 = \frac{(\frac{L}{p})^{\nu - w}}{\Gamma_{[p,q]}(\nu - w + 1)}. \tag{3.7}$$

We want to use a category of contractions to prove the first existence theorem in relation to the solutions of the Caputo-type $[p,q]$-based initial value problem in (1.1). These contractions are β-α-contractions in which we say that a function $\mathfrak{f} : \mathbb{K}_* \to \mathbb{K}_*$ is an β-α-contraction [35] if

$$\beta(z_1, z_2)d(\mathfrak{f}z_1, \mathfrak{f}z_2) \le \alpha(d(z_1, z_2)), \quad \forall\, z_1, z_2 \in \mathbb{K}_*,$$

so that $\beta : \mathbb{K}_*^2 \to \mathbb{R}_{\geq 0}$ and also if $\alpha : [0,\infty) \to [0,\infty)$ satisfies

$$\sum_{b=1}^{\infty} \alpha^b(\mathtt{t}) < \infty, \ \ \alpha(\mathtt{t}) < \mathtt{t}, \ \text{ for all } \mathtt{t} > 0.$$

Note that we use the symbol $\mathbb{V}$ for denoting the family of all such increasing mappings $\alpha : [0,\infty) \to [0,\infty)$.

Moreover, in the proof of our first existence theorem, β-admissible mappings are another group of the functions that play an important role [35]. These functions are defined by the following conditional proposition:

$$\text{if } \ \beta(\mathtt{z}_1, \mathtt{z}_2) \geq 1, \quad \text{then} \quad \beta(\mathfrak{f}\mathtt{z}_1, \mathfrak{f}\mathtt{z}_2) \geq 1.$$

By considering the definitions above, we will get help from the following theorem for the conclusion of our first existence theorem.

Theorem 3.2. *[35] Let* $(\mathbb{K}_*, \mathtt{d})$ *be a complete metric space,* $\beta : \mathbb{K}_* \times \mathbb{K}_* \to \mathbb{R}$*, and* $\alpha \in \mathbb{V}$*. Let* $\mathfrak{f} : \mathbb{K}_* \to \mathbb{K}_*$ *be an* β*-*α*-contraction. Assume that*

(1) $\mathfrak{f}$ *is* β*-admissible on* $\mathbb{K}_*$*;*
(2) there is $\mathtt{z}_0 \in \mathbb{K}_*$ *such that* $\beta(\mathtt{z}_0, \mathfrak{f}\mathtt{z}_0) \geq 1$*;*
(3) For every sequence $\{\mathtt{z}_i\}$ *in* $\mathbb{K}_*$ *with* $\mathtt{z}_i \to \mathtt{z}$*, if* $\beta(\mathtt{z}_i, \mathtt{z}_{i+1}) \geq 1$ *for all* $i \geq 1$*, then* $\beta(\mathtt{z}_i, \mathtt{z}) \geq 1$ *for each* $i \geq 1$*.*

Then $\mathfrak{f}$ *has a fixed point.*

Now, everything is ready to prove the theorem.

Theorem 3.3. *Assume that* $\mathfrak{f}_{\mathtt{z}} \in C([0, \frac{L}{\mathtt{p}}] \times \mathbb{K}_*^2, \mathbb{K}_*)$*,* $\Theta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$*,* $\alpha \in \mathbb{V}$*, and*

($\mathfrak{L}1$) *for all* $\mathtt{z}_1, \mathtt{z}_2, \mathtt{x}_1, \mathtt{x}_2 \in \mathbb{K}_*$ *and all* $\mathtt{t} \in [0, \frac{L}{\mathtt{p}}]$*,*

$$|\mathfrak{f}_{\mathtt{z}}(\mathtt{t}, \mathtt{z}_1, \mathtt{x}_1) - \mathfrak{f}_{\mathtt{z}}(\mathtt{t}, \mathtt{z}_2, \mathtt{x}_2)| \leq \tilde{A}\, \alpha(|\mathtt{z}_1 - \mathtt{z}_2| + |\mathtt{x}_1 - \mathtt{x}_2|),$$

and

$$\Theta((\mathtt{z}_1(\mathtt{t}), \mathtt{x}_1(\mathtt{t})), (\mathtt{z}_2(\mathtt{t}), \mathtt{x}_2(\mathtt{t}))) \geq 0,$$

with $\tilde{A} = \dfrac{1}{A_1 + A_2}$*.*

($\mathfrak{L}2$) *there is some* $\mathtt{z}_0 \in \mathbb{K}_*$ *so that for all* $\mathtt{t} \in [0, \frac{L}{\mathtt{p}}]$*,*

$$\Theta((\mathtt{z}_0(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_0(\mathtt{t})), (G\mathtt{z}_0(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}(G\mathtt{z}_0(\mathtt{t})))) \geq 0.$$

Moreover,

$$\Theta((\mathtt{z}_1(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_1(\mathtt{t})), (\mathtt{z}_2(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_2(\mathtt{t}))) \geq 0,$$

gives

$$\Theta((G\mathtt{z}_1(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}(G\mathtt{z}_1(\mathtt{t}))), (G\mathtt{z}_2(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}(G\mathtt{z}_2(\mathtt{t})))) \geq 0,$$

for every $\mathtt{z}_1, \mathtt{z}_2 \in \mathbb{K}_*$ *and* $\mathtt{t} \in [0, \frac{L}{\mathtt{p}}]$*.*

($\mathfrak{L}3$) *the inequality*

$$\Theta((\mathtt{z}_i(\mathtt{t}), {}^{c}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_i(\mathtt{t})), (\mathtt{z}_{i+1}(\mathtt{t}), {}^{c}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_{i+1}(\mathtt{t}))) \geq 0,$$

yields

$$\Theta((\mathtt{z}_i(\mathtt{t}), {}^{c}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_i(\mathtt{t})), (\mathtt{z}(\mathtt{t}), {}^{c}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t}))) \geq 0, \quad \forall\, i \geq 1, \forall\, \mathtt{t} \in [0, \frac{L}{\mathtt{p}}],$$

where $\{\mathtt{z}_i\}_{i\geq 1} \subseteq \mathbb{K}_*$ *is a sequence converging to* $\mathtt{z}$.

Then the Caputo-type $[\mathtt{p},\mathtt{q}]$*-based initial value problem in* (1.1) *has a solution on* $[0, \frac{L}{\mathtt{p}}]$.

Proof. Consider two elements $\mathtt{z}_1, \mathtt{z}_2 \in \mathbb{K}_*$ so that

$$\Theta((\mathtt{z}_1(\mathtt{t}), {}^{c}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_1(\mathtt{t})), (\mathtt{z}_2(\mathtt{t}), {}^{c}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_2(\mathtt{t}))) \geq 0,$$

for all $\mathtt{t} \in [0, \frac{L}{\mathtt{p}}]$. For the sake of simplicity in the calculations, assume

$$\begin{cases} \mathfrak{f}^{\nu}_{\mathtt{z},\mathtt{t},\mathtt{p}}(\mathtt{z}) = \mathfrak{f}_{\mathtt{z}}\Big[\dfrac{\mathtt{t}}{\mathtt{p}^{\nu-1}}, \mathtt{z}(\dfrac{\mathtt{t}}{\mathtt{p}^{\nu-1}}), {}^{c}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\dfrac{\mathtt{t}}{\mathtt{p}^{\nu-1}})\Big], \quad (\mathtt{t} \in [0, \dfrac{L}{\mathtt{p}}]), \\ \mathfrak{f}^{\nu,\mathtt{w}}_{\mathtt{z},\mathtt{t},\mathtt{p}}(\mathtt{z}) = \mathfrak{f}_{\mathtt{z}}\Big[\dfrac{\mathtt{t}}{\mathtt{p}^{\nu-\mathtt{w}-1}}, \mathtt{z}(\dfrac{\mathtt{t}}{\mathtt{p}^{\nu-\mathtt{w}-1}}), {}^{c}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\dfrac{\mathtt{t}}{\mathtt{p}^{\nu-\mathtt{w}-1}})\Big], \quad (\mathtt{t} \in [0, \dfrac{L}{\mathtt{p}}]). \end{cases} \tag{3.8}$$

The hypotheses of the theorem imply that

$$\begin{aligned} \Big|\mathfrak{f}^{\nu}_{\mathtt{z},\mathtt{t},\mathtt{p}}(\mathtt{z}_1) - \mathfrak{f}^{\nu}_{\mathtt{z},\mathtt{t},\mathtt{p}}(\mathtt{z}_2)\Big| &= \Big|\mathfrak{f}_{\mathtt{z}}\Big[\frac{\mathtt{t}}{\mathtt{p}^{\nu-1}}, \mathtt{z}_1(\frac{\mathtt{t}}{\mathtt{p}^{\nu-1}}), {}^{c}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_1(\frac{\mathtt{t}}{\mathtt{p}^{\nu-1}})\Big] \\ &\quad - \mathfrak{f}_{\mathtt{z}}\Big[\frac{\mathtt{t}}{\mathtt{p}^{\nu-1}}, \mathtt{z}_2(\frac{\mathtt{t}}{\mathtt{p}^{\nu-1}}), {}^{c}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_2(\frac{\mathtt{t}}{\mathtt{p}^{\nu-1}})\Big]\Big| \\ &\leq \tilde{A}\,\alpha\bigg(\Big|\mathtt{z}_1(\frac{\mathtt{t}}{\mathtt{p}^{\nu-1}}) - \mathtt{z}_2(\frac{\mathtt{t}}{\mathtt{p}^{\nu-1}})\Big| + \Big|{}^{c}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_1(\frac{\mathtt{t}}{\mathtt{p}^{\nu-1}}) - {}^{c}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_2(\frac{\mathtt{t}}{\mathtt{p}^{\nu-1}})\Big|\bigg) \\ &\leq \tilde{A}\,\alpha(\|\mathtt{z}_1 - \mathtt{z}_2\|_{\mathbb{K}_*}). \end{aligned} \tag{3.9}$$

Therefore, (3.9) yields

$$\begin{aligned} &\Big|G\mathtt{z}_1(\mathtt{t}) - G\mathtt{z}_2(\mathtt{t})\Big| \\ &\quad \leq \frac{1}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\nu)\mathtt{p}^{\binom{\nu}{2}}} \int_0^{\mathtt{t}} (\mathtt{t} - \mathtt{q}\mathtt{u})^{(\nu-1)}_{[\mathtt{p},\mathtt{q}]} \Big|\mathfrak{f}^{\nu}_{\mathtt{z},\mathtt{u},\mathtt{p}}(\mathtt{z}_1) - \mathfrak{f}^{\nu}_{\mathtt{z},\mathtt{u},\mathtt{p}}(\mathtt{z}_2)\Big|\, \mathtt{d}_{[\mathtt{p},\mathtt{q}]}\mathtt{u} \\ &\quad \leq \frac{\tilde{A}\,\alpha(\|\mathtt{z}_1 - \mathtt{z}_2\|_{\mathbb{K}_*})}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\nu)\mathtt{p}^{\binom{\nu}{2}}} \int_0^{\mathtt{t}} (\mathtt{t} - \mathtt{q}\mathtt{u})^{(\nu-1)}_{[\mathtt{p},\mathtt{q}]} \mathtt{d}_{[\mathtt{p},\mathtt{q}]}\mathtt{u} \\ &\quad \leq \frac{(\frac{L}{\mathtt{p}})^{\nu}}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\nu + 1)} \tilde{A}\,\alpha(\|\mathtt{z}_1 - \mathtt{z}_2\|_{\mathbb{K}_*}) \\ &\quad = \tilde{A}\,A_1\alpha(\|\mathtt{z}_1 - \mathtt{z}_2\|_{\mathbb{K}_*}). \end{aligned}$$

Moreover,

$$\begin{aligned}
&\left|({}^{c}\mathcal{D}^{w}_{[p,q]}Gz_1)(t) - ({}^{c}\mathcal{D}^{w}_{[p,q]}Gz_2)(t)\right| \\
&\leq \frac{1}{\Gamma_{[p,q]}(\nu - w)p^{\binom{\nu-w}{2}}} \int_0^t (t - qu)^{(\nu-w-1)}_{[p,q]} \left|\mathfrak{f}^{\nu,w}_{z,u,p}(z_1) - \mathfrak{f}^{\nu,w}_{z,u,p}(z_2)\right| d_{[p,q]}u \\
&\leq \frac{\tilde{A}\,\alpha(\|z_1 - z_2\|_{\mathbb{K}_*})}{\Gamma_{[p,q]}(\nu - w)p^{\binom{\nu-w}{2}}} \int_0^t (t - qu)^{(\nu-w-1)}_{[p,q]} d_{[p,q]}u \\
&\leq \frac{(\frac{L}{p})^{\nu-w}}{\Gamma_{[p,q]}(\nu - w + 1)} \tilde{A}\,\alpha(\|z_1 - z_2\|_{\mathbb{K}_*}) \\
&= \tilde{A}\,A_2\alpha(\|z_1 - z_2\|_{\mathbb{K}_*}).
\end{aligned}$$

By these estimates, we get $\|Gz_1 - Gz_2\|_{\mathbb{K}_*} \leq (A_1 + A_2)\tilde{A}\,\alpha(\|z_1 - z_2\|_{\mathbb{K}_*}) = \alpha(\|z_1 - z_2\|_{\mathbb{K}_*})$. We define $\beta : \mathbb{K}_* \times \mathbb{K}_* \to [0, \infty)$ by

$$\beta(z_1, z_2) = \begin{cases} 1, & \text{if } \Theta((z_1(t), {}^{c}\mathcal{D}^{w}_{[p,q]}z_1(t)), (z_2(t), {}^{c}\mathcal{D}^{w}_{[p,q]}z_2(t))) \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

for all $z_1, z_2 \in \mathbb{K}_*$. Then, for the same elements $z_1, z_2 \in \mathbb{K}_*$, we have

$$\beta(z_1, z_2)d(Gz_1, Gz_2) \leq \alpha(d(z_1, z_2)).$$

That is, G is a β-α-contraction. In the following, one can easily prove that G is β-admissible and that $\beta(z_0, Gz_0) \geq 1$. Lastly, a sequence $\{z_i\}_{i\geq 1} \subseteq \mathbb{K}_*$ is to be assumed that converges to z, and we let $\beta(z_i, z_{i+1}) \geq 1$ for every i. Accordingly, the definition of the non-negative function β implies that

$$\Theta((z_i(t), {}^{c}\mathcal{D}^{w}_{[p,q]}z_i(t)), (z_{i+1}(t), {}^{c}\mathcal{D}^{w}_{[p,q]}z_{i+1}(t))) \geq 0.$$

As a result,

$$\Theta((z_i(t), {}^{c}\mathcal{D}^{w}_{[p,q]}z_i(t)), (z(t), {}^{c}\mathcal{D}^{w}_{[p,q]}z(t))) \geq 0.$$

Thus, $\beta(z_i, z) \geq 1$ for all i. Note that all hypotheses of Theorem 3.2 now hold; therefore $G(z^{**}) = z^{**} \in \mathbb{K}_*$. This means that z^{**} is a solution of the Caputo-type $[p, q]$-based initial value problem in (1.1). □

For the second existence theorem, the standard contractions and compact operators are used in the framework of the hypotheses of Krasnoselskii's fixed point theorem.

Theorem 3.4. *[36] Let $Y \subseteq \mathbb{K}_*$ be a nonempty bounded, closed, convex set, and let $\mathfrak{f}_1$ and $\mathfrak{f}_2$ be defined on Y so that*

(1) $\mathfrak{f}_1 z_1 + \mathfrak{f}_2 z_2 \in Y$, for all $z_1, z_2 \in Y$;
(2) the continuous function $\mathfrak{f}_1$ is compact;
(3) $\mathfrak{f}_2$ is contraction.

Then there is $z \in Y$ such that $z = \mathfrak{f}_1 z + \mathfrak{f}_2 z$.

Theorem 3.5. *Assume that* $\mathfrak{f}_z \in C([0, \frac{L}{p}] \times \mathbb{K}_*^2, \mathbb{K}_*)$ *and*

($\mathfrak{L}4$) *there is* $\Bbbk \in C([0, \frac{L}{p}], \mathbb{R})$ *such that for all* $t \in [0, \frac{L}{p}]$ *and* $z_1, z_2, x_1, x_2 \in \mathbb{K}_*$,

$$|\mathfrak{f}_z(t, z_1, x_1) - \mathfrak{f}_z(t, z_2, x_2)| \le \Bbbk(t)(|z_1 - z_2| + |x_1 - x_2|);$$

($\mathfrak{L}5$) *there are* $\mathfrak{g} \in C([0, \frac{L}{p}], \mathbb{R}^+)$ *and a non-decreasing function* $\alpha \in C(\mathbb{R}^+, \mathbb{R}^+)$ *such that for all* $t \in [0, \frac{L}{p}]$ *and each* $z_1, z_2 \in \mathbb{K}_*$,

$$|\mathfrak{f}_z(t, z_1, z_2)| \le \mathfrak{g}(t)\alpha(|z_1| + |z_2|).$$

Then the Caputo-type $[p, q]$*-based initial value problem in* (1.1) *has a solution if*

$$\mathbb{A} = \|\Bbbk\|(A_1 + A_2) < 1, \tag{3.10}$$

so that A_1, A_2 *are considered in* (3.7) *and* $\|\Bbbk\| = \sup_{t\in[0,\frac{L}{p}]} |\Bbbk(t)|$.

Proof. To begin the proof, we select an approximate value $\varepsilon > 0$ and let $\|\mathfrak{g}\| = \sup_{t\in[0,\frac{L}{p}]} |\mathfrak{g}(t)|$ such that

$$\varepsilon \ge \chi + \alpha(\|z\|_{\mathbb{K}_*})\|\mathfrak{g}\|(A_1 + A_2), \tag{3.11}$$

where A_1 and A_2 are given in (3.7),

$$\chi = \lambda + \eta\frac{L}{p} + \frac{\eta p^{w-1} L}{\Gamma_{[p,q]}(2 - w)}(\frac{p}{L})^{w}.$$

We also define the bounded, closed and convex set $\mathbb{K}_\varepsilon = \{z \in \mathbb{K}_* : \|z\|_{\mathbb{K}_*} \le \varepsilon\}$ in $\mathbb{K}_*$. On this set $\mathbb{K}_\varepsilon$, the operators G_1 and G_2 are formulated as follows:

$$(G_1 z)(t) = \lambda + \eta t, \quad \forall\, t \in [0, \frac{L}{p}],$$

and

$$(G_2 z)(t) = \frac{1}{\Gamma_{[p,q]}(\nu)p^{\binom{\nu}{2}}} \int_0^t (t - qu)_{[p,q]}^{(\nu-1)} \mathfrak{f}^{\nu}_{z,u,p}(z)\, d_{[p,q]}u, \quad \forall\, t \in [0, \frac{L}{p}].$$

Let $\hat{\alpha} = \sup_{z\in\mathbb{K}_*} \alpha(\|z\|_{\mathbb{K}_*})$. For every element $z_1, z_2 \in \mathbb{K}_\varepsilon$, the following estimates can be obtained

$$\begin{aligned}
\left|(G_1 z_1 + G_2 z_2)(t)\right| &\le |\lambda| + |\eta| t + \frac{1}{\Gamma_{[p,q]}(\nu)p^{\binom{\nu}{2}}} \int_0^t (t - qu)_{[p,q]}^{(\nu-1)} \left|\mathfrak{f}^{\nu}_{z,u,p}(z_2)\right| d_{[p,q]}u \\
&\le |\lambda| + |\eta| t + \frac{1}{\Gamma_{[p,q]}(\nu)p^{\binom{\nu}{2}}} \int_0^t (t - qu)_{[p,q]}^{(\nu-1)} \mathfrak{g}(u)\alpha(|z_2(\frac{u}{p^{\nu-1}})| + |{}^{c}\mathcal{D}^{w}_{[p,q]} z_2(\frac{u}{p^{\nu-1}})|)\, d_{[p,q]}u \\
&\le |\lambda| + |\eta| t + \hat{\alpha}\|\mathfrak{g}\|\Big[\frac{(\frac{L}{p})^{\nu}}{\Gamma_{[p,q]}(\nu + 1)}\Big] \\
&= \lambda + \eta\frac{L}{p} + \hat{\alpha}\|\mathfrak{g}\| A_1,
\end{aligned}$$

and

$$\Big|({}^{c}\mathcal{D}^{w}_{[p,q]}G_1 z_1 + {}^{c}\mathcal{D}^{w}_{[p,q]}G_2 z_2)(t)\Big|$$

$$\leq \frac{|\eta| p^{w}}{\Gamma_{[p,q]}(2-w)} t^{1-w} + \frac{1}{\Gamma_{[p,q]}(\nu-w) p^{\binom{\nu-w}{2}}} \int_0^t (t-qu)^{(\nu-w-1)}_{[p,q]} \Big|\mathfrak{f}^{\nu,w}_{z,u,p}(z_2)\Big| \, d_{[p,q]}u$$

$$\leq \frac{\eta p^{w}}{\Gamma_{[p,q]}(2-w)} (\frac{L}{p})^{1-w} + \hat{\alpha}\|\mathfrak{g}\| \Big[\frac{(\frac{L}{p})^{\nu-w}}{\Gamma_{[p,q]}(\nu-w+1)}\Big]$$

$$= \frac{\eta p^{w-1} L}{\Gamma_{[p,q]}(2-w)} (\frac{p}{L})^{w} + \hat{\alpha}\|\mathfrak{g}\| A_2.$$

As a result,

$$\|G_1 z_1 + G_2 z_2\|_{\mathbb{K}_*} \leq \chi + \hat{\alpha}\|\mathfrak{g}\|(A_1 + A_2) \leq \varepsilon,$$

and

$$(G_1 z_1 + G_2 z_2) \in \mathbb{K}_{\varepsilon}.$$

Note that for every $z \in \mathbb{K}_{\varepsilon}$ and all $t \in [0, \frac{L}{p}]$, $(G_1 z)(t) = \lambda + \eta t$ is a linear function with respect to t, and this fact implies that G_1 is continuous. Furthermore, $G_1(\mathbb{K}_{\varepsilon})$ is contained in a compact set.

For the final step, we prove that G_2 is a contraction. Thus, for every $z_1, z_2 \in \mathbb{K}_{\varepsilon}$, we have

$$\Big|(G_2 z_1)(t) - (G_2 z_2)(t)\Big|$$

$$\leq \frac{1}{\Gamma_{[p,q]}(\nu) p^{\binom{\nu}{2}}} \int_0^t (t-qu)^{(\nu-1)}_{[p,q]} \Big|\mathfrak{f}^{\nu}_{z,u,p}(z_1) - \mathfrak{f}^{\nu}_{z,u,p}(z_2)\Big| \, d_{[p,q]}u$$

$$\leq \frac{1}{\Gamma_{[p,q]}(\nu) p^{\binom{\nu}{2}}} \int_0^t (t-qu)^{(\nu-1)}_{[p,q]}$$

$$\times \mathbb{k}(u)\Bigg(\Big|z_1(\frac{u}{p^{\nu-1}}) - z_2(\frac{u}{p^{\nu-1}})\Big| + \Big|{}^{c}\mathcal{D}^{w}_{[p,q]}z_1(\frac{u}{p^{\nu-1}}) - {}^{c}\mathcal{D}^{w}_{[p,q]}z_2(\frac{u}{p^{\nu-1}})\Big|\Bigg) d_{[p,q]}u$$

$$\leq \frac{(\frac{L}{p})^{\nu}}{\Gamma_{[p,q]}(\nu+1)} \|\mathbb{k}\| \|z_1 - z_2\|_{\mathbb{K}_*}$$

$$= \|\mathbb{k}\| A_2 \|z_1 - z_2\|_{\mathbb{K}_*},$$

and

$$\Big|({}^{c}\mathcal{D}^{w}_{[p,q]}G_2 z_1)(t) - ({}^{c}\mathcal{D}^{w}_{[p,q]}G_2 z_2)(t)\Big|$$

$$\leq \frac{1}{\Gamma_{[p,q]}(\nu-w) p^{\binom{\nu-w}{2}}} \int_0^t (t-qu)^{(\nu-w-1)}_{[p,q]} \Big|\mathfrak{f}^{\nu,w}_{z,u,p}(z_1) - \mathfrak{f}^{\nu,w}_{z,u,p}(z_2)\Big| \, d_{[p,q]}u$$

$$\leq \frac{1}{\Gamma_{[p,q]}(\nu-w) p^{\binom{\nu-w}{2}}} \int_0^t (t-qu)^{(\nu-w-1)}_{[p,q]}$$

$$\times \Bbbk(\mathrm{u})(|\mathrm{z}_1(\frac{\mathrm{u}}{\mathrm{p}^{\nu-\mathrm{w}-1}}) - \mathrm{z}_2(\frac{\mathrm{u}}{\mathrm{p}^{\nu-\mathrm{w}-1}})| + |{}^{\mathrm{c}}\mathcal{D}^{\mathrm{w}}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}_1(\frac{\mathrm{u}}{\mathrm{p}^{\nu-\mathrm{w}-1}}) - {}^{\mathrm{c}}\mathcal{D}^{\mathrm{w}}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}_2(\frac{\mathrm{u}}{\mathrm{p}^{\nu-\mathrm{w}-1}})|)\, \mathrm{d}_{[\mathrm{p},\mathrm{q}]}\mathrm{u}$$

$$\leq \frac{(\frac{L}{\mathrm{p}})^{\nu-\mathrm{w}}}{\Gamma_{[\mathrm{p},\mathrm{q}]}(\nu - \mathrm{w} + 1)}\|\Bbbk\|\|\mathrm{z}_1 - \mathrm{z}_2\|_{\mathbb{K}_*}$$

$$= \|\Bbbk\| A_2 \|\mathrm{z}_1 - \mathrm{z}_2\|_{\mathbb{K}_*}.$$

Thus,

$$\left\|G_2\mathrm{z}_1 - G_2\mathrm{z}_2\right\|_{\mathbb{K}_*} \leq \|\Bbbk\|(A_1 + A_2)\|\mathrm{z}_1 - \mathrm{z}_2\|_{\mathbb{K}_*} = \mathbb{A}\|\mathrm{z}_1 - \mathrm{z}_2\|_{\mathbb{K}_*},$$

so that $\mathbb{A} < 1$ is the Lipschitz constant. Therefore, G_2 is a contraction on $\mathbb{K}_\varepsilon$. The conclusion of Theorem 3.4 tells us that the Caputo-type $[\mathrm{p},\mathrm{q}]$-based initial value problem in (1.1) has a solution. □

4. On solutions of the $[\mathrm{p},\mathrm{q}]$-based inclusion problem in (1.2)

Before starting the proofs for the existence theorems in relation to the solutions of the Caputo-type $[\mathrm{p},\mathrm{q}]$-based inclusion problem in (1.2), we recall some families of sets, like all bounded, closed, compact, and convex sets in $\mathbb{K}_*$, which are denoted by $\mathrm{P}_{\mathfrak{B}}(\mathbb{K}_*)$, $\mathrm{P}_{\mathfrak{CL}}(\mathbb{K}_*)$, $\mathrm{P}_{\mathfrak{CM}}(\mathbb{K}_*)$, and $\mathrm{P}_{\mathfrak{Cx}}(\mathbb{K}_*)$, respectively. Here, $(\mathbb{K}_*, \|\cdot\|_{\mathbb{K}_*})$ is a normed space.

For the Caputo-type $[\mathrm{p},\mathrm{q}]$-based inclusion problem in (1.2), we take $\mathrm{z} \in C_{\mathbb{K}_*}(\Im^L_{[\mathrm{p},\mathrm{q}]}, \mathbb{K}_*)$ as a solution if the conditions $\mathrm{z}(\mathrm{t})\big|_{\mathrm{t}=0} = \lambda$ and $\mathcal{D}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(\mathrm{t})\big|_{\mathrm{t}=0} = \eta$ are satisfied for such a function z, and there is also an integrable function like $F \in \mathcal{L}^1(\Im^L_{[\mathrm{p},\mathrm{q}]})$ such that $F(\mathrm{t}) \in \mathfrak{F}_{\mathrm{z}}(\mathrm{t}, \mathrm{z}(\mathrm{t}), {}^{\mathrm{c}}\mathcal{D}^{\mathrm{w}}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(\mathrm{t}))$ for almost all $\mathrm{t} \in \Im^L_{[\mathrm{p},\mathrm{q}]}$ and

$$\mathrm{z}(\mathrm{t}) = \lambda + \eta\mathrm{t} + \frac{1}{\Gamma_{[\mathrm{p},\mathrm{q}]}(\nu)\mathrm{p}^{\binom{\nu}{2}}}\int_0^{\mathrm{t}} (\mathrm{t} - \mathrm{q}\mathrm{u})^{(\nu-1)}_{[\mathrm{p},\mathrm{q}]} F\Big[\frac{\mathrm{u}}{\mathrm{p}^{\nu-1}}\Big]\, \mathrm{d}_{[\mathrm{p},\mathrm{q}]}\mathrm{u}, \quad \forall\, \mathrm{t} \in \Im^L_{[\mathrm{p},\mathrm{q}]}.$$

Regarding the Caputo-type $[\mathrm{p},\mathrm{q}]$-based inclusion problem in (1.2), for the existing multivalued function $\mathfrak{F}_{\mathrm{z}}$, we can define the collection of the selections as

$$(\mathbb{S})_{\mathfrak{F}_{\mathrm{z}},\mathrm{z}} = \{F \in \mathcal{L}^1(\Im^L_{[\mathrm{p},\mathrm{q}]}) : F(\mathrm{t}) \in \mathfrak{F}_{\mathrm{z}}(\mathrm{t}, \mathrm{z}(\mathrm{t}), {}^{\mathrm{c}}\mathcal{D}^{\mathrm{w}}_{[\mathrm{p},\mathrm{q}]}\mathrm{z}(\mathrm{t})), \quad \mathrm{t} \in \Im^L_{[\mathrm{p},\mathrm{q}]}\},$$

for every $\mathrm{z} \in \mathbb{K}_*$. Moreover, the operator $\mathcal{F} : \mathbb{K}_* \to \mathrm{P}(\mathbb{K}_*)$ is given by

$$\mathcal{F}(\mathrm{z}) = \{\bar{F} \in \mathbb{K}_* : \ \exists F \in (\mathbb{S})_{\mathfrak{F}_{\mathrm{z}},\mathrm{z}} \ \text{s.t.}\ \bar{F}(\mathrm{t}) = \hbar(\mathrm{t}), \quad \mathrm{t} \in \Im^L_{[\mathrm{p},\mathrm{q}]}\}, \tag{4.1}$$

so that

$$\hbar(\mathrm{t}) = \lambda + \eta\mathrm{t} + \frac{1}{\Gamma_{[\mathrm{p},\mathrm{q}]}(\nu)\mathrm{p}^{\binom{\nu}{2}}}\int_0^{\mathrm{t}} (\mathrm{t} - \mathrm{q}\mathrm{u})^{(\nu-1)}_{[\mathrm{p},\mathrm{q}]} F\Big[\frac{\mathrm{u}}{\mathrm{p}^{\nu-1}}\Big]\, \mathrm{d}_{[\mathrm{p},\mathrm{q}]}\mathrm{u}.$$

Put

$$\zeta_1 = \|\breve{g}\| A_1 \text{ and } \zeta_2 = \|\breve{g}\| A_2. \tag{4.2}$$

In the present position, we apply a specific type of the β-admissible β-α-contractive multi-valued functions [37] for proving the third existence theorem related to the solutions of the Caputo-type $[\mathrm{p},\mathrm{q}]$-based inclusion problem in (1.2).

For reminding this type of the generalized contractions, we assume $\mathfrak{F}_{\mathtt{z}} : \mathbb{K}_\star \to \mathrm{P}_{\mathfrak{C}\mathfrak{L},\mathfrak{B}}(\mathbb{K}_\star)$, $\beta : \mathbb{K}_\star^2 \to [0, +\infty)$ and $\alpha \in \mathbb{V}$. Then $\mathfrak{F}_{\mathtt{z}}$ is β-admissible if for each $\mathtt{z}_1 \in \mathbb{K}_\star$ and $\mathtt{z}_2 \in \mathfrak{F}_{\mathtt{z}}\mathtt{z}_1$,

$$\beta(\mathtt{z}_1, \mathtt{z}_2) \geq 1 \;\; \Rightarrow \;\; \beta(\mathtt{z}_2, \mathtt{z}_3) \geq 1, \;\; \forall\, \mathtt{z}_3 \in \mathfrak{F}_{\mathtt{z}}\mathtt{z}_2.$$

Moreover, the multivalued function $\mathfrak{F}_{\mathtt{z}}$ is an β-α-contraction if

$$\beta(\mathtt{z}_1, \mathtt{z}_2)\mathbb{H}_d(\mathfrak{F}_{\mathtt{z}}\mathtt{z}_1, \mathfrak{F}_{\mathtt{z}}\mathtt{z}_2) \leq \alpha(d(\mathtt{z}_1, \mathtt{z}_2)), \quad \forall\, \mathtt{z}_1, \mathtt{z}_2 \in \mathbb{K}_\star.$$

Here, $\mathbb{H}_d$ is the Pompeiu-Hausdorff metric [37].

The following theorem will be useful in this regard.

Theorem 4.1. *[37] Let $(\mathbb{K}_*, \mathrm{d})$ be a complete metric space, $\alpha \in \mathbb{V}$, and $\beta : \mathbb{K}_* \times \mathfrak{A}_* \to [0, \infty)$ is increasing in a strict manner. Let $\mathfrak{F}_{\mathtt{z}} : \mathbb{K}_* \to \mathrm{P}_{\mathfrak{C}\mathfrak{L},\mathfrak{B}}(\mathbb{K}_*)$ be a β-α-contraction, and assume that*

1) *$\mathfrak{F}_{\mathtt{z}}$ is β-admissible;*
2) *$\beta(\mathtt{z}_0, \mathtt{z}_1) \geq 1$ for some $\mathtt{z}_0 \in \mathfrak{A}_*$ and $\mathtt{z}_1 \in \mathfrak{F}_{\mathtt{z}}\mathtt{z}_0$;*
3) *for every sequence $\{\mathtt{z}_i\}$ in $\mathbb{K}_*$ with $\beta(\mathtt{z}_i, \mathtt{z}_{i+1}) \geq 1$ and $\mathtt{z}_i \to \mathtt{z}$ for all $i \in \mathbb{N}$, there is a subsequence $\{\mathtt{z}_{i_r}\}$ of $\{\mathtt{z}_i\}$ so that $\beta(\mathtt{z}_{i_r}, \mathtt{z}) \geq 1$ for each $r \in \mathbb{N}$.*

Then $\mathfrak{F}_{\mathtt{z}}$ has a fixed point.

The third main existence theorem is stated in the following.

Theorem 4.2. *Let $\mathfrak{F}_{\mathtt{z}} : \Im^L_{[\mathrm{p},\mathrm{q}]} \times \mathbb{K}_*^2 \to \mathrm{P}_{\mathfrak{C}\mathfrak{M}}(\mathbb{K}_*)$, and assume the following:*

($\mathfrak{L}6$) *$\mathfrak{F}_{\mathtt{z}}$ is bounded and integrable, and $\mathfrak{F}_{\mathtt{z}}(\cdot, \mathtt{z}_1, \mathtt{z}_2) : \Im^L_{[\mathrm{p},\mathrm{q}]} \to \mathrm{P}_{\mathfrak{C}\mathfrak{M}}$ is measurable for both arbitrary elements $\mathtt{z}_1, \mathtt{z}_2 \in \mathbb{K}_*$;*

($\mathfrak{L}7$) *there is a $\breve{g} \in C(\Im^L_{[\mathrm{p},\mathrm{q}]}, [0, \infty))$ and $\alpha \in \mathbb{V}$ such that*

$$\mathbb{H}_{\mathrm{d}}(\mathfrak{F}_{\mathtt{z}}(\mathtt{t}, \mathtt{z}_1, \mathtt{z}_2), \mathfrak{F}_{\mathtt{z}}(\mathtt{t}, \tilde{\mathtt{z}}_1, \tilde{\mathtt{z}}_2)) \leq \breve{g}(\mathtt{t})\left(\frac{\tilde{A}}{\|\breve{g}\|}\right)\alpha(|\mathtt{z}_1 - \tilde{\mathtt{z}}_1| + |\mathtt{z}_2 - \tilde{\mathtt{z}}_2|), \tag{4.3}$$

for all $\mathtt{t} \in \Im^L_{[\mathrm{p},\mathrm{q}]}$ and each $\mathtt{z}_1, \mathtt{z}_2, \tilde{\mathtt{z}}_1, \tilde{\mathtt{z}}_2 \in \mathbb{K}_*$, where $\sup_{\mathtt{t}\in\Im^L_{[\mathrm{p},\mathrm{q}]}} |\breve{g}(\mathtt{t})| = \|\breve{g}\|$, $\tilde{A} = \dfrac{1}{A_1 + A_2}$ and A_1, A_2 are denoted in (3.7);

($\mathfrak{L}8$) *there is a function $\Theta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ so that $\Theta((\mathtt{z}_1, \mathtt{z}_2), (\tilde{\mathtt{z}}_1, \tilde{\mathtt{z}}_2)) \geq 0$ for each $\mathtt{z}_1, \mathtt{z}_2, \tilde{\mathtt{z}}_1, \tilde{\mathtt{z}}_2 \in \mathbb{K}_*$;*

($\mathfrak{L}9$) *there is a sequence $\{\mathtt{z}_i\}_{i\geq 1} \subseteq \mathbb{K}_*$ converging to $\mathtt{z}$ such that*

$$\Theta((\mathtt{z}_i(\mathtt{t}), {}^{\mathrm{c}}\mathcal{D}^{\mathtt{w}}_{[\mathrm{p},\mathrm{q}]}\mathtt{z}_i(\mathtt{t})), (\mathtt{z}_{i+1}(\mathtt{t}), {}^{\mathrm{c}}\mathcal{D}^{\mathtt{w}}_{[\mathrm{p},\mathrm{q}]}\mathtt{z}_{i+1}(\mathtt{t}))) \geq 0, \quad \forall\, \mathtt{t} \in \Im^L_{[\mathrm{p},\mathrm{q}]}, \; i \geq 1,$$

gives the existence of a subsequence $\{\mathtt{z}_{i_r}\}_{r\geq 1}$ of $\{\mathtt{z}_i\}$ with

$$\Theta((\mathtt{z}_{i_r}(\mathtt{t}), {}^{\mathrm{c}}\mathcal{D}^{\mathtt{w}}_{[\mathrm{p},\mathrm{q}]}\mathtt{z}_{i_r}(\mathtt{t})), (\mathtt{z}(\mathtt{t}), {}^{\mathrm{c}}\mathcal{D}^{\mathtt{w}}_{[\mathrm{p},\mathrm{q}]}\mathtt{z}(\mathtt{t}))) \geq 0,$$

for all $\mathtt{t} \in \Im^L_{[\mathrm{p},\mathrm{q}]}$ and $r \geq 1$;

($\mathfrak{L}10$) *there are* $\mathtt{z}_0 \in \mathbb{K}_*$ *and* $\bar{F} \in \mathcal{F}(\mathtt{z}_0)$ *so that*

$$\Theta((\mathtt{z}_0(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_0(\mathtt{t})), (\bar{F}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\bar{F}(\mathtt{t}))) \geq 0,$$

for all $\mathtt{t} \in \Im^L_{[\mathtt{p},\mathtt{q}]}$*, where* $\mathcal{F} : \mathbb{K}_* \to \mathrm{P}(\mathbb{K}_*)$ *has been considered in* (4.1)*;*

($\mathfrak{L}11$) *for every* $\mathtt{z} \in \mathbb{K}_*$ *and* $\bar{F} \in \mathcal{F}(\mathtt{z})$ *such that*

$$\Theta((\mathtt{z}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})), (\bar{F}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\bar{F}(\mathtt{t}))) \geq 0,$$

there is some $\hbar \in \mathcal{F}(\mathtt{z})$ *such that*

$$\Theta((\bar{F}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\bar{F}(\mathtt{t})), (\hbar(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\hbar(\mathtt{t}))) \geq 0,$$

for all $\mathtt{t} \in \Im^L_{[\mathtt{p},\mathtt{q}]}$.

Then the Caputo-type $[\mathtt{p},\mathtt{q}]$*-based inclusion problem in* (1.2) *has a solution.*

Proof. As we saw in the previous section, a fixed point of $\mathcal{F} : \mathbb{K}_* \to \mathrm{P}(\mathbb{K}_*)$ is the same solution of the Caputo-type $[\mathtt{p},\mathtt{q}]$-based inclusion problem in (1.2). Since the closed-valued mapping

$$\mathtt{t} \to \mathfrak{F}_{\mathtt{z}}(\mathtt{t}, \mathtt{z}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t}))$$

is measurable for every $\mathtt{z} \in \mathbb{K}_*$, there is a measurable selection for $\mathfrak{F}_{\mathtt{z}}$; in other words, $(\mathbb{S})_{\mathfrak{F}_{\mathtt{z}},\mathtt{z}} \neq \emptyset$.

As $(\mathbb{S})_{\mathfrak{F}_{\mathtt{z}},\mathtt{z}}$ is nonempty, we can now show that for each $\mathtt{z} \in \mathbb{K}_*$, $\mathcal{F}(\mathtt{z}) \subseteq \mathbb{K}_*$ is closed. To do this, consider $\{\mathtt{z}_i\}_{i\geq 1}$ as a sequence in $\mathcal{F}(\mathtt{z})$ with $\mathtt{z}_i \to \mathtt{z}$. For all i, some $F_i \in (\mathbb{S})_{\mathfrak{F}_{\mathtt{z}},\mathtt{z}}$ exists which satisfies

$$\mathtt{z}_i(\mathtt{t}) = \lambda + \eta\mathtt{t} + \frac{1}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\nu)\mathtt{p}^{\binom{\nu}{2}}} \int_0^{\mathtt{t}} (\mathtt{t} - \mathtt{q}\mathtt{u})^{(\nu-1)}_{[\mathtt{p},\mathtt{q}]} F_i\Big[\frac{\mathtt{u}}{\mathtt{p}^{\nu-1}}\Big]\, \mathrm{d}_{[\mathtt{p},\mathtt{q}]}\mathtt{u},$$

for almost all $\mathtt{t} \in \Im^L_{[\mathtt{p},\mathtt{q}]}$. But $\mathfrak{F}_{\mathtt{z}}$ is compact. As a result, there is a subsequence $\{F_i\}_{i\geq 1}$ that approaches some $F \in \mathcal{L}^1(\Im^L_{[\mathtt{p},\mathtt{q}]})$. Accordingly, $F \in (\mathbb{S})_{\mathfrak{F}_{\mathtt{z}},\mathtt{z}}$ can be selected so that

$$\begin{aligned}\mathtt{z}_i(\mathtt{t}) &\to \mathtt{z}(\mathtt{t}) \\ &= \lambda + \eta\mathtt{t} + \frac{1}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\nu)\mathtt{p}^{\binom{\nu}{2}}} \int_0^{\mathtt{t}} (\mathtt{t} - \mathtt{q}\mathtt{u})^{(\nu-1)}_{[\mathtt{p},\mathtt{q}]} F\Big[\frac{\mathtt{u}}{\mathtt{p}^{\nu-1}}\Big]\, \mathrm{d}_{[\mathtt{p},\mathtt{q}]}\mathtt{u},\end{aligned}$$

for all $\mathtt{t} \in \Im^L_{[\mathtt{p},\mathtt{q}]}$. This means that $\mathtt{z} \in \mathcal{F}(\mathtt{z})$ and $\mathcal{F}$ is closed-valued. Again, when $\mathfrak{F}_{\mathtt{z}}$ is compact-valued, the proof of the boundedness of $\mathcal{F}(\mathtt{z})$ will be straightforward for each $\mathtt{z} \in \mathbb{K}_*$.

In the following, we prove that $\mathcal{F}$ is a β-α-contraction. We first need a non-negative function like β on $\mathbb{K}_* \times \mathbb{K}_*$. We define it as

$$\beta(\mathtt{z}, \tilde{\mathtt{z}}) = \begin{cases} 1 & \text{if } \Theta((\mathtt{z}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})), (\tilde{\mathtt{z}}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\tilde{\mathtt{z}}(\mathtt{t}))) \geq 0; \\ 0 & \text{otherwise,} \end{cases}$$

for each $\mathtt{z}, \tilde{\mathtt{z}} \in \mathbb{K}_*$. Let $\mathtt{z}, \tilde{\mathtt{z}} \in \mathbb{K}_*$ and $\bar{F}_1 \in \mathcal{F}(\tilde{\mathtt{z}})$. Select $F_1 \in (\mathbb{S})_{\mathfrak{F}_{\mathtt{z}},\tilde{\mathtt{z}}}$ so that it satisfies

$$\bar{F}_1(\mathtt{t}) = \lambda + \eta\mathtt{t} + \frac{1}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\nu)\mathtt{p}^{\binom{\nu}{2}}} \int_0^{\mathtt{t}} (\mathtt{t} - \mathtt{q}\mathtt{u})^{(\nu-1)}_{[\mathtt{p},\mathtt{q}]} F_1\Big[\frac{\mathtt{u}}{\mathtt{p}^{\nu-1}}\Big]\, \mathrm{d}_{[\mathtt{p},\mathtt{q}]}\mathtt{u},$$

for all $\mathtt{t} \in \mathfrak{I}^{L}_{[\mathtt{p},\mathtt{q}]}$. The inequality (4.3), under condition ($\mathfrak{L}7$), gives

$$\mathbb{H}_{\mathtt{d}}(\mathfrak{F}_{\mathtt{z}}(\mathtt{t},\mathtt{z},{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}),\mathfrak{F}_{\mathtt{z}}(\mathtt{t},\tilde{\mathtt{z}},{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\tilde{\mathtt{z}})) \leq \breve{g}(\mathtt{t})\Big(\frac{\tilde{A}}{\|\breve{g}\|}\Big)\alpha\big(|\mathtt{z}-\tilde{\mathtt{z}}| + \big|{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z} - {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\tilde{\mathtt{z}}\big|\big),$$

for each $\mathtt{z},\tilde{\mathtt{z}} \in \mathbb{K}_*$, with

$$\Theta((\mathtt{z}(\mathtt{t}),{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})),(\tilde{\mathtt{z}}(\mathtt{t}),{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\tilde{\mathtt{z}}(\mathtt{t}))) \geq 0,$$

for almost all $\mathtt{t} \in \mathfrak{I}^{L}_{[\mathtt{p},\mathtt{q}]}$. Some $\hbar \in \mathfrak{F}_{\mathtt{z}}(\mathtt{t},\mathtt{z}(\mathtt{t}),{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t}))$ exists such that

$$|F_1(\mathtt{t}) - \hbar| \leq \breve{g}(\mathtt{t})\Big(\frac{\tilde{A}}{\|\breve{g}\|}\Big)\alpha\big(|\mathtt{z}(\mathtt{t})-\tilde{\mathtt{z}}(\mathtt{t})| + \big|{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t}) - {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\tilde{\mathtt{z}}(\mathtt{t})\big|\big).$$

Define $\vartheta : \mathfrak{I}^{L}_{[\mathtt{p},\mathtt{q}]} \to \mathrm{P}(\mathbb{K}_*)$ as

$$\vartheta(\mathtt{t}) = \Big\{\hbar \in \mathbb{K}_* : |F_1(\mathtt{t}) - \hbar| \leq \breve{g}(\mathtt{t})\Big(\frac{\tilde{A}}{\|\breve{g}\|}\Big)\alpha\big(|\mathtt{z}(\mathtt{t})-\tilde{\mathtt{z}}(\mathtt{t})| + \big|{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t}) - {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\tilde{\mathtt{z}}(\mathtt{t})\big|\big)\Big\},$$

for all $\mathtt{t} \in \mathfrak{I}^{L}_{[\mathtt{p},\mathtt{q}]}$. Since F_1 and $\omega = \breve{g}\Big(\frac{\tilde{A}}{\|\breve{g}\|}\Big)\alpha\big(|\mathtt{z}-\tilde{\mathtt{z}}| + \big|{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z} - {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\tilde{\mathtt{z}}\big|\big)$ are measurable, accordingly, $\vartheta(\cdot) \cap \mathfrak{F}_{\mathtt{z}}(\cdot,\mathtt{z}(\cdot),{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\cdot))$ is measurable too. In this step, select $F_2 \in \mathfrak{F}_{\mathtt{z}}(\mathtt{t},\mathtt{z}(\mathtt{t}),{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t}))$ which satisfies

$$|F_1(\mathtt{t}) - F_2(\mathtt{t})| \leq \breve{g}(\mathtt{t})\Big(\frac{\tilde{A}}{\|\breve{g}\|}\Big)\alpha\big(|\mathtt{z}(\mathtt{t})-\tilde{\mathtt{z}}(\mathtt{t})| + \big|{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t}) - {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\tilde{\mathtt{z}}(\mathtt{t})\big|\big), \quad \forall\, \mathtt{t} \in \mathfrak{I}^{L}_{[\mathtt{p},\mathtt{q}]}.$$

Consider $\bar{F}_2 \in \mathcal{F}(\mathtt{z})$ to be

$$\bar{F}_2(\mathtt{t}) = \lambda + \eta\mathtt{t} + \frac{1}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\nu)\mathtt{p}^{\binom{\nu}{2}}}\int_0^{\mathtt{t}} (\mathtt{t}-\mathtt{q}\mathtt{u})^{(\nu-1)}_{[\mathtt{p},\mathtt{q}]} F_2\Big[\frac{\mathtt{u}}{\mathtt{p}^{\nu-1}}\Big]\,\mathtt{d}_{[\mathtt{p},\mathtt{q}]}\mathtt{u},$$

for all $\mathtt{t} \in \mathfrak{I}^{L}_{[\mathtt{p},\mathtt{q}]}$. In this case,

$$\begin{aligned}
&|\bar{F}_1(\mathtt{t}) - \bar{F}_2(\mathtt{t})| \\
&\quad \leq \frac{1}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\nu)\mathtt{p}^{\binom{\nu}{2}}}\int_0^{\mathtt{t}} (\mathtt{t}-\mathtt{q}\mathtt{u})^{(\nu-1)}_{[\mathtt{p},\mathtt{q}]} \Big|F_1\Big[\frac{\mathtt{u}}{\mathtt{p}^{\nu-1}}\Big] - F_2\Big[\frac{\mathtt{u}}{\mathtt{p}^{\nu-1}}\Big]\Big|\,\mathtt{d}_{[\mathtt{p},\mathtt{q}]}\mathtt{u} \\
&\quad \leq \frac{(\frac{L}{\mathtt{p}})^{\nu}}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\nu+1)}\|\breve{g}\|\Big(\frac{\tilde{A}}{\|\breve{g}\|}\Big)\alpha(\|\mathtt{z}-\tilde{\mathtt{z}}\|_{\mathbb{K}_*}) \\
&\quad = \tilde{A}A_1\alpha(\|\mathtt{z}-\tilde{\mathtt{z}}\|_{\mathbb{K}_*}),
\end{aligned}$$

and

$$\big|{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\bar{F}_1(\mathtt{t}) - {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\bar{F}_2(\mathtt{t})\big| \leq \Big[\frac{(\frac{L}{\mathtt{p}})^{\nu-\mathtt{w}}}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\nu-\mathtt{w}+1)}\Big]\|\breve{g}\|\Big(\frac{\tilde{A}}{\|\breve{g}\|}\Big)\alpha(\|\mathtt{z}-\tilde{\mathtt{z}}\|_{\mathbb{K}_*})$$

$$= \tilde{A} A_2 \alpha(\|\mathtt{z} - \tilde{\mathtt{z}}\|_{\mathbb{K}_*}),$$

for all $\mathtt{t} \in \mathfrak{I}^L_{[\mathtt{p},\mathtt{q}]}$. Therefore,

$$\|\bar{F}_1 - \bar{F}_2\|_{\mathbb{K}_*} = \sup_{\mathtt{t}\in\mathfrak{I}^L_{[\mathtt{p},\mathtt{q}]}} |\bar{F}_1(\mathtt{t}) - \bar{F}_2(\mathtt{t})| + \sup_{\mathtt{t}\in\mathfrak{I}^L_{[\mathtt{p},\mathtt{q}]}} \left|{}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\bar{F}_1(\mathtt{t}) - {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\bar{F}_2(\mathtt{t})\right|$$

$$\leq (A_1 + A_2)\tilde{A}\,\alpha(\|\mathtt{z} - \tilde{\mathtt{z}}\|_{\mathbb{K}_*}) = \alpha(\|\mathtt{z} - \tilde{\mathtt{z}}\|_{\mathbb{K}_*}).$$

Hence,

$$\beta(\mathtt{z}, \tilde{\mathtt{z}})\mathbb{H}_{\mathtt{d}}(\mathcal{F}(\mathtt{z}) - \mathcal{F}(\tilde{\mathtt{z}})) \leq \alpha(\|\mathtt{z} - \tilde{\mathtt{z}}\|_{\mathbb{K}_*}),$$

is fulfilled for each $\mathtt{z}, \tilde{\mathtt{z}} \in \mathbb{K}_*$. We thus see that $\mathcal{F}$ is an β-α-contraction.

Consider $\mathtt{z} \in \mathbb{K}_*$ and $\tilde{\mathtt{z}} \in \mathcal{F}(\mathtt{z})$ satisfying $\mathtt{w}(\mathtt{z}, \tilde{\mathtt{z}}) \geq 1$ and

$$\Theta((\mathtt{z}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t})), (\tilde{\mathtt{z}}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\tilde{\mathtt{z}}(\mathtt{t}))) \geq 0.$$

Some $\hbar \in \mathcal{F}(\tilde{\mathtt{z}})$ exists such that

$$\Theta((\tilde{\mathtt{z}}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\tilde{\mathtt{z}}(\mathtt{t})), (\hbar(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\hbar(\mathtt{t}))) \geq 0.$$

Hence, $\beta(\tilde{\mathtt{z}}, \hbar) \geq 1$, and therefore, $\mathcal{F}$ is β-admissible. Now, take $\mathtt{z}_0 \in \mathbb{K}_*$ and $\tilde{\mathtt{z}} \in \mathcal{F}(\mathtt{z}_0)$ such that $\forall\, \mathtt{t} \in \mathfrak{I}^L_{[\mathtt{p},\mathtt{q}]}$,

$$\Theta((\mathtt{z}_0(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_0(\mathtt{t})), (\tilde{\mathtt{z}}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\tilde{\mathtt{z}}(\mathtt{t})) \geq 0.$$

Evidently, $\beta(\mathtt{z}_0, \tilde{\mathtt{z}}) \geq 1$. Next, let $\{\mathtt{z}_i\}_{i\geq 1} \subseteq \mathbb{K}_*$ be a sequence converging to $\mathtt{z}$ and consider $\beta(\mathtt{z}_i, \mathtt{z}_{i+1}) \geq 1$ for all i. In this case,

$$\Theta((\mathtt{z}_i(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_i(\mathtt{t})), (\mathtt{z}_{i+1}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_{i+1}(\mathtt{t}))) \geq 0.$$

The hypothesis in ($\mathfrak{L}$9) implies the existence of a subsequence $\{\mathtt{z}_{i_r}\}_{r\geq 1}$ in the sequence $\{\mathtt{z}_i\}$ such that

$$\Theta((\mathtt{z}_{i_r}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}_{i_r}(\mathtt{t})), (\mathtt{z}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{\mathtt{w}}_{[\mathtt{p},\mathtt{q}]}\mathtt{z}(\mathtt{t}))) \geq 0,$$

for all $\mathtt{t} \in \mathfrak{I}^L_{[\mathtt{p},\mathtt{q}]}$. Hence $\beta(\mathtt{z}_{i_r}, \mathtt{z}) \geq 1$ for all r. Finally, the abovementioned conclusions in each part of the proof show that Theorem 4.1 can be used to indicate the existence of some fixed point for $\mathcal{F}$. Therefore, there is a solution for the Caputo-type $[\mathtt{p},\mathtt{q}]$-based inclusion problem in (1.2). □

The fourth existence theorem will be proved, via the approximate end-point property, to show the existence of the end-point [38]. We review these notions here once again.

Let $\mathfrak{F}_{\mathtt{z}} : \mathbb{K}_* \to \mathrm{P}(\mathbb{K}_*)$ be a multivalued function. $\mathtt{z} \in \mathbb{K}_*$ is termed an end-point of $\mathfrak{F}_{\mathtt{z}}$ if $\mathfrak{F}_{\mathtt{z}}(\mathtt{z}) = \{\mathtt{z}\}$ [38].

Moreover, we say that the approximate end-point property holds for $\mathfrak{F}_{\mathtt{z}}$ whenever

$$\inf_{\mathtt{z}_1\in\mathbb{K}_*} \Big[\sup_{\mathtt{z}_2\in\mathfrak{F}_{\mathtt{z}}(\mathtt{z}_1)} d(\mathtt{z}_1, \mathtt{z}_2)\Big] = 0.$$

Theorem 4.3. *[38] Let* $(\mathbb{K}_*, \mathtt{d})$ *be a complete metric space, and assume the following:*

1) *the upper semi-continuous function* $\alpha : [0,\infty) \to [0,\infty)$ *is such that* $\liminf_{t\to\infty}(t - \alpha(t)) > 0$ *and* $\alpha(t) < t$ *for all* $t > 0$*;*
2) $\mathfrak{F}_z : \mathbb{K}_* \to P_{\mathfrak{CL},\mathfrak{B}}(\mathbb{K}_*)$ *is such that* $\mathbb{H}_d(\mathfrak{F}_z z_1, \mathfrak{F}_z z_2) \le \alpha(d(z_1, z_2))$ *for all* $z_1, z_2 \in \mathbb{K}_*$.

Then $\mathfrak{F}_z$ *has a unique end-point if and only if it has an approximate end-point property.*

Now, the last theorem can be established.

Theorem 4.4. *Let* $\mathfrak{F}_z : \Im^L_{[p,q]} \times \mathbb{K}^2_* \to P_{\mathfrak{CM}}(\mathbb{K}_*)$*, and assume the following:*

($\mathfrak{L}$12) *there is* $\alpha : [0,\infty) \to [0,\infty)$ *such that* $\liminf_{t\to\infty}(t - \alpha(t)) \ge 0$ *and* $\alpha(t) \le t$ *for all* $t > 0$. *Note that* α *is nondecreasing and upper semicontinuous;*

($\mathfrak{L}$13) $\mathfrak{F}_z : \Im^L_{[p,q]} \times \mathbb{K}^2_* \to P_{\mathfrak{CM}}(\mathbb{K}_*)$ *is integrable and bounded such that* $\mathfrak{F}_z(\cdot, z_1, z_2) : \Im^L_{[p,q]} \to P_{\mathfrak{CM}}(\mathbb{K}_*)$ *is measurable for every* $z_1, z_2 \in \mathbb{K}_*$*;*

($\mathfrak{L}$14) *some* $\breve{g} \in C(\Im^L_{[p,q]}, [0,\infty))$ *exists such that*

$$\mathbb{H}_d(\mathfrak{F}_z(t, z_1, z_2), \mathfrak{F}_z(t, \tilde{z_1}, \tilde{z_2})) \le \breve{g}(t)\tilde{A}_\star \alpha(|z_1 - \tilde{z_1}| + |z_2 - \tilde{z_2}|) \tag{4.4}$$

which holds for every $t \in \Im^L_{[p,q]}$ *and* $z_1, z_2, \tilde{z_1}, \tilde{z_2} \in \mathbb{K}_*$*, where* $\tilde{A}_\star = \frac{1}{\zeta_1+\zeta_2}$ *and* ζ_1, ζ_2 *are denoted in* (4.2)*;*

($\mathfrak{L}$15) $\mathcal{F}$*, introduced by* (4.1)*, has the approximate end-point property.*

Then the Caputo-type $[p, q]$*-based inclusion problem in* (1.2) *has a solution.*

Proof. The main purpose in this proof is that $\mathcal{F} : \mathbb{K}_* \to P(\mathbb{K}_*)$ admits an end-point. To achieve this goal, we should prove that $\mathcal{F}(z) \subseteq \mathbb{K}_*$ is closed in the first step for every $z \in \mathbb{K}_*$. But consider the fact that the multivalued function $t \to \mathfrak{F}_z(t, z(t), {}^c\mathcal{D}^w_{[p,q]}z(t))$ is closed-valued and measurable for every $z \in \mathbb{K}_*$. Hence, one can find a measurable selection for $\mathfrak{F}_z$, and therefore, $(\mathbb{S})_{\mathfrak{F}_z,z} \ne \emptyset$. Similar to the proof of Theorem 4.2, we can easily confirm that $\mathcal{F}(z)$ is closed-valued. Moreover, $\mathcal{F}(z)$ is bounded for each $z \in \mathbb{K}_*$ because $\mathfrak{F}_z$ is compact-valued.

The establishment of the inequality

$$\mathbb{H}_d(\mathcal{F}(z), \mathcal{F}(\hbar)) \le \alpha(\|z - \hbar\|_{\mathbb{K}_*}),$$

is our next purpose. For this, let $z, \hbar \in \mathbb{K}_*$, $\bar{F}_1 \in \mathcal{F}(\hbar)$ and choose $F_1 \in (\mathbb{S})_{\mathfrak{F}_z,\hbar}$ such that

$$\bar{F}_1(t) = \lambda + \eta t + \frac{1}{\Gamma_{[p,q]}(\nu)p^{\binom{\nu}{2}}} \int_0^t (t - qu)^{(\nu-1)}_{[p,q]} F_1\Big[\frac{u}{p^{\nu-1}}\Big]\, d_{[p,q]}u,$$

for almost all $t \in \Im^L_{[p,q]}$. By the condition in ($\mathfrak{L}$14) and by (4.4) in it, we have

$$\begin{aligned}\mathbb{H}_d(&\mathfrak{F}_z(t, z(t), {}^c\mathcal{D}^w_{[p,q]}z(t)), \mathfrak{F}_z(t, \hbar(t), {}^c\mathcal{D}^w_{[p,q]}\hbar(t)))\\ &\le \breve{g}(t)\tilde{A}_\star \alpha(|z(t) - \hbar(t)| + \big|{}^c\mathcal{D}^w_{[p,q]}z(t) - {}^c\mathcal{D}^w_{[p,q]}\hbar(t)\big|),\end{aligned}$$

for all $t \in \Im^L_{[p,q]}$. Now, some $\bar{\mu} \in \mathfrak{F}_z(t, z(t), {}^c\mathcal{D}^w_{[p,q]}z(t))$ exists such that

$$|F_1(t) - \bar{\mu}| \le \breve{g}(t)\tilde{A}_\star \alpha(|z(t) - \hbar(t)| + \big|{}^c\mathcal{D}^w_{[p,q]}z(t) - {}^c\mathcal{D}^w_{[p,q]}\hbar(t)\big|).$$

Define $\sigma : \Im^{L}_{[p,q]} \to P(\mathbb{K}_*)$ as

$$\sigma(t) = \{\bar{\mu} \in \mathbb{K}_* : |F_1(t) - \bar{\mu}| \leq \breve{g}(t)\tilde{A}_{\star}\alpha(|z(t) - \hbar(t)| + \left|{}^{c}\mathcal{D}^{w}_{[p,q]}z(t) - {}^{c}\mathcal{D}^{w}_{[p,q]}\hbar(t)\right|)\},$$

for all $t \in \Im^{L}_{[p,q]}$. It is known that F_1 and $b_* = \breve{g}\tilde{A}_{\star}\alpha(|z - \hbar| + \left|{}^{c}\mathcal{D}^{w}_{[p,q]}z - {}^{c}\mathcal{D}^{w}_{[p,q]}\hbar\right|)$ are measurable. Therefore, $\sigma(\cdot) \cap \mathfrak{F}_z(\cdot, z(\cdot), {}^{c}\mathcal{D}^{w}_{[p,q]}z(\cdot))$ is measurable.

In the next step, we consider $F_2 \in \mathfrak{F}_z(t, z(t), {}^{c}\mathcal{D}^{w}_{[p,q]}z(t))$ s.t.

$$|F_1(t) - F_2(t)| \leq \breve{g}(t)\tilde{A}_{\star}\alpha(|z(t) - \hbar(t)| + \left|{}^{c}\mathcal{D}^{w}_{[p,q]}z(t) - {}^{c}\mathcal{D}^{w}_{[p,q]}\hbar(t)\right|),$$

for all $t \in \Im^{L}_{[p,q]}$. We also consider $\bar{F}_2 \in \mathcal{F}(z)$ such that

$$\bar{F}_2(t) = \lambda + \eta t + \frac{1}{\Gamma_{[p,q]}(\nu)p^{\binom{\nu}{2}}} \int_0^t (t - qu)^{(\nu-1)}_{[p,q]} F_2\Big[\frac{u}{p^{\nu-1}}\Big]\, d_{[p,q]}u,$$

for all $t \in \Im^{L}_{[p,q]}$. By continuing the steps conducted in Theorem 4.2, we get

$$\begin{aligned}\|\bar{F}_1 - \bar{F}_2\|_{\mathbb{K}_*} &= \sup_{t\in\Im^{L}_{[p,q]}} |\bar{F}_1(t) - \bar{F}_2(t)| + \sup_{t\in\Im^{L}_{[p,q]}} \left|{}^{c}\mathcal{D}^{w}_{[p,q]}\bar{F}_1(t) - {}^{c}\mathcal{D}^{w}_{[p,q]}\bar{F}_2(t)\right| \\ &\leq (\zeta_1 + \zeta_2)\tilde{A}_{\star}\alpha(\|z - \hbar\|_{\mathbb{K}_*}) = \alpha(\|z - \hbar\|_{\mathbb{K}_*}).\end{aligned}$$

Therefore,

$$\mathbb{H}_d(\mathcal{F}(z), \mathcal{F}(\hbar)) \leq \alpha(\|z - \hbar\|_{\mathbb{K}_*}),$$

for every $z, \hbar \in \mathbb{K}_*$. Finally, note that the hypothesis in ($\mathfrak{L}15$) tells us that $\mathcal{F}$ has the approximate end-point property. As a result, the existence of $z^{**} \in \mathbb{K}_*$ is confirmed by Theorem 4.3 s.t. $\mathcal{F}(z^{**}) = \{z^{**}\}$. In fact, as we expected, z^{**} will be an end-point for $\mathcal{F}$. Therefore, the Caputo-type $[p, q]$-based inclusion problem in (1.2) has a solution like z^{**}. □

5. Application: $[p, q]$-based model of a vibrating eardrum

One of the most important parts of the human ear is the eardrum, which is a vibrating membrane and is located inside the ear. When sound waves enter the human ear, these waves vibrate the eardrum and cause the eardrum to send these vibrations in the form of nerve signals to the brain. The vibrations produced by the impact of these sound waves create a frequency response that is called the hearing range of the human ear. Note that in a normal state, the natural range of hearing is between 20 and 20,000 Hz, and with age, this range becomes smaller and smaller. Since the majority of biological processes or events follow Newton's second law, the mathematical model of the vibrating eardrum also follows this law [39]. When the eardrum vibrates, a mechanical system is redesigned in which the sound wave is introduced as a pressure agent. The causes of a vibrating ear include exposure to excessive loud noises, infections and other diseases. Exposure to frequently loud noises can damage the auditory system and lead to hearing loss.

In the form of such a system, it is easy to mathematically model the transmission of different frequencies of sound waves in a nonlinear way and, in this way, analyze the qualitative and quantitative behavior of the vibrating eardrum. From a mathematical point of view, the concepts of the existence of a solution and its uniqueness for the given system can be checked for such a nonlinear vibration equation that is created by a driving force. For the first time, we aimed to investigate this system in the form of new two-parameteric $[\mathtt{p},\mathtt{q}]$-based operators. When modeling how sound wave vibrations translate into mechanical vibrations across different frequencies or amplitudes, $\mathtt{p}$ can relate inversely with frequency, while $\mathtt{q}$ may represent amplitude scaling. Thus, using a derivative operator like ${}^{\mathtt{c}}\mathcal{D}_{[\mathtt{p},\mathtt{q}]}$ where the responses are characterized via both rate scales helps simulate auditory responses under varying auditory conditions effectively. In a general manner, the two parameters $\mathtt{p}$ and $\mathtt{q}$ play a controlling role in this model, and we can examine different values for these parameters when we analyze the model numerically. Now, we pay attention to the following examples, in which the existence of the solutions is studied theoretically, not numerically. We again emphasize that these examples state the abstract form of a nonlinear vibration equation, and the source nonlinear terms and coefficients have been chosen arbitrarily to validate the correctness of our theoretical results. For analyzing the structure of the model and the values of the parameters exactly, we need numerical algorithms in the context of $[\mathtt{p},\mathtt{q}]$-functions and $[\mathtt{p},\mathtt{q}]$-transforms, although the theory of $[\mathtt{p},\mathtt{q}]$-calculus is very young and we should extend some basic concepts and tools in future studies.

Example 5.1. *We consider the Caputo-type* $[\mathtt{p},\mathtt{q}]$*-based initial value problem in* (1.1) *and formulate a mathematical model of the nonlinear vibrating eardrum as follows:*

$$\begin{cases} ({}^{\mathtt{c}}\mathcal{D}^{1.5}_{[0.5,0.4]}\mathtt{z})(\mathtt{t}) - 0.002\mathtt{t}({}^{\mathtt{c}}\mathcal{D}^{0.75}_{[0.5,0.4]}\mathtt{z})(\mathtt{t}) - \dfrac{2\mathtt{t}|\sin(\mathtt{z}(\mathtt{t}))|}{1000(1+|\sin(\mathtt{z}(\mathtt{t}))|)} = 0, \\ \mathtt{z}(\mathtt{t})\Big|_{\mathtt{t}=0} = 0.075, \qquad \mathcal{D}_{[0.5,0.4]}\mathtt{z}(\mathtt{t})\Big|_{\mathtt{t}=0} = 0.35, \end{cases} \tag{5.1}$$

where $\mathtt{q} = 0.4$*,* $\mathtt{p} = 0.5$*,* $\nu = 1.5$*,* $\mathtt{w} = 0.75$*,* $\lambda = 0.075$*,* $\eta = 0.35$*, and* $L = 0.01$ *with* $\frac{L}{\mathtt{p}} = 0.02$ *and* $\mathtt{t} \in \Im^{0.01}_{[0.5;0.4]} = [0, 0.02]$*. A continuous function* $\mathfrak{f}_{\mathtt{z}} : [0, 0.02] \times \mathbb{R}^2 \to \mathbb{R}$ *is given by*

$$\mathfrak{f}_{\mathtt{z}}(\mathtt{t}, \mathtt{z}(\mathtt{t}), \mathtt{z}^*(\mathtt{t})) = \frac{2\mathtt{t}|\sin(\mathtt{z}(\mathtt{t}))|}{1000(1+|\sin(\mathtt{z}(\mathtt{t}))|)} + 0.002\mathtt{t}\mathtt{z}^*(\mathtt{t}).$$

For each $\mathtt{z}_1, \mathtt{z}_2, \mathtt{z}_1^*, \mathtt{z}_2^* \in \mathbb{R}$*, we have*

$$\begin{aligned} \Big|\mathfrak{f}_{\mathtt{z}}(\mathtt{t}, \mathtt{z}_1(\mathtt{t}), \mathtt{z}_2(\mathtt{t})) - \mathfrak{f}_{\mathtt{z}^*}(\mathtt{t}, \mathtt{z}_1^*(\mathtt{t}), \mathtt{z}_2^*(\mathtt{t}))\Big| &\le 0.002\mathtt{t}(|\sin(\mathtt{z}_1(\mathtt{t})) - \sin(\mathtt{z}_1^*(\mathtt{t}))| + |\mathtt{z}_2(\mathtt{t}) - \mathtt{z}_2^*(\mathtt{t})|) \\ &\le 0.002\mathtt{t}(|\mathtt{z}_1(\mathtt{t}) - \mathtt{z}_1^*(\mathtt{t})| + |\mathtt{z}_2(\mathtt{t}) - \mathtt{z}_2^*(\mathtt{t})|). \end{aligned}$$

Put $\mathbb{k}(\mathtt{t}) = 0.002\mathtt{t}$ *for all* $\mathtt{t}$*. Then* $\|\mathbb{k}\| = \sup_{\mathtt{t}\in[0,0.02]} |0.002\mathtt{t}| = 0.00004$*. In the sequel, we aim to define an increasing function* $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $\alpha(\varsigma) = \varsigma$ *for each* $\varsigma \in \mathbb{R}^+$*. In this case,*

$$\begin{aligned} \Big|\mathfrak{f}_{\mathtt{z}}(\mathtt{t}, \mathtt{z}(\mathtt{t}), {}^{\mathtt{c}}\mathcal{D}^{0.75}_{[0.5,0.4]}\mathtt{z}(\mathtt{t}))\Big| &\le 0.002\mathtt{t}(|\mathtt{z}(\mathtt{t})| + \Big|{}^{\mathtt{c}}\mathcal{D}^{0.75}_{[0.5,0.4]}\mathtt{z}(\mathtt{t})\Big|) \\ &= 0.002\mathtt{t}\alpha(|\mathtt{z}(\mathtt{t})| + \Big|{}^{\mathtt{c}}\mathcal{D}^{0.75}_{[0.5,0.4]}\mathtt{z}(\mathtt{t})\Big|). \end{aligned}$$

Evidently, $\mathbb{g} : \Im^{0.01}_{[0.5,0.4]} = [0, 0.02] \to \mathbb{R}^+$ *given by* $\mathbb{g}(\mathtt{t}) = 0.002\mathtt{t}$ *is continuous. By* (3.7)*, calculate the following constant*

$$A_1 = \frac{(\frac{L}{\mathtt{p}})^{\nu}}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\nu+1)} \approx \frac{0.0028}{\Gamma_{[0.5,0.4]}(2.5)} \approx 0.0009899415,$$

$$A_2 = \frac{(\frac{L}{\mathtt{p}})^{\nu-\mathtt{w}}}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\nu-\mathtt{w}+1)} \approx \frac{0.05318}{\Gamma_{[0.5,0.4]}(1.75)} \approx 0.03162107.$$

By (3.10)*, we get* $\mathbb{A} = \|\mathbb{k}\|(A_1 + A_2) \approx 0.00000130444046 < 1$. *In accordance with the conclusion of Theorem 3.5, the Caputo-type* $[\mathtt{p},\mathtt{q}]$*-based initial value problem in* (5.1) *of a vibrating eardrum has a solution in* $\mathfrak{I}^{0.01}_{[0.5,0.4]}$.

Example 5.2. *We consider the Caputo-type* $[\mathtt{p},\mathtt{q}]$*-based inclusion problem in* (1.2)*, and formulate an inclusion-based mathematical model of the nonlinear vibrating eardrum. We apply the same parameters given in Example 5.1, i.e.,* $\mathtt{q} = 0.4$*,* $\mathtt{p} = 0.5$*,* $\nu = 1.5$*,* $\mathtt{w} = 0.75$*,* $\lambda = 0.075$*,* $\eta = 0.35$*, and* $L = 0.01$ *with* $\frac{L}{\mathtt{p}} = 0.02$ *and* $\mathtt{t} \in \mathfrak{I}^{0.01}_{[0.5;0.4]} = [0, 0.02]$. *According to these data, we formulate*

$$\begin{cases} ({}^{c}\mathcal{D}^{1.5}_{[0.5,0.4]}\mathtt{z})(\mathtt{t}) \in \left[0, \dfrac{0.02\mathtt{t}|\tan(\mathtt{z}(\mathtt{t}))|}{5(1+|\tan(\mathtt{z}(\mathtt{t}))|)} + \dfrac{4}{1000}\mathtt{t}\sin\left(({}^{c}\mathcal{D}^{0.75}_{[0.5,0.4]}\mathtt{z})(\mathtt{t})\right)\right], \\ \mathtt{z}(\mathtt{t})\big|_{\mathtt{t}=0} = 0.075, \qquad \mathcal{D}_{[0.5,0.4]}\mathtt{z}(\mathtt{t})\big|_{\mathtt{t}=0} = 0.35. \end{cases} \tag{5.2}$$

Therefore, the set-valued function $\mathfrak{F}_{\mathtt{z}} : \mathfrak{I}^{0.01}_{[0.5,0.4]} = [0, 0.02] \times \mathbb{R}^2 \to \mathrm{P}(\mathbb{R})$ *is given by*

$$\mathfrak{F}_{\mathtt{z}}(\mathtt{t}, \mathtt{z}_1(\mathtt{t}), \mathtt{z}_2(\mathtt{t})) = \left[0, \frac{0.02\mathtt{t}|\tan(\mathtt{z}_1(\mathtt{t}))|}{5(1+|\tan(\mathtt{z}_1(\mathtt{t}))|)} + \frac{4}{1000}\mathtt{t}\sin(\mathtt{z}_2(\mathtt{t}))\right],$$

for all $\mathtt{t} \in [0, 0.02]$. *Now,* $\breve{g} \in C([0, 0.02], [0,\infty))$ *is selected such that* $\breve{g}(\mathtt{t}) = \frac{\mathtt{t}}{10}$ *for each* $\mathtt{t} \in [0, 0.02]$. *Thus,* $\|\breve{g}\| = \sup_{\mathtt{t}\in[0,0.02]} |\frac{\mathtt{t}}{10}| = \frac{0.02}{10} = 0.002$. *Also, define* $\alpha : [0,\infty) \to [0,\infty)$ *as* $\alpha(\mathtt{t}) = \frac{4\mathtt{t}}{100}$ *for almost all* $\mathtt{t} > 0$*;* α *is clearly increasing and upper semicontinuous. In this case,* $\liminf_{\mathtt{t}\to\infty}(\mathtt{t} - \alpha(\mathtt{t})) > 0$ *and* $\alpha(\mathtt{t}) < \mathtt{t}$ *for all* $\mathtt{t} > 0$. *Now,* (3.7) *and* (4.2) *give*

$$A_1 = \frac{(\frac{L}{\mathtt{p}})^{\nu}}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\nu+1)} \approx \frac{0.0028}{\Gamma_{[0.5,0.4]}(2.5)} \approx 0.0009899415,$$

$$A_2 = \frac{(\frac{L}{\mathtt{p}})^{\nu-\mathtt{w}}}{\Gamma_{[\mathtt{p},\mathtt{q}]}(\nu-\mathtt{w}+1)} \approx \frac{0.05318}{\Gamma_{[0.5,0.4]}(1.75)} \approx 0.03162107.$$

and

$$\zeta_1 = \|\breve{g}\|A_1 \approx 0.000001979883 \quad \text{and} \quad \zeta_2 = \|\breve{g}\|A_2 \approx 0.00006324214.$$

For every $\mathtt{z}_1, \mathtt{z}_2, \tilde{\mathtt{z}_1}, \tilde{\mathtt{z}_2} \in \mathbb{R}$, *we get*

$$\begin{aligned} \mathbb{H}_{\mathtt{d}}(\mathfrak{F}_{\mathtt{z}}(\mathtt{t}, \mathtt{z}_1(\mathtt{t}), \mathtt{z}_2(\mathtt{t})), \mathfrak{F}_{\mathtt{z}}(\mathtt{t}, \tilde{\mathtt{z}_1}(\mathtt{t}), \tilde{\mathtt{z}_2}(\mathtt{t}))) &\le \frac{\mathtt{t}}{10}.\frac{4}{100}(|\mathtt{z}_1(\mathtt{t}) - \tilde{\mathtt{z}_1}(\mathtt{t})| + |\mathtt{z}_2(\mathtt{t}) - \tilde{\mathtt{z}_2}(\mathtt{t})|) \\ &= \frac{\mathtt{t}}{10}\alpha(|\mathtt{z}_1(\mathtt{t}) - \tilde{\mathtt{z}_1}(\mathtt{t})| + |\mathtt{z}_2(\mathtt{t}) - \tilde{\mathtt{z}_2}(\mathtt{t})|) \\ &\le \breve{g}(\mathtt{t})\alpha(|\mathtt{z}_1(\mathtt{t}) - \tilde{\mathtt{z}_1}(\mathtt{t})| + |\mathtt{z}_2(\mathtt{t}) - \tilde{\mathtt{z}_2}(\mathtt{t})|)\left[\frac{1}{\zeta_1 + \zeta_2}\right]. \end{aligned}$$

In the following, the set-valued function $\mathcal{F} : \mathbb{K}_* \to \mathrm{P}(\mathbb{K}_*)$ *is defined by*

$$\mathcal{F}(\mathtt{z}) = \left\{\bar{F} \in \mathbb{K}_* : \ \exists F \in (\mathbb{S})_{\mathfrak{F}_{\mathtt{z}},\mathtt{z}} \text{ so that } \bar{F}(\mathtt{t}) = \hbar(\mathtt{t}), \ \forall\, \mathtt{t} \in \mathfrak{I}^{0.01}_{[0.5,0.4]} = [0.0.02]\right\},$$

such that

$$\hbar(t) = 0.075 + 0.35t + \frac{1}{\Gamma_{[0.5,0.4]}(1.5)0.5^{\binom{1.5}{2}}} \int_0^t (t - 0.4u)_{[0.5,0.4]}^{(1.5-1)} F\Big[\frac{u}{0.5^{1.5-1}}\Big] d_{[0.5,0.4]}u.$$

Lastly, in accordance with the conclusion of Theorem 4.4, the Caputo-type $[p, q]$*-based inclusion problem in* (5.2) *of a vibrating eardrum has a solution.*

6. Conclusions

Two nonlinear single-valued and set-valued mappings were used, in this paper, to model the behaviour of a vibrating eardrum in the framework of a Caputo-type $[p, q]$-based initial value problem in (1.1) and the Caputo-type $[p, q]$-based inclusion problem in (1.2). In fact, for the first time, we used the $[p, q]$-derivatives for modeling it. To prove the theoretical aspects of this study, we used some fixed point theorems and end-point theorems under some special contractions and compact mappings. In this study, β-α-contractions played an important role, we could establish the existence property for the solutions of two given $[p, q]$-based dynamic systems. Note that if we take $p = 1$ and if $p = 1$ and $q \to 1$, then our given models reduce to the Caputo-type q-based model and the standard Caputo-type model of the vibrating eardrum, respectively. For subsequent studies, it is necessary to find some numerical $[p, q]$-based algorithms to obtain the approximate solutions for such nonlinear $[p, q]$-based initial value problems. In this direction, one can study other dynamical behaviors of solutions more accurately.

Author contributions

R.G.: Conceptualization, Investigation, Writing-original draft, Writing-review & editing; S.E.: Conceptualization, Formal analysis, Methodology, Software, Writing-original draft; I.S.: Formal analysis, Investigation, Software, Writing-review & editing; R.A.: Formal analysis, Methodology, Writing-original draft. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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