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Research article

Unveiling new reverse Hilbert-type dynamic inequalities within the framework of Delta calculus on time scales

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Abstract: Some new reverse versions of Hilbert-type inequalities are studied in this paper. The results are established by applying the time scale versions of reverse Hölder's inequality, reverse Jensen's inequality, chain rule on time scales, and the mean inequality. As applications, some particular results (when $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \mathbb{R}$) are considered. Our results provide some new estimates for these types of inequalities and improve some of those recently published in the literature.

Keywords: Hilbert-type inequalities; reverse Hilbert-type inequality; delta calculus; time scales
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1. Introduction

Hilbert-type inequalities play a crucial role in mathematical theory, especially in areas such as complex analysis, numerical analysis, and the qualitative theory of differential equations, along with their applications. In recent years, numerous refinements, generalizations, extensions, and practical uses of Hilbert's inequality have been widely discussed in the literature. Both Hilbert's discrete inequality and its integral version [1, Theorem 316] have been extended in various directions (see, for instance, [2–5]). More recently, Zhao and Debnath [6] introduced novel inverse Hilbert integral inequalities. For example, they demonstrated that if $k, l \geq 1$, $f(\varrho) > 0$ for $0 < \varrho < x$, and $g(\nu) > 0$ for

$0 < \nu < y$, then

$$\int_0^x \int_0^y \frac{F^k(s)G^l(t)}{(st)^{\frac{1}{\beta}}} dtds \geq \frac{1}{2} kl(xy)^{\frac{1}{\beta}} \left(\int_0^x (x-s) \left[F^{k-1}(s) f(s) \right]^\alpha ds \right)^{\frac{1}{\alpha}} \times \left(\int_0^y (y-t) \left[G^{l-1}(t) g(t) \right]^\alpha dt \right)^{\frac{1}{\alpha}}, \tag{1.1}$$

where $\beta < 0$, or $0 < \beta < 1$ with $1/\beta + 1/\alpha = 1$,

$$F(s) = \int_0^s f(\varrho) d\varrho \text{ and } G(t) = \int_0^t g(\nu) d\nu \text{ for } s \in (0, x), \ t \in (0, y). \tag{1.2}$$

In a related development, Y. H. Kim [7] introduced further generalizations of these inverse Hilbert-Pachpatte integral inequalities. Under the conditions $k, l \geq 1$, $r \leq 0$, $f(\varrho) \geq 0$ for $0 < \varrho < x$, and $g(\nu) \geq 0$ for $0 < \nu < y$, he established that for $\beta < 0$ or $0 < \beta < 1$ and $1/\beta + 1/\alpha = 1$, the following inequality holds:

$$\int_0^x \int_0^y \frac{F^k(s)G^l(t)}{(\frac{s^r+t^r}{2})^{\frac{2}{\beta r}}} dtds \geq kl(xy)^{\frac{1}{\beta}} \left(\int_0^x (x-s) \left[F^{k-1}(s) f(s) \right]^\alpha ds \right)^{\frac{1}{\alpha}} \times \left(\int_0^y (y-t) \left[G^{l-1}(t) g(t) \right]^\alpha dt \right)^{\frac{1}{\alpha}}, \tag{1.3}$$

with $F(s)$ and $G(t)$ as defined in (3.14).

Additionally, Z. Changjian and M. Bencze [8] established that the following inverse Hilbert integral inequality: If $k, l \geq 1$, $r \leq 0$, $f(\varrho) > 0$ for $0 < \varrho < x$, and $g(\nu) > 0$ for $0 < \nu < y$, then for $\beta < 0$ or $0 < \beta < 1$ with $1/\beta + 1/\alpha = 1$, the following inequality holds:

$$\int_0^x \int_0^y \frac{F^k(s)G^l(t)}{C(s,t,\beta)} dtds \geq kl(xy)^{\frac{1}{\beta}} \left(\int_0^x (x-s) \left[F^{k-1}(s) \right]^\alpha ds \right)^{\frac{1}{\alpha}} \times \left(\int_0^y (y-t) \left[G^{l-1}(t) \right]^\alpha dt \right)^{\frac{1}{\alpha}}, \tag{1.4}$$

where $C(s,t,\beta) = \left(\int_0^s f^\beta(\varrho) d\varrho \right)^{1/\beta} \left(\int_0^t g^\beta(\nu) d\nu \right)^{1/\beta}$, $F(s)$ and $G(t)$ are as defined as above.

In recent years, the theory of time scales has emerged as a fascinating and rapidly growing branch of mathematics, attracting significant attention. This theory was established by the renowned mathematician Stefan Hilger [9], whose primary goal was to unify discrete and continuous analysis [10]. The general idea of this theory is to derive new results for dynamic equations or dynamic inequalities where the domain of the unknown function is what is known as a time scale $\mathbb{T}$, defined as an arbitrary closed subset of $\mathbb{R}$ (see [11, 12]). As a result, there has been a substantial increase in the number of results obtained in this field in recent years. Among the most important topics studied within this theory is the investigation of classical inequalities, with many researchers working to derive more generalized forms of these inequalities using time scale calculus. These inequalities have since become known as dynamic inequalities. Among them are the reverse Hilbert-type inequalities. Below, we highlight some of these results, which have motivated the writing of this paper.

In [13], the authors derived new dynamic inequalities of the inverse Hilbert-type as follows: For $h, l \geq 1$, $r \leq 0$ and $p < 0$ or $0 < p < 1$ with $1/p + 1/q = 1$. If $f(\eta) \geq 0$ for $\eta \in [a, x]_{\mathbb{T}}$ and $g(\tau) \geq 0$ for $\tau \in [a, y]_{\mathbb{T}}$ are right-dense continuous functions, then the following inequality holds for $s \in [a, x]_{\mathbb{T}}$ and $t \in [a, y]_{\mathbb{T}}$:

$$\begin{aligned}
&\int_a^x \int_a^y \frac{\left(\int_a^s f(\eta)\Delta\eta\right)^h \left(\int_a^t g(\tau)\Delta\tau\right)^l}{[(s-a)^r + (t-a)^r]^{\frac{2}{pr}}} \Delta t \Delta s \\
\geq\ & hl\left(\frac{1}{2}\right)^{\frac{2}{pr}} (x-a)^{\frac{1}{p}} (y-a)^{\frac{1}{p}} \left(\int_a^x (x-\sigma(s))\left(f(s)\left[\int_a^s f(\eta)\Delta\eta\right]^{h-1}\right)^q \Delta s\right)^{\frac{1}{q}} \\
& \left(\int_a^y (y-\sigma(t))\left(g(t)\left[\int_a^t g(\tau)\Delta\tau\right]^{l-1}\right)^q \Delta t\right)^{\frac{1}{q}}. \qquad (1.5)
\end{aligned}$$

Additionally, they showed that if $h(\eta) > 0$ for $\eta \in [a, x]_{\mathbb{T}}$ and $k(\tau) > 0$ for $\tau \in [a, y]_{\mathbb{T}}$ are right-dense continuous functions and if Φ and Ψ are non-negative, concave, and supermultiplicative functions, then, for $s \in [a, x]_{\mathbb{T}}$ and $t \in [a, y]_{\mathbb{T}}$, the following inequality is valid:

$$\begin{aligned}
&\int_a^x \int_a^y \frac{\Phi(\int_a^s f(\eta)\Delta\eta)\Psi(\int_a^t g(\tau)\Delta\tau)}{[(s-a)^r + (t-a)^r]^{\frac{2}{pr}}} \Delta t \Delta s \\
\geq\ & E(p, r, x, y)\left(\int_a^x (x-\sigma(s))\left(h(s)\Phi\left[\frac{f(s)}{h(s)}\right]\right)^q \Delta s\right)^{\frac{1}{q}} \\
& \times\left(\int_a^y (y-\sigma(t))\left(k(t)\Psi\left[\frac{g(t)}{k(t)}\right]\right)^q \Delta t\right)^{\frac{1}{q}}, \qquad (1.6)
\end{aligned}$$

where

$$E(p, r, x, y) = \left(\frac{1}{2}\right)^{\frac{2}{pr}} \left(\int_a^x \left(\frac{\Phi\left(\int_a^s h(\eta)\Delta\eta\right)}{\int_a^s h(\eta)\Delta\eta}\right)^p \Delta s\right)^{\frac{1}{p}} \left(\int_a^y \left(\frac{\Psi\left(\int_a^t k(\tau)\Delta\tau\right)}{\int_a^t k(\tau)\Delta\tau}\right)^p \Delta t\right)^{\frac{1}{p}}.$$

In [14], further results on the reverse Hilbert-type inequalities were established: For $h, l \geq 1$, $p < 0$, or $0 < p < 1$ with $1/p + 1/q = 1$, and for non-negative functions $f \in C_{rd}([a, x]_{\mathbb{T}}, \mathbb{R}^+)$ and $g \in C_{rd}([a, y]_{\mathbb{T}}, \mathbb{R}^+)$, then, for $s \in [a, x]_{\mathbb{T}}$ and $t \in [a, y]_{\mathbb{T}}$, the following inequality holds:

$$\begin{aligned}
&\int_a^x \int_a^y \frac{\left(\int_a^s f(\eta)\Delta\eta\right)^h \left(\int_a^t g(\tau)\Delta\tau\right)^l}{\left(q(s-a)^{p-1} + p\,(t-a)^{q-1}\right)^{\frac{p+q}{pq}}} \Delta t \Delta s \\
\geq\ & \frac{hl}{(p+q)^{\frac{p+q}{pq}}} (x-a)^{\frac{p-1}{p}} (y-a)^{\frac{q-1}{q}} \left(\int_a^x (\rho(x)-\sigma(s))\left(f(s)\left[\int_a^s f(\eta)\Delta\eta\right]^{h-1}\right)^p \Delta s\right)^{\frac{1}{p}} \\
& \left(\int_a^y (\rho(y)-\sigma(t))\left(g(t)\left[\int_a^t g(\tau)\Delta\tau\right]^{l-1}\right)^q \Delta t\right)^{\frac{1}{q}}. \qquad (1.7)
\end{aligned}$$

Additionally, they proved that if $h \in C_{rd}([a, x]_{\mathbb{T}}, \mathbb{R}^+)$, $k \in C_{rd}([a, y]_{\mathbb{T}}, \mathbb{R}^+)$ are positive and Φ, Ψ are non-negative, concave, and supermultiplicative functions, then for $s \in [a, x]_{\mathbb{T}}$ and $t \in [a, y]_{\mathbb{T}}$, the following inequality is valid:

$$\begin{aligned}
&\int_a^x \int_a^y \frac{\Phi(\int_a^s f(\eta)\Delta\eta)\Psi(\int_a^t g(\tau)\Delta\tau)}{\left(q(s-a)^{p-1} + p\,(t-a)^{q-1}\right)^{\frac{p+q}{pq}}}\Delta t \Delta s \\
\geq\ & \mathcal{G}(p,q,x,y)\left(\int_a^x (\rho(x)-\sigma(s))\left(h(s)\Phi\left[\frac{f(s)}{h(s)}\right]\right)^p \Delta s\right)^{\frac{1}{p}} \\
&\times\left(\int_a^y (\rho(y)-\sigma(t))\left(k(t)\Psi\left[\frac{g(t)}{k(t)}\right]\right)^q \Delta t\right)^{\frac{1}{q}}, \qquad (1.8)
\end{aligned}$$

where

$$\mathcal{G}(p,q,x,y) = \left(\frac{1}{p+q}\right)^{\frac{p+q}{pq}} \left(\int_a^x \left(\frac{\Phi\left(\int_a^s h(\eta)\Delta\eta\right)}{\int_a^s h(\eta)\Delta\eta}\right)^{\frac{p}{p-1}} \Delta s\right)^{\frac{p-1}{p}} \left(\int_a^y \left(\frac{\Psi\left(\int_a^t k(\tau)\Delta\tau\right)}{\int_a^t k(\tau)\Delta\tau}\right)^{\frac{q}{q-1}} \Delta t\right)^{\frac{q-1}{q}}.$$

Recently, several authors have successfully extended both continuous and discrete integral inequalities of Hilbert-type and other types to arbitrary time scales, as demonstrated in works like [15–17] and the references they cite.

The purpose of this paper is to establish a new set of reverse inequalities of Hilbert's type within the framework of time scales. These results are analogous to those demonstrated in [13, 14]. The key distinction in our results is that the numerator on the left-hand side is expressed as a sum, whereas previous studies utilized a product of functions.

The structure of this paper is as follows: Section 2 introduces the necessary preliminaries regarding time scale theory and presents the fundamental lemmas required for the proofs. Section 3 is dedicated to proving the principal results of our study. Several examples are given to support our results in Section 4.

2. Preliminaries

We outline the essential concepts and properties on time scales that are relevant to deriving our main results.
For $\tau \in \mathbb{T}$, the forward jump operator $\sigma: \mathbb{T} \to \mathbb{T}$ is defined as:

$$\sigma(\tau) = \inf\{\theta \in \mathbb{T} : \theta > \tau\},$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is given by:

$$\rho(\tau) = \sup\{\theta \in \mathbb{T} : \theta < \tau\}.$$

Based on these, a point $\tau \in \mathbb{T}$ with $\inf \mathbb{T} < \tau < \sup \mathbb{T}$ is classified as:

- **Right-scattered** if $\sigma(\tau) > \tau$.
- **Right-dense** if $\sigma(\tau) = \tau$.
- **Left-scattered** if $\rho(\tau) < \tau$.

• **Left-dense** if $\rho(\tau) = \tau$.
If $\mathbb{T}$ has a left-scattered maximum s_m, then $\mathbb{T}^k = \mathbb{T} - \{s_m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$. The function $\mu : \mathbb{T} \to [0, \infty)$ defined by $\mu(\tau) = \sigma(\tau) - \tau$ is known as the graininess. For a function $\chi : \mathbb{T} \to \mathbb{R}$, the notation $\chi^\sigma(\varsigma)$ refers to $\chi(\sigma(\varsigma))$. In time scales calculus, the notation $\mathcal{I}_\mathbb{T}$ is given by $\mathcal{I}_\mathbb{T} = \mathcal{I} \cap \mathbb{T}$.

For a function $\chi : \mathbb{T} \to \mathbb{R}$, the delta derivative of χ at $\tau \in \mathbb{T}^k$ is defined as follows: $\forall \varepsilon > 0$, where a neighborhood U of τ exists such that

$$\left|\chi(\sigma(\tau)) - \chi(\theta) - \chi^\Delta(\tau)[\sigma(\tau) - \theta]\right| \leq \varepsilon \left|\sigma(\tau) - \theta\right|, \quad \forall \theta \in U.$$

A function χ is delta-differentiable on $\mathbb{T}^k$ if it is delta-differentiable at every $\tau \in \mathbb{T}^k$.

A function $\chi : \mathbb{T} \to \mathbb{R}$ is right-dense continuous (*rd*-continuous) if it is continuous at all right-dense points in $\mathbb{T}$ and has finite left-sided limits at all left-dense points in $\mathbb{T}$. The class of *rd*-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

For $a, b \in \mathbb{T}$, and a delta differentiable function χ, the delta integral of χ^Δ is defined as

$$\int_a^b \chi^\Delta(\tau)\Delta\tau = \chi(b) - \chi(a).$$

The integration by the parts formula is:

$$\int_a^b \chi(\tau)\xi^\Delta(\tau)\Delta\tau = \chi(\tau)\xi(\tau)|_a^b - \int_a^b \chi^\Delta(\tau)\xi^\sigma(\tau)\Delta\tau, \; a, b \in \mathbb{T} \text{ and } \chi, \xi \in C_{rd}([a, b]_\mathbb{T}, \mathbb{R}). \tag{2.1}$$

Note:

• If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(\tau) = \tau, \; \mu(\tau) = 0, \; \chi^\Delta(\tau) = \chi'(\tau) \text{ and } \int_a^b \chi(\tau)\Delta\tau = \int_a^b \chi(\tau)d\tau. \tag{2.2}$$

• If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(\tau) = \tau + 1, \; \mu(\tau) = 1, \; \chi^\Delta(\tau) = \Delta\chi(\tau) \text{ and } \int_a^b \chi(\tau)\Delta\tau = \sum_{\tau=a}^{b-1} \chi(\tau). \tag{2.3}$$

Next, we state the key inequalities and rules.

(1) Hölder's inequality [18, 19] *For $a, b \in \mathbb{T}$, and $\zeta, \chi \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, we have*

$$\int_a^b \zeta(t)\chi(t)\Delta t \leq \left[\int_a^b \zeta^\beta(t)\Delta t\right]^{\beta^{-1}} \left[\int_a^b \chi^\alpha(t)\Delta t\right]^{\alpha^{-1}}, \tag{2.4}$$

where $\beta > 1$ and $\beta^{-1} + \alpha^{-1} = 1$. The inequality reverses for $\beta < 0$ or $0 < \beta < 1$.

(2) Jensen's inequality [18, 20] *For $a, b \in \mathbb{T}$, and $c, d \in \mathbb{R}$. Suppose that $\zeta \in C_{rd}([a, b]_\mathbb{T}, (c, d))$, and $\eta \in C_{rd}([a, b]_\mathbb{T}, \mathbb{R}^+)$ with $\int_a^b \eta(s)\Delta s > 0$. If $\Phi \in C((c, d), \mathbb{R})$ is convex, in which case*

$$\Phi\left(\frac{\int_a^b \eta(s)\zeta(s)\Delta s}{\int_a^b \eta(s)\Delta s}\right) \leq \frac{\int_a^b \eta(s)\Phi(\zeta(s))\Delta s}{\int_a^b \eta(s)\Delta s}. \tag{2.5}$$

The inequality reverses if Φ is concave.

(3) Power rule for integration [21] *For $a, s \in T$ with $s \geq a$, and $\zeta \in C_{rd}([a,s]_{\mathbb{T}}, R)$. If $0 < m \leq 1$, then*

$$\left(\int_a^{\sigma(s)} \zeta(\tau)\Delta\tau\right)^m \geq m\int_a^{\sigma(s)} \zeta(\tau)\left(\int_a^{\sigma(\tau)} \zeta(\varrho)\Delta\varrho\right)^{m-1}\Delta\tau. \tag{2.6}$$

(4) Inequality of means [22] *Let $\lambda > 0$, $w_i > 0$ for $i = 1, 2, ..., n$, and $\sum_{i=1}^m w_i = 1$. In that case*

$$\prod_{i=1}^n s_i^{w_i} \leq \left(\sum_{i=1}^n w_i s_i^\lambda\right)^{\frac{1}{\lambda}}, \quad s_i \geq 0. \tag{2.7}$$

The inequality reverses for $\lambda < 0$.

3. Main results

In this section, we establish our main results.

Theorem 3.1. *Let $a, s, t, x, y \in \mathbb{T}$, $0 < p, q \leq 1$, $r > 0$, $l < 0$, and $\beta < 0$ or $0 < \beta < 1$ with $1/\beta + 1/\alpha = 1$, $f \in C_{rd}([a, x]_{\mathbb{T}}, \mathbb{R}^+)$, and $g \in C_{rd}([a, y]_{\mathbb{T}}, \mathbb{R}^+)$. Suppose that $F(s)$ and $G(t)$ are defined as follows:*

$$F(s) = \int_a^s f(\varrho)\Delta\varrho \text{ for } s \in [a, x]_{\mathbb{T}}, \text{ and } G(t) = \int_a^t g(\tau)\Delta\tau \text{ for } t \in [a, y]_{\mathbb{T}}, \tag{3.1}$$

where $\varrho \in [a, s]_{\mathbb{T}}$ and $\tau \in [a, t]_{\mathbb{T}}$. In this case

$$\begin{aligned} &\int_a^x \int_a^y \frac{\left([F^\sigma(s)]^{pr} + [G^\sigma(t)]^{qr}\right)^{\frac{2}{r}}}{\left((\sigma(s) - a)^{\frac{2l}{\beta}} + (\sigma(t) - a)^{\frac{2l}{\beta}}\right)^{\frac{1}{l}}}\Delta t\Delta s \\ \geq\ & D(r, l, p, q, \beta, x, y)\left(\int_a^x (x - s)\left(f(s)\,[F^\sigma(s)]^{p-1}\right)^\alpha \Delta s\right)^{\frac{1}{\alpha}} \\ & \left(\int_a^y (y - t)\left(g(t)\,[G^\sigma(t)]^{q-1}\right)^\alpha \Delta t\right)^{\frac{1}{\alpha}}, \end{aligned} \tag{3.2}$$

where

$$D(r, l, p, q, \beta, x, y) = (2)^{\frac{2l-r}{lr}}\, pq\,(x - a)^{\frac{1}{\beta}}\,(y - a)^{\frac{1}{\beta}}.$$

Proof. Using inequality (2.6), it follows that

$$\begin{aligned} F^p(\sigma(s)) &= \left(\int_a^{\sigma(s)} f(\varrho)\Delta\varrho\right)^p \geq p\int_a^{\sigma(s)} f(\varrho)\left(\int_a^{\sigma(\varrho)} f(s)\Delta s\right)^{p-1}\Delta\varrho \\ &= p\int_a^{\sigma(s)} f(\varrho)\,[F(\sigma(\varrho))]^{p-1}\,\Delta\varrho, \end{aligned} \tag{3.3}$$

and

$$G^q(\sigma(t)) = \left(\int_a^{\sigma(t)} g(\tau)\Delta\tau\right)^q \geq q\int_a^{\sigma(t)} g(\tau)\left(\int_a^{\sigma(\tau)} g(t)\Delta t\right)^{q-1}\Delta\tau$$

$$= \quad q\int_{a}^{\sigma(t)} g(\tau)\,[G(\sigma(\tau))]^{q-1}\,\Delta\tau. \tag{3.4}$$

Then, we have

$$[F^{p}(\sigma(s))]^{r} \geq \left(p\int_{a}^{\sigma(s)} f(\varrho)\,[F(\sigma(\varrho))]^{p-1}\,\Delta\varrho\right)^{r}, \tag{3.5}$$

and

$$[G^{q}(\sigma(t))]^{r} \geq \left[q\int_{a}^{\sigma(t)} g(\tau)\,[G(\sigma(\tau))]^{q-1}\,\Delta\tau\right]^{r}. \tag{3.6}$$

Adding (3.5) and (3.6), and raising the result to the $(2/r)$-th power $(r > 0)$, we observe that

$$\begin{aligned}
&[F^{pr}(\sigma(s)) + G^{qr}(\sigma(t))]^{\frac{2}{r}} \\
\geq\ & \left\{\left(\left[p\int_{a}^{\sigma(s)} f(\varrho)\,[F^{\sigma}(\varrho)]^{p-1}\,\Delta\varrho\right]^{r} + \left[q\int_{a}^{\sigma(t)} g(\tau)\,[G^{\sigma}(\tau)]^{q-1}\,\Delta\tau\right]^{r}\right)^{\frac{1}{r}}\right\}^{2} \\
=\ & 2^{\frac{2}{r}}\left\{\left(\frac{1}{2}\left[p\int_{a}^{\sigma(s)} f(\varrho)\,[F^{\sigma}(\varrho)]^{p-1}\,\Delta\varrho\right]^{r} + \frac{1}{2}\left[q\int_{a}^{\sigma(t)} g(\tau)\,[G^{\sigma}(\tau)]^{q-1}\,\Delta\tau\right]^{r}\right)^{\frac{1}{r}}\right\}^{2}.
\end{aligned} \tag{3.7}$$

Using the inequality (2.7) of power means, we deduce that

$$s_1^{\omega_1} s_2^{\omega_2} \leq \left(\omega_1 s_1^{\lambda} + \omega_2 s_2^{\lambda}\right)^{\frac{1}{\lambda}}. \tag{3.8}$$

Now, by setting $s_1 = p\int_{a}^{\sigma(s)} f(\varrho)\,[F^{\sigma}(\varrho)]^{p-1}\,\Delta\varrho$, $s_2 = q\int_{a}^{\sigma(t)} g(\tau)\,[G^{\sigma}(\tau)]^{q-1}\,\Delta\tau$, $\omega_1 = \omega_2 = 1/2$, and $\lambda = r > 0$ in (3.8), we get

$$\begin{aligned}
&\left(\frac{1}{2}\left[p\int_{a}^{\sigma(s)} f(\varrho)\,[F^{\sigma}(\varrho)]^{p-1}\,\Delta\varrho\right]^{r} + \frac{1}{2}\left[q\int_{a}^{\sigma(t)} g(\tau)\,[G^{\sigma}(\tau)]^{q-1}\,\Delta\tau\right]^{r}\right)^{\frac{1}{r}} \\
\geq\ & \left(p\int_{a}^{\sigma(s)} f(\varrho)\,[F^{\sigma}(\varrho)]^{p-1}\,\Delta\varrho\right)^{\frac{1}{2}}\left(q\int_{a}^{\sigma(t)} g(\tau)\,[G^{\sigma}(\tau)]^{q-1}\,\Delta\tau\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.7), we have

$$\begin{aligned}
[F^{pr}(\sigma(s)) + G^{qr}(\sigma(t))]^{\frac{2}{r}} \quad \geq \quad & 2^{\frac{2}{r}} pq\left(\int_{a}^{\sigma(s)} f(\varrho)\,[F^{\sigma}(\varrho)]^{p-1}\,\Delta\varrho\right) \\
& \times\left(\int_{a}^{\sigma(t)} g(\tau)\,[G^{\sigma}(\tau)]^{q-1}\,\Delta\tau\right).
\end{aligned} \tag{3.10}$$

Applying the reverse of (2.4) on $\int_{a}^{\sigma(s)} f(\varrho)\,[F^{\sigma}(\varrho)]^{p-1}\,\Delta\varrho$, with β and α, we get

$$\int_{a}^{\sigma(s)} f(\varrho)\,[F^{\sigma}(\varrho)]^{p-1}\,\Delta\varrho \geq (\sigma(s) - a)^{\frac{1}{\beta}}\left(\int_{a}^{\sigma(s)} \left(f(\varrho)\,[F^{\sigma}(\varrho)]^{p-1}\right)^{\alpha}\,\Delta\varrho\right)^{\frac{1}{\alpha}}. \tag{3.11}$$

Similarly, we have

$$\int_a^{\sigma(t)} g(\tau)\,[G^{\sigma}(\tau)]^{q-1}\,\Delta\tau \geq (\sigma(t)-a)^{\frac{1}{\beta}}\left(\int_a^{\sigma(t)}\left(g(\tau)\,[G^{\sigma}(\tau)]^{q-1}\right)^{\alpha}\Delta\tau\right)^{\frac{1}{\alpha}}. \tag{3.12}$$

Using (3.11) and (3.12) in (3.10), we have

$$\begin{aligned}[F^{pr}(\sigma(s))+G^{qr}(\sigma(t))]^{\frac{2}{r}} &\geq 2^{\frac{2}{r}}pq\,(\sigma(s)-a)^{\frac{1}{\beta}}\left(\int_a^{\sigma(s)}\left(f(\varrho)\,[F^{\sigma}(\varrho)]^{p-1}\right)^{\alpha}\Delta\varrho\right)^{\frac{1}{\alpha}} \\ &\quad\times(\sigma(t)-a)^{\frac{1}{\beta}}\left(\int_a^{\sigma(t)}\left(g(\tau)\,[G^{\sigma}(\tau)]^{q-1}\right)^{\alpha}\Delta\tau\right)^{\frac{1}{\alpha}}. \end{aligned}\tag{3.13}$$

Now, setting $s_1=(\sigma(s)-a)^{\frac{2}{\beta}}$, $s_2=(\sigma(t)-a)^{\frac{2}{\beta}}$, $\omega_1=\omega_2=1/2$, and $\lambda=l<0$ in (3.8), we get

$$\left((\sigma(s)-a)^{\frac{2}{\beta}}(\sigma(t)-a)^{\frac{2}{\beta}}\right)^{\frac{1}{2}} \geq \left(\frac{(\sigma(s)-a)^{\frac{2l}{\beta}}}{2}+\frac{(\sigma(t)-a)^{\frac{2l}{\beta}}}{2}\right)^{\frac{1}{l}}.$$

Hence,

$$(\sigma(s)-a)^{\frac{1}{\beta}}(\sigma(t)-a)^{\frac{1}{\beta}} \geq \left(\frac{(\sigma(s)-a)^{\frac{2l}{\beta}}}{2}+\frac{(\sigma(t)-a)^{\frac{2l}{\beta}}}{2}\right)^{\frac{1}{l}}. \tag{3.14}$$

From (3.14) in (3.13), we get

$$\begin{aligned}[F^{pr}(\sigma(s))+G^{qr}(\sigma(t))]^{\frac{2}{r}} &\geq 2^{\frac{2}{r}}pq\left(\frac{1}{2}\right)^{\frac{1}{l}}\left((\sigma(s)-a)^{\frac{2l}{\beta}}+(\sigma(t)-a)^{\frac{2l}{\beta}}\right)^{\frac{1}{l}} \\ &\quad\times\left(\int_a^{\sigma(s)}\left(f(\varrho)\,[F^{\sigma}(\varrho)]^{p-1}\right)^{\alpha}\Delta\varrho\right)^{\frac{1}{\alpha}} \\ &\quad\times\left(\int_a^{\sigma(t)}\left(g(\tau)\,[G^{\sigma}(\tau)]^{q-1}\right)^{\alpha}\Delta\tau\right)^{\frac{1}{\alpha}}. \end{aligned}\tag{3.15}$$

Dividing (3.15) by $\left((\sigma(s)-a)^{\frac{2l}{\beta}}+(\sigma(t)-a)^{\frac{2l}{\beta}}\right)^{\frac{1}{l}}>0$, we get

$$\begin{aligned} &\frac{[F^{pr}(\sigma(s))+G^{qr}(\sigma(t))]^{\frac{2}{r}}}{\left((\sigma(s)-a)^{\frac{2l}{\beta}}+(\sigma(t)-a)^{\frac{2l}{\beta}}\right)^{\frac{1}{l}}} \\ &\geq (2)^{\frac{2l-r}{lr}}pq\left(\int_a^{\sigma(s)}\left(f(\varrho)\,[F^{\sigma}(\varrho)]^{p-1}\right)^{\alpha}\Delta\varrho\right)^{\frac{1}{\alpha}} \\ &\quad\times\left(\int_a^{\sigma(t)}\left(g(\tau)\,[G^{\sigma}(\tau)]^{q-1}\right)^{\alpha}\Delta\tau\right)^{\frac{1}{\alpha}}. \end{aligned}\tag{3.16}$$

Integrating both sides of (3.16), we obtain

$$\int_a^x\int_a^y \frac{[F^{pr}(\sigma(s))+G^{qr}(\sigma(t))]^{\frac{2}{r}}}{\left((\sigma(s)-a)^{\frac{2l}{\beta}}+(\sigma(t)-a)^{\frac{2l}{\beta}}\right)^{\frac{1}{l}}}\Delta t\Delta s$$

$$\geq \quad (2)^{\frac{2l-r}{lr}}\, pq\left(\int_a^x \left[\int_a^{\sigma(s)} \left(f(\varrho)\,[F^\sigma(\varrho)]^{p-1}\right)^\alpha \Delta\varrho\right]^{\frac{1}{\alpha}} \Delta s\right)$$
$$\times\left(\int_a^y \left[\int_a^{\sigma(t)} \left(g(\tau)\,[G^\sigma(\tau)]^{q-1}\right)^\alpha \Delta\tau\right]^{\frac{1}{\alpha}} \Delta t\right). \qquad (3.17)$$

Applying the reverse of (2.4) to $\int_a^x \left[\int_a^{\sigma(s)} \left(f(\varrho)\,[F^\sigma(\varrho)]^{p-1}\right)^\alpha \Delta\varrho\right]^{\frac{1}{\alpha}} \Delta s$ with β and α, we get

$$\int_a^x \left[\int_a^{\sigma(s)} \left(f(\varrho)\,[F^\sigma(\varrho)]^{p-1}\right)^\alpha \Delta\varrho\right]^{\frac{1}{\alpha}} \Delta s \geq (x-a)^{\frac{1}{\beta}} \left(\int_a^x \int_a^{\sigma(s)} \left(f(\varrho)\,[F^\sigma(\varrho)]^{p-1}\right)^\alpha \Delta\varrho\Delta s\right)^{\frac{1}{\alpha}}. \qquad (3.18)$$

Similarly, we have

$$\int_a^y \left[\int_a^{\sigma(t)} \left(g(\tau)\,[G^\sigma(\tau)]^{q-1}\right)^\alpha \Delta\tau\right]^{\frac{1}{\alpha}} \Delta t \geq (y-a)^{\frac{1}{\beta}} \left(\int_a^y \int_a^{\sigma(t)} \left(g(\tau)\,[G^\sigma(\tau)]^{q-1}\right)^\alpha \Delta\tau\Delta t\right)^{\frac{1}{\alpha}}. \qquad (3.19)$$

Using (3.18) and (3.19) in (3.17), we get

$$\int_a^x \int_a^y \frac{[F^{pr}(\sigma(s)) + G^{qr}(\sigma(t))]^{\frac{2}{r}}}{\left((\sigma(s)-a)^{\frac{2l}{\beta}} + (\sigma(t)-a)^{\frac{2l}{\beta}}\right)^{\frac{1}{l}}} \Delta t\Delta s$$
$$\geq \quad (2)^{\frac{2l-r}{lr}}\, pq\,(x-a)^{\frac{1}{\beta}} \left(\int_a^x \int_a^{\sigma(s)} \left(f(\varrho)\,[F^\sigma(\varrho)]^{p-1}\right)^\alpha \Delta\varrho\Delta s\right)^{\frac{1}{\alpha}}$$
$$\times\,(y-a)^{\frac{1}{\beta}} \left(\int_a^y \int_a^{\sigma(t)} \left(g(\tau)\,[G^\sigma(\tau)]^{q-1}\right)^\alpha \Delta\tau\Delta t\right)^{\frac{1}{\alpha}}. \qquad (3.20)$$

Applying (2.1) to $\int_a^x \left(\int_a^{\sigma(s)} \left(f(\varrho)\,[F^\sigma(\varrho)]^{p-1}\right)^\alpha \Delta\varrho\right)\Delta s$ with

$$u^\sigma(s) = \int_a^{\sigma(s)} \left(f(\varrho)\,[F^\sigma(\varrho)]^{p-1}\right)^\alpha \Delta\varrho \quad \text{and} \quad v^\Delta(s) = 1,$$

we get

$$\int_a^x \int_a^{\sigma(s)} \left(f(\varrho)\,[F^\sigma(\varrho)]^{p-1}\right)^\alpha \Delta\varrho\Delta s$$
$$= \quad v(s)\int_a^s \left(f(\varrho)\,[F^\sigma(\varrho)]^{p-1}\right)^\alpha \Delta\varrho\Big|_a^x - \int_a^x (s-x)\left(f(s)\,[F^\sigma(s)]^{p-1}\right)^\alpha \Delta s$$
$$= \quad \int_a^x (x-s)\left(f(s)\,[F^\sigma(s)]^{p-1}\right)^\alpha \Delta s, \qquad (3.21)$$

where $v(s) = s - x$. Similarly, we have

$$\int_a^y \int_a^{\sigma(t)} \left(g(\tau)\,[G^\sigma(\tau)]^{q-1}\right)^\alpha \Delta\tau\Delta t = \int_a^y (y-t)\left(g(t)\,[G^\sigma(t)]^{q-1}\right)^\alpha \Delta t. \qquad (3.22)$$

Using (3.21) and (3.22) in (3.20), we have

$$\begin{aligned}
&\int_a^x \int_a^y \frac{\left[F^{pr}(\sigma(s)) + G^{qr}(\sigma(t))\right]^{\frac{2}{r}}}{\left[(\sigma(s)-a)^{\frac{2l}{\beta}} + (\sigma(t)-a)^{\frac{2l}{\beta}}\right]^{\frac{1}{l}}} \Delta t \Delta s \\
\geq\ & (2)^{\frac{2l-r}{lr}} pq\,(x-a)^{\frac{1}{\beta}} \left(\int_a^x (x-s)\left(f(s)\left[F^{\sigma}(s)\right]^{p-1}\right)^{\alpha} \Delta s\right)^{\frac{1}{\alpha}} \\
& \times (y-a)^{\frac{1}{\beta}} \left(\int_a^y (y-t)\left(g(t)\left[G^{\sigma}(t)\right]^{q-1}\right)^{\alpha} \Delta t\right)^{\frac{1}{\alpha}} \\
=\ & D(r,l,p,q,\beta,x,y) \left(\int_a^x (x-s)\left(f(s)\left[F^{\sigma}(s)\right]^{p-1}\right)^{\alpha} \Delta s\right)^{\frac{1}{\alpha}} \\
& \times \left(\int_a^y (y-t)\left(g(t)\left[G^{\sigma}(t)\right]^{q-1}\right)^{\alpha} \Delta t\right)^{\frac{1}{\alpha}},
\end{aligned}$$

which is (3.2). □

In the subsequent theorems, we assume the existence of two functions, Φ and Ψ, which are real-valued, non-negative, concave, and supermultiplicative functions defined on $\mathbb{R}^+$. (A function G is termed supermultiplicative if it satisfies: $G(\varsigma\tau) \geq G(\varsigma)\, G(\tau)$, $\forall \varsigma, \tau \in \mathbb{R}^+$.)

Theorem 3.2. *Let $a, s, t, x, y \in \mathbb{T}$, $r, l, \beta, \alpha, f, g, F,$ and G be as in Theorem 3.1. Suppose that $h(\varrho)$ for $\varrho \in [a, s]_{\mathbb{T}}$, and $k(\tau)$ for $\tau \in [a, t]_{\mathbb{T}}$ are positive rd-continuous functions. Define*

$$H(s) = \int_a^s h(\varrho)\Delta\varrho \quad \text{and} \quad K(t) = \int_a^t k(\tau)\Delta\tau. \tag{3.23}$$

Then, for $s \in [a, x]_{\mathbb{T}}$ and $t \in [a, y]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_a^x \int_a^y \frac{\left(\left[\Phi(F^{\sigma}(s))\right]^r + \left[\Psi(G^{\sigma}(t))\right]^r\right)^{\frac{2}{r}}}{\left[(\sigma(s)-a)^{\frac{2l}{\beta}} + (\sigma(t)-a)^{\frac{2l}{\beta}}\right]^{\frac{1}{l}}} \Delta t \Delta s \\
\geq\ & E(r,l,\beta,x,y) \left(\int_a^x (x-s)\left(h(s)\Phi\left[\frac{f(s)}{h(s)}\right]\right)^{\alpha} \Delta s\right)^{\frac{1}{\alpha}} \\
& \times \left(\int_a^y (y-t))\left(k(t)\Psi\left[\frac{g(t)}{k(t)}\right]\right)^{\alpha} \Delta t\right)^{\frac{1}{\alpha}},
\end{aligned} \tag{3.24}$$

where

$$E(r,l,\beta,x,y) = (2)^{\frac{2l-r}{lr}} \left(\int_a^x \left(\frac{\Phi(H^{\sigma}(s))}{H^{\sigma}(s)}\right)^{\beta} \Delta s\right)^{\frac{1}{\beta}} \left(\int_a^y \left(\frac{\Psi(K^{\sigma}(t))}{K^{\sigma}(t)}\right)^{\beta} \Delta t\right)^{\frac{1}{\beta}}. \tag{3.25}$$

Proof. On the basis of the given assumptions and applying inverse Jensen's inequality, we have

$$\Phi(F^{\sigma}(s)) = \Phi\left(\frac{H^{\sigma}(s) \int_a^{\sigma(s)} h(\varrho)\frac{f(\varrho)}{h(\varrho)}\Delta\varrho}{\int_a^{\sigma(s)} h(\varrho)\Delta\varrho}\right)$$

$$\geq \Phi(H^\sigma(s))\Phi\left(\frac{\int_a^{\sigma(s)} h(\varrho)\frac{f(\varrho)}{h(\varrho)}\Delta\varrho}{\int_a^{\sigma(s)} h(\varrho)\Delta\varrho}\right)$$

$$\geq \frac{\Phi(H^\sigma(s))}{H^\sigma(s)}\int_a^{\sigma(s)} h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\Delta\varrho. \tag{3.26}$$

Similarly,

$$\Psi(G^\sigma(t)) \geq \frac{\Psi(K^\sigma(t))}{K^\sigma(t)}\int_a^{\sigma(t)} k(\tau)\Psi\left[\frac{g(\tau)}{k(\tau)}\right]\Delta\tau. \tag{3.27}$$

Using (3.26) and (3.27), and applying the mean inequality and the reverse of (2.4), we get for $r > 0$ and $l < 0$

$$\begin{aligned}
\left([\Phi(F^\sigma(s))]^r + [\Psi(G^\sigma(t))]^r\right)^{\frac{2}{r}} &\geq \left[\begin{array}{c}\left(\frac{\Phi(H^\sigma(s))}{H^\sigma(s)}\int_a^{\sigma(s)} h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\Delta\varrho\right)^r \\ +\left(\frac{\Psi(K^\sigma(t))}{K(\sigma(t))}\int_a^{\sigma(t)} k(\tau)\Psi\left[\frac{g(\tau)}{k(\tau)}\right]\Delta\tau\right)^r\end{array}\right]^{\frac{2}{r}} \\
&\geq 2^{\frac{2}{r}}\left(\frac{\Phi(H^\sigma(s))}{H^\sigma(s)}\int_a^{\sigma(s)} h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\Delta\varrho\right) \\
&\quad\times\left(\frac{\Psi(K^\sigma(t))}{K^\sigma(t)}\int_a^{\sigma(t)} k(\tau)\Psi\left[\frac{g(\tau)}{k(\tau)}\right]\Delta\tau\right) \\
&\geq 2^{\frac{2}{r}}\frac{\Phi(H^\sigma(s))}{H^\sigma(s)}(\sigma(s)-a)^{\frac{1}{\beta}}\left(\int_a^{\sigma(s)}\left(h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\right)^\alpha\Delta\varrho\right)^{\frac{1}{\alpha}} \\
&\quad\times\frac{\Psi(K^\sigma(t))}{K^\sigma(t)}(\sigma(t)-a)^{\frac{1}{\beta}}\left(\int_a^{\sigma(t)}\left(k(\tau)\Psi\left[\frac{g(\tau)}{k(\tau)}\right]\right)^\alpha\Delta\tau\right)^{\frac{1}{\alpha}} \\
&\geq 2^{\frac{2}{r}}\left(\frac{1}{2}\right)^{\frac{1}{l}}\left[(\sigma(s)-a)^{\frac{2l}{\beta}} + (\sigma(t)-a)^{\frac{2l}{\beta}}\right]^{\frac{1}{l}} \\
&\quad\times\frac{\Phi(H^\sigma(s))}{H^\sigma(s)}\left(\int_a^{\sigma(s)}\left(h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\right)^\alpha\Delta\varrho\right)^{\frac{1}{\alpha}} \\
&\quad\times\frac{\Psi(K^\sigma(t))}{K^\sigma(t)}\left(\int_a^{\sigma(t)}\left(k(\tau)\Psi\left[\frac{g(\tau)}{k(\tau)}\right]\right)^\alpha\Delta\tau\right)^{\frac{1}{\alpha}}.
\end{aligned} \tag{3.28}$$

Dividing both sides of (3.28) by $\left[(\sigma(s)-a)^{\frac{2l}{\beta}} + (\sigma(t)-a)^{\frac{2l}{\beta}}\right]^{\frac{1}{l}} > 0$, we obtain

$$\begin{aligned}
\frac{\left([\Phi(F^\sigma(s))]^r + [\Psi(G^\sigma(t))]^r\right)^{\frac{2}{r}}}{\left[(\sigma(s)-a)^{\frac{2l}{\beta}} + (\sigma(t)-a)^{\frac{2l}{\beta}}\right]^{\frac{1}{l}}} &\geq 2^{\frac{2l-r}{rl}}\frac{\Phi(H^\sigma(s))}{H^\sigma(s)}\left(\int_a^{\sigma(s)}\left(h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\right)^\alpha\Delta\varrho\right)^{\frac{1}{\alpha}} \\
&\quad\times\frac{\Psi(K^\sigma(t))}{K^\sigma(t)}\left(\int_a^{\sigma(t)}\left(k(\tau)\Psi\left[\frac{g(\tau)}{k(\tau)}\right]\right)^\alpha\Delta\tau\right)^{\frac{1}{\alpha}}.
\end{aligned} \tag{3.29}$$

Integrating both sides of (3.29), we get

$$\int_a^x\int_a^y \frac{\left([\Phi(F^\sigma(s))]^r + [\Psi(G^\sigma(t))]^r\right)^{\frac{2}{r}}}{\left[(\sigma(s)-a)^{\frac{2l}{\beta}} + (\sigma(t)-a)^{\frac{2l}{\beta}}\right]^{\frac{1}{l}}}\Delta t\Delta s$$

$$
\begin{aligned}
&\geq 2^{\frac{2l-r}{rl}}\left(\int_a^x \frac{\Phi(H^\sigma(s))}{H^\sigma(s)}\left(\int_a^{\sigma(s)}\left(h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\right)^\alpha \Delta\varrho\right)^{\frac{1}{\alpha}}\Delta s\right)\\
&\quad\times\left(\int_a^y \frac{\Psi(K^\sigma(t))}{K^\sigma(t)}\left(\int_a^{\sigma(t)}\left(k(\tau)\Psi\left[\frac{g(\tau)}{k(\tau)}\right]\right)^\alpha \Delta\tau\right)^{\frac{1}{\alpha}}\Delta t\right). \qquad (3.30)
\end{aligned}
$$

Applying the reverse of (2.4) to $\int_a^x \frac{\Phi(H^\sigma(s))}{H^\sigma(s)}\left(\int_a^{\sigma(s)}\left(h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\right)^\alpha \Delta\varrho\right)^{\frac{1}{\alpha}}\Delta s$, with α and β, we obtain

$$
\begin{aligned}
&\int_a^x \frac{\Phi(H^\sigma(s))}{H^\sigma(s)}\left(\int_a^{\sigma(s)}\left(h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\right)^\alpha \Delta\varrho\right)^{\frac{1}{\alpha}}\Delta s\\
\geq\ &\int_a^x\left(\left(\frac{\Phi(H^\sigma(s))}{H^\sigma(s)}\right)^\beta \Delta s\right)^{\frac{1}{\beta}}\left(\int_a^x\left(\int_a^{\sigma(s)}\left(h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\right)^\alpha \Delta\varrho\right)\Delta s\right)^{\frac{1}{\alpha}}. \qquad (3.31)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_a^y \frac{\Psi(K^\sigma(t))}{K^\sigma(t)}\left(\int_a^{\sigma(t)}\left(k(\tau)\Psi\left[\frac{g(\tau)}{k(\tau)}\right]\right)^\alpha \Delta\tau\right)^{\frac{1}{\alpha}}\Delta t\\
\geq\ &\int_a^y\left(\left(\frac{\Psi(K^\sigma(t))}{K^\sigma(t)}\right)^\beta \Delta t\right)^{\frac{1}{\beta}}\left(\int_a^y\left(\int_a^{\sigma(t)}\left(k(\tau)\Psi\left[\frac{g(\tau)}{k(\tau)}\right]\right)^\alpha \Delta\tau\right)\Delta t\right)^{\frac{1}{\alpha}}. \qquad (3.32)
\end{aligned}
$$

Using (3.31) and (3.32) in (3.30), we get

$$
\begin{aligned}
&\int_a^x\int_a^y \frac{\left([\Phi(F^\sigma(s))]^r+[\Psi(G^\sigma(t))]^r\right)^{\frac{2}{r}}}{\left[(\sigma(s)-a)^{\frac{2l}{\beta}}+(\sigma(t)-a)^{\frac{2l}{\beta}}\right]^{\frac{1}{l}}}\Delta t\Delta s\\
\geq\ &2^{\frac{2l-r}{rl}}\int_a^x\left(\left(\frac{\Phi(H^\sigma(s))}{H^\sigma(s)}\right)^\beta \Delta s\right)^{\frac{1}{\beta}}\left(\int_a^x\left(\int_a^{\sigma(s)}\left(h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\right)^\alpha \Delta\varrho\right)\Delta s\right)^{\frac{1}{\alpha}}\\
&\times\int_a^y\left(\left(\frac{\Psi(K^\sigma(t))}{K^\sigma(t)}\right)^\beta \Delta t\right)^{\frac{1}{\beta}}\left(\int_a^y\left(\int_a^{\sigma(t)}\left(k(\tau)\Psi\left[\frac{g(\tau)}{k(\tau)}\right]\right)^\alpha \Delta\tau\right)\Delta t\right)^{\frac{1}{\alpha}}\\
=\ &E(r,l,\beta,x,y)\left(\int_a^x\left(\int_a^{\sigma(s)}\left(h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\right)^\alpha \Delta\varrho\right)\Delta s\right)^{\frac{1}{\alpha}}\\
&\times\left(\int_a^y\left(\int_a^{\sigma(t)}\left(k(\tau)\Psi\left[\frac{g(\tau)}{k(\tau)}\right]\right)^\alpha \Delta\tau\right)\Delta t\right)^{\frac{1}{\alpha}}. \qquad (3.33)
\end{aligned}
$$

Applying integration by parts on $\int_a^x\left(\int_a^{\sigma(s)}\left(h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\right)^\alpha \Delta\varrho\right)\Delta s$ with

$$
u(\sigma(s)) = \int_a^{\sigma(s)}\left(h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\right)^\alpha \Delta\varrho \quad\text{and}\quad v^\Delta(s)=1,
$$

we obtain

$$
\int_a^x\left(\int_a^{\sigma(s)}\left(h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\right)^\alpha \Delta\varrho\right)\Delta s \;=\; v(s)\left(\int_a^s\left(h(\varrho)\Phi\left[\frac{f(\varrho)}{h(\varrho)}\right]\right)^\alpha \Delta\varrho\right)\Bigg|_a^x
$$

$$- \int_a^x (s-x)\left(h(s)\Phi\left[\frac{f(s)}{h(s)}\right]\right)^{\alpha}\Delta s$$
$$= \int_a^x (x-s)\left(h(s)\Phi\left[\frac{f(s)}{h(s)}\right]\right)^{\alpha}\Delta s, \quad (3.34)$$

where $v(s) = s - x$.

Similarly, we have

$$\int_a^y \left(\int_a^{\sigma(t)} \left(k(\tau)\Psi\left[\frac{g(\tau)}{k(\tau)}\right]\right)^{\beta}\Delta\tau\right)\Delta t = \int_a^y (y-t)\left(k(t)\Psi\left[\frac{g(t)}{k(t)}\right]\right)^{\beta}\Delta t. \quad (3.35)$$

Using (3.34) and (3.35) in (3.33), we have

$$\int_a^x\int_a^y \frac{([\Phi(F^{\sigma}(s))]^r + [\Psi(G^{\sigma}(t))]^r)^{\frac{2}{r}}}{\left[(\sigma(s)-a)^{\frac{2l}{\beta}} + (\sigma(t)-a)^{\frac{2l}{\alpha}}\right]^{\frac{1}{l}}}\Delta t\Delta s$$
$$\geq E(r,l,\beta,x,y)\left(\int_a^x (x-s)\left(h(s)\Phi\left[\frac{f(s)}{h(s)}\right]\right)^{\alpha}\Delta s\right)^{\frac{1}{\alpha}}$$
$$\times\left(\int_a^y (y-t)\left(k(t)\Psi\left[\frac{g(t)}{k(t)}\right]\right)^{\beta}\Delta t\right)^{\frac{1}{\beta}},$$

which is (3.24). □

The following theorems deal with slight variants of the inequality (3.24) given in Theorem 3.2.

Theorem 3.3. *Let $s, t, x, y, a \in \mathbb{T}$, r, l, β, α, f, and g be as in Theorem 3.1, and let H, K, $h(\varrho)$, and $k(\tau)$ be as in Theorem 3.2. Define*

$$F(s) = \frac{1}{H(s)}\int_a^s h(\varrho)f(\varrho)\Delta\varrho \quad \text{and} \quad G(t) = \frac{1}{K(t)}\int_a^t k(\tau)g(\tau)\Delta\tau. \quad (3.36)$$

Then for $s \in [a,x]_{\mathbb{T}}$ and $t \in [a,y]_{\mathbb{T}}$, we have

$$\int_a^x\int_a^y \frac{([\Phi(F^{\sigma}(s))H^{\sigma}(s)]^r + [\Psi(G^{\sigma}(t)K^{\sigma}(t)]^r)^{\frac{2}{r}}}{\left((\sigma(s)-a)^{\frac{2l}{\beta}} + (\sigma(t)-a)^{\frac{2l}{\beta}}\right)^{\frac{1}{l}}}\Delta t\Delta s$$
$$\geq Q(r,l,\beta,x,y)\left(\int_a^x (x-s)\,[h(s)\Phi(f(s))]^{\alpha}\,\Delta s\right)^{\frac{1}{\alpha}}$$
$$\times\left(\int_a^y (y-t)\,[k(t)\Psi(g(t))]^{\alpha}\,\Delta t\right)^{\frac{1}{\alpha}}, \quad (3.37)$$

where

$$Q(r,l,\beta,x,y) = (2)^{\frac{2l-r}{lr}}(x-a)^{\frac{1}{\beta}}(y-a)^{\frac{1}{\beta}}.$$

Proof. Using the hypotheses of Theorem 3.3 and the inverse Jensen's inequality, we find that

$$\begin{aligned}\Phi(F^{\sigma}(s)) &= \Phi\left(\frac{1}{H^{\sigma}(s)}\int_a^{\sigma(s)} h(\varrho)f(\varrho)\Delta\varrho\right)\\ &\geq \frac{1}{H^{\sigma}(s)}\int_a^{\sigma(s)} h(\varrho)\Phi(f(\varrho))\Delta\varrho. \end{aligned}\tag{3.38}$$

From (3.38), we get

$$\Phi(F^{\sigma}(s))H^{\sigma}(s) \geq \int_a^{\sigma(s)} h(\varrho)\Phi(f(\varrho))\Delta\varrho. \tag{3.39}$$

Analogously,

$$\Psi(G^{\sigma}(t)K^{\sigma}(t) \geq \int_a^{\sigma(t)} k(\tau)\Psi(g(\tau))\Delta\tau. \tag{3.40}$$

We then have

$$[\Phi(F^{\sigma}(s))H^{\sigma}(s)]^r \geq \left(\int_a^{\sigma(s)} h(\varrho)\Phi(f(\varrho))\Delta\varrho\right)^r, \tag{3.41}$$

and

$$[\Psi(G^{\sigma}(t)K^{\sigma}(t)]^r \geq \left(\int_a^{\sigma(t)} k(\tau)\Psi(g(\tau))\Delta\tau\right)^r. \tag{3.42}$$

Combining (3.41) and (3.42) and raising the result to the $(2/r)$-th power $(r > 0)$, we get

$$\begin{aligned}&([\Phi(F^{\sigma}(s))H^{\sigma}(s)]^r + [\Psi(G^{\sigma}(t)K^{\sigma}(t)]^r)^{\frac{2}{r}}\\ \geq\ & \left\{\left[\left(\int_a^{\sigma(s)} h(\varrho)\Phi(f(\varrho))\Delta\varrho\right)^r + \left(\int_a^{\sigma(t)} k(\tau)\Psi(g(\tau))\Delta\tau\right)^r\right]^{\frac{1}{r}}\right\}^2\\ =\ & 2^{\frac{2}{r}}\left\{\left[\frac{1}{2}\left(\int_a^{\sigma(s)} h(\varrho)\Phi(f(\varrho))\Delta\varrho\right)^r + \frac{1}{2}\left(\int_a^{\sigma(t)} k(\tau)\Psi(g(\tau))\Delta\tau\right)^r\right]^{\frac{1}{r}}\right\}^2. \end{aligned}\tag{3.43}$$

By applying (3.8) to the right-hand side of (3.43), we observe

$$\begin{aligned}&([\Phi(F^{\sigma}(s))H^{\sigma}(s)]^r + [\Psi(G^{\sigma}(t)K^{\sigma}(t)]^r)^{\frac{2}{r}}\\ \geq\ & 2^{\frac{2}{r}}\left(\int_a^{\sigma(s)} h(\varrho)\Phi(f(\varrho))\Delta\varrho\right)\left(\int_a^{\sigma(t)} k(\tau)\Psi(g(\tau))\Delta\tau\right). \end{aligned}\tag{3.44}$$

Applying the reverse of (2.4) to $\int_a^{\sigma(s)} h(\varrho)\Phi(f(\varrho))\Delta\varrho$, with β and α, we get

$$\int_a^{\sigma(s)} h(\varrho)\Phi(f(\varrho))\Delta\varrho \geq (\sigma(s)-a)^{\frac{1}{\beta}}\left(\int_a^{\sigma(s)} [h(\varrho)\Phi(f(\varrho))]^{\alpha}\Delta\varrho\right)^{\frac{1}{\alpha}}. \tag{3.45}$$

Similarly, we obtain

$$\int_a^{\sigma(t)} k(\tau)\Psi(g(\tau))\Delta\tau \geq (\sigma(t)-a)^{\frac{1}{\beta}}\left(\int_a^{\sigma(t)} [k(\tau)\Psi(g(\tau))]^{\alpha}\Delta\tau\right)^{\frac{1}{\alpha}}. \tag{3.46}$$

Using (3.45) and (3.46) in (3.44), we have

$$\begin{aligned}
&\left([\Phi(F^{\sigma}(s))H^{\sigma}(s)]^{r}+[\Psi(G^{\sigma}(t)K^{\sigma}(t)]^{r}\right)^{\frac{2}{r}} \\
\geq\ & 2^{\frac{2}{r}}(\sigma(s)-a)^{\frac{1}{\beta}}\left(\int_{a}^{\sigma(s)}[h(\varrho)\Phi(f(\varrho))]^{\alpha}\Delta\varrho\right)^{\frac{1}{\alpha}} \\
&\times(\sigma(t)-a)^{\frac{1}{\beta}}\left(\int_{a}^{\sigma(t)}[k(\tau)\Psi(g(\tau))]^{\alpha}\Delta\tau\right)^{\frac{1}{\alpha}}. \qquad (3.47)
\end{aligned}$$

Using (3.14) in (3.47), we get

$$\begin{aligned}
&\left([\Phi(F^{\sigma}(s))H^{\sigma}(s)]^{r}+[\Psi(G^{\sigma}(t)K^{\sigma}(t)]^{r}\right)^{\frac{2}{r}} \\
\geq\ & 2^{\frac{2}{r}}\left(\frac{1}{2}\right)^{\frac{1}{l}}\left((\sigma(s)-a)^{\frac{2l}{\beta}}+(\sigma(t)-a)^{\frac{2l}{\beta}}\right)^{\frac{1}{l}} \\
&\times\left(\int_{a}^{\sigma(s)}[h(\varrho)\Phi(f(\varrho))]^{\alpha}\Delta\varrho\right)^{\frac{1}{\alpha}}\left(\int_{a}^{\sigma(t)}[k(\tau)\Psi(g(\tau))]^{\alpha}\Delta\tau\right)^{\frac{1}{\alpha}}. \qquad (3.48)
\end{aligned}$$

Dividing (3.48) by $\left((\sigma(s)-a)^{\frac{2l}{\beta}}+(\sigma(t)-a)^{\frac{2l}{\beta}}\right)^{\frac{1}{l}}>0$, we have

$$\begin{aligned}
&\frac{\left([\Phi(F^{\sigma}(s))H^{\sigma}(s)]^{r}+[\Psi(G^{\sigma}(t)K^{\sigma}(t)]^{r}\right)^{\frac{2}{r}}}{\left((\sigma(s)-a)^{\frac{2l}{\beta}}+(\sigma(t)-a)^{\frac{2l}{\beta}}\right)^{\frac{1}{l}}} \\
\geq\ & (2)^{\frac{2l-r}{lr}}\left(\int_{a}^{\sigma(s)}[h(\varrho)\Phi(f(\varrho))]^{\alpha}\Delta\varrho\right)^{\frac{1}{\alpha}}\left(\int_{a}^{\sigma(t)}[k(\tau)\Psi(g(\tau))]^{\alpha}\Delta\tau\right)^{\frac{1}{\alpha}}. \qquad (3.49)
\end{aligned}$$

Integrating both sides of (3.49), we obetain

$$\begin{aligned}
&\int_{a}^{x}\int_{a}^{y}\frac{\left([\Phi(F^{\sigma}(s))H^{\sigma}(s)]^{r}+[\Psi(G^{\sigma}(t)K^{\sigma}(t)]^{r}\right)^{\frac{2}{r}}}{\left((\sigma(s)-a)^{\frac{2l}{\beta}}+(\sigma(t)-a)^{\frac{2l}{\beta}}\right)^{\frac{1}{l}}}\Delta t\Delta s \\
\geq\ & (2)^{\frac{2l-r}{lr}}\left(\int_{a}^{x}\left(\int_{a}^{\sigma(s)}[h(\varrho)\Phi(f(\varrho))]^{\alpha}\Delta\varrho\right)^{\frac{1}{\alpha}}\Delta s\right) \\
&\times\left(\int_{a}^{y}\left(\int_{a}^{\sigma(t)}[k(\tau)\Psi(g(\tau))]^{\alpha}\Delta\tau\right)^{\frac{1}{\alpha}}\Delta t\right). \qquad (3.50)
\end{aligned}$$

By again applying the reveres of (2.4) to $\int_{a}^{x}\left(\int_{a}^{\sigma(s)}[h(\varrho)\Phi(f(\varrho))]^{\alpha}\Delta\varrho\right)^{\frac{1}{\alpha}}\Delta s$ with β and α, we get

$$\int_{a}^{x}\left(\int_{a}^{\sigma(s)}[h(\varrho)\Phi(f(\varrho))]^{\alpha}\Delta\varrho\right)^{\frac{1}{\alpha}}\Delta s\geq(x-a)^{\frac{1}{\beta}}\left(\int_{a}^{x}\int_{a}^{\sigma(s)}[h(\varrho)\Phi(f(\varrho))]^{\alpha}\Delta\varrho\Delta s\right)^{\frac{1}{\alpha}}. \qquad (3.51)$$

Similarly, we have

$$\int_{a}^{y}\left(\int_{a}^{\sigma(t)}[k(\tau)\Psi(g(\tau))]^{\alpha}\Delta\tau\right)^{\frac{1}{\alpha}}\Delta t\geq(y-a)^{\frac{1}{\beta}}\left(\int_{a}^{y}\int_{a}^{\sigma(t)}[k(\tau)\Psi(g(\tau))]^{\alpha}\Delta\tau\Delta t\right)^{\frac{1}{\alpha}}. \qquad (3.52)$$

Using (3.51) and (3.52) in (3.50), we obtain

$$\begin{aligned}
&\int_a^x \int_a^y \frac{\left([\Phi(F^\sigma(s))H^\sigma(s)]^r + [\Psi(G^\sigma(t)K^\sigma(t)]^r\right)^{\frac{2}{r}}}{\left((\sigma(s)-a)^{\frac{2l}{\beta}} + (\sigma(t)-a)^{\frac{2l}{\beta}}\right)^{\frac{1}{l}}} \Delta t \Delta s \\
\geq\ & (2)^{\frac{2l-r}{lr}} (x-a)^{\frac{1}{\beta}} \left(\int_a^x \int_a^{\sigma(s)} [h(\varrho)\Phi(f(\varrho))]^\alpha \Delta\varrho \Delta s \right)^{\frac{1}{\alpha}} \\
& \times (y-a)^{\frac{1}{\beta}} \left(\int_a^y \int_a^{\sigma(t)} [k(\tau)\Psi(g(\tau))]^\alpha \Delta\tau \Delta t \right)^{\frac{1}{\alpha}}. \qquad (3.53)
\end{aligned}$$

Applying intgration by parts on $\int_a^x \int_a^{\sigma(s)} [h(\varrho)\Phi(f(\varrho))]^\alpha \Delta\varrho\Delta s$ with

$$u^\sigma(s) = \int_a^{\sigma(s)} [h(\varrho)\Phi(f(\varrho))]^\alpha \Delta\varrho \quad \text{and} \quad v^\Delta(s) = 1,$$

we get

$$\begin{aligned}
&\int_a^x \int_a^{\sigma(s)} [h(\varrho)\Phi(f(\varrho))]^\alpha \Delta\varrho\Delta s \\
=\ & v(s) \int_a^s [h(\varrho)\Phi(f(\varrho))]^\alpha \Delta\varrho \Big|_a^x - \int_a^x (s-x) [h(s)\Phi(f(s))]^\alpha \Delta s \\
=\ & \int_a^x (x-s) [h(s)\Phi(f(s))]^\alpha \Delta s, \qquad (3.54)
\end{aligned}$$

where $v(s) = s - x$. Similarly, we have

$$\int_a^y \int_a^{\sigma(t)} [k(\tau)\Psi(g(\tau))]^\alpha \Delta\tau\Delta t = \int_a^y (y-t) [k(t)\Psi(g(t))]^\alpha \Delta t. \qquad (3.55)$$

Using (3.54) and (3.55) in (3.53), we find

$$\begin{aligned}
&\int_a^x \int_a^y \frac{\left([\Phi(F^\sigma(s))H^\sigma(s)]^r + [\Psi(G^\sigma(t)K^\sigma(t)]^r\right)^{\frac{2}{r}}}{\left((\sigma(s)-a)^{\frac{2l}{\beta}} + (\sigma(t)-a)^{\frac{2l}{\beta}}\right)^{\frac{1}{l}}} \Delta t \Delta s \\
\geq\ & (2)^{\frac{2l-r}{lr}} (x-a)^{\frac{1}{\beta}} \left(\int_a^x (x-s) [h(s)\Phi(f(s))]^\alpha \Delta s \right)^{\frac{1}{\alpha}} \\
& \times (y-t_0)^{\frac{1}{\beta}} \left(\int_a^y (y-t) [k(t)\Psi(g(t))]^\alpha \Delta t \right)^{\frac{1}{\alpha}} \\
=\ & Q(r, l, \beta, x, y) \left(\int_a^x (x-s) [h(s)\Phi(f(s))]^\alpha \Delta s \right)^{\frac{1}{\alpha}} \\
& \times \left(\int_a^y (y-t) [k(t)\Psi(g(t))]^\alpha \Delta t \right)^{\frac{1}{\alpha}}.
\end{aligned}$$

This completes the proof. □

4. Examples

To illustrate the results, we state the corresponding theorems given in the previous sections for the special cases of $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

Example 4.1. *Given $\mathbb{T} = \mathbb{R}$ in Theorem 3.1, $a, s, t, x, y \in \mathbb{R}$, $r > 0$, $l < 0$, $0 < p, q \le 1$, and $\beta < 0$ or $0 < \beta < 1$ such that $1/\beta + 1/\alpha = 1$. Suppose that $F(s)$ and $G(t)$ are given by:*

$$F(s) = \int_a^s f(\varrho)d\varrho \text{ for } s \in [a, x] \text{ and } G(t) = \int_a^t g(\tau)d\tau \text{ for } t \in [a, y],$$

where $\varrho \in [a, s]$ and $\tau \in [a, t]$. In that case

$$\begin{aligned} & \int_a^x \int_a^y \frac{\left[\left(\frac{F^{pr}(s)+G^{qr}(t)}{2}\right)^{\frac{1}{r}}\right]^2}{\left[\frac{(s-a)^{\frac{2l}{\beta}}+(t-a)^{\frac{2l}{\beta}}}{2}\right]^{\frac{1}{l}}} dtds \\ \geq \;& pq\,(x-a)^{\frac{1}{\beta}} \left(\int_a^x (x-s)\left(f(s)\,[F(s)]^{p-1}\right)^{\alpha} ds\right)^{\frac{1}{\alpha}} \\ & \times (y-a)^{\frac{1}{\beta}} \left(\int_a^y (y-t)\left(g(t)\,[G(t)]^{q-1}\right)^{\alpha} dt\right)^{\frac{1}{\alpha}}. \end{aligned} \tag{4.1}$$

Remark 4.2. *In the context of inequality (4.1), consider the limits as $r \to 0^+$ and $l \to 0^-$. We find that*

$$\begin{aligned} & \lim_{r\to 0}\left(\frac{F^{pr}(s)+G^{qr}(t)}{2}\right)^{\frac{1}{r}} \\ = \;& \exp\left[\lim_{r\to 0}\left(\frac{1}{r}\ln\frac{F^{pr}(s)+G^{qr}(t)}{2}\right)\right] \\ = \;& \exp\left[\lim_{r\to 0}\left(\frac{1}{F^{pr}(s)+G^{qr}(t)}.\frac{d}{dr}(F^{pr}(s)+G^{qr}(t))\right)\right] \\ = \;& [F^p(s).G^q(t)]^{\frac{1}{2}}, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} & \lim_{l\to 0}\left(\frac{(s-a)^{\frac{2l}{\beta}}+(t-a)^{\frac{2l}{\beta}}}{2}\right)^{\frac{1}{l}} \\ = \;& \exp\left[\lim_{l\to 0}\left(\frac{1}{l}\ln\frac{(s-a)^{\frac{2l}{\beta}}+(t-a)^{\frac{2l}{\beta}}}{2}\right)\right] \\ = \;& \exp\left[\lim_{l\to 0}\left(\frac{1}{(s-a)^{\frac{2l}{\beta}}+(t-a)^{\frac{2l}{\beta}}}.\frac{d}{dl}((s-a)^{\frac{2l}{\beta}}+(t-a)^{\frac{2l}{\beta}})\right)\right] \\ = \;& (s-a)^{\frac{1}{\beta}}.(t-a)^{\frac{1}{\beta}}. \end{aligned} \tag{4.3}$$

Thus, incorporating these results into (4.1), we get:

$$\begin{aligned}
&\int_a^x \int_a^y \frac{F^p(s)\cdot G^q(t)}{(s-a)^{\frac{1}{\beta}}\cdot (t-a)^{\frac{1}{\beta}}} dtds \\
\geq\ & pq\,(x-a)^{\frac{1}{\beta}} \left[\int_a^x (x-s)\left(f(s)\,[F(s)]^{p-1}\right)^{\alpha} ds\right]^{\frac{1}{\alpha}} \\
&\times (y-a)^{\frac{1}{\beta}} \left[\int_a^y (y-t)\left(g(t)\,[G(t)]^{q-1}\right)^{\alpha} dt\right]^{\frac{1}{\alpha}}.
\end{aligned}$$

This result can be recognized as an inverse inequality similar to the reverse Hilbert-type inequality (1.1) established by Zhao [6, Theorem 2.1, for $a = 0$].

Example 4.3. *Consider $\mathbb{T} = \mathbb{Z}$ in Theorem 3.1, with $a, s, t, x, y \in \mathbb{Z}$, $r > 0$, $l < 0$, $0 < p, q \leq 1$, and $\beta < 0$ or $0 < \beta < 1$ such that $1/\beta + 1/\alpha = 1$. Let $\{f_s\}$ and $\{g_t\}$ be non-negative sequences of real numbers, and define*

$$F_s = \sum_{n=a}^{s-1} f_n \quad \text{and} \quad G_t = \sum_{\tau=a}^{t-1} g_\tau.$$

Then

$$\begin{aligned}
&\sum_{s=a}^{x-1}\sum_{t=a}^{y-1} \frac{[F^{pr}(s+1)+G^{qr}(t+1)]^{\frac{2}{r}}}{\left[(s+1-a)^{\frac{2l}{\beta}}+(t+1-a)^{\frac{2l}{\beta}}\right]^{\frac{1}{l}}} \\
\geq\ & (2)^{\frac{2l-r}{lr}}\, pq\,(x-a)^{\frac{1}{\beta}}\,(y-a)^{\frac{1}{\beta}} \left(\sum_{s=a}^{x-1}(x-s)\left(f(s)\,[F(s+1)]^{p-1}\right)^{\alpha}\right)^{\frac{1}{\alpha}} \\
&\times\left(\sum_{t=a}^{y-1}(y-t)\left(g(t)\,[G(t+1)]^{q-1}\right)^{\alpha}\right)^{\frac{1}{\alpha}}. \qquad (4.4)
\end{aligned}$$

Example 4.4. *Let $\mathbb{T} = \mathbb{R}$ in Theorem 3.2, where $s, t, x, y, a \in \mathbb{R}$, r, l, β, α, f, g, F, and G are as in Example 4.1. Assume that $h(\varrho)$, for $\varrho \in [a, s]$ and $k(\tau)$ for $\tau \in [a, t]$ are two positive functions. Define*

$$H(s) = \int_a^s h(\varrho)d\varrho \quad \text{and} \quad K(t) = \int_a^t k(\tau)d\tau.$$

Then for $s \in [a, x]$ and $t \in [a, y]$, we have

$$\begin{aligned}
&\int_a^x\int_a^y \frac{\left[\left(\frac{[\Phi(F(s))]^r+[\Psi(G(t))]^r}{2}\right)^{\frac{1}{r}}\right]^2}{\left[\frac{(s-a)^{\frac{2l}{\beta}}+(t-a)^{\frac{2l}{\beta}}}{2}\right]^{\frac{1}{l}}} dtds \\
\geq\ & E_0(\beta, x, y)\left(\int_a^x (x-s)\left(h(s)\Phi\left[\frac{f(s)}{h(s)}\right]\right)^{\alpha} ds\right)^{\frac{1}{\alpha}} \\
&\times\left(\int_a^y (y-t))\left(k(t)\Psi\left[\frac{g(t)}{k(t)}\right]\right)^{\alpha} dt\right)^{\frac{1}{\alpha}}, \qquad (4.5)
\end{aligned}$$

where

$$E_0(\beta,x,y)=\left(\int_a^x\left(\frac{\Phi(H(s))}{H(s)}\right)^{\beta}ds\right)^{\frac{1}{\beta}}\left(\int_a^y\left(\frac{\Psi(K(t))}{K(t)}\right)^{\beta}dt\right)^{\frac{1}{\beta}}.$$

Remark 4.5. *In (4.5), let $r\to 0^+$ and $l\to 0^-$. We then note that*

$$\begin{aligned}
&\lim_{r\to 0}\left(\frac{[\Phi(F(s))]^r+[\Psi(G(t))]^r}{2}\right)^{\frac{1}{r}}\\
=\;&\exp\left[\lim_{r\to 0}\left(\frac{1}{r}\ln\frac{[\Phi(F(s))]^r+[\Psi(G(t))]^r}{2}\right)\right]\\
=\;&\exp\left[\lim_{r\to 0}\frac{1}{[\Phi(F(s))]^r+[\Psi(G(t))]^r}\cdot\frac{d}{dr}\left([\Phi(F(s))]^r+[\Psi(G(t))]^r\right)\right]\\
=\;&[\Phi(F(s))\Psi(G(t)]^{\frac{1}{2}}.
\end{aligned}\tag{4.6}$$

Hence, from (4.3) and (4.6), we get

$$\begin{aligned}
&\int_a^x\int_a^y\frac{\Phi(F(s))\cdot\Psi(G(t)}{(s-a)^{\frac{1}{\beta}}\cdot(t-a)^{\frac{1}{\beta}}}dtds\\
\geq\;&E_0(\beta,x,y)\left(\int_a^x(x-s)\left(h(s)\Phi\left[\frac{f(s)}{h(s)}\right]\right)^{\alpha}ds\right)^{\frac{1}{\alpha}}\\
&\times\left(\int_a^y(y-t))\left(k(t)\Psi\left[\frac{g(t)}{k(t)}\right]\right)^{\alpha}dt\right)^{\frac{1}{\alpha}}.
\end{aligned}$$

This is just a reversed version of the Hilbert-type inequality established by Zhao [6, Theorem 2.2, for $a=0$].

Example 4.6. *Suppose that $\mathbb{T}=\mathbb{Z}$ in Theorem 3.2, where $s,t,x,y,a\in\mathbb{Z}$, $r,l,\beta,\alpha,\{f_s\},\{g_t\},F_s$, and G_t are as in Example 4.3. Assume that $\{h_s\}$ and $\{k_t\}$ are two positive sequences of real numbers. Define*

$$H_s=\sum_{n=a}^{s-1}h_n\quad\text{and}\quad K_t=\sum_{\tau=a}^{t-1}k_\tau.$$

Then

$$\begin{aligned}
&\sum_{s=a}^{x-1}\sum_{t=a}^{y-1}\frac{([\Phi(F(s+1))]^r+[\Psi(G(t+1))]^r)^{\frac{2}{r}}}{\left[(s+1-a)^{\frac{2l}{\beta}}+(t+1-a)^{\frac{2l}{\alpha}}\right]^{\frac{1}{l}}}\\
\geq\;&E_1(r,l,\alpha,\beta,x,y)\left(\sum_{s=a}^{x-1}(x-s)\left(h(s)\Phi\left[\frac{f(s)}{h(s)}\right]\right)^{\alpha}\right)^{\frac{1}{\alpha}}\\
&\times\left(\sum_{t=a}^{y-1}(y-t))\left(k(t)\Psi\left[\frac{g(t)}{k(t)}\right]\right)^{\alpha}\right)^{\frac{1}{\alpha}},
\end{aligned}$$

where

$$E_1(r,l,\beta,x,y)=(2)^{\frac{2l-r}{lr}}\left(\sum_{s=a}^{x}\left(\frac{\Phi(H(s+1))}{H(s+1)}\right)^{\beta}\right)^{\frac{1}{\beta}}\left(\sum_{t=a}^{y}\left(\frac{\Psi(K(t+1))}{K(t+1)}\right)^{\beta}\right)^{\frac{1}{\beta}}.$$

Example 4.7. *Given $\mathbb{T}=\mathbb{R}$ in Theorem 3.3, where $s,t,x,y,a\in\mathbb{R}$, r, l, β, α, f, and g, are as in Example 4.1. Assume that H, K, $h(\varrho)$, and $k(\tau)$ are as in Corollary 4.4 and define*

$$F(s)=\frac{1}{H(s)}\int_a^s h(\varrho)f(\varrho)d\varrho \quad \text{and} \quad G(t)=\frac{1}{K(t)}\int_a^t k(\tau)g(\tau)d\tau.$$

Then, for $s\in[a,x]$ and $t\in[a,y]$, we have

$$\begin{aligned}
&\int_a^x\int_a^y \frac{\left[\left(\frac{[\Phi(F(s))H(s)]^r+[\Psi(G(t)K(t)]^r}{2}\right)^{\frac{1}{r}}\right]^2}{\left[\frac{(s-a)^{\frac{2l}{\beta}}+(t-a)^{\frac{2l}{\beta}}}{2}\right]^{\frac{1}{l}}}dtds\\
\geq\;& (x-a)^{\frac{1}{\beta}}(y-a)^{\frac{1}{\beta}}\left(\int_a^x (x-s)\left[h(s)\Phi(f(s))\right]^{\alpha}ds\right)^{\frac{1}{\alpha}}\\
&\times\left(\int_a^y (y-t)\left[k(t)\Psi(g(t))\right]^{\alpha}dt\right)^{\frac{1}{\alpha}}. \qquad (4.7)
\end{aligned}$$

Remark 4.8. *In the context of inequality (4.7), as $r\to 0^+$ and $l\to 0^-$, we observe that*

$$\begin{aligned}
&\lim_{r\to 0}\left(\frac{[\Phi(F(s))H(s)]^r+[\Psi(G(t)K(t)]^r}{2}\right)^{\frac{1}{r}}\\
=\;&\exp\left[\lim_{r\to 0}\left(\frac{1}{r}\ln\frac{[\Phi(F(s))H(s)]^r+[\Psi(G(t)K(t)]^r}{2}\right)\right]\\
=\;&\exp\left[\lim_{r\to 0}\frac{1}{[\Phi(F(s))H(s)]^r+[\Psi(G(t)K(t)]^r}\cdot\frac{d}{dr}\left([\Phi(F(s))H(s)]^r+[\Psi(G(t)K(t)]^r\right)\right]\\
=\;&[H(s)K(t)\Phi(F(s))\Psi(G(t)]^{\frac{1}{2}}. \qquad (4.8)
\end{aligned}$$

Thus, from (4.3) and (4.8), we derive

$$\begin{aligned}
&\int_a^x\int_a^y \frac{H(s)K(t)\Phi(F(s))\Psi(G(t)}{(s-a)^{\frac{1}{\beta}}\cdot(t-a)^{\frac{1}{\beta}}}dtds\\
\geq\;& (x-a)^{\frac{1}{\beta}}(y-a)^{\frac{1}{\beta}}\left(\int_a^x (x-s)\left[h(s)\Phi(f(s))\right]^{\alpha}ds\right)^{\frac{1}{\alpha}}\\
&\times\left(\int_a^y (y-t)\left[k(t)\Psi(g(t))\right]^{\alpha}dt\right)^{\frac{1}{\alpha}}.
\end{aligned}$$

This result corresponds to a reversed version of the Hilbert-type inequality, as established by Zhao [6, Theorem 2.3, for $a=0$.].

Example 4.9. *Given $\mathbb{T}=\mathbb{Z}$ in Theorem 3.3, with $s,t,x,y,a\in\mathbb{Z}$ and r, l, β, α, $\{f_s\}$, and $\{g_t\}$ are as described Example 4.3. Assume the sequences $\{h_s\}$, $\{k_t\}$, H_s, and K_t are as outlined in Example 4.6. Define*

$$F_s=\frac{1}{H_s}\sum_{n=a}^{s-1}h_nf_n \quad \text{and} \quad G_t=\frac{1}{K_t}\sum_{\tau=a}^{t-1}k_\tau g_\tau.$$

Then

$$\sum_{s=a}^{x}\sum_{t=a}^{y}\frac{\left([\Phi(F(s+1))H(s+1)]^{r}+[\Psi(G(t+1)K(t+1)]^{r}\right)^{\frac{2}{r}}}{\left((s+1-a)^{\frac{2l}{\beta}}+(t+1-a)^{\frac{2l}{\beta}}\right)^{\frac{1}{l}}}$$
$$\leq (2)^{\frac{2l-r}{lr}}(x-a)^{\frac{1}{\beta}}(y-a)^{\frac{1}{\beta}}\left(\sum_{s=a}^{x}(x-s)\left[h_s\Phi(f_s)\right]^{\alpha}\right)^{\frac{1}{\alpha}}$$
$$\times\left(\sum_{t=a}^{y}(y-t)\left[k_t\Psi(g_t)\right]^{\alpha}\right)^{\frac{1}{\alpha}},$$

where

$$E_1(r,l,\beta,x,y)=(2)^{\frac{2l-r}{lr}}\left(\sum_{s=a}^{x}\left(\frac{\Phi(H(s+1))}{H(s+1)}\right)^{\beta}\right)^{\frac{1}{\beta}}\left(\sum_{t=a}^{y}\left(\frac{\Psi(K(t+1))}{K(t+1)}\right)^{\beta}\right)^{\frac{1}{\beta}}.$$

5. Conclusions

In this article, we utilized the inverse Hölder's inequality and inverse Jensen's inequality to discuss and prove several novel generalizations of inverse Hilbert-type inequalities. Our exploration covered both discrete and continuous forms of these inequalities, providing a unified approach that bridges the gap between different calculus settings.

For future work, we plan to further generalize these results by applying alpha-conformable fractional derivatives on time scales, allowing for a deeper exploration of fractional calculus within this framework. Additionally, we aim to extend our findings to the context of diamond-alpha calculus.

Author contributions

Haytham M. Rezk: Investigation, software, supervision, writing-original draft; Mohammed Zakarya: Writing-review editing, funding; Amirah Ayidh I Al-Thaqfan: Writing-review editing, funding; Maha Ali: Writing-review editing, funding; Belal A. Glalah: Investigation, software, writing-original draft. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

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