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Research article

Exploring weighted Tsallis extropy: Insights and applications to human health

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Abstract: This article presents the notion of the continuous case of the weighted Tsallis extropy function as an information measure that follows the framework of continuous distribution. We introduce this concept from two perspectives, depending on the extropy and weighted Tsallis entropy. Various examples to illustrate the two perspectives of the weighted Tsallis extropy by examining a few of its characteristics are presented. Some features and stochastic orders of those measures, including the maximum value, are introduced. An alternative depiction of the proposed models concerning the hazard rate function is provided. Furthermore, the order statistics of the weighted Tsallis extropy and their lower bounds are considered. Moreover, the bivariate Tsallis extropy and its weighted version are derived. Non-parametric estimators are also derived for the new measures under cancer-related fatalities in the European Union countries data. Additionally, a pattern recognition comparison between Tsallis extropy and weighted Tsallis extropy is presented.

Keywords: extropy; weighted Tsallis entropy; hazard rate function; stochastic orders; order statistics; non-parametric estimation
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1. Introduction

In the continuous case, weighted entropy extends the concept of entropy to probability density functions (PDFs), where each data point or event is associated with a weight or importance factor. The formulas for the entropy and weighted entropy measures of a continuous random variable (RV) X, which follows a corresponding PDF $g(x)$ and weight function $w(x)$, are given respectively, as shown in

Shannon [1] and Guiasu [2], by:

$$H(X) = -\int_{-\infty}^{\infty} g(x)\ln(g(x))\,dx,$$

$$H_w(X) = -\int_{-\infty}^{\infty} w(x)g(x)\ln(g(x))\,dx.$$

A weighted measure of entropy in the continuous case is used in a number of disciplines, including information theory, statistics, and data analysis, to quantify uncertainty while considering the importance or significance of different data points or events within a continuous distribution.

The weighted Tsallis entropy represents an extended measure of the Tsallis entropy that incorporates weights or importance factors for different events or outcomes in a probability distribution. The Tsallis entropy, introduced by Tsallis [3], is a model extension of Shannon entropy [1], and is widely employed in information theory and statistical physics (e.g., see Mohamed et al. [4], Behera et al. [5], and Nicolis et al. [6]). In the setting of an RV X established across $\mathbb{R}$, the continuous status of Tsallis measure of entropy is obtained by

$$Tn_\beta(X) = \frac{1}{\beta - 1}\left(1 - \int_0^{\infty} g^{\beta}(x)dx\right),\ 1 \neq \beta > 0.$$

Recently, much literature has considered the Tsallis entropy in many applications. A non-parametric kernel-class estimator for the Tsallis entropy was suggested by Maya et al. [7], where the observations in question manifest a ρ-mixing dependence requirement. Moreover, under appropriate regularity conditions, they demonstrated the estimator's asymptotic characteristics. In order to harmonize the definitions of Shannon, fractional entropies, and Tsallis, Balakrishnan et al. [8] put out a new concept. This was referred to as the unified formulation of entropy or fractional Tsallis entropy.

In the continuous case, with weighted PDF $g_w(x) = \frac{x}{\mathcal{E}(X)}g(x)$, where $\mathcal{E}(X) = \int_0^{\infty} xg(x)dx$ is the expected value, Das [9] has presented the notion of weighted Tsallis entropy as follows:

$$\begin{aligned} WTn_\lambda(X) &= \frac{1}{\lambda - 1}\left(\int_0^{\infty} g_w(x)\,dx - \int_0^{\infty} (g_w(x))^{\lambda}\ dx\right) \\ &= \frac{1}{\lambda - 1}\left(1 - \int_0^{\infty}\left(\frac{x}{\mathcal{E}(X)}g(x)\right)^{\lambda}\ dx\right). \end{aligned} \tag{1.1}$$

The weighted Tsallis entropy is used in many branches to quantify the degree of uncertainty measure or information content in a distribution while considering the relative importance of different data points or events. The selection of the entropic index λ can affect the sensitivity of the entropy measure to the tails of the distribution, making it a flexible tool for analyzing various types of data.

For the case of continuous RV X, Lad et al. [10] presented entropy's complementary dual, which is known as extropy, by replacing the PDF $g(x)$ with $1 - g(x)$, which is given by

$$Ex^*(X) = -\int_0^{\infty} (1 - g(x))\ln(1 - g(x))dx. \tag{1.2}$$

Lad et al. [10] approximated (1.2) to the following

$$Ex(X) = -\frac{1}{2}\int_0^{\infty} g^2(x)dx, \tag{1.3}$$

which is the most commonly used form for the extropy measure in the literature for the case of a continuous RV X; see, for example, Raqab and Qiu [11] and Qiu [12].

The extropy-based form of Tsallis entropy, or Tsallis extropy measure, as thoroughly examined by Balakrishnan et al. [13], can also be explored. The Tsallis extropy under the discrete situation of an RV X with probability vector $P = (p_1, \ldots, p_N)$ and support S of cardinality N is provided by

$$\begin{aligned} DTx_\lambda(P) &= \frac{1}{\lambda - 1} \sum_{j=1}^{N} (1 - p_j) \left[1 - (1 - p_j)^{\lambda - 1}\right] \\ &= \frac{1}{\lambda - 1} \left(N - 1 - \sum_{j=1}^{N} (1 - p_j)^\lambda \right), \end{aligned} \tag{1.4}$$

where $\lambda > 0$, $\lambda \neq 1$. Moreover, Buono et al. [14] presented a unified formulation of extropy known as fractional Tsallis extropy, which is given by

$$DFTx_{\lambda,r}(P) = \frac{1}{\lambda - 1} \sum_{j=1}^{N} (1 - p_j) \left[1 - (1 - p_j)^{\lambda - 1}\right] \left[-\log(1 - p_j)\right]^{r-1},$$

where $\lambda > 0$, $\lambda \neq 1$, and $0 < r \leq 1$. If $r = 1$, then $DFTx_{\lambda,1}(P)$ returns to $DTx_\lambda(P)$ given in (1.4). Furthermore, if $r = 1$ and $\lambda \to 1$, then $DFTx_{\lambda,1}(P)$ returns to the discrete case of extropy measure.

Mohamed et al. [15] (see also Mohamed et al. [16]) introduced the Tsallis extropy follows the continuous case of an RV X with the support (α,β), where $-\infty < \alpha < \beta < \infty$, by the expression

$$\begin{aligned} Tx_\lambda(X) &= \frac{1}{\lambda - 1} \left(\int_\alpha^\beta (1 - g(x))dx - \int_\alpha^\beta (1 - g(x))^\lambda dx \right) \\ &= \frac{1}{\lambda - 1} \left(\beta - \alpha - 1 - \int_\alpha^\beta (1 - g(x))^\lambda dx \right). \end{aligned} \tag{1.5}$$

The conditions concerning λ can be encapsulated as detailed below

$$\Lambda = \begin{cases} \lambda \neq 1, \lambda > 0 \ when \ g(x) \ \leq 1, \\ \lambda \in \mathbb{Z}^+ \backslash \{1\} \ when \ g(x) > 1. \end{cases} \tag{1.6}$$

Studying the weighted version of uncertainty measures holds a prominent place in the literature, especially the weighted extropy and its extensions. Abdul Sathar and Nair [17] have introduced the weighted extropy by

$$WEx(X) = -\frac{1}{2} \int_0^\infty x g^2(x) dx. \tag{1.7}$$

Moreover, they discussed the weighted past and residual extropy measures. Additionally, the weighted extropy, weighted past, and weighted residual extropies were defined by Balakrishnan et al. [18]. They provided various characterization findings and limits under reversed hazard functions and monotonicity of hazard. Bivariate and weighted variants of extropy were also shown. Gupta and Chaudhary [19] introduced the characterization results, monotone properties, and stochastic comparison of the measure of general weighted extropy of ranked set sampling. Chaudhary et al. [20] proposed the weighted versions of negative cumulative extropy and cumulative residual extropy measures. The non-parametric estimation of weighted extropy and its extensions also take attention. Irshad et al. [21] considered some non-parametric estimators of weighted extropy based on kernel density estimator; see also Abdul Sathar and Nair [17]. In the case of ordered variables such as order statistics as well as k-record values, Bansal and Gupta [22] investigated the weighted residual extropy and weighted extropy and examined their monotone qualities, characterization findings, and a few other properties. This research for the past extropy measure was also examined.

This paper introduces the weighted Tsallis extropy, a measure of uncertainty for continuous RVs that incorporates both the PDF and qualitative weights of values. It presents two formulations: one based on the extropy measure (Eq (1.2)) and another on a modified Tsallis entropy (Eq (1.1)). The paper includes examples comparing Tsallis extropy and its weighted variant, discusses convergence limitations, and explores stochastic order comparisons. Key results include the derivation of maximum weighted Tsallis extropy using Lagrange multipliers, expressions for k-th order statistics based on the beta distribution, and extensions to bivariate models. Applications also involve kernel-based non-parametric estimation and classification problems.

1.1. Key innovations and applications

The weighted Tsallis extropy and its bivariate extensions provide insights into complex systems with applications in pattern recognition, reliability analysis, financial modeling, survival analysis, information theory, and signal processing, showcasing their versatility across diverse domains.

The structure of the rest portion of this consideration is as follows. In Section 2, we extract the two concepts of the weighted Tsallis extropy from two different perspectives. Furthermore, different examples of the two perspectives of the weighted Tsallis exteropies are presented. The properties of those measures are studied together and separately. In Section 3, the weighted Tsallis extropy of order statistics is given. Section 4 deals with bivariate Tsallis extropy and its weighted version. In Section 5, we examine non-parametric estimators for the given measures. In addition, we compare the Tsallis extropy and weighted Tsallis extropy in addressing the classification problem using pattern recognition.

2. Weighted Tsallis extropy

In parallel with the ideas of both weighted entropy and extropy, we can define the weighted Tsallis extropy based on two perspectives.

Definition 2.1. *Given that X is an RV defined in* (α,β)*, as* $-\infty < \alpha < \beta < \infty$*, follows a PDF* $g(.)$*. In that case, the weighted version for Tsallis extropy can be provided in two manners as outlined below:*

1) *According to the extropy perspective given in (1.2), the weighted Tsallis extropy could be obtained by replacing the PDF* $g(x)$ *in (1.5) by* $g_w(x) = \frac{xg(x)}{\mathcal{E}(X)}$ *as follows:*

$$\begin{aligned} WT_\lambda^1(X) &= \frac{1}{\lambda-1}\left(\int_\alpha^\beta (1-g_w(x))dx - \int_\alpha^\beta (1-g_w(x))^\lambda dx\right) \\ &= \frac{1}{\lambda-1}\left(\beta-\alpha-1-\int_\alpha^\beta \left(1-\frac{xg(x)}{\mathcal{E}(X)}\right)^\lambda dx\right). \end{aligned} \tag{2.1}$$

The conditions concerning λ can be encapsulated as detailed below

$$\Lambda_1 = \begin{cases} \lambda \neq 1, \lambda > 0 \text{ when } g_w(x) \ \leq 1, \\ \lambda \in \mathbb{Z}^+ \backslash \{1\} \text{ when } g_w(x) > 1. \end{cases} \tag{2.2}$$

2) *The weighted Tsallis entropy presented in (1.1) can be reformulated, for simplicity, as follows:*

$$WTn^*_\lambda(X) = \frac{1}{\lambda - 1}\left(\int_0^\infty \frac{x}{\mathcal{E}(X)} g(x)\, dx - \int_0^\infty \frac{x}{\mathcal{E}(X)} g^\lambda(x)\, dx\right), \tag{2.3}$$

noting that this form is in the same manner as defining the weighted extropy in (1.7), where the weight is not raised to a power. According to the reformulated weighted Tsallis entropy given in (2.3), the weighted Tsallis extropy could be obtained by replacing the PDF $g(x)$ in (2.3) by $1 - g(x)$ as follows

$$\begin{aligned} WT^2_\lambda(X) &= \frac{1}{\lambda - 1}\left(\int_\alpha^\beta \frac{x(1 - g(x))}{\mathcal{E}(X)} dx - \int_\alpha^\beta \frac{x(1 - g(x))^\lambda}{\mathcal{E}(X)} dx\right) \\ &= \frac{1}{\lambda - 1}\left(\frac{\beta^2 - \alpha^2}{2\mathcal{E}(X)} - 1 - \int_\alpha^\beta \frac{x}{\mathcal{E}(X)} (1 - g(x))^\lambda dx\right), \end{aligned} \tag{2.4}$$

where the conditions on λ are given in (1.6).

In the next examples, we provide two instances of distributions possessing identical Tsallis extropy and study the varying of weighted Tsallis extropy.

Example 2.1. *Consider that the RV X follows a power distribution function, and its PDF is provided as follows:*

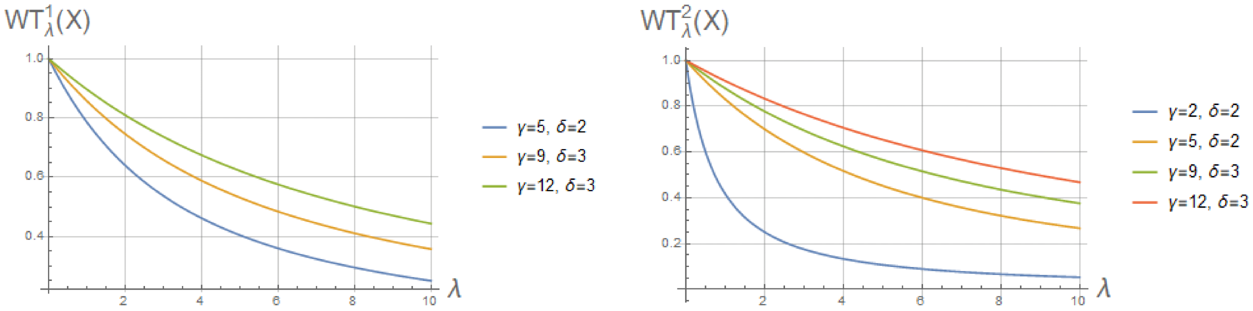
$$g(x) = \frac{\delta x^{(\delta - 1)}}{\gamma^\delta}, 0 \leq x \leq \gamma, \text{ and } \delta, \gamma > 0.$$

With noting that $\mathcal{E}(X) = \frac{\gamma\,\delta}{1+\delta}$. Then, from (2.1) and (2.4), we can see that

$$WT^1_\lambda(X) = \frac{1}{\lambda - 1}\left(\gamma - 1 - \int_0^\gamma \left(1 - x^\delta \gamma^{(-1-\delta)}(1 + \delta)\right)^\lambda dx\right),$$

$$WT^2_\lambda(X) = \frac{1}{\lambda - 1}\left(\frac{\gamma(1 + \delta)}{2\delta} - 1 - \int_0^\gamma \frac{x(1 + \delta)}{\gamma\,\delta}\left(1 - \frac{\delta x^{(\delta - 1)}}{\gamma^\delta}\right)^\lambda dx\right).$$

Figure 1 shows the weighted Tsallis extropies $WT^1_\lambda(X)$ and $WT^2_\lambda(X)$ for a power function distribution across various γ and δ values. Additionally, it is observable that as the disparity between γ and δ expands, the weighted Tsallis extropies also increase.



(a) (b)

Figure 1. The weighted Tsallis extropies of power function distribution (a) $WT^1_\lambda(X)$, and (b) $WT^2_\lambda(X)$.

Example 2.2. *Provided that the two RVs X_1 and X_2 follow the uniform distributions $U(0,\beta)$ and $U(\alpha,\alpha+\beta)$, respectively, in considering that $\alpha,\beta > 0$, the PDFs $g_{X_1}(x) = \frac{1}{\beta}$ and $g_{X_2}(x) = \frac{1}{\beta}$. Thus, from (1.5), we have*

$$Tx_\lambda(X_1) = Tx_\lambda(X_2) = \frac{1}{\lambda-1}\left(\beta - 1 - \beta\left(1-\frac{1}{\beta}\right)^\lambda\right).$$

Moreover, for the weighted Tsallis extropy given in (2.4), we note that

$$WT^2_\lambda(X_1) = WT^2_\lambda(X_2) = \frac{1}{\lambda-1}\left(\beta - 1 - \beta\left(1-\frac{1}{\beta}\right)^\lambda\right),$$

which are equal. But for the weighted Tsallis extropy given in (2.1) with $X_1 \sim U(0,4)$ and $X_2 \sim U(2,4)$, we note that $WT^1_\lambda(X_1) = \frac{9}{16}$ and $WT^1_\lambda(X_2) = \frac{81}{128}$ which are different.

Remark 2.1. *For the uniform distribution $U(\alpha,\beta)$, we note that*

$$Tx_\lambda(X) = WT^2_\lambda(X) = \frac{1}{\lambda-1}\left(\beta-\alpha-1+(\beta-\alpha)\left(1+\frac{1}{\beta-\alpha}\right)^\lambda\right),$$

which are equal.

Example 2.3. *Suppose that the two RVs X_1 and X_2 follow the PDFs $g_{X_1}(x) = \frac{1+x}{4}$, $0 \le x \le 2$, and $g_{X_2}(x) = 1 - \frac{1+x}{4}$, $0 \le x \le 2$, respectively. Thus, from (1.5), we have $Tx_\lambda(X_1) = Tx_\lambda(X_2) = \frac{11}{24}$, which are equal. But for the weighted Tsallis extropies given in (2.1) and (2.4), we note that $WT^1_\lambda(X_1) = \frac{53}{245}$, $WT^1_\lambda(X_2) = \frac{53}{125}$, $WT^2_\lambda(X_1) = \frac{11}{28}$ and $WT^2_\lambda(X_2) = \frac{11}{20}$ which are different.*

Example 2.4. *Provided that the RV X follows a piece-wise constant PDF*

$$g(x) = \sum_{q=1}^{n} d_q \boldsymbol{I}_{[q-1,q)}(x),$$

in considering that $d_q \geq 0$, $q = 1, ..., n$, $\sum_{q=1}^{n} d_q = 1$ and the function of the indicator of x on the interval $[q-1, q)$ is $\boldsymbol{1}_{[q-1,q)}(x)$. Thus, the Tsallis form of extropy and the weighted Tsallis extropy measure of X are, respectively,

$$\begin{aligned} Tx_\lambda(X) &= \frac{1}{\lambda - 1}\left[n - 1 - \int_0^n \left(1 - \sum_{q=1}^{n} d_q \boldsymbol{1}_{[q-1,q)}(x)\right)^\lambda dx\right] \\ &= \frac{1}{\lambda - 1}\left[n - 1 - \sum_{k=0}^{\Delta 2_\lambda} \binom{\lambda}{k}(-1)^k \int_0^n \left(\sum_{q=1}^{n} d_q \boldsymbol{1}_{[q-1,q)}(x)\right)^k dx\right] \\ &= \frac{1}{\lambda - 1}\left[n - 1 - \sum_{k=0}^{\Delta 2_\lambda} \binom{\lambda}{k}(-1)^k \sum_{q=1}^{n} \int_{q-1}^{q} d_q^k dx\right] \\ &= \frac{1}{\lambda - 1}\left[n - 1 - \sum_{k=0}^{\Delta 2_\lambda} \binom{\lambda}{k}(-1)^k \sum_{q=1}^{n} d_q^k\right], \end{aligned}$$

$$WT_\lambda^1(X) = \frac{1}{\lambda - 1}\left[n - 1 - \sum_{k=0}^{\Delta 1_\lambda} \binom{\lambda}{k}(-1)^k \frac{\sum_{q=1}^{n} \frac{d_q^k}{k}\left[q^{k+1} - (q-1)^{k+1}\right]}{\left[\sum_{q=1}^{n} \frac{d_q}{2}[2q-1]\right]^k}\right],$$

$$WT_\lambda^2(X) = \frac{1}{\lambda - 1}\left[\frac{n^2}{2\sum_{q=1}^{n} d_q[2q-1]} - 1 - \sum_{k=0}^{\Delta 2_\lambda} \binom{\lambda}{k}(-1)^k \frac{\sum_{q=1}^{n} d_q^k\,[2q-1]}{\left[\sum_{q=1}^{n} d_q[2q-1]\right]}\right],$$

where

$$\Delta 1_\lambda = \begin{cases} \lambda, & \lambda \in \mathbb{Z}^+ \backslash \{1\}; \\ \infty, & \lambda \neq 1, \lambda > 0 \text{ when } \frac{xg(x)}{\mathcal{E}(X)} < 1, \end{cases} \tag{2.5}$$

and

$$\Delta 2_\lambda = \begin{cases} \lambda, & \lambda \in \mathbb{Z}^+ \backslash \{1\}; \\ \infty, & \lambda \neq 1, \lambda > 0 \text{ when } g(x) < 1. \end{cases} \tag{2.6}$$

Through the permutation $d_1, d_2, ..., d_n$ we obtain different distributions. Therefore, the Tsallis extropies are equal, but, except in special cases, the weighted Tsallis extropies differ.

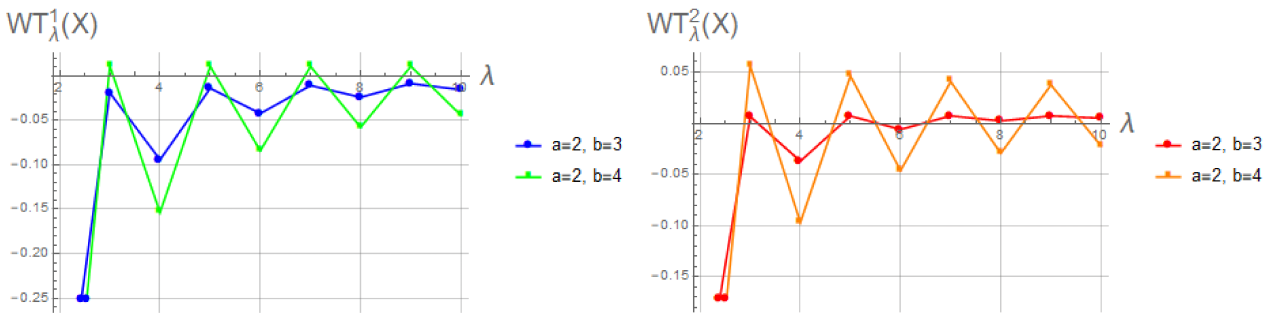
Example 2.5. *Provided that the RV X follows a beta distribution with a and b as parameters, and alongside PDF $g(x) = \frac{x^{a-1}(1-x)^{b-1}}{Beta(a,b)}$, $0 < x < 1$, where $Beta(a, b)$ is the function of beta. Thus, the weighted Tsallis extropies from (2.1) and (2.4) are given, respectively, by*

$$\begin{aligned} WT_\lambda^1(X) &= -\frac{1}{\lambda - 1}\sum_{k=0}^{\Delta_\lambda} \binom{\lambda}{k}(-1)^k \int_0^1 \frac{x^k g^k(x)}{\mathcal{E}^k(X)} dx \\ &= -\frac{1}{\lambda - 1}\sum_{k=0}^{\Delta_\lambda} \binom{\lambda}{k}(-1)^k \frac{Beta(ka+1, kb-k+1)}{\mathcal{E}^k(X) Beta^k(a, b)} \end{aligned} \tag{2.7}$$

and

$$
\begin{aligned}
WT_\lambda^2(X) &= -\frac{1}{\lambda-1}\sum_{k=0}^{\Delta_\lambda}\binom{\lambda}{k}(-1)^k\int_0^1\frac{xg^k(x)}{\mathcal{E}(X)}dx \\
&= -\frac{1}{\lambda-1}\sum_{k=0}^{\Delta_\lambda}\binom{\lambda}{k}(-1)^k\frac{Beta(ka-k+2,kb-k+1)}{\mathcal{E}(X)Beta^k(a,b)}.
\end{aligned}
\tag{2.8}
$$

Figure 2 shows the plots of the weighted Tsallis extropies in (2.7) and (2.8) for beta distribution across various a and b values.



(a) (b)

Figure 2. The weighted Tsallis extropies of beta distribution (a) $WT_\lambda^1(X)$, and (b) $WT_\lambda^2(X)$.

In the next subsection, we will discuss some features of the proposed two versions of the weighted Tsallis extropy.

2.1. *Features of the first and second perspectives $WT_\lambda^1(X)$ and $WT_\lambda^2(X)$*

The next result shows the conditions of the non-negativity of weighted Tsallis extropy.

Proposition 2.1. *Suppose that X is a non-negative RV (N-RV) supported with (α,β), $0<\alpha<\beta<\infty$, and PDF $g(.)$ and cumulative distribution function (CDF) $G(.)$. Therefore,*

1) *According to Eq (2.1), whenever $\frac{xg(x)}{\mathcal{E}(X)}\le 1$, then the weighted Tsallis extropy retains a non-negative value.*

2) *According to Eq (2.4), whenever $g(x)\le 1$, then the weighted Tsallis extropy retains a non-negative value.*

Proof. 1) Given $\frac{xg(x)}{\mathcal{E}(X)}\le 1$, then, if $\lambda>1(\lambda<1)$, we obtain

$$0\le\int_\alpha^\beta(1-\frac{xg(x)}{\mathcal{E}(X)})^\lambda dx\le(\ge)\int_\alpha^\beta(1-\frac{xg(x)}{\mathcal{E}(X)})dx=\beta-\alpha-1.$$

Thus, when $\lambda>1$ $(\lambda<1)$ in Eq (2.1), we obtain

$$WT_\lambda^1(X)=\frac{1}{\lambda-1}\left(\beta-\alpha-1-\int_\alpha^\beta\left(1-\frac{xg(x)}{\mathcal{E}(X)}\right)^\lambda dx\right)\ge 0.$$

2) Given $g(x) \leq 1$, then, if $\lambda > 1(\lambda < 1)$, we obtain

$$0 \leq \int_\alpha^\beta \frac{x(1-g(x))^\lambda}{\mathcal{E}(X)}dx \leq (\geq) \int_\alpha^\beta \frac{x(1-g(x))}{\mathcal{E}(X)}dx = \frac{\beta^2-\alpha^2}{2\mathcal{E}(X)} - 1.$$

Thus, when $\lambda > 1$ ($\lambda < 1$) in Eq (2.4), we obtain

$$WT_\lambda^2(X) = \frac{1}{\lambda-1}\left(\frac{\beta^2-\alpha^2}{2\mathcal{E}(X)} - 1 - \int_\alpha^\beta \frac{x\,(1-g(x))^\lambda}{\mathcal{E}(X)}dx\right) \geq 0.$$

□

In what follows, the behavior of $WT_\lambda^1(X)$ and $WT_\lambda^2(X)$ as λ approaches 1 can be examined using L'Hôpital's rule, as outlined below:

1) From (2.1), we obtain

$$\begin{aligned}\lim_{\lambda\to1} WT_\lambda^1(X) &= \lim_{\lambda\to1}\frac{1}{\lambda-1}\left(\beta-\alpha-1-\int_\alpha^\beta\left(1-\frac{xg(x)}{\mathcal{E}(X)}\right)^\lambda dx\right)\\ &= \lim_{\lambda\to1} -\int_\alpha^\beta\left(1-\frac{xg(x)}{\mathcal{E}(X)}\right)^\lambda \ln\left(1-\frac{xg(x)}{\mathcal{E}(X)}\right)dx \\ &= -\int_\alpha^\beta\left(1-\frac{xg(x)}{\mathcal{E}(X)}\right)\ln\left(1-\frac{xg(x)}{\mathcal{E}(X)}\right)dx = WEx1(X),\end{aligned} \tag{2.9}$$

where $WEx1(X) = -\int_\alpha^\beta \left(1-\frac{xg(x)}{\mathcal{E}(X)}\right)\ln\left(1-\frac{xg(x)}{\mathcal{E}(X)}\right)dx$.

2) From (2.4), we obtain

$$\begin{aligned}\lim_{\lambda\to1} WT_\lambda^2(X) &= \lim_{\lambda\to1}\frac{1}{\lambda-1}\left(\frac{\beta^2-\alpha^2}{2\mathcal{E}(X)}-1-\int_\alpha^\beta\frac{x(1-g(x))^\lambda}{\mathcal{E}(X)}dx\right)\\ &= \lim_{\lambda\to1} -\int_\alpha^\beta\frac{x(1-g(x))^\lambda}{\mathcal{E}(X)}\ln\frac{x(1-g(x))^\lambda}{\mathcal{E}(X)}dx \\ &= -\int_\alpha^\beta\frac{x(1-g(x))}{\mathcal{E}(X)}\ln\frac{x(1-g(x))}{\mathcal{E}(X)}dx = WEx2(X),\end{aligned} \tag{2.10}$$

where $WEx2(X) = -\int_\alpha^\beta \frac{x(1-g(x))}{\mathcal{E}(X)}\ln\frac{x(1-g(x))}{\mathcal{E}(X)}dx$.

Remark 2.2. *To present the weighted case of the extropy measure in (1.2), we can do two cases: First, we can replace $1-g(x)$ with $1-xg(x)$. Second, we can replace $1-g(x)$ with $x(1-g(x))$. Therefore, we can express the weighted extropy in the following two ways:*

1) First perspective

$$WEx^*(X) = -\int_0^\infty (1-xg(x))\ln(1-xg(x))dx. \tag{2.11}$$

2) Second perspective

$$WEx^*(X) = -\int_0^\infty x(1-g(x))\ln x(1-g(x))dx. \tag{2.12}$$

Moreover, we can see that when $\mathcal{E}(X) = 1$, the two weighted Tsallis extropy presented in (2.9) and (2.10) will turn to weighted extropy given in (2.11) and (2.12), respectively. The literature deals with the weighted extropy given in (1.7). Therefore, according to the definition of extropy, we can consider that $WEx1(X)$ and $WEx2(X)$ are alternative perspectives of the weighted extropy when $\mathcal{E}(X) \neq 1$.

Proposition 2.2. *Consider an N-RV X supported in (α,β), $0 < \alpha < \beta < \infty$.*

1) *From (2.1), under the condition $\frac{xg(x)}{\mathcal{E}(X)} \leq 1$ and $1 \neq \lambda > 0$, then $WT^1_\lambda(X) \leq 1$.*
2) *From (2.4), under the condition $g(x) \leq 1$ and $1 \neq \lambda > 0$, then $WT^2_\lambda(X) \leq 1$.*

Proof. 1) From (2.1), by using Bernoulli's inequality, and under the conditions $\frac{xg(x)}{\mathcal{E}(X)} \leq 1$ and $1 \neq \lambda > 0$, we have

$$\begin{aligned} WT^1_\lambda(X) &= \frac{1}{\lambda-1}\left(\beta-\alpha-1-\int_\alpha^\beta\left(1-\frac{xg(x)}{\mathcal{E}(X)}\right)^\lambda dx\right) \\ &\leq \frac{1}{\lambda-1}\left(\beta-\alpha-1-\int_\alpha^\beta\left(1-\lambda\frac{xg(x)}{\mathcal{E}(X)}\right) dx\right) = 1. \end{aligned}$$

2)

$$\begin{aligned} WT^2_\lambda(X) &= \frac{1}{\lambda-1}\left(\frac{\beta^2-\alpha^2}{2\mathcal{E}(X)}-1-\int_\alpha^\beta \frac{x(1-g(x))^\lambda}{\mathcal{E}(X)} dx\right) \\ &\leq \frac{1}{\lambda-1}\left(\frac{\beta^2-\alpha^2}{2\mathcal{E}(X)}-1-\int_\alpha^\beta \frac{x(1-\lambda g(x))}{\mathcal{E}(X)} dx\right) = 1. \end{aligned}$$

□

In the following proposition, we can represent the two versions of weighted Tsallis extropy in relation to the function of hazard rate $\xi(x) = \frac{g(x)}{\overline{G}(x)}$, $\overline{G}(x) = 1 - G(x)$.

Proposition 2.3. *Consider a N-RV X with the supporting (α,β), $0 < \alpha < \beta < \infty$. Then, we have*

1) *An alternative way of expressing the weighted Tsallis extropy as presented in (2.1), using $\xi(x)$, is provided by*

$$\begin{aligned} WT^1_\lambda(X) &= \frac{1}{\lambda-1}\left[\beta-\alpha-1-\sum_{k=0}^{\Delta 1_\lambda}\binom{\lambda}{k}\int_\alpha^\beta\left(\frac{-xg(x)}{\mathcal{E}(X)}\right)^k dx\right] \\ &= \frac{-1}{\lambda-1}\left[1+\sum_{k=1}^{\Delta 1_\lambda}\binom{\lambda}{k}\frac{(-1)^k}{k\mathcal{E}^k(X)}\mathcal{E}(X_k^k\xi^{k-1}(X_k)))\right]. \end{aligned} \tag{2.13}$$

2) *An alternative way of expressing the weighted Tsallis extropy given in (2.4), using $\xi(x)$, is provided by*

$$\begin{aligned} WT^2_\lambda(X) &= \frac{1}{\lambda-1}\left[\frac{\beta^2-\alpha^2}{2\mathcal{E}(X)}-1-\sum_{k=0}^{\Delta 2_\lambda}\binom{\lambda}{k}\int_\alpha^\beta \frac{x(-g(x))^k}{\mathcal{E}(X)} dx\right] \\ &= \frac{-1}{\lambda-1}\left[1+\sum_{k=1}^{\Delta 2_\lambda}\binom{\lambda}{k}\frac{(-1)^k}{k\mathcal{E}(X)}\mathcal{E}(X_k\xi^{k-1}(X_k)))\right], \end{aligned} \tag{2.14}$$

such that $\Delta 1_\lambda$ and $\Delta 2_\lambda$ are defined in (2.5) and (2.6), respectively, and the PDF of the RV X_k is $k\overline{G}^{k-1}(x)g(x)$ for all $1 \leq k < \infty$.

Definition 2.2. *Consider N-RVs X_1 and X_2 with the supporting (α,β), with noting that $0 < \alpha < \beta < \infty$. Therefore, X_1 is considered smaller than X_2 in weighted Tsallis form of extropy-j of order λ, $X_1 \leq_{WTj_\lambda} X_2$, if $WTj_\lambda(X_1) \leq WTj_\lambda(X_2)$, $j = 1, 2$.*

As per the findings by Shaked and Shanthikumar [23], we will employ certain stochastic orders, namely stochastic order ($\leq_{ST}$) and dispersive order ($\leq_{DIS}$) (representing the variability distribution order). Additionally, the preceding orders signify the following:

Lemma 2.1. *Consider N-RVs X_1 and X_2 with the supporting (α,β), $0 < \alpha < \beta < \infty$, which follow CDFs G_1 and G_2, respectively. Furthermore, let g_1 as well as g_2 be the corresponding PDFs and G_1^{-1} as well as G_2^{-1} be their right continuous inverses, respectively (quantile functions). Then*

(1) If X_1 and X_2 share a common finite left endpoint in their supports, then the condition $X_1 \leq_{DIS} X_2$ implies $X_1 \leq_{ST} X_2$ (cf. Jeon et al. [24]).

(2) If $X_1 \leq_{ST} X_2$, then $\overline{G}_1(x) \leq \overline{G}_2(x)$ and $G_1(x) \geq G_2(x)$, for all x, and $G_1^{-1}(u) \leq G_2^{-1}(u)$, for all $u \in (0,1)$. Moreover, $X_1 \leq_{ST} X_2$ if and only if $\mathcal{E}(\varphi(X_1)) \leq \mathcal{E}(\varphi(X_2))$, which remains valid for all increasing functions $\varphi(.)$, assuming the expectations exist (cf. Qiu [12]).

(3) If $X_1 \leq_{DIS} X_2$, then $g_1(G_1^{-1}(u)) \geq g_2(G_2^{-1}(u)$, $\forall\, u \in (0.1)$ (cf. Qiu [12]).

Theorem 2.1. *Consider N-RVs X_1 and X_2 with the supporting (α,β), $0 < \alpha < \beta < \infty$, which follow CDFs G_1 and G_2, respectively. Finally, let $X_1 \leq_{DIS} X_2$. Then, under the conditions $\frac{xg_1(x)}{\mathcal{E}(X_1)} \leq 1$, and $0 \leq \frac{G_2^{-1}(u)}{\mathcal{E}(X_2)} \leq \frac{G_1^{-1}(u)}{\mathcal{E}(X_1)}$, we get $X_1 \geq_{WT_\lambda^1} (\leq_{WT_\lambda^1}) X_2$, if $\lambda_1 \geq (\leq) 1$. Moreover, under the condition $g_1(x) \leq 1$ and $0 \leq \frac{G_1^{-1}(u)}{\mathcal{E}(X_1)} \leq \frac{G_2^{-1}(u)}{\mathcal{E}(X_2)}$, we get $X_1 \geq_{WT_\lambda^2} (\leq_{WT_\lambda^2}) X_2$, if $\lambda_1 \geq (\leq) 1$.*

Proof. From (2.1), we get

$$\begin{aligned}(\lambda-1)WT_\lambda^1(X_1) &= \beta - \alpha - 1 - \int_\alpha^\beta \left(1 - \frac{xg_1(x)}{\mathcal{E}(X_1)}\right)^\lambda dx \\ &= \beta - \alpha - 1 - \int_0^1 \left(1 - \frac{G_1^{-1}(u)g_1(G_1^{-1}(u))}{\mathcal{E}(X_1)}\right)^\lambda \frac{1}{g_1(G_1^{-1}(u))} du.\end{aligned}$$

Under the conditions of the theorem, we conclude that the 1st and 2nd integrals are positive. Thus, if $X_1 \leq_{DIS} X_2$, then Lemma 2.1 guarantees that

$$\begin{aligned}(\lambda-1)WT_\lambda^1(X_1) &= \beta - \alpha - 1 - \int_0^1 \left(1 - \frac{G_1^{-1}(u)g_1(G_1^{-1}(u))}{\mathcal{E}(X_1)}\right)^\lambda \frac{1}{g_1(G_1^{-1}(u))} du \\ &\geq \beta - \alpha - 1 - \int_0^1 \left(1 - \frac{G_1^{-1}(u)g_1(G_1^{-1}(u))}{\mathcal{E}(X_1)}\right)^\lambda \frac{1}{g_2(G_2^{-1}(u))} du \\ &\geq \beta - \alpha - 1 - \int_0^1 \left(1 - \frac{G_2^{-1}(u)g_2(G_2^{-1}(u))}{\mathcal{E}(X_2)}\right)^\lambda \frac{1}{g_2(G_2^{-1}(u))} du \\ &= (\lambda-1)WT_\lambda^1(X_2).\end{aligned}$$

Therefore, $X_1 \geq_{WT^1_\lambda} (\leq_{WT^1_\lambda}) X_2$, if $\lambda_1 \geq (\leq) 1$. Similarly, from (2.4), we get

$$\begin{aligned}(\lambda-1)WT^2_\lambda(X_1) &= \beta-\alpha-1-\int_\alpha^\beta \frac{x}{\mathcal{E}(X_1)}(1-g_1(x))^\lambda\,dx\\ &= \beta-\alpha-1-\int_0^1 \frac{G_1^{-1}(u)}{\mathcal{E}(X_1)}\left(1-g_1(G_1^{-1}(u))\right)^\lambda du.\end{aligned}$$

Under the conditions of the theorem, we conclude that the 1st and 2nd integrals are positive. Thus, if $X_1 \leq_{DIS} X_2$, then Lemma 2.1 guarantees that

$$\begin{aligned}(\lambda-1)WT^2_\lambda(X_1) &= \beta-\alpha-1-\int_0^1 \frac{G_1^{-1}(u)}{\mathcal{E}(X_1)}\left(1-g_1(G_1^{-1}(u))\right)^\lambda \frac{1}{g_1(G_1^{-1}(u))}du\\ &\geq \beta-\alpha-1-\int_0^1 \frac{G_2^{-1}(u)}{\mathcal{E}(X_2)}\left(1-g_1(G_1^{-1}(u))\right)^\lambda \frac{1}{g_2(G_2^{-1}(u))}du\\ &\geq \beta-\alpha-1-\int_0^1 \frac{G_2^{-1}(u)}{\mathcal{E}(X_2)}\left(1-g_2(G_2^{-1}(u))\right)^\lambda \frac{1}{g_2(G_2^{-1}(u))}du\\ &= (\lambda-1)WT^2_\lambda(X_2).\end{aligned}$$

Therefore, $X_1 \geq_{WT^2_\lambda} (\leq_{WT^2_\lambda}) X_2$, if $\lambda_1 \geq (\leq) 1$. □

Remark 2.3. *The two mutually exclusive conditions* $0 \leq \frac{G_2^{-1}(u)}{\mathcal{E}(X_2)} \leq \frac{G_1^{-1}(u)}{\mathcal{E}(X_1)}$ *and* $0 \leq \frac{G_1^{-1}(u)}{\mathcal{E}(X_1)} \leq \frac{G_2^{-1}(u)}{\mathcal{E}(X_2)}$ *mean that either* $X_1 \geq_{WT^1_\lambda} (\leq_{WT^1_\lambda}) X_2$, *or* $X_1 \geq_{WT^2_\lambda} (\leq_{WT^2_\lambda}) X_2$, *or both measures of information do not satisfy this stochastic order.*

The subsequent theorem illustrates how a change (modification) influences the weighted Tsallis extropies of an RV.

Theorem 2.2. *Consider N-RVs* X_1 *and* X_2 *with the supporting* (α,β), $0 < \alpha < \beta < \infty$, *which follow CDFs* G_1 *and* G_2, *respectively. If* $X_2 = \psi(X_1)$ *as* ψ *is the status of a continuous function with the derivative* $\psi'(x)$ *where* $\mathcal{E}(X_2^2) < 1$. *If* $|\psi'(x)| \geq 1$, *then* $WT^1_\lambda(X_1) \leq WT^1_\lambda(X_2)$ *and* $WT^2_\lambda(X_1) \leq WT^2_\lambda(X_2)$, *respectively.*

Proof. Using the Jacobian transformation for $X_2 = \psi(X_1)$ and $X_1 = \psi^{-1}(X_2)$, we have $J_\psi(X_2) = |\frac{dX_1}{dX_2}| = |\frac{d\psi^{-1}(X_2)}{dX_2}|$. Using the chain rule for the function ψ and its inverse ψ^{-1}, we have: $\frac{dX_1}{dX_2} = \frac{1}{\frac{d\psi(X_1)}{dX_1}} = \frac{1}{\psi'(\psi^{-1}(X_2))}$. Hence, $g_{X_2}(x) = g_{X_1}(\psi^{-1}(x))|\frac{1}{\psi'(\psi^{-1}(x))}|$. Thus, from (2.1), we get

$$\begin{aligned}(\lambda-1)WT^1_\lambda(X_2) &= \beta-\alpha-1-\sum_{k=0}^{\Delta 1_\lambda}\binom{\lambda}{k}\int_\alpha^\beta \left(\frac{-xg_2(x)}{\mathcal{E}(X_2)}\right)^k dx\\ &= \beta-\alpha-1-\sum_{k=0}^{\Delta 1_\lambda}\binom{\lambda}{k}\frac{(-1)^k}{\mathcal{E}^k(X_2)}\int_\alpha^\beta x^k g_2^k(x)dx\\ &= \beta-\alpha-1-\sum_{k=0}^{\Delta 1_\lambda}\binom{\lambda}{k}\frac{(-1)^k}{\mathcal{E}^k(\psi(X_1))}\int_0^1 u^k g_1^{k-1}(u)\left(\frac{1}{|\psi'(u)|}\right)^{k-1} du,\end{aligned}$$

similarly, from (2.4), we get

$$\begin{aligned}(\lambda-1)WT_\lambda^2(X_2) &= \frac{\beta^2-\alpha^2}{2\mathcal{E}(X)} - 1 - \sum_{k=0}^{\Delta 2_\lambda}\binom{\lambda}{k}\int_\alpha^\beta \frac{x(-g_2(x))^k}{\mathcal{E}(X_2)}dx \\ &= \frac{\beta^2-\alpha^2}{2\mathcal{E}(X)} - 1 - \sum_{k=0}^{\Delta 2_\lambda}\binom{\lambda}{k}\frac{(-1)^k}{\mathcal{E}(X_2)}\int_\alpha^\beta x g_2^k(x)dx \\ &= \frac{\beta^2-\alpha^2}{2\mathcal{E}(X)} - 1 - \sum_{k=0}^{\Delta 2_\lambda}\binom{\lambda}{k}\frac{(-1)^k}{\mathcal{E}(\psi(X_1))}\int_0^1 u g_1^{k-1}(u)\left(\frac{1}{|\psi'(u)|}\right)^{k-1}du,\end{aligned}$$

and the remaining part of the proof follows a similar approach to Theorem 1 in Ebrahimi et al. [25]. □

2.2. *Further features of the first perspective $WT_\lambda^1(X)$*

In this subsection, the maximum weighted Tsallis extropy given in (2.1) can be obtained in the following result.

Theorem 2.3. *Consider an N-RV X with the support (α,β), with the consideration that $0<\alpha<\beta<\infty$ follows PDF f. Hence, from (2.1) with $g_w(x)=\frac{xg(x)}{\mathcal{E}(X)}$, X has the maximum weighted Tsallis extropy $WT_\lambda^2(X)$ if and only if the PDF $g(x)=\frac{\mathcal{E}(X)}{x(\beta-\alpha)}=\frac{1}{x(\ln\beta-\ln\alpha)}$.*

Proof. From (2.1), we obtain

$$WT_\lambda^1(X)=\frac{1}{\lambda-1}\left(\int_\alpha^\beta(1-g_w(x))dx-\int_\alpha^\beta(1-g_w(x))^\lambda dx\right),$$

constraint to

$$\int_\alpha^\beta g_w(x)dx=1. \tag{2.15}$$

We can achieve the maximization of $WT_\lambda^1(X)$ by employing the Lagrange multipliers method in the following manner:

$$G(X)=\frac{1}{\lambda-1}\left(\int_\alpha^\beta(1-g_w(x))dx-\int_\alpha^\beta(1-g_w(x))^\lambda dx\right)+\eta\left(\int_\alpha^\beta g_w(x)dx-1\right).$$

Differentiating $G(X)$ with respect to $g_w(x)$ and then equating to zero, we obtain

$$\frac{dG(X)}{dg_w(x)}=0=\frac{1}{\lambda-1}\left(-1+\lambda(1-g_w(x))^{\lambda-1}\right)+\eta,$$

therefore, we get

$$g_w(x)=1-\left(\frac{1}{\lambda}+\eta\frac{1-\lambda}{\lambda}\right)^{\frac{1}{\lambda-1}}. \tag{2.16}$$

To find the value of η, we substitute (2.16) in the constraint (2.15), thus

$$\eta=\frac{\lambda}{1-\lambda}\left(\left(1-\frac{1}{\beta-\alpha}\right)^{\lambda-1}-\frac{1}{\lambda}\right). \tag{2.17}$$

Substituting (2.17) in (2.16), it holds that $g(x)=\frac{\mathcal{E}(X)}{x(\beta-\alpha)}=\frac{1}{x(\ln\beta-\ln\alpha)}$. □

Theorem 2.4. *Consider an N-RV X with the supporting* (α, β)*, with considering that* $0 < \alpha < \beta < \infty$ *follows PDF* f*. Then, from (2.1) and (1.1), assuming that* $\beta - \alpha \geq 2$*, we obtain*

1) $WT^1_\lambda(X) \leq WTn_\lambda(X)$, *when* $0 < \lambda < 2$.

2) $WT^1_\lambda(X) \geq WTn_\lambda(X)$, *when* $\lambda > 2$.

Proof. From (2.1) and (1.1), we get

$$WTn_\lambda(X) - WT^1_\lambda(X) = \frac{1}{\lambda - 1}\left(2 - (\beta - \alpha) - \int_\alpha^\beta g_w^\lambda(x)dx + \int_\alpha^\beta (1 - g_w(x))^\lambda dx\right).$$

Hence, the Lagrange function can be expressed as

$$G(X) = WTn_\lambda(X) - WT^1_\lambda(X) + \eta\left(\int_\alpha^\beta g_w(x)dx - 1\right).$$

Subsequently, the derivative concerning $g_w(x)$ is $\frac{dG(X)}{dg_w(x)} = \frac{-\lambda}{\lambda-1}\left(g_w^{\lambda-1}(x) + (1 - g_w(x))^{\lambda-1}\right) + \eta$, with the vanishing equation $g_w^{\lambda-1}(x) + (1 - g_w(x))^{\lambda-1} = \tau$, τ is a constant, and the remaining part of the proof shall follow a similar approach as presented in Balakrishnan et al. [13]. □

Figure 3 shows the comparison between $WT^1_\lambda(X)$, $WT^2_\lambda(X)$, and $TEn_\lambda(X)$ of the power function distribution, which ensures the results in Theorem 2.4.

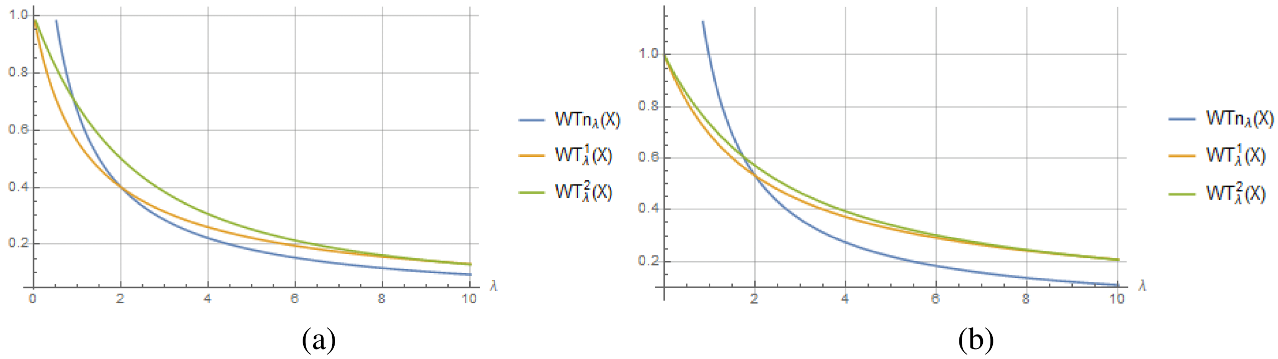


Figure 3. $WT^1_\lambda(X)$, $WT^2_\lambda(X)$ and $TEn_\lambda(X)$ of power function distributions with (a) $\gamma = 3$ and $\delta = 2$, and (b) $\gamma = 7$ and $\delta = 5$.

Now, for the specific choice of $\lambda = 2$, we can obtain the following result.

Theorem 2.5. *Consider an N-RV X with the supporting* (α, β), $0 < \alpha < \beta < \infty$*, which follows PDF g and weighted PDF* $g_w(x) = \frac{xg(x)}{\mathcal{E}(X)}$*. Then, from (2.1), when* $\lambda = 2$*, we have*

$$WT^1_2(X) \approx 1 - g_w(\mu_w) + \frac{1}{2}g''_w(\mu_w)Var(X_w),$$

where $\mu_w = \mathcal{E}(X_w) = \int_\alpha^\beta x g_w(x)dx < \infty$*, and* $Var(X_w) = \mathcal{E}(X_w - \mu_w)^2 < \infty$.

Proof. From (2.1), when $\lambda = 2$, we get

$$\begin{aligned} WT_2^1(X) &= \frac{1}{2-1}\left(\beta - \alpha - 1 - \int_\alpha^\beta (1 - g_w(x))^2 dx\right) \\ &= \beta - \alpha - 1 - \int_\alpha^\beta (1 + g_w^2(x) - 2g_w(x))dx \\ &= \beta - \alpha - 1 - (\beta - \alpha) - \int_\alpha^\beta g_w^2(x)dx + 2 \\ &= 1 - \int_\alpha^\beta g_w^2(x)dx \\ &= 1 - \mathcal{E}(g_w(X_w)). \end{aligned} \tag{2.18}$$

In what follows, an approximate expression for the weighted Tsallis extropy is derived using the Taylor series. To achieve this, the approximation of $\mathcal{E}(g_w(X_w))$ is obtained as follows:

$$\begin{aligned} \mathcal{E}(g_w(X_w)) &\approx \mathcal{E}\left[g_w(\mu_w) + (X_w - \mu_w)g_w'(\mu_w) + \frac{1}{2}(X_w - \mu_w)^2 g_w''(\mu_w)\right] \\ &= g_w(\mu_w) + \frac{1}{2}g_w''(\mu_w)Var(X_w). \end{aligned} \tag{2.19}$$

Thus, using (2.18) and (2.19), we have

$$WT_2^1(X) \approx 1 - g_w(\mu_w) + \frac{1}{2}g_w''(\mu_w)Var(X_w).$$

□

2.3. Further features of the second perspective $WT_\lambda^2(X)$

This subsection will discuss the results of the weighted Tsallis extropy given in (2.4) when $\lambda = 2$.

Theorem 2.6. *Given independent N-RVs X and Y with the supporting* (α,β), $0 < \alpha < \beta < \infty$ *follow PDF* g_X *and* g_Y*, respectively. Then, from (2.4), when* $\lambda = 2$*, we have*

$$WT_2^2(X + Y) \geq 1 - 2\left[(Ex(X))(WEx(Y)) + (Ex(Y))(WEx(X))\right],$$

where $Ex(X)$ *and* $WEx(X)$ *are defined in (1.3) and (1.7), respectively.*

Proof. From (2.4), when $\lambda = 2$, we have

$$\begin{aligned} WT_2^2(X) &= \frac{1}{2-1}\left(\frac{(\beta^2 - \alpha^2)}{2\mathcal{E}(X)} - 1 - \int_\alpha^\beta \frac{x(1 - g(x))^2}{\mathcal{E}(X)}dx\right) \\ &= \frac{(\beta^2 - \alpha^2)}{2\mathcal{E}(X)} - 1 - \int_\alpha^\beta \frac{x}{\mathcal{E}(X)}(1 + g^2(x) - 2g(x))dx \\ &= \frac{(\beta^2 - \alpha^2)}{2\mathcal{E}(X)} - 1 - \frac{(\beta^2 - \alpha^2)}{2\mathcal{E}(X)} - \int_\alpha^\beta xg^2(x)dx + 2 \\ &= 1 - \int_\alpha^\beta xg^2(x)dx \\ &= 1 + 2WEx(X). \end{aligned} \tag{2.20}$$

Given that X as well as Y are independent N-RVs, the density function of $R = X + Y$ is provided for $r > 0$ as follows: $g_R(r) = \int_\alpha^r g_X(x)g_Y(r-x)dx$. From (2.20), utilizing Jensen's inequality (see Kharazmi et al. [26]), we get

$$\begin{aligned} WT_2^2(X+Y) &= 1 - \int_\alpha^\beta r\left[\int_\alpha^r g_X(x)g_Y(r-x)dx\right]^2 dr \\ &\geq 1 - \int_\alpha^\beta r\int_\alpha^r g_X^2(x)g_Y^2(r-x)dxdr \\ &= 1 - \int_\alpha^\beta g_X^2(x)\int_x^\beta rg_Y^2(r-x)drdx \\ &= 1 - \int_\alpha^\beta g_X^2(x)\int_\alpha^\beta (r+x)g_Y^2(r)drdx \\ &= 1 - \left[(\int_\alpha^\beta g_X^2(x)dx)(\int_\alpha^\beta rg_Y^2(r)dr) + (\int_\alpha^\beta xg_X^2(x)dx)(\int_\alpha^\beta g_Y^2(r)dr)\right] \\ &= 1 - 2\left[(Ex(X))(WEx(Y)) + (Ex(Y))(WEx(X))\right]. \end{aligned}$$

□

Remark 2.4. *Specifically, when X and Y exhibit independence and identical distribution as outlined in Theorem 2.6, we obtain that*

$$WT_2^2(X) \geq 1 - 4(Ex(X))(WEx(X)).$$

3. Weighted Tsallis extropy of order statistics

This section will explore the two suggested models of weighted Tsallis extropy of order statistics. Consider the n independently and identically in distribution (i.i.d) N-RVs $X_1, X_2, ..., X_n$ with the support (α_1,β_1), $0 < \alpha_1 < \beta_1 < \infty$, following PDF g and CDF F. The ordered RVs $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ are known as order statistics, with the k-th order statistic denoted as $X_{k:n}$, where $1 \leq k \leq n$. For more details, see David and Nagaraja [27] and Wang et al. [28]. Then, from (2.1) and (1.1), the weighted Tsallis extropies of the k-th order statistic are provided, respectively, as follows:

1) Similarly to (2.1), the weighted Tsallis extropy measure of the k-th order statistic is expressed as

$$\begin{aligned} WT_\lambda^1(X_{k:n}) &= \frac{1}{\lambda-1}\left(\int_{\alpha_1}^{\beta_1}(1-g_{w_{k:n}}(x))dx - \int_{\alpha_1}^{\beta_1}(1-g_{w_{k:n}}(x))^\lambda dx\right) \\ &= \frac{1}{\lambda-1}\left(\beta_1-\alpha_1-1-\int_{\alpha_1}^{\beta_1}\left(1-\frac{xg_{k:n}(x)}{\mathcal{E}(X_{k:n})}\right)^\lambda dx\right), \end{aligned} \tag{3.1}$$

where the conditions on λ can be summarized by

$$\Lambda_1 = \begin{cases} \lambda \neq 1, \lambda > 0 \text{ } when \text{ } g_{w_{k:n}}(x) \leq 1, \\ \lambda \in \mathbb{Z}^+\backslash\{1\} \text{ } when \text{ } g_{w_{k:n}}(x) > 1, \end{cases} \tag{3.2}$$

and $g_{k:n}(x) = \frac{G^{k-1}(x)\overline{G}^{n-k}(x)g(x)}{Beta(k,n-k+1)}$, $\mathcal{E}(X_{k:n}) = \int_{\alpha_1}^{\beta_1} xg_{k:n}(x)dx$.

2) Similarly to (2.4), the weighted Tsallis extropy measure of the k-th order statistic is expressed as

$$\begin{aligned} WT^2_\lambda(X_{k:n}) &= \frac{1}{\lambda-1}\left(\int_{\alpha_1}^{\beta_1}\frac{x(1-g_{k:n}(x))}{\mathcal{E}(X_{k:n})}dx - \int_{\alpha_1}^{\beta_1}\frac{x(1-g_{k:n}(x))^\lambda}{\mathcal{E}(X_{k:n})}dx\right) \\ &= \frac{1}{\lambda-1}\left(\frac{\beta_1^2-\alpha_1^2}{2\mathcal{E}(X_{k:n})} - 1 - \int_{\alpha_1}^{\beta_1}\frac{x}{\mathcal{E}(X_{k:n})}(1-g_{k:n}(x))^\lambda dx\right), \end{aligned} \tag{3.3}$$

where the conditions on λ are

$$\Lambda = \begin{cases} \lambda \neq 1, \lambda > 0 \; when \; g_{k:n}(x) \; \leq 1, \\ \lambda \in \mathbb{Z}^+\backslash\{1\} \; when \; g_{k:n}(x) > 1. \end{cases} \tag{3.4}$$

In the subsequent theorems, we will present the weighted Tsallis extropy of the k-th order statistic in another perspective.

Theorem 3.1. *Consider the n i.i.d N-RVs $X_1, X_2, ..., X_n$ with the support (α_1,β_1), $0 < \alpha_1 < \beta_1 < \infty$, following PDF g and CDF F. Then, from (3.1), the weighted Tsallis extropy of the k-th order statistic is expressed as*

$$\begin{aligned} WT^1_\lambda(X_{k:n}) = \frac{1}{\lambda-1}&\left(\beta_1 - \alpha_1 - 1 - \sum_{j=0}^{\Delta 1_\lambda}\binom{\lambda}{j}\frac{(-1)^j Beta(kj-j+1, nj-kj+1)}{(Beta(k,n-k+1)\mathcal{E}(G^{-1}(U_{k:n})))^j}\right. \\ &\left.\times\, \mathcal{E}((G^{-1}(V_n))^j(g(G^{-1}(V_n)))^{j-1})\right), \end{aligned}$$

where the RV V_n follows a beta distribution with parameters $kj-j+1$ and $nj-kj+1$ (i.e, $Beta(kj-j+1, nj-kj+1)$). Moreover, for the weighted Tsallis extropy which follows a $U(0,1)$ distribution, we get

$$WT^1_\lambda(U_{k:n}) = \frac{-1}{\lambda-1}\sum_{j=0}^{\Delta 1_\lambda}\binom{\lambda}{j}\frac{(-1)^j Beta(kj+1, nj-kj+1)}{(Beta(k,n-k+1)\mathcal{E}(U_{k:n}))^j},$$

where $\Delta 1_\lambda$ is defined in (2.5).

Proof. From (3.1), the weighted Tsallis extropy of the k-th order statistic can be derived as

$$\begin{aligned} WT^1_\lambda(X_{k:n}) &= \frac{1}{\lambda-1}\left(\beta_1-\alpha_1-1-\int_{\alpha_1}^{\beta_1}\left(1-\frac{x g_{k:n}(x)}{\mathcal{E}(X_{k:n})}\right)^\lambda dx\right) \\ &= \frac{1}{\lambda-1}\left(\beta_1-\alpha_1-1-\sum_{j=0}^{\Delta 1_\lambda}\binom{\lambda}{j}\frac{(-1)^j}{(Beta(k,n-k+1)\mathcal{E}(G^{-1}(U_{k:n})))^j}\right. \\ &\quad\left.\times\int_0^1 (G^{-1}(u))^j(g(G^{-1}(u)))^{j-1}u^{kj-j}(1-u)^{nj-kj}du\right), \end{aligned} \tag{3.5}$$

then the result follows. Furthermore, from (3.1) and under $U(0,1)$ distribution, we get

$$\begin{aligned} WT^1_\lambda(U_{k:n}) &= \frac{-1}{\lambda-1}\sum_{j=0}^{\Delta 1_\lambda}\binom{\lambda}{j}\frac{(-1)^j}{(Beta(k,n-k+1)\mathcal{E}(U_{k:n}))^j}\int_0^1\left(u\,u^{k-1}(1-u)^{n-k}\right)^j du \\ &= \frac{-1}{\lambda-1}\sum_{j=0}^{\Delta 1_\lambda}\binom{\lambda}{j}\frac{(-1)^j}{(Beta(k,n-k+1)\mathcal{E}(U_{k:n}))^j}\int_0^1 u^{kj}(1-u)^{nj-kj}du, \end{aligned}$$

then the result follows. □

Theorem 3.2. *Consider the n i.i.d N-RVs* $X_1, X_2, ..., X_n$ *with the support* (α_1,β_1), $0 < \alpha_1 < \beta_1 < \infty$, *following PDF g and CDF F. Then, from (3.3), the weighted Tsallis extropy of the k-th order statistic is expressed as*

$$WT_\lambda^2(X_{k:n}) = \frac{1}{\lambda-1}\left(\frac{\beta_1^2-\alpha_1^2}{2\mathcal{E}(X_{k:n})} - 1 - \sum_{j=0}^{\Delta 2_\lambda}\binom{\lambda}{j}\frac{(-1)^j Beta(kj-j+1, nj-kj+1)}{\mathcal{E}(G^{-1}(U_{k:n}))(Beta(k,n-k+1))^j}\right.$$
$$\left.\times\,\mathcal{E}((G^{-1}(V_n))(g(G^{-1}(V_n)))^{j-1})\right),$$

where the RV V_n *follows a beta distribution with parameters* $kj-j+1$ *and* $nj-kj+1$ *(i.e,* $Beta(kj-j+1, nj-kj+1)$*). Moreover, for the weighted Tsallis extropy which follows a* $U(0,1)$ *distribution, we get*

$$WT_\lambda^2(U_{k:n}) = \frac{-1}{\lambda-1}\sum_{j=0}^{\Delta 2_\lambda}\binom{\lambda}{j}\frac{(-1)^j Beta(kj+2, nj-kj+1)}{\mathcal{E}(U_{k:n})(Beta(k,n-k+1))^j},$$

where $\Delta 2_\lambda$ *is defined in (2.6).*

Proof. The proof proceeds in a manner analogous to Theorem 3.1. □

The following findings establish a lower bound for the weighted Tsallis extropy of order statistics.

Theorem 3.3. *Consider the n i.i.d N-RVs* $X_1, X_2, ..., X_n$ *with the support* (α_1,β_1), $0 < \alpha_1 < \beta_1 < \infty$, *following PDF g and CDF F. Then, from (3.1) and under the conditions* $g_{w_{k:n}}(x) \le 1$ *and* $\lambda < 1$, *we obtain*

$$WT_\lambda^1(X_{k:n}) \ge \frac{1}{\lambda-1}\left(\beta_1-\alpha_1 - 1 - \sum_{j=0}^{\Delta 1_\lambda}\binom{\lambda}{j}\frac{(-1)^j Beta(kj-j+1, nj-kj+1)}{(Beta(i,n-i+1)\mathcal{E}(G^{-1}(U_{k:n})))^j}\right.$$
$$\left.\times\, D^{j-1}\mathcal{E}((G^{-1}(V_n))^j)\right),$$

where $D = g(d)$ *and* $d = \sup\{x : g(x) \le D\}$ *is the mode. On the other hand, under the conditions* $g_{w_{k:n}}(x) \le 1$ *and* $\lambda > 1$, *we obtain*

$$WT_\lambda^1(X_{k:n}) \le \frac{1}{\lambda-1}\left(\beta_1-\alpha_1 - 1 - \sum_{j=0}^{\Delta 1_\lambda}\binom{\lambda}{j}\frac{(-1)^j Beta(kj-j+1, nj-kj+1)}{(Beta(i,n-i+1)\mathcal{E}(G^{-1}(U_{k:n})))^j}\right.$$
$$\left.\times\, D^{j-1}\mathcal{E}((G^{-1}(V_n))^j)\right).$$

Proof. From (3.1) and under the conditions $g_{w_{k:n}}(x) \le 1$ and $\lambda < 1$, we have

$$\begin{aligned}
WT_\lambda^1(X_{k:n}) &= \frac{1}{\lambda-1}\left(\int_{\alpha_1}^{\beta_1}(1-g_{w_{k:n}}(x))dx - \int_{\alpha_1}^{\beta_1}(1-g_{w_{k:n}}(x))^\lambda dx\right)\\
&= \frac{1}{\lambda-1}\left(\beta_1-\alpha_1-1-\int_{\alpha_1}^{\beta_1}\left(1-\frac{x g_{k:n}(x)}{\mathcal{E}(X_{k:n})}\right)^\lambda dx\right)\\
&= \frac{1}{\lambda-1}\left(\beta_1-\alpha_1-1-\int_{\alpha_1}^{\beta_1}\left(1-\frac{xG^{k-1}(x)\overline{G}^{n-k}(x)g(x)}{Beta(k,n-k+1)\mathcal{E}(X_{k:n})}\right)^\lambda \frac{g(x)}{g(x)}dx\right).
\end{aligned}$$

Put $G(x) = u$, then

$$WT_\lambda^1(X_{k:n}) = \frac{1}{\lambda-1}\left(\beta_1 - \alpha_1 - 1 - \int_0^1 \left(1 - \frac{G^{-1}(u)u^{k-1}(1-u)^{n-k}g(G^{-1}(u))}{Beta(k,n-k+1)\mathcal{E}(X_{k:n})}\right)^\lambda \times \frac{du}{g(G^{-1}(u))}\right).$$

If $g(x) \leq D$ and $\lambda < 1$ (i.e. $\frac{1}{\lambda-1} < 0$), we obtain

$$\begin{aligned} WT_\lambda^1(X_{k:n}) &\geq \frac{1}{\lambda-1}\left(\beta_1 - \alpha_1 - 1 - \int_0^1 \left(1 - \frac{G^{-1}(u)u^{k-1}(1-u)^{n-k}D}{Beta(k,n-k+1)\mathcal{E}(X_{k:n})}\right)^\lambda \frac{du}{D}\right) \\ &= \frac{1}{\lambda-1}\left(\beta_1 - \alpha_1 - 1 - \sum_{j=0}^{\Delta 1_\lambda} \binom{\lambda}{j} \frac{(-1)^j}{(Beta(k,n-k+1)\mathcal{E}(G^{-1}(U_{k:n})))^j}\right. \\ &\quad \left. \times D^{j-1} \int_0^1 (G^{-1}(u))^j u^{kj-j}(1-u)^{nj-kj} du\right) \\ &= \frac{1}{\lambda-1}(\beta_1 - \alpha_1 - 1 - \sum_{j=0}^{\Delta 1_\lambda} \binom{\lambda}{j} \frac{(-1)^j Beta(kj-j+1, nj-kj+1)}{(Beta(k,n-k+1)\mathcal{E}(G^{-1}(U_{k:n})))^j} D^{j-1}\mathcal{E}((G^{-1}(V_n))^j)). \end{aligned}$$

The proof of the second part of the theorem, i.e., when $\lambda > 1$, is obvious. This proves the theorem. □

Theorem 3.4. *Consider the n i.i.d N-RVs $X_1, X_2, ..., X_n$ with the support (α_1,β_1), $0 < \alpha_1 < \beta_1 < \infty$, following PDF g and CDF F. Then, from (3.3) and under the conditions $g_{k:n}(x) \leq 1$ and $\lambda < 1$, we have*

$$\begin{aligned} WT_\lambda^2(X_{k:n}) \geq \frac{1}{\lambda-1}&\left(\frac{\beta_1^2 - \alpha_1^2}{2\mathcal{E}_{k:n}(X)} - 1 - \sum_{j=0}^{\Delta 2_\lambda} \binom{\lambda}{j} \frac{(-1)^j}{\mathcal{E}(G^{-1}(U_{k:n}))(Beta(k,n-k+1))^j}\right. \\ &\times D^{j-1}\mathcal{E}((G^{-1}(V_n)))), \end{aligned}$$

where $D = g(d)$ and $d = \sup\{x : g(x) \leq D\}$ is the mode. On the other hand, under the conditions $g_{k:n}(x) \leq 1$ and $\lambda > 1$, we obtain

$$\begin{aligned} WT_\lambda^1(X_{k:n}) \leq \frac{1}{\lambda-1}&\left(\frac{\beta_1^2 - \alpha_1^2}{2\mathcal{E}_{k:n}(X)} - 1 - \sum_{j=0}^{\Delta 2_\lambda} \binom{\lambda}{j} \frac{(-1)^j}{\mathcal{E}(G^{-1}(U_{k:n}))(Beta(k,n-k+1))^j}\right. \\ &\times D^{j-1}\mathcal{E}((G^{-1}(V_n)))). \end{aligned}$$

Proof. The proof proceeds in a manner analogous to Theorem 3.3. □

4. Bivariate Tsallis extropy and its weighted version

In this section, the bivariate Tsallis extropy and its weighted counterpart will be introduced. Consider independent N-RVs X and Y with the support (α_1,β_1) and (α_2,β_2), where $0 < \alpha_1 < \beta_1 < \infty$, $0 < \alpha_2 < \beta_2 < \infty$, follow PDF g_X and g_Y, respectively, and joint PDF $g_{X,Y}$. Then, then bivariate Tsallis extropy of (X, Y) is provided by

$$\begin{aligned} Tx_\lambda(X,Y) &= \frac{1}{\lambda-1}\left(\int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}(1-g_{X,Y}(x,y))dxdy - \int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}(1-g_{X,Y}(x,y))^\lambda dxdy\right) \\ &= \frac{1}{\lambda-1}\left((\beta_1-\alpha_1)(\beta_2-\alpha_2) - 1 - \int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}(1-g_{X,Y}(x,y))^\lambda dxdy\right), \end{aligned} \tag{4.1}$$

where the conditions on λ can be summarized by

$$\Lambda = \begin{cases} \lambda \neq 1, \lambda > 0 \ when \ g_{X,Y}(x,y) \ \leq 1, \\ \lambda \in \mathbb{Z}^{+} \backslash \{1\} \ when \ g_{X,Y}(x,y) > 1. \end{cases} \tag{4.2}$$

Remark 4.1. *We can obtain the multi-Tsallis extropy of the N-dimensional vector in the same manner given in (4.1), as follows*

$$\begin{aligned} Tx_{\lambda}(X_1, X_2, ..., X_N) = & \frac{1}{\lambda - 1}\Bigg(\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}...\int_{\alpha_N}^{\beta_N}(1 - g_{X_1,X_2,...,X_N}(x_1, x_2, ..., x_N))dx_N...dx_2dx_1 \\ & - \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}...\int_{\alpha_N}^{\beta_N}(1 - g_{X_1,X_2,...,X_N}(x_1, x_2, ..., x_N))^{\lambda}dx_N...dx_2dx_1\Bigg) \\ = & \frac{1}{\lambda - 1}\big((\beta_1 - \alpha_1)(\beta_2 - \alpha_2)...(\beta_N - \alpha_N) - 1 \\ & - \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}...\int_{\alpha_N}^{\beta_N}(1 - g_{X_1,X_2,...,X_N}(x_1, x_2, ..., x_N))^{\lambda}dx_N...dx_2dx_1\Big), \end{aligned} \tag{4.3}$$

where the conditions on λ can be summarized by

$$\Lambda = \begin{cases} \lambda \neq 1, \lambda > 0 \ when \ g_{X_1,X_2,...,X_N}(x_1, x_2, ..., x_N) \ \leq 1, \\ \lambda \in \mathbb{Z}^{+} \backslash \{1\} \ when \ g_{X_1,X_2,...,X_N}(x_1, x_2, ..., x_N) > 1. \end{cases} \tag{4.4}$$

Noting that $\int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1} xyg_{X,Y}(x,y), dx, dy = \mathcal{E}(XY)$, and similarly to Section 2, the bivariate weighted Tsallis extropy can be considered from two perspectives as follows:

1) According to the extropy perspective, the bivariate weighted Tsallis extropy is expressed as

$$\begin{aligned} WT_{\lambda}^{1}(X, Y) = & \frac{1}{\lambda - 1}\left(\int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}\left(1 - \frac{xyg_{X,Y}(x,y)}{\mathcal{E}(X\ Y)}\right)dxdy - \int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}\left(1 - \frac{xyg_{X,Y}(x,y)}{\mathcal{E}(X\ Y)}\right)^{\lambda}dxdy\right) \\ = & \frac{1}{\lambda - 1}\Bigg(\int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}dxdy - \int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}\frac{xyg_{X,Y}(x,y)}{\mathcal{E}(X\ Y)}dxdy \\ & - \int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}\left(1 - \frac{xyg_{X,Y}(x,y)}{\mathcal{E}(X\ Y)}\right)^{\lambda}dxdy\Bigg) \\ = & \frac{1}{\lambda - 1}\left((\beta_1 - \alpha_1)(\beta_2 - \alpha_2) - 1 - \int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}\left(1 - \frac{xyg_{X,Y}(x,y)}{\mathcal{E}(X\ Y)}\right)^{\lambda}dxdy\right), \end{aligned} \tag{4.5}$$

where the conditions on λ can be summarized by

$$\Lambda_1 = \begin{cases} \lambda \neq 1, \lambda > 0 \ when \ g_{X,Y}(x,y) \ \leq 1, \\ \lambda \in \mathbb{Z}^{+} \backslash \{1\} \ when \ g_{X,Y}(x,y) > 1. \end{cases} \tag{4.6}$$

2) According to the weighted Tsallis entropy (weighted generalized entropy) perspective, the weighted Tsallis extropy is expressed as

$$\begin{aligned}
WT_\lambda^2(X,Y) &= \frac{1}{\lambda-1}\left(\int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}\frac{xy(1-g_{X,Y}(x,y))}{\mathcal{E}(X\ Y)}dxdy-\int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}\frac{xy(1-g_{X,Y}(x,y))^\lambda}{\mathcal{E}(X\ Y)}dxdy\right)\\
&= \frac{1}{\lambda-1}\left(\int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}xydxdy-\int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}\frac{xyg_{X,Y}(x,y)}{\mathcal{E}(X\ Y)}dxdy\right.\\
&\quad\left.-\int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}\frac{xy(1-g_{X,Y}(x,y))^\lambda}{\mathcal{E}(X\ Y)}dxdy\right)\\
&= \frac{1}{\lambda-1}\left(\frac{(\beta_1^2-\alpha_1^2)(\beta_2^2-\alpha_2^2)}{4\mathcal{E}(X\ Y)}-1-\int_{\alpha_2}^{\beta_2}\int_{\alpha_1}^{\beta_1}\frac{xy(1-g_{X,Y}(x,y))^\lambda}{\mathcal{E}(X\ Y)}dxdy\right),
\end{aligned} \tag{4.7}$$

where the conditions on λ are given in (4.2).

Example 4.1. *Let us examine (X, Y) as a bivariate beta RV and a joint PDF*

$$g_{X,Y}(x,y)=\frac{1}{Beta(a,b,c)}x^{a-1}(y-x)^{b-1}(1-y)^{c-1}, 0<x<y<1, a,b,c>0,$$

where the bivariate beta function $Beta(a,b,c)=\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$. Then, from (4.1) and (4.5), the bivariate Tsallis extropy and its weighted version are given, respectively, by

$$\begin{aligned}
Tx_\lambda(X,Y) &= \frac{1}{\lambda-1}\left(-\int\int_{0<x<y<1}(1-g_{X,Y}(x,y))^\lambda dxdy\right)\\
&= \frac{-1}{\lambda-1}\sum_{k=0}^{\Delta 2_\lambda}\binom{\lambda}{k}\frac{(-1)^k}{Beta^k(a,b,c)}\int\int_{0<x<y<1}g_{X,Y}^k(x,y)dxdy\\
&= \frac{-1}{\lambda-1}\sum_{k=0}^{\Delta 2_\lambda}\binom{\lambda}{k}\frac{(-1)^k}{Beta^k(a,b,c)}\int\int_{0<x<y<1}x^{ka-k}(y-x)^{kb-k}(1-y)^{kc-k}dxdy\\
&= \frac{-1}{\lambda-1}\sum_{k=0}^{\Delta 2_\lambda}\binom{\lambda}{k}\frac{(-1)^k Beta(ka-k+1,kb-k+1,kc-k+1)}{Beta^k(a,b,c)}.
\end{aligned}$$

$$
\begin{aligned}
WT_\lambda^2(X,Y) &= \frac{1}{\lambda-1}\left(-\int\int_{0<x<y<1}\frac{xy(1-g_{X,Y}(x,y))^\lambda}{\mathcal{E}(X\ Y)}dxdy\right)\\
&= \frac{-1}{\lambda-1}\sum_{k=0}^{\Delta 2_\lambda}\binom{\lambda}{k}\frac{(-1)^k}{\mathcal{E}(X\ Y)Beta^k(a,b,c)}\int\int_{0<x<y<1}xyg_{X,Y}^k(x,y)dxdy\\
&= \frac{-1}{\lambda-1}\sum_{k=0}^{\Delta 2_\lambda}\binom{\lambda}{k}\frac{(-1)^k}{\mathcal{E}(X\ Y)Beta^k(a,b,c)}\int\int_{0<x<y<1}x^{ka-k+1}y(y-x)^{kb-k}\times(1-y)^{kc-k}dxdy\\
&= \frac{-1}{\lambda-1}\sum_{k=0}^{\Delta 2_\lambda}\binom{\lambda}{k}\frac{(-1)^k}{\mathcal{E}(X\ Y)Beta^k(a,b,c)}\int\int_{0<x<y<1}x^{ka-k+1}(y-x+x)(y-x)^{kb-k}\\
&\quad\times(1-y)^{kc-k}dxdy\\
&= \frac{-1}{\lambda-1}\sum_{k=0}^{\Delta 2_\lambda}\binom{\lambda}{k}\frac{(-1)^k}{\mathcal{E}(X\ Y)Beta^k(a,b,c)}\left[\int\int_{0<x<y<1}x^{ka-k+1}(y-x)^{kb-k+1}(1-y)^{kc-k}dxdy\right.\\
&\quad\left.+\int\int_{0<x<y<1}x^{ka-k+2}(y-x)^{kb-k}(1-y)^{kc-k}dxdy\right]\\
&= \frac{-1}{\lambda-1}\sum_{k=0}^{\Delta 2_\lambda}\binom{\lambda}{k}\frac{(-1)^k}{Beta^k(a,b,c)}\left[Beta(ka-k+2,kb-k+2,kc-k+1)\right.\\
&\quad\left.+Beta(ka-k+3,kb-k+1,kc-k+1)\right].
\end{aligned}
$$

Where $\Delta 1_\lambda$ and $\Delta 2_\lambda$ are specified in (2.5) and (2.6), respectively. Figure 4 shows the plots of the bivariate Tsallis extropy and its weights given in (4.1), (4.5), and (4.7) for the bivariate beta distribution with parameters $a = 2$, $b = 3$, and $c = 4$.

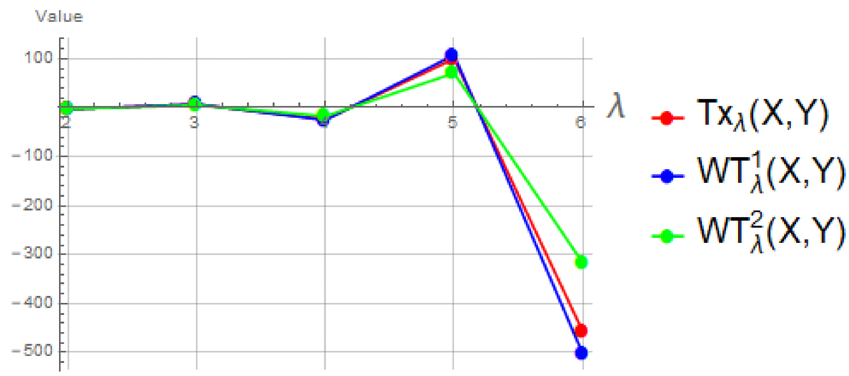


Figure 4. The Tsallis extropy and its weights of the bivariate beta distribution with $a = 2$, $b = 3$, and $c = 4$.

5. Applications

In this section, we will apply the two measures of weighted Tsallis extropy to non-parametric estimation using a kernel function. Additionally, we will demonstrate their application in a pattern recognition context.

5.1. Non-parametric estimation

The estimation with a non-parametric procedure of the weighted measures of information has been presented in many references, for example, Chakraborty et al. [29], which utilized kernel and non-kernel procedures. In this section, the non-parametric estimators for the weighted Tsallis extropy are proposed. To achieve this, we applied the kernel density estimation for the PDF as defined in [30] and [31]. Therefore, from (2.1) and (2.4), the non-parametric kernel estimation for $WT^1_\lambda(X)$ and $WT^2_\lambda(X)$ are given, respectively, by

$$WT^1_{\lambda,n}(R) = \frac{1}{\lambda - 1}\left(\beta - \alpha - 1 - \int_\alpha^\beta \left(1 - \frac{rg_n(r)}{\mathcal{E}_n(R)}\right)^\lambda dr\right), \tag{5.1}$$

$$WT^2_{\lambda,n}(R) = \frac{1}{\lambda - 1}\left(\frac{\beta^2 - \alpha^2}{2\mathcal{E}_n(R)} - 1 - \int_\alpha^\beta \frac{r(1 - g_n(r))^\lambda}{\mathcal{E}_n(R)} dr\right), \tag{5.2}$$

where $\mathcal{E}_n(R) = \int_\alpha^\beta rg_n(r)dr$, $g_n(r) = \frac{1}{nh_n}\sum_{i=1}^n k\left(\frac{r-R_i}{h_n}\right)$, $k(r)$ is the kernel function, and h_n is the bandwidth, where $n\ h_n \longrightarrow \infty$ when $n \longrightarrow \infty$.

In our study, we choose the Gaussian kernel $k(r) = \frac{1}{\sqrt{2\pi}}e^{-\frac{r^2}{2}}$ with examining the performance of the kernel estimator for the weighted Tsallis extropy defined in (5.1) and (5.2), involves evaluating it with a power distribution characterized by parameters $\gamma = 3$ and $\delta = 1$ with theoretical values $WT^1_\lambda(X) = \frac{5}{9}$ and $WT^2_\lambda(X) = \frac{2}{3}$. Moreover, we generated the sample n from the autoregressive model of order 1 in the time series (AR(1)) with a correlation coefficient of 0.5; see Figure 5.

Table 1 and Figure 6 show the weighted Tsallis extropy estimators $WT^1_{\lambda,n}(R)$ and $WT^2_{\lambda,n}(R)$ with $\lambda = 2$. Therefore, we observe that as the sample size n increases, the estimators tend towards the theoretical values.

Table 1. Weighted Tsallis extropy estimators $WT^1_{\lambda,n}(R)$ and $WT^2_{\lambda,n}(R)$ with $\lambda = 2$.

n	$WT^1_{\lambda,n}(R)$				$WT^2_{\lambda,n}(R)$			
	$h_n = 0.5$	$h_n = 1$	$h_n = 1.5$	$h_n = 2$	$h_n = 0.5$	$h_n = 1$	$h_n = 1.5$	$h_n = 2$
5	0.208199	0.62635	0.603008	0.586261	0.584359	0.803122	0.828114	0.848743
10	0.590792	0.615449	0.599275	0.585895	0.701696	0.784638	0.8169	0.843064
15	0.575866	0.607945	0.595744	0.583639	0.671134	0.764188	0.806439	0.837324
20	0.568561	0.610035	0.599936	0.587147	0.665514	0.76413	0.807878	0.838456
25	0.548026	0.606418	0.598457	0.586151	0.640561	0.754436	0.803231	0.835926
30	0.540067	0.606657	0.600135	0.587579	0.632793	0.753208	0.803392	0.836166

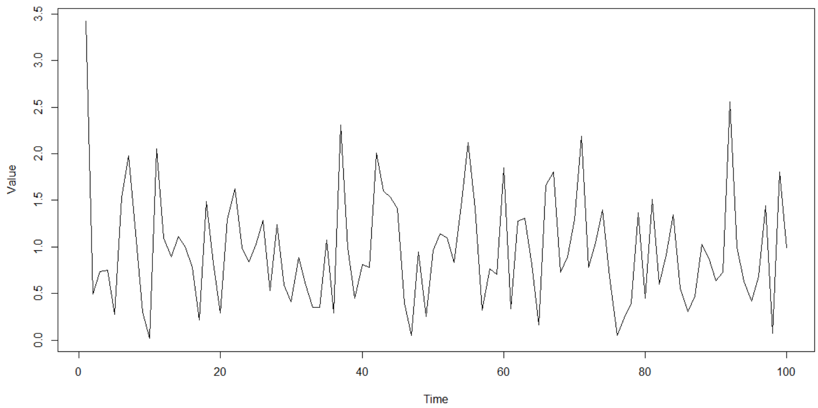


Figure 5. Generated AR(1) sample with correlation coefficient 0.5.

(a)

(b)

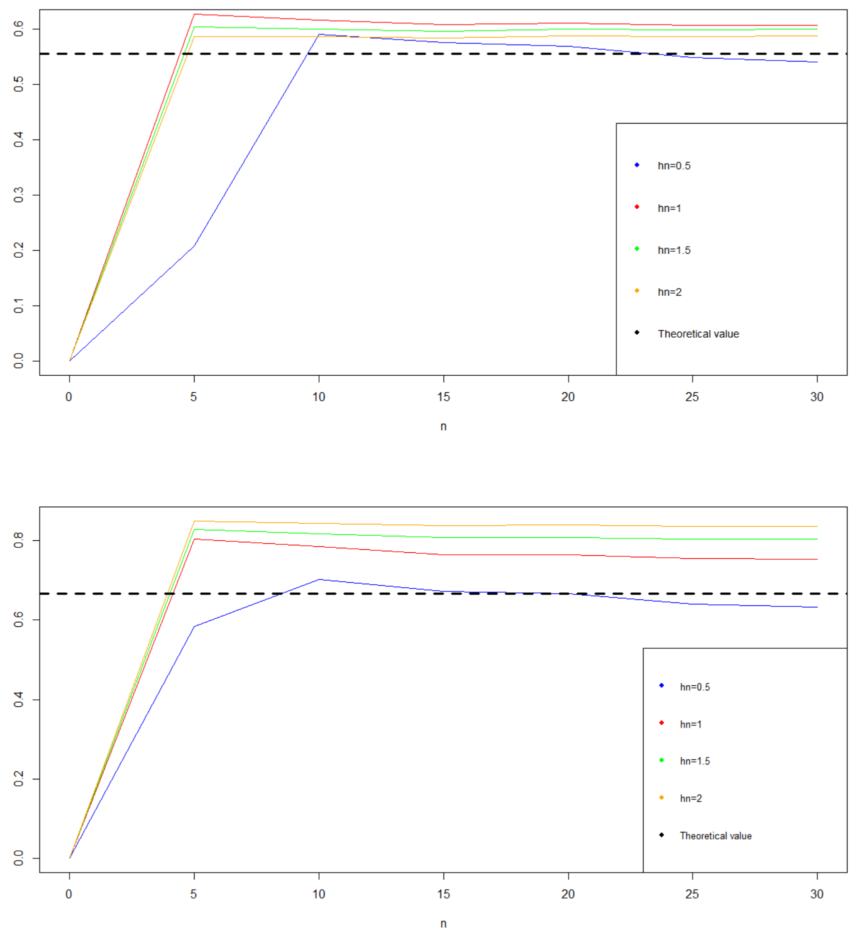


Figure 6. Weighted Tsallis extropy estimators with $\lambda = 2$: (a) $WT^1_{\lambda,n}(R)$, and (b) $WT^2_{\lambda,n}(R)$.

5.2. *Real data set*

In this subsection, we explore the practical applicability of the suggested estimator for weighted Tsallis extropy in real-life scenarios. The data set provides information concerning cancer-related fatalities in the European Union countries; see [32]. Figures 7 and 8 show the data and analysis of European Union countries' deaths from cancer. We choose the Gaussian kernel $k(r) = \frac{1}{\sqrt{2\pi}}e^{-\frac{r^2}{2}}$ with examining the performance of the kernel estimator for the weighted Tsallis extropy defined in (5.1) and (5.2), which involves evaluating it with a power distribution characterized by parameters $\gamma = 300$ and $\delta = 1$ with theoretical values $WT_\lambda^1(X) = 0.9955$ and $WT_\lambda^2(X) = 0.9966$. Figure 9 shows the weighted Tsallis extropy estimators $WT_{\lambda,n}^1(R)$ and $WT_{\lambda,n}^2(R)$ with $\lambda = 2$. Therefore, we observe that as the bandwidth h_n increases, the estimators tend towards the theoretical values. In addition, we can see how the two estimators are close to each other.

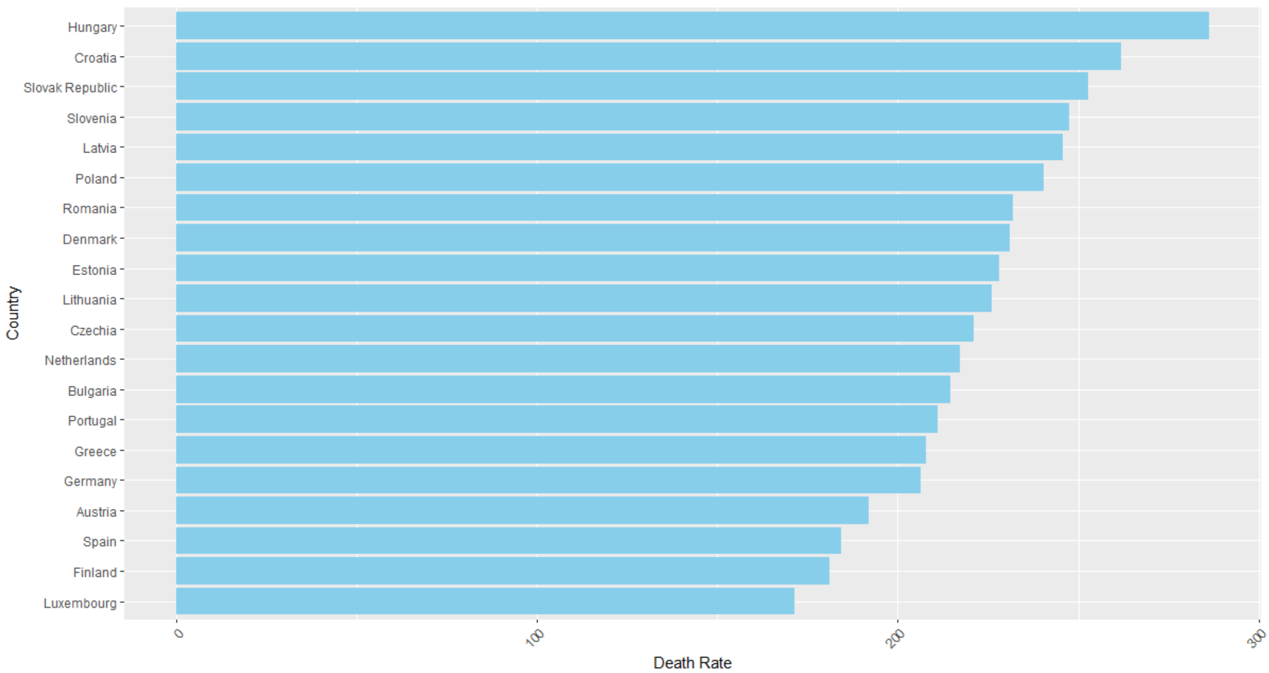


Figure 7. Fatalities resulting from cancer across European Union nations.

Figure 8. Analysis of the Fatalities resulting from cancer across European Union nations.

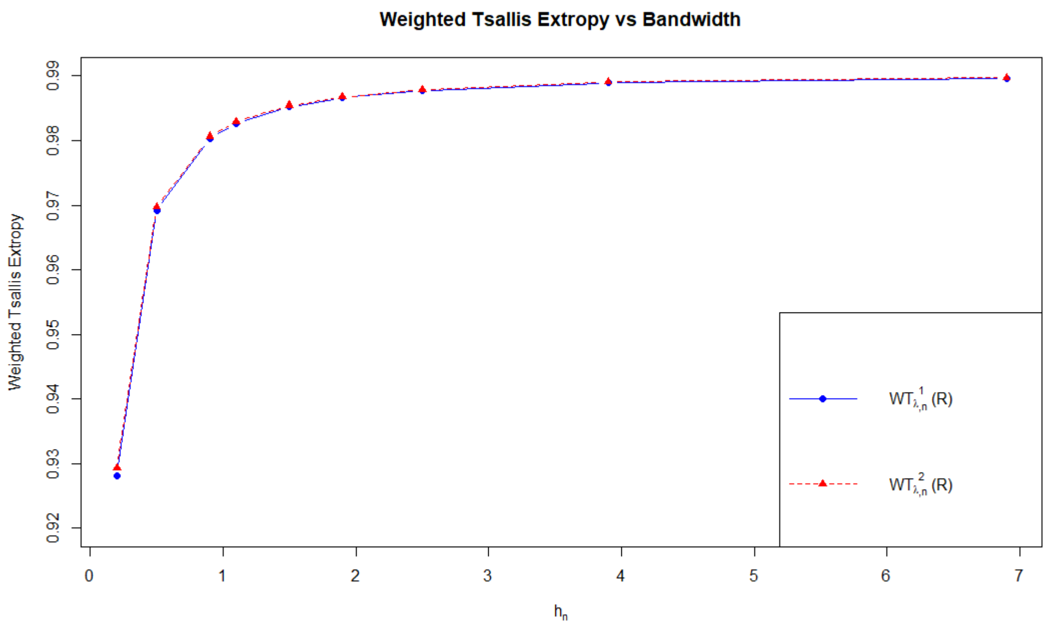


Figure 9. Weighted Tsallis extropy estimators of the deaths from cancer in countries within the European Union, with $\lambda = 2$.

5.3. Classification issue through the identification of patterns

In this subsection, we will use the discrete form of weighted Tsallis extropy and compare it with the Tsallis extropy given in (1.4) to address the classification problem using pattern recognition. In the discrete case of an RV X with a probability vector $(p_1, \ldots, p_N)$, support S of cardinality N, and weight w_p, we can define the two perspectives of weighted Tsallis extropy as follows:

$$DWT_\lambda^1(P) = \frac{1}{\lambda - 1}\left(\sum_{j=1}^{N}(1 - w_p p_j) - \sum_{j=1}^{N}(1 - w_p p_j)^\lambda\right), \tag{5.3}$$

$$DWT_\lambda^2(P) = \frac{1}{\lambda - 1}\left(\sum_{j=1}^{N} w_p(1 - p_j) - \sum_{j=1}^{N} w_p(1 - p_j)^\lambda\right), \tag{5.4}$$

where $\lambda > 0$, and $\lambda \neq 1$.

We tackle a classification problem using weighted Tsallis extropy models. The Iris dataset, as reported in [33], is the subject of our investigation. Furthermore, we compare our findings to the methods for the Tsallis extropy put forward by Balakrishnan et al. [13]. Three flower varieties—Iris Setosa, Iris Versicolor, and Iris Virginica—need to be categorized. A dataset consisting of 150 samples is employed, with 50 examples distributed equally across the categories. Each flower's sepal length (B_1), sepal width (B_2), petal length (B_3), and petal width (B_4) are measured in centimeters. As indicated in Table 2, 40 specimens of each iris species are chosen, and a sample with the greatest and lowest values is found to establish an interval number model. An unidentified test sample is represented by each entry in the dataset. It is believed that $(6.5, 3, 4.9, 1.8)$, which originates from the Virginica species, is the selected singleton sample data.

Table 2. (i) The statistical model's interval numbers; (ii) Kang's method-based probability distributions.

(i) Item	B_1	B_2	B_3	B_4
Setosa	[4.4,5.8]	[2.3,4.4]	[1.0,1.9]	[0.1,0.6]
Versicolour	[4.9,7.0]	[2.0,3.4]	[3.0,5.1]	[1.0,1.7]
Virginica	[4.9,7.9]	[2.2,3.8]	[4.5,6.9]	[1.4,2.5]
(ii) Item	B_1	B_2	B_3	B_4
P(Setosa)	0.270571	0.27477	0.145978	0.156327
P(Versicolour)	0.419584	0.351607	0.429446	0.373669
P(Virginica)	0.309845	0.373622	0.424576	0.470003

Then, using the method of Kang et al. [34], which is based on the similarity in interval numbers, we produce four distinct probability distributions. Furthermore, the equation defines how similar they are.

$$\tau(Y_1, Y_2) = \frac{1}{1 + M\, Iv(\Omega_1, \Omega_2)}, \tag{5.5}$$

where τ is the backing coefficient. The gap between the intervals $\Omega_1 = [r_1, r_2]$ was calculated in one case with M set to 5.The formula $\Omega_2 = [t_1, t_2]$ is acquired by

$$Iv(\Omega_1, \Omega_2) = \left[\left(\frac{r_1 + r_2}{2}\right) - \left(\frac{t_1 + t_2}{2}\right)\right]^2 + \frac{1}{3}\left[\left(\frac{r_1 - r_2}{2}\right)^2 + \left(\frac{t_1 - t_2}{2}\right)^2\right]. \tag{5.6}$$

For interval Ω_2, we use individual values from the selected sample; for interval Ω_1, we use the intervals given in Table 2(i) to create probability distributions. Each of the four detected qualities produces three resemblance values, which are further normative to provide a distribution of probabilities, as seen in Table 2(ii). Our measures for these probability distributions are then evaluated throughout $\lambda = 4$, as shown in Table 3(i). Because the exponential function is monotonic, we choose $\Xi(y) = e^{-y}$ as the baseline weight function, from which we normalize the weights. For instance, the process yields Table 3(ii) for the petal length in relation to the Tsallis extropy and weighted Tsallis extropy in (1.4), (5.3), and (5.4) correspondingly (notice that we take the weight $w_p = \frac{1}{3}$) as:

$$\Xi(B_3) = \frac{e^{-DTx_\lambda(B_4)}}{e^{-DTx_\lambda(B_1)} + e^{-DTx_\lambda(B_2)} + e^{-DTx_\lambda(B_3)} + e^{-DTx_\lambda(B_4)}},$$
$$\Xi(B_3) = \frac{e^{-DWT^1_\lambda(B_4)}}{e^{-DWT^1_\lambda(B_1)} + e^{-DWT^1_\lambda(B_2)} + e^{-DWT^1_\lambda(B_3)} + e^{-DWT^1_\lambda(B_4)}},$$
$$\Xi(B_3) = \frac{e^{-DWT^2_\lambda(B_4)}}{e^{-DWT^2_\lambda(B_1)} + e^{-DWT^2_\lambda(B_2)} + e^{-DWT^2_\lambda(B_3)} + e^{-DWT^2_\lambda(B_4)}}.$$

Table 3. (i) Tsallis extropy and weighted Tsallis extropy; (ii) the weights with $\lambda = 4$.

(i) Item	B_1	B_2	B_3	B_4
$DTx_\lambda(P)$	0.458847	0.464228	0.417479	0.42019
$DWT^1_\lambda(P)$	0.262516	0.263642	0.255125	0.255393
$DWT^2_\lambda(P)$	0.152949	0.154743	0.13916	0.140063
(ii) Item	$\Xi(B_1)$	$\Xi(B_2)$	$\Xi(B_3)$	$Q(B_4)$
$DTx_\lambda(P)$	0.245322	0.244005	0.255683	0.254991
$DWT^1_\lambda(P)$	0.249163	0.248882	0.251011	0.250944
$DWT^1_\lambda(P)$	0.248443	0.247998	0.251893	0.251665

Consequently, the final probability distribution for $DTx_\lambda(P)$ is as follows when $\lambda = 4$ is chosen:

$$\mathbb{P}(Setosa) = 0.210608, \ \mathbb{P}(Versicolour) = 0.393811, \ \mathbb{P}(Virginica) = 0.395581,$$

and for $DWT^1_\lambda(P)$:

$$\mathbb{P}(Setosa) = 0.211673, \ \mathbb{P}(Versicolour) = 0.393619, \ \mathbb{P}(Virginica) = 0.394708,$$

and for $DWT^2_\lambda(P)$:

$$\mathbb{P}(Setosa) = 0.211477, \ \mathbb{P}(Versicolour) = 0.393655, \ \mathbb{P}(Virginica) = 0.394868,$$

The selected flower was then determined to belong to the class with the biggest probabilities, which is Iris Virginica. Consequently, in this instance, a precise conclusion was reached.

Using this approach, we analyzed all 150 samples across $\lambda = 4$, with fifty specimens from each species. Our findings indicate that, according to our evaluations, the method's overall recognition rates stay at 94.66%, as shown in Table 4. It is clear that when matched to the Tsallis extropy technique, our method performs similarly.

Table 4. The recognition rates of the Tsallis extropy and weighted Tsallis extropy methods.

Approach	Setosa	Versicolour	Virginica	Overall
$DTx_\lambda(P)$	100%	98%	86%	94.66%
$DWT^1_\lambda(P)$	100%	98%	86%	94.66%
$DWT^2_\lambda(P)$	100%	98%	86%	94.66%

6. Conclusions and future work

In this study, we have built upon the concepts of extropy, as the dual of entropy, and weighted Tsallis entropy to derive two perspectives of weighted Tsallis extropy. We have provided several examples to illustrate the principles introduced in this work. Some examples show a comparison between Tsallis extropy and weighted Tsallis extropy. Key features of the first and second perspectives of weighted Tsallis extropy, such as non-negativity, bounds, and stochastic orders, have been discussed. Additionally, we have explored the relationship between weighted Tsallis extropy and weighted extropy. We proposed an alternative representation of the models in terms of the hazard rate function. Furthermore, we investigated the maximum weighted Tsallis extropy for the first perspective, $WT^1_\lambda(X)$. Our models were extended to order statistics, and their bounds were analyzed in terms of the beta distribution. We also introduced bivariate Tsallis extropy and its weighted version. Moreover, we discussed non-parametric estimators for the new measures using simulations and real data to demonstrate their performance and effectiveness. Finally, we performed a comparison between Tsallis extropy and weighted Tsallis extropy in the context of classification problems using pattern recognition.

There are practical circumstances when this measure is also connected to the past to assess the uncertainty of the previous lifetime of an item that failed at a certain period, as the weighted Tsallis extropy cannot be applied to an RV that has already survived for a while. Therefore, we can extend this study to the residual and past cumulative cases of weighted Tsallis extropy. Additionally, it is possible to build the weighted Tsallis extropy-based goodness of fitting test and compare its results with those of other well-known tests. The weighted Tsallis extropy for the order statistics has been covered. However, more research is needed on its application in identifying other ordered variables, such as the record values, which could lead to additional properties. In addition, exploring the properties of the proposed measure in the concomitants of order statistics and record values setup remains another area of interest. Moreover, in the continuous case, we can derive the fractional Tsallis extropy measure proposed by Buono et al. [14] and discuss its weighted version. Research on the aforementioned subjects is still in progress; the findings will be released in a different publication.

The quality and accuracy of weighted Tsallis extropy depend significantly on the selection of weights, with poorly defined or subjective weights potentially leading to biased or unreliable results. Additionally, calculating weighted Tsallis extropy, particularly in higher dimensions or bivariate extensions, can be computationally intensive due to complex integrals and the need for precise kernel-based or numerical methods. Convergence to theoretical limits may also be slow or uncertain, especially with small sample sizes, sparse data, or extreme weights. Overcoming these challenges requires careful data preparation, robust methodologies for weight selection, computational optimization, and domain-specific adjustments to enhance the utility and applicability of weighted Tsallis extropy and its extensions.

Author contributions

Ramy Abdelhamid Aldallal, Haroon M. Barakat and Mohamed Said Mohamed: Methodology, conceptualization, investigation, software, resources, writing-original draft, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declares that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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